tensor $T_{\lambda\mu}$ appears as a function of the velocity vector u^ν and its derivative. Finally the equation (15) corresponds to the principle of the conservation of momentum and energy. Following still the analogy with the old general theory of relativity we may regard $-T_{\lambda\mu}h^{\lambda\mu} = -Q_{\lambda\mu}h^{\lambda\mu}$ as the mass and get in this way a physical interpretation of g/h.[5]

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[1] Hlavatý, V., "The Elementary Basic Principles of the Unified Theory of Relativity," these PROCEEDINGS, **38**, 243–247 (1952). This paper will be denoted in text by UI. Its notation is used also here and in particular $g_{(\lambda\mu)} = h_{\lambda\mu}$, $g_{[\lambda\mu]} = k_{\lambda\mu}$, $S_{\lambda\mu}{}^\nu = \Gamma_{[\lambda\mu]}{}^\nu$, while ∇_μ is the symbol of the covariant derivative with respect to $h_{\lambda\mu}$.

[2] All metric notions as well as the lowering or raising of indices are based on $h_{\lambda\mu}$. We assume $gh \neq 0$, where g and h are the determinants of $g_{\lambda\mu}$ and $h_{\lambda\mu}$, respectively. The determinant of $k_{\lambda\mu}$ will be denoted by k.

[3] Substituting for p from (9) into (5) one obtains four conditions for $g_{\lambda\mu}$ in terms of $u^\nu\eta$.

[4] The law of inverse squares cannot be applied here.

[5] The requirement $F_{\lambda\mu} \neq 0$ is essential for this unified theory. If $F_{\lambda\mu} = 0$ then $\Gamma_{\lambda\mu}{}^\nu = \left\{ {}^{\nu}_{\lambda\mu} \right\}$. While (13b) is identically satisfied in this case, (13a) leads to $H_{\lambda\mu} = 0$.

SOME REMARKS ON REPRESENTATIONS OF CONNECTED GROUPS

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1. *Introduction.*—The purpose of this note is to bring to light a fact which has escaped notice, viz., in the direct integral reduction of the regular representation of a connected separable[1] locally compact group, factors of Type II_1 occur almost nowhere[2] (cf. Corollary 3). This proof is carried out by the following scheme of argument. We show first that a connected locally compact group which has sufficiently many unitary representations which generate rings of finite type is the group direct product of a compact group and an abelian group[3] (cf. Theorem 1). From this it follows quite easily that a unitary representation of a connected locally compact group generates a ring of operators which has no summand of Type II_1 (cf. Theorem 1) and, in particular, is not itself a factor of Type II_1. Employing a theorem of Mautner,[4] to the effect that, for almost every factor in the direct integral reduction of the regular representation of a group, there exists a strongly continuous representation of the group which generates the factor, we obtain the final result.

The theorem of Segal and von Neumann[5] stating that a real or complex connected semisimple Lie group without compact constituents has no non-trivial strongly continuous representations in a finite factor follows easily from our results (cf. Corollary 2) as does a recent result of the second named author[6] which states that a connected Lie group which has a faithful uniformly continuous representation is the direct product of a compact and abelian group (cf. Corollary 4).

2. Representations.—The following theorem contains the main argument.

THEOREM 1. *A connected locally compact group G which has sufficiently many strongly continuous unitary representations into rings of finite type (I_n and II_1) has the form $K \times E_n$ with K a compact connected group and E_n a vector group. Each strongly continuous unitary representation of a connected locally compact group generates a ring of operators which has no summand of Type* II_1.

Proof: We show that G possesses a fundamental system of neighborhoods of the identity e which have a compact closure and which are invariant under inner automorphisms of G. In fact, let g_0 be an element of G distinct from e and let $g \rightarrow U_g$ be a unitary representation of G into a ring of operators $\mathfrak{M}$ of finite type such that $U_{g_0} \neq I$ (there exists such a representation by hypothesis). We shall employ the results of Dixmier[7] as regards the existence of a trace in $\mathfrak{M}$. Then for A in $\mathfrak{M}$, $A^{\#}$ is the trace of A and is an operator in the center of $\mathfrak{M}$. We impose the following topology on the group $\mathfrak{M}_U$ of unitary operators in $\mathfrak{M}$: a subbasic open neighborhood of I is given by a vector x and a positive ϵ and consists of all operators U in $\mathfrak{M}$ such that $[[I\text{-}U]]_x < \epsilon$ where $[[A]]_x = ((A, A))_x^{1/2}$ and $((A, B))_x = ((B^*A)^{\#} x, x)$. Note that $((A, B))_x$ is a positive semi-definite "inner product" so that $[[A]]_x$ behaves like a Hilbert space norm with the exception that $[[A]]_x$ may be 0 without A being 0. Observe that

$$[[UA]]_x = ((A^*U^*UA)^{\#}x, x)^{1/2} = ((A^*A)^{\#}x, x)^{1/2} = [[A]]_x$$

and that

$$[[B^*]]_x = ((BB^*)^{\#}x, x)^{1/2} = ((B^*B)^{\#}x, x)^{1/2} = [[B]]_x$$

so that

$$[[AU]]_x = [[U^*A^*]]_x = [[A^*]]_x = [[A]]_x,$$

for all unitary U in $\mathfrak{M}$. It is easy to verify, though we shall not need the fact, that $\mathfrak{M}_U$ is a topological group in the given topology.

We shall prove now that the function $[[U_g - I]]_x$ is a continuous function on G (which shows that the map $g \rightarrow U_g$ is a (continuous) representation of G into $\mathfrak{M}_U$ in the given topology). Indeed $[[U - I]]_x$ is a strongly continuous function of U at I, for $U \rightarrow U^*$ is strongly continuous at I on unitary operators ($\|U^*x - x\| = \|Ux - x\|$), AB is strongly continuous jointly in A and B on the sphere of radius 2, and the trace is strongly continuous on the sphere of radius 4 (cf. Dixmier[7]). The continuity of $[[U_g - I]]_x$ follows from this fact and the strong continuity of the representation $g \to U_g$, for

$$[[U_{g_1} - I]]_x - [[U_{g_2} - I]]_x \leq [[U_{g_1} - U_{g_2}]]_x = [[U_{g_1 g_2^{-1}} - I]]_x$$

The functions we have just shown to be continuous are real non-negative, 0 at e, constant on conjugate classes (i.e., $[[U_{g_1 g g_1^{-1}} - I]]_x = [[U_g - I]]_x$), and, for our given g_0, $U_{g_0} \neq I$. Since the trace of a positive operator is 0 only if the operator is 0, there exists a vector x such that $[[U_{g_0} - I]]_x \neq 0$. These functions form a family $\mathfrak{F}$ satisfying the conditions of the following lemma, so that the proof of this lemma completes the proof of the first part of the theorem.

LEMMA 1. *If G is a connected locally compact group which possesses a family $\mathfrak{F}$ of continuous functions which* (1) *are constant on conjugate classes,* (2) *for $g \neq e$ there exists a function φ_g in $\mathfrak{F}$ such that $\varphi_g(g) \neq \varphi_g(e)$, then G is the direct product of a connected compact group and a vector group.*

Proof: We establish this result by proving the existence of arbitrarily small compact invariant neighborhoods of the identity e and then employing a well-known result[8] to the effect that a connected group with such neighborhoods is the direct product of a connected compact group and a vector group.

In fact, let N be a neighborhood of e in G with compact closure N^-. For each point g on the boundary of N, choose a function φ_g in $\mathfrak{F}$ such that $\varphi_g(g) \neq 0 = \varphi_g(e)$ (we can normalize so that $\varphi(e) = 0$ for all functions φ in $\mathfrak{F}$ by subtracting constants). Let N_g be the (open) set of points g' in G such that $|\varphi_g(g')| > \frac{1}{2}|\varphi_g(g)|$. The sets N_g form an open covering of the boundary of N (which is compact). Select a finite subcovering $N_{g_1}, \ldots, N_{g_n}$; and let $m = \min_i\{|\varphi_{g_i}(g_i)|\} > 0$. Let M be the set of points g of N such that $|\varphi_{g_i}(g)| < m/2$ for $i = 1, \ldots, n$. Then each conjugate to an element of M lies in N, for if $g'hg'^{-1}$ lies outside of N, with h in M, there exists a $\bar{g}$ such that $\bar{g}h\bar{g}^{-1}$ lies on the boundary of N. Indeed, the map $g \to ghg^{-1}$ is continuous, so that the image of our connected group G under this map is connected. But this image contains the point h in N and $g'hg'^{-1}$ outside of N, so that this image must meet the boundary of N. Since $\bar{g}h\bar{g}^{-1}$ is on the boundary of N, it lies in some N_{g_i}, and, thus,

$$|\varphi_{g_i}(\bar{g}h\bar{g}^{-1})| > |\varphi_{g_i}(g_i)|/2 \geq m/2.$$

But

$$|\varphi_{g_i}(\bar{g}h\bar{g}^{-1})| = |\varphi_{g_i}(h)| < m/2,$$

since h is in M. From this contradiction, it follows that the transforms of M lie in N. The union N_1 of all the transforms of M is then an open invariant neighborhood of e contained in N, and the lemma is proved.

Suppose now that G is a connected locally compact group with a strongly continuous representation $g \rightarrow U_g$ which generates a ring $\mathfrak{M}$, and suppose that $\mathfrak{M}$ contains a direct summand $\mathfrak{N}$ of Type II_1 (as a ring). The projection of $\mathfrak{M}$ into $\mathfrak{N}$ gives a strongly continuous representation of G which generates $\mathfrak{N}$. Let G_1 be the kernel of this last representation. The induced representation of G/G_1 into $\mathfrak{N}$ is faithful, strongly continuous, and generates $\mathfrak{N}$. Now G/G_1 is connected so that, by our preceding argument, $G/G_1 = K \times E_n$ with K compact connected and E_n a vector group. However, the subring of $\mathfrak{N}$ generated by the image of K is of Type I, since the Hilbert space decomposes into a direct sum of finite dimensional invariant subspaces under the image of K (Peter-Weyl theory), while the subring generated by E_n is abelian, in the center of $\mathfrak{N}$, and is therefore of Type I_1. These two commuting subrings of Type I generate $\mathfrak{N}$ which, consequently, is not of Type II_1.[9] This contradiction shows that $\mathfrak{M}$ cannot have a summand of Type II_1, and the proof is complete.

Since the regular representation of a group is faithful, we can state:

COLLARY 1. *If G is a connected locally compact group and the weakly closed group algebra of G (i.e., the left or right regular representation of G) is of finite type then $G = K \times E_n$, K compact connected and E_n a vector group.*

COROLLARY 2. *There are no strongly continuous, non-trivial, unitary representations of a connected locally compact group G, which is (topologically) generated by non-compact connected simple Lie groups, into a finite factor.*

Proof: Since G is generated by non-compact, connected, simple Lie groups, the representation of G must be non-trivial on some such subgroup, say G_1. The kernel G_2 in G_1, being an invariant subgroup of G_1, is discrete, in the center of G_1, and G_1/G_2 is a connected simple Lie group. The induced representation of G_1/G_2 is faithful, so that, by Theorem 1, G_1/G_2 is the product of a connected compact group and a vector group. Since G_1/G_2 is simple, the vector group must be trivial and G_1/G_2 is a compact simple Lie group. Thus G_1, being a covering group of G_1/G_2, is compact, by Weyl's Theorem, contradicting the nature of G_1.

We note, in passing, that the condition that G be generated by connected simple Lie groups can be altered so that G is generated by non-compact simple locally compact groups if we now understand "simple" to mean "no closed invariant subgroups."

COROLLARY 3. *If G is a connected locally compact group there is no part of Type II_1 in its weakly closed group algebra. If G is, in addition, separable, the direct integral reduction of the weakly closed group algebra contains factors of Type II_1 only on a set of measure zero.*

Proof: The first statement is an immediate consequence of the last statement of Theorem 1. Suppose G is separable. If factors of Type II_1 occurred at a set of measure greater than zero, then G would have a strongly continuous, unitary representation into one of these factors which generates the factor by Mautner's theorem,[4] contradicting Theorem 1.

COLLARY 4. *If G is a connected locally compact group which has sufficiently many unitary representations continuous in the uniform operator topology, then $G = K \times E$ with K a compact connected group and E_n a vector group.*

Proof: The function $\|U_g - I\|$ for each of the given representations form a family $\mathfrak{F}$ of continuous functions satisfying (1), (2) of Lemma 1.

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[1] The reduction theory of rings of operators to factors is carried out only in the case of a separable Hilbert space. There is, however, a global study of rings of operators (carried out without recourse to the factor reduction) which is valid for non-separable spaces (cf. Kaplansky, I., "Projections in Banach Algebras," *Ann. Math.*, **53**, 235–249 (1951)). The corresponding global statement for non-separable groups is also contained in Corollary 3.

[2] von Neumann, J., "On Rings of Operators. *Reduction Theory*," *Ann. Math.*, **50**, 401–485 (1949); Mautner, F. I., "Unitary Representations of Locally Compact Groups I," *Ibid.*, **51**, 1–25 (1950).

[3] It was pointed out to us by I. E. Segal, after our results were obtained, that the first part of our Theorem 1 is very closely related to R. Godement's, Theoreme 6, "Mémoier sur la théorie des caractères dans les groupes localement compacts unimodulaires," *J. Math.*, **47** (1951). In fact, Godement's theorem is essentially our Theorem 1 applied to the particular case where the regular representation of the group is of finite type (cf. Corollary 1). Even the proofs are similar; however, the slightly more general procedure we employ allows us to conclude the interesting corollaries following Theorem 1.

[4] Mautner, F. I., "Unitary Representations of Locally Compact Groups II," *Ann. Math.*, **52**, 528–556 (1950); see, especially, pp. 520–535.

[5] Segal, I. E., and von Neumann, J., "A Theorem on Unitary Representations of Semisimple Lie Groups," *Ibid.*, **52**, 509–517 (1950).

[6] Singer, I. M., "Uniformly Continuous Representations of Lie Groups," to appear in *Ann. Math.*

[7] Dixmier, J., "Les anneaux d'operateurs de classe finie," *Ann. École Norm. Sup.*, pp. 209–261 (1949).

[8] The clearest reference to this result in the non-separable case seems to be Theorem 3 in K. Iwasawa's, "Topological Groups with Invariant Compact Neighborhoods of the Identity," *Ann. Math.*, **54**, 345–348 (1951). The result we desire follows trivially from Iwasawa's much stronger result. The result was first obtained by H. Freudenthal, "Topologische Gruppen mit genugend vielen fastperiodischen Funktionen," *Ibid.*, **37**, 57–77 (1936); in the separable case.

[9] If $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are of Type I and commute, an abelian projection for the ring generated by $\mathfrak{N}_1$ and $\mathfrak{N}_2$ is obtained by taking the product of an abelian projection in $\mathfrak{N}_1$ and an abelian projection in $\mathfrak{N}_2$. All such products cannot be 0, since $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are generated by their abelian projections.