

⁹ Since (3.1) is homogeneous of degree zero in v_m, v^m , and a^m in (3.2) can be taken to be dy^m/dy^4 , $d^2y^m/(dy^4)^2$, resp. Then $v^4 \equiv 1, a^4 \equiv 0$.

¹⁰ On the nullspheres ($\lambda = 0$) one gets $R^p = 0$, where $R^p = 0$ —i.e., just the Coulomb infinity.

¹¹ E.g., λ acts like a sort of *gauge*, or numerical length assigned to some standard physical rod by the observer at x^m . This rod can be assigned any positive value, but not zero.

¹² The left member is the covariant acceleration $(d^2x^\alpha/d\theta^2) + \left\{ \begin{smallmatrix} \alpha \\ \beta\gamma \end{smallmatrix} \right\} (dx^\beta/d\theta)(dx^\gamma/d\theta)$.

¹³ $ds > 0$ assumed, whence $\text{Sgn } \lambda_0 = \text{Sgn } d\theta$.

¹⁴ The identity $dx^4/ds = \sqrt{\lambda^2 + (\lambda_0/\lambda)^2}/\sqrt{1 - v^2/c^2}$ following from (2.2) has been used.

MULTIPLICITY THEORY FOR OPERATOR ALGEBRAS

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Communicated by John von Neumann, December 3, 1954

1. *Introduction.*—We shall outline, in this note, a multiplicity or unitary-invariants theory for adjoint-preserving homomorphisms of self-adjoint algebras of operators closed in the uniform operator topology (briefly, “representations of C*-algebras”). A detailed account of this theory containing a historical survey together with complete bibliographical references will be published elsewhere. We take the liberty of omitting from this note all bibliographical references.

In its general aspect, the central problem in the study of operator algebras is that of associating with each operator algebra a set of entities (invariants) which completely determine it, i.e., such that two operator algebras are structurally the same if and only if they have identical sets of invariants. The province in which one seeks invariants for any structure is indicated by experience with the structure and the special theories developed for special cases of the structure. Again, experience and applicability to the special cases usually indicates when a set of invariants for a structure is “the correct set” in the sense of relevance and simplicity. In the case of operator algebras, the central question just mentioned has two main parts. On the one hand, we may ask for invariants for the operator algebra as an algebraic object (algebraic invariants), and, on the other, we may ask for invariants for the operator algebra together with its action on some Hilbert space (spatial, multiplicity, or unitary invariants). In an earlier paper we presented what we felt were “the correct algebraic invariants” for the general C*-algebra (and noted there that, of course, one would insist on more detailed invariants for special classes of C*-algebras such as the factors of Type II₁). In this note we give the spatial invariants for a C*-algebra.

Some remarks concerning the relation of these results to previous work in multiplicity theory are in order. The finite-dimensional C*-algebras are completely described by the Wedderburn structure theory. The first infinite-dimensional results were those of Hahn-Hellinger, describing when two self-adjoint operators are unitarily equivalent. Later technical improvements in their description made it possible to apply their theory directly to commutative C*-algebras. This last theory contains that of the single self-adjoint operator. The next step in the commutative theory was the study by Nakano-Segal of Abelian rings of operators (a

ring of operators is a C^* -algebra closed in the weak operator topology). The class of Abelian rings of operators is a subclass of the class of commutative C^* -algebras and admits a very precise set of multiplicity invariants. At first sight, it might seem that the theory of Abelian rings of operators contains that of the single operator, as does the general theory of commutative C^* -algebras, but this is not the case. Roughly speaking, the multiplicity theory for a single self-adjoint operator (and a commutative C^* -algebra) involves its spectrum together with an ideal of Borel sets in the spectrum, the null sets of some measure, while the multiplicity theory for an Abelian ring of operators involves the measure algebra of some measure (actually, an equivalence class of measures) on its spectrum, i.e., the Boolean algebra of measurable sets modulo the null sets. In the case of noncommutative rings of operators, Murray and von Neumann took the decisive step by investigating factors (those rings of operators whose centers consist of scalar multiples of the identity) and developing a multiplicity theory for all but the factors of Type III. Using techniques of Dixmier, which generalize those of Murray and von Neumann and apply to general rings of operators (with no portion of Type III), Dye and Griffin generalized the Murray-von Neumann multiplicity theory for factors to rings of operators (with no portion of Type III). Recently, Griffin developed the spatial theory for rings of Type III, and his results show that, in reality, this is the simplest case, as far as the spatial theory is concerned. Thus the spatial invariants for non-Abelian rings have been determined, carrying the multiplicity theory for Abelian rings over to general rings of operators. The theory outlined in this note completes self-adjoint multiplicity theory by listing the spatial invariants for noncommutative C^* -algebras. This is the noncommutative (but self-adjoint) extension of the classical spectral multiplicity theory for a single self-adjoint operator.

The invariants we set down are of the same general character as those obtained in the commutative (or single-operator) case and yield the commutative theory immediately upon specialization. These invariants are the pure-state space of the C^* -algebra together with a descending chain of ideals of Borel sets in the pure-state space and a distinguished class of such ideals. There is at present no technique for determining which systems of the above type are associated with C^* -algebras, but this defect also appears in the classical spectral multiplicity theory of a single self-adjoint operator. In the single-operator situation the question is that of which ideals of Borel sets on a bounded subset of the real line can serve as the null sets of some Borel measure.

Since spatial equivalence of operator algebras implies, in particular, algebraic equivalence, a set of spatial invariants must include, in some form, the algebraic invariants. We actually set down spatial invariants for representations of abstract C^* -algebras as concrete C^* -algebras, something slightly more general (and useful) than locating invariants for concrete C^* -algebras (acting on a Hilbert space). By so doing, we gain the technical advantage of introducing the assumption of algebraic equivalence without the necessity of employing a bewildering array of useless algebraic isomorphisms. We have developed the theory without separability restrictions, although the results are new in the separable case, and although our natural inclination is to present the somewhat less involved picture permitted by the assumption of separability. Our choice was guided by the desire to have this theory serve as the final step in self-adjoint multiplicity theory.

2. *Basic Notions and Definitions.*—We recall that an abstract C^* -algebra is a Banach algebra over the complexes with a $*$ -operation which is a conjugate-linear, involutory antiautomorphism satisfying the condition $\|A^*A\| = \|A\|^2$, for each element A in the Banach algebra, with $\|\cdot\|$ the norm on the Banach algebra. Each such algebra has a faithful (one-one) representation as an algebra of operators on a Hilbert space, closed in the uniform (norm) topology (a concrete C^* -algebra). We shall assume, throughout, that our concrete and abstract C^* -algebras contain a unit element. The linear functionals which are real and nonnegative on the positive operators in the algebra and 1 at the identity are the states of the C^* -algebra. The states form a (compact) convex subset of the space of all continuous linear functionals, and the extreme points of this convex set are “the pure states” of the algebra. The closure of the set of pure states is a compact-Hausdorff space called “the pure-state space of the algebra.” The obvious map from the elements of the C^* -algebra to functions on the pure-state space is a linear, order-preserving isomorphism of the C^* -algebra with a linear subspace of the set of continuous functions on the pure-state space. (The topology on the pure-state space is the weakest in which the functions of this set are continuous.) We refer to this linear space of functions as “the representing function system of the algebra.”

DEFINITION 1. If \mathfrak{A} is a C^* -algebra and φ is a representation of \mathfrak{A} as an algebra \mathfrak{A}_0 of operators on the Hilbert space \mathcal{H} , we denote by \mathfrak{N}_φ the ideal of Borel subsets of X , the pure-state space of \mathfrak{A} , consisting of those Borel sets S which are null sets of the Borel measure induced on X by each positive extension to $C(X)$, the family of continuous functions on X , of $f\varphi\lambda$, for each countably additive (equivalently, strongly sequentially continuous; equivalently, strongly continuous) state of \mathfrak{A}_0^- , the weak closure of \mathfrak{A}_0 , where λ is the canonical isomorphism of \mathfrak{A} with its representing function system. We call the sets in \mathfrak{N}_φ “the permanent null sets of φ .”

In view of results of Dixmier, Dye, and Griffin, we could employ, in place of the strongly sequentially continuous states f of \mathfrak{A}_0^- , those states g of \mathfrak{A}_0^- arising from unit vectors x in \mathcal{H} ($g(A) = (Ax, x)$). The strongly sequentially continuous states are those strongly continuous on the unit sphere of \mathfrak{A}_0^- . A projection E (and its range) are said to be cyclic under a C^* -algebra \mathfrak{A} when there exists a vector x such that $[\mathfrak{A}x]$, the closure of the manifold consisting of the vectors Ax , with A in \mathfrak{A} , is the range of E . If this is the case, then E lies in \mathfrak{A}' , the set of operators which commute with \mathfrak{A} . We call this set “the commutant of \mathfrak{A} ” and employ the notation \mathfrak{A}' , throughout, for the commutant of \mathfrak{A} . We say that a projection E in a ring of operators \mathfrak{A} is σ -finite when each orthogonal family of projections in \mathfrak{A} contained in E has at most a countable number of members (we say that E is σ -finite relative to \mathfrak{A} when the ring to which we are referring is not clearly indicated by the context). The ring \mathfrak{A} is σ -finite when its unit element is σ -finite.

DEFINITION 2. A ring of operators \mathfrak{A} is said to have decomposability character a when a is the least cardinal number such that the unit element in \mathfrak{A} is the sum of a family of mutually orthogonal projections with cardinal a , each projection cyclic under \mathfrak{A}' . The ring \mathfrak{A} is said to be purely decomposable with character a when the ring $\mathfrak{A}P$ has decomposability character a for each central projection P in \mathfrak{A} .

The following lemma provides a decomposition of rings of operators which we shall employ later.

LEMMA 1. If \mathfrak{A} is a ring of operators acting on a Hilbert space of dimension a , then,

corresponding to each cardinal number b not exceeding a , there is a central projection P_b in \mathfrak{A} such that, for each central projection Q which is σ -finite relative to the center of \mathfrak{A} , either $QP_b = 0$ or $\mathfrak{A}QP_b$ has decomposability character b . If (Q_b) , $b \leq a$, is a set of central projections in \mathfrak{A} such that Q_b has the decomposability property of P_b and $\bigcup_{b \leq a} Q_b = I$, then $Q_b = P_b$, for each $b \leq a$.

We refer to the projection P_b constructed in the above lemma as "the central portion of \mathfrak{A} of character b ."

DEFINITION 3. If \mathfrak{A} is a C^* -algebra and φ is a representation of \mathfrak{A} as an algebra \mathfrak{A}_0 of operators on the Hilbert space \mathfrak{H} , then, for each unit vector x in \mathfrak{H} , we call the ideal, \mathfrak{N}_x , of Borel sets, S , in the pure-state space, X , of \mathfrak{A} such that S is a null set of the Borel measure induced on X by each positive extension of $f\varphi\lambda$, where λ is the canonical isomorphism of \mathfrak{A} with its representing function system and f is the state of \mathfrak{A}_0 defined by $f(A) = (Ax, x)$, for each A in \mathfrak{A}_0 , a null ideal of φ . If the projection in \mathfrak{A}' with range $[\mathfrak{N}_x]$ is a maximal cyclic projection in \mathfrak{A}' , we call \mathfrak{N}_x "a characteristic null ideal of φ ."

In addition to the decomposition of the lemma above, we will have use for a part of the usual type decomposition of rings of operators. We recall the fact that a ring of operators \mathfrak{A} contains a central projection P such that $\mathfrak{A}Q$ is infinite for each nonzero, central projection Q contained in P , and $\mathfrak{A}(I - P)$ is finite (i.e., Q is an infinite projection in \mathfrak{A} and $I - P$ is finite). A projection P in \mathfrak{A} , with these properties, is unique. We call P "the infinite central portion of \mathfrak{A} " and $I - P$ "the finite central portion."

DEFINITION 4. With the notation of Definition 3, if P is the infinite central portion of \mathfrak{A}_0^- , Q the finite central portion of \mathfrak{A}_0' , and we let R be PQ , then we call the family of characteristic null ideals of the representation φ_1 of \mathfrak{A} defined by $\varphi_1(A) = \varphi(A)R$ "the ideal class of φ ."

The following definitions are intended for that portion of the theory which involves finite rings with finite commutants. We could have accomplished their effect by employing Griffin's coupling operator appropriately. This reference, however, involves the use of Dixmier's generalization of the Murray-von Neumann trace, a device unnatural to the present theory, which requires, in this connection, nothing more than the elementary dimension theory of projections in a finite ring of operators.

DEFINITION 5. If E is a projection in a finite ring of operators, \mathfrak{A} , of Type II_1 or I_m , we call the infimum of all ratios, a/b , such that E is contained in a orthogonal, equivalent copies of a projection which has b orthogonal, equivalent copies, "the upper dimension of E ." We denote this number by " $u(E)$ " and define $u(0)$ to be 0. The infimum, $l(E)$, of the numbers $u(PE)$ as P ranges over the nonzero central projections in \mathfrak{A} is called "the lower dimension of E ." For an arbitrary finite ring \mathfrak{A} and projection E in \mathfrak{A} , we take as $u(E)$ the supremum of the numbers $u(PE)$ relative to $\mathfrak{A}P$, where P is the maximal central projection in \mathfrak{A} such that $\mathfrak{A}P$ is of Type II_1 or I_m , for some m , with $l(E)$ defined formally as above.

The above definition leads to a convenient, trace-free dimension theory for projections.

DEFINITION 6. If \mathfrak{A} is a finite ring of operators with a finite commutant \mathfrak{A}' and σ -finite center, we define "the upper coupling number of \mathfrak{A} , \mathfrak{A}' " to be $1/l(E')$, where E' is a maximal cyclic projection in \mathfrak{A}' , provided that $E' \neq I$, and to be $u(E)$, where E is a

maximal cyclic projection in \mathfrak{A} , if $E' = I$. In case \mathfrak{A} does not have a σ -finite center, we define the upper coupling number of \mathfrak{A} , \mathfrak{A}' to be the supremum of the upper coupling numbers of $\mathfrak{A}P$, $\mathfrak{A}'P$ as P ranges over the nonzero, σ -finite, central projections in \mathfrak{A} . We denote the upper coupling number of \mathfrak{A} , \mathfrak{A}' by " $(\mathfrak{A}, \mathfrak{A}')$."

LEMMA 2. If \mathfrak{A} is a finite ring of operators with finite commutant \mathfrak{A}' , then, corresponding to each positive real number r , there is a central projection P_r such that $(\mathfrak{A}Q, \mathfrak{A}'Q) \leq r$, for each nonzero, central projection Q contained in P_r . In addition, P_r is equal to the unit element in \mathfrak{A} for r not less than $(\mathfrak{A}, \mathfrak{A}')$.

We call P_r of the above lemma "the central portion of \mathfrak{A} , \mathfrak{A}' with upper coupling number r ."

3. *Principal Results.*—The following extension theorem is basic to the theory. The detailed account of this theory will contain a stronger result both with respect to the generality of maps considered and with regard to the systems being mapped. The stronger result will be more in the spirit of a measure-theoretic proposition involving a noncommutative measure space with a noncommutative range for the integral; for the present, however, the simpler statement will suffice.

THEOREM 1. If φ is a representation of the C^* -algebra \mathfrak{A}_1 acting on the Hilbert space \mathfrak{H}_1 into the C^* -algebra \mathfrak{A}_2 acting on \mathfrak{H}_2 , then φ has an extension (strongly continuous) mapping \mathfrak{A}_1^- into \mathfrak{A}_2^- if and only if $\mathfrak{N}_\lambda \subseteq \mathfrak{N}_\varphi$, where λ is the canonical isomorphism of \mathfrak{A}_1 with its representing function system. If $\mathfrak{N}_\lambda = \mathfrak{N}_\varphi$, then the extension of φ is an isomorphism.

DEFINITION 7. Let \mathfrak{A} be a C^* -algebra, φ a representation of \mathfrak{A} as an algebra \mathfrak{A}_0 of operators on the Hilbert space \mathfrak{H} , P the infinite central portion of \mathfrak{A}_0' , Q the infinite central portion of $\mathfrak{A}_0(I - P)$, and $R = I - P - Q$. Let P_a be the central portion of $\mathfrak{A}_0'P$ of character a , for each cardinal a not exceeding the dimension of \mathfrak{H} . For each positive real number or infinite cardinal a , define $P_a = Q + R_a + \sum_{b \leq a} P_b'$, where R_a is the central portion of $\mathfrak{A}_0'R$, $\mathfrak{A}_0'R$ with upper coupling number a (we use the obvious convention that $R_a = R$ with a an infinite cardinal). We associate with φ a "multiplicity function," f_φ , which assigns to 0 the ideal class of φ and to each positive real number or infinite cardinal a the ideal of permanent null sets of the representation φ_a of \mathfrak{A} defined by $\varphi_a(A) = \varphi(A)P_a$.

Concerning this definition, we remark that it is appropriate to use the number 0 for the Q portion of the representation, for this is the portion which is infinite with finite commutant and, so, has "coupling" 0.

THEOREM 2. Two representations φ_1, φ_2 of a C^* -algebra \mathfrak{A} as algebras of operators $\mathfrak{A}_1, \mathfrak{A}_2$ acting on Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$ are unitarily equivalent (i.e., there exists a unitary transformation U of \mathfrak{H}_1 onto \mathfrak{H}_2 such that $U\varphi_1(A)U^{-1} = \varphi_2(A)$, for each A in \mathfrak{A}) if and only if their associated multiplicity functions are identical.

* The author is a Fulbright grantee.