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DETERMINANT THEORY IN FINITE FACTORS*

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1. Introduction

In a series of fundamental memoirs on rings of operators [6, 7, 8, 9], F. J. Murray and J. v. Neumann classified factors (i.e., central rings of operators) by means of a "relative dimension function." They studied, extensively, those factors for which this dimension function has a finite range ("finite factors") and showed that these factors (and these factors alone) admit a trace function[1] with the standard algebraic properties.

In an attempt to establish, what seems to us to be a further important algebraic property of the trace, viz., the trace of a generalized nilpotent operator is zero, and, more generally, the trace of an operator lies in the convex hull of its spectrum, we were led to the introduction of a determinant theory for finite factors. This paper will be concerned, principally, with the development of this theory.

We might note that it is a simple algebraic matter to prove that the trace $T(N)$ of a *proper* nilpotent N is zero. In fact, if $N^n = 0$ and E is the projection on the closure of the range of N, then $EN = N$ so that $(NE)^{n-1} = N^{n-1}E = 0$. Then $T(N) = T(EN) = T(NE) = 0$, by induction on n. That the normalized trace lies in the convex hull of the spectrum of a finite-dimensional matrix follows at once by bringing the matrix to super-diagonal form, whereby the normalized trace appears as the "center of gravity" of the spectrum. This fact together with the theory developed in R.O. IV, Chapter IV, yields the same result for operators in an approximately finite factor. Furthermore, it is an immediate consequence of the spectral theorem that the trace of a normal operator in an arbitrary finite factor lies in the convex hull of its spectrum. None of these easily proved facts enabled us to conclude the result for arbitrary non-normal operators in non-approximately finite factors. However, the general result was established as a byproduct of the determinant theory.

In §2 we define the determinant on regular operators in a factor of type II_1, and establish the properties of this determinant. The proof that the trace lies in the convex hull of the spectrum is given in §3 as an application of the results of §2. The uniqueness of the determinant is established in §4 by means of an algebraic characterization. The final section, §5, begins with a discussion of the normalization which has taken place in the definition of the determinant. A

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* The results of this paper were outlined in a note by the same authors [4].

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[1] The term "trace" refers, throughout this paper, to the normalized trace which takes the value 1 at the identity operator. For a complete account of the theory of factors, the reader is referred to the original papers on this subject [6, 7, 8, 9]. In making reference to these papers, we shall use the abbreviation R.O., I, II, III, and IV.

justification for considering only a *positive-valued* determinant is given in the form of various results on the non-existence of characters on the group of unitary operators in a factor. The paper concludes with a study of the possible extensions of the determinant of §2 to singular operators in the factor.

2. Definition and properties of the determinant

Let $\mathfrak{M}$ be a factor of type II_1, let T and D be the normalized trace and dimension function, respectively, in $\mathfrak{M}$ (cf. R.O. I and II), and let X be a regular operator in $\mathfrak{M}$ (i.e., X has a bounded inverse). Then X has a unique decomposition, $X = UH$, where U is unitary and $H = (X^*X)^{\frac{1}{2}}$ is positive and regular; U and H both belong to $\mathfrak{M}$ (cf. R.O. I, Lemma 44.1).

DEFINITION. *With X and H as above, we define "determinant of X" by*[2]

$$\Delta(X) = \Delta(H) = \exp\,[T(\log H)] = \exp\left[\int \log\lambda\, dD(E_\lambda)\right],$$

where $\int \lambda\, dE_\lambda$ $(=H)$ *is the spectral representation of* H.

We establish, in the following lemma, the most elementary properties of the determinant.

LEMMA 1. *The determinant satisfies the following relations:*

(1°) $\| X^{-1} \|^{-1} \leqq \Delta(X) \leqq \| X \|$, *for regular* X, *and, in particular* $\Delta(I) = 1$.

(2°) $\Delta(\lambda X) = |\lambda|\,\Delta(X)$, *for non-zero* λ *and regular* X.

(3°) $\Delta(\exp A) = |\exp T(A)| = \exp \operatorname{Re} T(A)$, *for normal* A.

(4°) $\Delta[f(A)] = \exp\left[\int \log |f(z)|\, dD(E_z)\right]$, *where* $A\left(=\int z\, dE_z\right)$ *is normal and* $f(z)$ *is continuous and non-zero on the spectrum of* A.

(5°) $\Delta(AB) = \Delta(A)\Delta(B)$, *for normal, commuting, regular* A *and* B.

(6°) $\Delta(U_1XU_2) = \Delta(X)$, *for unitary* U_1 *and* U_2 *and regular* X.

(7°) $\Delta(X^*) = \Delta(X) = [\Delta(X^*X)]^{\frac{1}{2}}$, *for regular* X.

(8°) $\Delta(X^{-1}) = 1/\Delta(X)$, *for regular* X.

PROOF. Ad (1°): Let $H = (X^*X)^{\frac{1}{2}}$, $\alpha = \| X^{-1} \|^{-1} = \| H^{-1} \|^{-1}$, $\beta = \| X \| = \| H \|$, and $H = \int \lambda\, dE_\lambda \left(= \int_\alpha^\beta \lambda\, dE_\lambda\right)$. Then $\log \Delta(X) = T(\log H) = \int_\alpha^\beta \log\lambda\, dD(E_\lambda)$; so that $\log\alpha \leqq \log\Delta(X) \leqq \log\beta$.

Ad (2°): $\Delta(\lambda X) = \Delta\{[(\lambda X)^*\lambda X]^{\frac{1}{2}}\} = \Delta(|\lambda| H) = \exp T[\log(|\lambda| H)] = \exp\,[T(\log|\lambda| I) + T(\log H)] = |\lambda|\cdot\Delta(H)$.

Ad (3°): Put $A = A_1 + iA_2 (A_1A_2 = A_2A_1$, A_1 and A_2 self-adjoint). Then $\exp A = \exp iA_2 \exp A_1$ is the polar decomposition of $\exp A$; so that $\Delta(\exp A) = \Delta(\exp A_1) = \exp T(A_1) = \exp \operatorname{Re} T(A)$.

[2] Throughout this paper, "log" refers to the principal value of the logarithm.

Ad (4°): Since $f(A)$ is normal, we have $\Delta[f(A)] = \Delta(|f(A)|)$. Now $\log|f(A)| = \int \log|f(z)|\, dE_z$, so that

$$\Delta(|f(A)|) = \exp\left[T\left(\int \log\ |f(z)|\, dE_z\right)\right] = \exp \int \log|f(z)|\, dD(E_z).$$

Ad (5°): Since $AB = BA$, we have $AB^* = B^*A$ (cf. [3]), and hence there exist functions f, g and a self-adjoint operator $H = \int_0^1 \lambda\, dE_\lambda$ such that $A = f(H)$, $B = g(H)$. Then $AB = (fg)(H)$, from which the statement follows by application of (4°).

Ad (6°): Note first that $\Delta(W^*XW) = \Delta(X)$ for unitary W in $\mathfrak{M}$, since all operations (notably the trace) employed in forming the determinant are invariant under unitary transformations from the factor. Now

$$[(U_1XU_2)^*(U_1XU_2)]^{\frac{1}{2}} = U_2^*(X^*X)^{\frac{1}{2}}U_2,$$

so that $\Delta(U_1XU_2) = \Delta[U_2^*(X^*X)^{\frac{1}{2}}U_2] = \Delta(X)$.

Ad (7°): Let $X = UH$ be the polar decomposition of X. Then $X^* = HU^*$, hence, by (6°), $\Delta(X^*) = \Delta(H) = \Delta(X)$. Further, $\Delta(X^*X) = \Delta(H^2) = [\Delta(H)]^2$ by (5°).

Ad (8°): As above, $X = UH$, so that $X^{-1} = H^{-1}U^*$. Thus $\Delta(X^{-1}) = \Delta(H^{-1}) = 1/\Delta(H)$, by (5°) and (6°).

THEOREM 1. *The determinant has the following properties in addition to those listed in Lemma* 1:

(1°) $\Delta(XY) = \Delta(X)\Delta(Y)$, *for arbitrary regular* X *and* Y.

(2°) $\Delta(\exp A) = |\exp T(A)| = \exp \operatorname{Re} T(A)$ *for arbitrary* A *in* $\mathfrak{M}$.

(3°) *The determinant is continuous on regular elements, in the uniform topology.*

(4°) $\Delta(H_1) \geqq \Delta(H_2)$ *if* $H_1 \geqq H_2 > 0$ *and* H_2 *is regular.*

(5°) $\Delta(X)$ *does not exceed the sprectral radius of* X.

Before beginning the proof of this theorem, we establish the lemma which is the basic tool for dealing with the non-commutative situation. We view $\mathfrak{M}$ as a Banach algebra with the operator bound "$\|\ \ \|$" as norm, and consider certain aspects of the theory of analytic functions from $\mathfrak{M}$ to the complex numbers (cf. [2]).

LEMMA 2. *Let* $f(\lambda)$ *be analytic in a domain*[3] Λ *bounded by a curve* Γ *in the complex* λ*-plane, and let* $X(t)$, $0 \leqq t \leqq 1$, *be a differentiable family of operators in* $\mathfrak{M}$, *such that the spectrum of each* $X(t)$ *lies in* Λ. *Then* $f[X(t)]$ *is differentiable with respect to* t, *and*[4]

$$T\left\{\frac{d}{dt}f[X(t)]\right\} = T\ \{g[X(t)]\cdot X'(t)\}$$

where $g(\lambda) = df(\lambda)/d\lambda$ *and* $X'(t) = dX(t)/dt$.

[3] For our purposes, we need only deal with the simplest types of domains Λ, for example, rectangles; our considerations are valid, however, in much more general circumstances.

[4] The formula obtained from that of this lemma by omitting the trace, T, is not, in general, valid, as illustrated by the example $f(\lambda) = \lambda^2$.

PROOF. The Cauchy formula yields (cf. [2, p. 641])

$$f[X(t)] = \frac{1}{2\pi i}\int_\Gamma f(\lambda)\,(\lambda - X(t))^{-1}\,d\lambda$$

for each t, $0 \leqq t \leqq 1$. By the second resolvent equation [5, p. 115] and the fact that $(\lambda - X(t))^{-1}$ is uniformly continuous in the uniform topology on the (compact) product space of the boundary curve Γ and the interval $0 \leqq t \leqq 1$, we obtain

$$\frac{d}{dt}f[X(t)] = \frac{1}{2\pi i}\int_\Gamma f(\lambda)\,(\lambda - X(t))^{-1}X'(t)\,(\lambda - X(t))^{-1}\,d\lambda.$$

In fact,

$$\frac{1}{h}f[X(t+h)] - f[X(t)] = \frac{1}{2\pi i}\int_\Gamma f(\lambda)\,\frac{1}{h}\,[(\lambda - X(t+h))^{-1} - (\lambda - X(t))^{-1}]\,d\lambda$$

$$= \frac{1}{2\pi i}\int_\Gamma f(\lambda)\,(\lambda - X(t+h))^{-1}\left(\frac{1}{h}\right)(X(t+h) - X(t))\,(\lambda - X(t))^{-1}\,d\lambda.$$

As h tends to 0, $(\lambda - X(t+h))^{-1}$ tends to $(\lambda - X(t))^{-1}$, uniformly with respect to λ on Γ, and $(1/h)[X(t+h) - X(t)]$ tends to $X'(t)$.

On the other hand, a partial integration establishes

$$g[X(t)] = \frac{1}{2\pi i}\int_\Gamma g(\lambda)\,(\lambda - X(t))^{-1}\,d\lambda = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - X(t))^{-2}\,d\lambda$$

for each t (note that $d(\lambda - X)^{-1}/d\lambda = -(\lambda - X)^{-2}$ for λ in the resolvent set of X, by the first resolvent equation [5, p. 99]).

The application of the trace to the integral expression for $df[X(t)]/dt$ and $g[X(t)]X'(t)$ can be performed under the integral sign by virtue of the continuity of the trace (in the uniform topology). Moreover,

$$T[(\lambda - X(t))^{-1}X'(t)(\lambda - X(t))^{-1}] = T[(\lambda - X(t))^{-2}X'(t)]$$

and the proof of the lemma is complete.

We shall derive statements (1°) and (2°) of Theorem 1 from the following Lemma:

LEMMA 3. *If H is self-adjoint, then $\Delta(\exp A^* \exp H \exp A) = \exp T(A^* + A)\cdot \exp T(H)$ for arbitrary A in $\mathfrak{M}$.*

PROOF.[5] For $0 \leqq t \leqq 1$, define $X(t) = \exp(tA^*) \exp H \exp(tA)$. Each $X(t)$ is positive and regular. Moreover, $X(t)$ is differentiable in t. The functions

[5] The formal mechanism behind this lemma, and, in fact, behind the multiplicativity of the determinant is contained in the Campbell-Baker-Hausdorff formula (cf., for example, H. F. Baker, *Alternants and continuous groups*, Proc. London Math. Soc., (2) 3 (1905), pp. 24–47) $\exp x \exp y = \exp z$, where $z = x + y +$ commutators (note that the trace of a commutator is zero). The convergence precaution necessary to apply this formula to our situation makes the approach we indicate as short, however, and preferable in the sense that it does not rely upon this algebraic result.

$\| X(t)^{-1} \|^{-1}$ and $\| X(t) \|$ are positive, continuous functions of t. Thus we can choose constants α and β such that $0 < \alpha < \| X(t)^{-1} \|^{-1} < \| X(t) \| < \beta < \infty$ for all t. In other words, the interval (α, β) contains the spectrum of each $X(t)$.

We now apply Lemma 2 with $f(\lambda) = \log \lambda$, which is analytic in some (rectangular) neighborhood of (α, β). Since $X'(t) = A^*X(t) + X(t)A$, we conclude

$$T\left[\frac{d}{dt} \log X(t)\right] = T[X(t)^{-1}X'(t)] = T[X(t)^{-1}A^*X(t) + A] = T(A^* + A);$$

whence,

$$T[\log X(1)] - T[\log X(0)] = \int_0^1 \frac{d}{dt} T[\log X(t)]\, dt$$

$$= \int_0^1 T\left[\frac{d}{dt} \log X(t)\right] dt = T(A^* + A);$$

i.e., $\log \Delta(\exp A^* \exp H \exp A) - T(H) = T(A^* + A)$.

PROOF OF THEOREM 1. Ad (1°): Write $X = U_1H_1$, $Y = H_2U_2$ with U_1, U_2 unitary, H_1, H_2 positive, and introduce the self-adjoint operators $A_1 = \log H_1$, $A_2 = \log H_2$. By application of Lemma 3, we see that $\Delta(XY) = \Delta(U_1H_1H_2U_2) = \Delta(H_1H_2) = [\Delta(H_2H_1^2H_2)]^{\frac{1}{2}} = [\Delta(\exp A_2 \exp (2A_1) \exp A_2)]^{\frac{1}{2}} = [\exp T(2A_2) \exp T(2A_1)]^{\frac{1}{2}} = \Delta(\exp A_2)\Delta(\exp A_1) = \Delta(X)\Delta(Y)$.

Ad (2°): Put $H = 0$ in Lemma 3 and note that $\Delta (\exp A) = [\Delta(\exp A^* \exp A)]^{\frac{1}{2}}$ by Lemma 1, (7°).

Ad (3°): The continuity of the determinant is implied by the following inequality:

$$| \Delta(Y) - \Delta(X) | \leqq \| X \| \| X^{-1} \| \| Y - X \| .$$

We deal first with the special case, $X = I$, and prove that $| \Delta(I + A) - 1 | \leqq \| A \|$. This follows from Lemma 1, (1°), since $\| I + A \| \leqq 1 + \| A \|$, and $\| (I + A)^{-1} \|^{-1} \geqq 1 - \| A \|$ (note that, for arbitrary x in Hilbert space $\mathcal{H}$, $\| (I + A)x \| \geqq \| x \| - \| A \| \| x \|$). In the general case, put $A = (Y - X)X^{-1}$. Then, by (1°), $| \Delta(Y) - \Delta(X) | = | \Delta(YX^{-1}) - 1 | \Delta(X) = | \Delta(I + A) - 1 | \Delta(X) \leqq \| A \| \Delta(X) \leqq \| Y - X \| \| X^{-1} \| \| X \|$.

Ad (4°): One has $H_1^{-\frac{1}{2}}H_2H_1^{-\frac{1}{2}} \leqq I$, so that, by Lemma 1, (1°), $\Delta(H_1^{-\frac{1}{2}}H_2H_1^{-\frac{1}{2}}) \leqq 1$, and thus, by (1°), $\Delta(H_1) \geqq \Delta(H_2)$.

Ad (5°): By application of (1°) and Lemma 1, (1°),

$$\Delta(X) = [\Delta(X^n)]^{1/n} \leqq \| X^n \|^{1/n}$$

for all positive integers n, so that $\Delta(X) \leqq \lim \| X^n \|^{1/n}$, the spectral radius of X.

3. Location of the trace

In this section we shall apply the theory developed above to establish the following result.

THEOREM 2. *The trace $T(A)$ of an arbitrary operator A in $\mathfrak{M}$ is located in the convex hull of the spectrum of A. In particular, $T(A) = 0$ when A is a generalized nilpotent operator in $\mathfrak{M}$.*

PROOF. It suffices to prove that $T(A)$ lies in each closed half-plane Π which contains the spectrum Σ of A. We may even assume that Π is the left half-plane ($\mathrm{Re}\,\lambda \leqq 0$). Indeed, consider in place of A the operator $\alpha A + \beta$, where the complex numbers α and β are so chosen that the transformation $\lambda \rightarrow \alpha\lambda + \beta$ maps Π onto the left half plane, $\mathrm{Re}\,\lambda \leqq 0$. Then $T(A)$ and Σ are replaced by the trace $\alpha T(A) + \beta$ and the spectrum $\alpha\Sigma + \beta$, respectively, of the operator $\alpha A + \beta$.

In order, now, to prove that $\mathrm{Re}\, T(A) \leqq 0$ when $\mathrm{Re}\,\Sigma \leqq 0$, we introduce the regular operator $\exp A$, whose spectrum is $\exp \Sigma$ (cf. [1, p. 195]) and hence lies in the unit disc. It follows from this remark, by Theorem 1, (2°) and (5°), that $\exp \mathrm{Re}\, T(A) = \Delta(\exp A) \leqq 1$, and hence $\mathrm{Re}\, T(A) \leqq 0$.

As immediate consequences of this result, we have $| T(A) | \leqq r$, the spectral radius of A, and, in particular, a generalized nilpotent operator in $\mathfrak{M}$ has trace 0 (since its spectrum consists solely of the number 0).

4. Uniqueness of the determinant

In this section we shall characterize the determinant in a factor $\mathfrak{M}$ of type II_1 by means of some of the algebraic properties listed in §2.

THEOREM 3. *A numerically valued function Δ_1, which is defined on the group of regular operators in a factor $\mathfrak{M}$ of type II_1 and which possesses the properties:*

(1°) $\Delta_1(XY) = \Delta_1(X)\Delta_1(Y)$, *for regular operators X and Y,*

(2°) $\Delta_1(X^*) = \Delta_1(X)$, *for arbitrary regular X,*

(3°) $\Delta_1(\lambda I) = \lambda$, *for some positive $\lambda \neq 1$,*

(4°) $\Delta_1(X) \leqq 1$ *if $0 < X \leqq I$ and X is regular,*

coincides with the determinant Δ defined in §2.

PROOF. From (1°) and (3°) it follows that $\Delta_1(I) = 1$. If U is unitary then $[\Delta_1(U)]^2 = \Delta_1(U)\Delta_1(U^*) = \Delta_1(UU^*) = \Delta_1(I) = 1$. Since every unitary operator $U\left(= \int_0^{2\pi} e^{i\theta}\, dE_\theta\right)$ has a square root, for example $V = \int_0^{2\pi} e^{i\theta/2}\, dE_\theta$, we have $\Delta_1(U) = [\Delta_1(V)]^2 = 1$. For an arbitrary regular operator X, introduce the polar decomposition $X = UH$, where U is unitary and H is positive and regular. Then $\Delta_1(X) = \Delta_1(U)\Delta_1(H) = \Delta_1(H)$. Hence, it remains to prove that $\Delta_1(H) = \Delta(H)$ for arbitrary *positive* regular H in $\mathfrak{M}$.

Condition (1°), and the fact that $\Delta_1(I) = 1$, implies that the set of positive numbers λ for which $\Delta_1(\lambda I) = \lambda$, is a subgroup of the positive reals. This subgroup differs from $\{1\}$, by (3°), and contains, therefore, arbitrarily small numbers. For any given positive regular H, we can choose a number λ in this subgroup such that $\lambda H \leqq I$. One has $\Delta_1(\lambda I)\Delta_1(H) = \Delta_1(\lambda H) \leqq 1$, by (1°) and (4°). In particular, $\Delta_1(H)$ is real. Thus $\Delta_1(H) = [\Delta_1(H^{\frac{1}{2}})]^2 \geqq 0$. Moreover $1 = \Delta_1(I) = \Delta_1(H)\Delta_1(H^{-1})$ so that $\Delta_1(H) > 0$.

Since each positive, regular operator H has the form $\exp A$, A self-adjoint, it remains to prove that $\Delta_1(\exp A) = \Delta(\exp A)$, $(= \exp T(A))$. To prove this, we consider the function $T_1(A) = \log \Delta_1(\exp A)$, defined on the self-adjoint operators A in $\mathfrak{M}$. We shall show that $T_1(A) = T(A)$. For this purpose, we appeal to the uniqueness of the trace (cf. R.O. I, Theorem XIII, p. 219). In fact, if A and B are commuting, self-adjoint operators, then $\exp(A + B) = \exp A \exp B$, so that, by (1°),

$$T_1(A + B) = \log \Delta_1[\exp(A + B)]$$
$$= \log \Delta_1(\exp A) + \log \Delta_1(\exp B) = T_1(A) + T_1(B).$$

In order to prove that $T_1(aA) = aT_1(A)$ for arbitrary real a, consider first the case where A is positive. The real-valued function $\phi(a) = T_1(aA)$ satisfies the functional equation, $\phi(a + b) = \phi(a) + \phi(b)$, and is monotone, since, for $a \leqq 0$, $\phi(a) = \log \Delta_1(\exp aA) \leqq 0 = \phi(0)$ (note that $\exp aA \leqq I$, so that $\Delta_1(\exp aA) \leqq 1$). Thus $\phi(a) = \phi(1)a$, i.e., $T_1(aA) = aT_1(A)$. Since $T_1(-A) = -T_1(A)$, we have the same relationship if $A \leqq 0$. For arbitrary self-adjoint A, write $A = A_1 + A_2$ with $A_1 \geqq 0$, $A_2 \leqq 0$, and use the additivity of T_1 to obtain

$$T_1(aA) = aT_1(A).$$

From the monotonicity of $\phi(a)$, choosing $a = -1$, we have $\phi(-1) = T_1(-A) \leqq 0$ for positive A, i.e.,

$$T_1(A) \geqq 0 \text{ if } A \geqq 0.$$

By (3°), we can choose a positive number λ, $\lambda \neq 1$, such that $\Delta_1(\lambda I) = \lambda$. Hence $(\log \lambda)T_1(I) = T_1[(\log \lambda)I] = T_1[\log(\lambda I)] = \log \Delta_1(\lambda I) = \log \lambda$. Since $\log \lambda \neq 0$ this implies that

$$T_1(I) = 1.$$

Finally, for arbitrary unitary U in $\mathfrak{M}$,

$$T_1(U^*AU) = \log \Delta_1[\exp(U^*AU)]$$
$$= \log \Delta_1(U^*(\exp A)U) = \log \Delta_1(\exp A) = T_1(A).$$

From the properties of T_1, noted above, it follows, now, that $T_1(A) = T(A)$, for arbitrary self-adjoint A in $\mathfrak{M}$, and we conclude that $\Delta_1(X) = \Delta(X)$ for arbitrary regular X in $\mathfrak{M}$.

5. Related questions

In this section we shall indicate the reasons which compel one to consider *positive*-valued determinants. We shall also discuss the possibilities of extending the determinant to the singular operators in $\mathfrak{M}$.

Concerning the first of these questions, it is natural to ask whether or not a notion of determinant can be developed which, in the classical finite-dimensional case reduces to the usual determinant. Clearly one must introduce a normalization in order to avoid having an infinite determinant for, say, $2I$. In a manner analogous to the normalization of the trace, such a normalization for determinants, in the n-dimensional case, should be accomplished by passing to an n^{th} root of the usual determinant. One encounters, at this point, the difficulty of making a coherent selection of n^{th} roots.

Even if we ignore the problems arising in connection with this normalization, we are faced, in the development of a "signed" determinant theory, with the problem of constructing a non-trivial character, viz., the signum of the determinant, on the group of unitary operators in the factor. This character must satisfy certain additional conditions if the determinant theory is to be at all reasonable.

The following theorem demonstrates the impossibility of constructing a character satisfying the barest minimum of such conditions.

THEOREM 4. *In a factor* $\mathfrak{M}$ *of type* II_1 *or* I_n $(n \geqq 2)$ *there exists on* $\mathfrak{M}_U$, *the group of unitary operators in* $\mathfrak{M}$, *no character* χ *with the property* $\chi(\lambda U) = \lambda \cdot \chi(U)$, *for all* λ *of modulus* 1.

If $\mathfrak{M}$ *is a factor not of type* I_n, n *finite, then* 1 *is the only character on* $\mathfrak{M}_U$, *which is continuous in the uniform topology.*

PROOF. Let χ be a character on $\mathfrak{M}_U$ such that $\chi(\lambda U) = \lambda \cdot \chi(U)$ for all complex numbers λ of modulus 1. Write $I = E_1 + \cdots + E_n$, where $E_1, \cdots, E_n$ are equivalent, orthogonal projections in $\mathfrak{M}$; and consider the unitary operator

$$U = \zeta E_1 + \zeta^2 E_2 + \cdots + \zeta^n E_n,$$

where ζ is a primitive n^{th} root of unity. The operator $V = \zeta U$ is clearly equivalent to U, i.e., $V = W^{-1}UW$, for some W in $\mathfrak{M}_U$. Hence $\chi(V) = \chi(W^{-1})\chi(U)\chi(W) = \chi(U)$. But, by assumption, $\chi(V) = \chi(\zeta U) = \zeta \cdot \chi(U)$, which is impossible since $\zeta \neq 1$.

Let $\mathfrak{M}$ be a factor not of type I_n, n finite, and let ξ be a uniformly continuous character on $\mathfrak{M}_U$. Choose a projection E in $\mathfrak{M}$ and consider the unitary operators $U(\lambda) = \lambda E + (I - E)$, $|\lambda| = 1$. The function $\phi(\lambda) = \xi(U(\lambda))$ is a continuous character on the unit circle, and hence $\phi(\lambda) = \lambda^m$ for some integer m. The integer m depends only on the relative dimensions of E and $I - E$ (write $m(E)$ in place of m). In fact, if F has the same relative dimension and co-dimension as E, then $F = W^{-1}EW$, W in $\mathfrak{M}_U$ so that $W^{-1}U(\lambda)W = \lambda F + (I - F)$ $(= V(\lambda))$, and therefore $\xi(V(\lambda)) = \xi(U(\lambda))$. We show that $m(E) = 0$. Choose an integer $p > |m(E)|$ and decompose E into p equivalent, orthogonal projections $E_1, \cdots, E_p$ in $\mathfrak{M}$ (in case $\mathfrak{M}$ is of type I_∞, we assume $D(E)$ to be infinite). Then, $U(\lambda) = U_1(\lambda) \cdots U_p(\lambda)$, where $U_q(\lambda) = \lambda E_q + (I - E_q)$. Since the E_q all have the same relative dimension and co-dimension,

$$\lambda^{m(E)} = \xi(U(\lambda)) = \xi(U_1(\lambda)) \cdots \xi(U_p(\lambda)) = (\lambda^{m(E_1)})^p,$$

for each complex number λ of modulus 1. This implies that $p \cdot m(E_1) = m(E)$. Now $m(E)$, $m(E_1)$ are integers and $p > |m(E)|$, so that $m(E) = 0$. If $\mathfrak{M}$ is of type I_∞ and $D(E)$ is finite, write $\lambda E + I - E = (\lambda I)(\bar{\lambda}(I - E) + E)$. The above results show that $\xi(\lambda I) = \xi(\bar{\lambda}(I - E) + E) = 1$ so that $\xi(\lambda E + I - E) = 1$.

Thus ξ is 1 on the group generated by all $U(\lambda)$ ($|\lambda| = 1$, E in $\mathfrak{M}$). This group is dense in $\mathfrak{M}_U$, in the uniform topology, by the spectral theorem. Since ξ is uniformly continuous, $\xi(U) = 1$ for each U in $\mathfrak{M}_U$.

With regard to the question of extending the notion of determinant to the singular operators in $\mathfrak{M}$, two different possibilities present themselves. On the one hand, guided by classical determinant theory, we can extend the determinant "algebraically" merely by requiring that it be zero on all singular operators in $\mathfrak{M}$. On the other hand, we can extend the determinant in an analytical manner by maintaining the definition in §2 with the understanding that $\Delta(H) = 0$ when $\int \log \lambda \, dD(E_\lambda) = -\infty$; in particular, $\Delta(H) = 0$ if H has a nullspace.

Except for continuity, both of these extensions preserve all the properties of the determinant noted in §2 (with obvious modifications). The relation $\Delta(XY) = \Delta(X)\Delta(Y)$, for arbitrary X and Y in $\mathfrak{M}$, is proved with the aid of the following lemmas, the first of which refers to the algebraic extension, the second to the analytic extension.

LEMMA 4. *In a factor M of type* II_1, *the product of two operators is singular unless both operators are regular.*[6]

PROOF. Suppose $A'A = C$ with C regular. Then $BA = I$, with $B = C^{-1}A'$. We show that A and hence A' are regular. Clearly A has nullspace $\{0\}$ so that the range of A^* is dense. Now the closure of the range of A is equivalent (in $\mathfrak{M}$) to the closure $\mathfrak{K}$ of the range of A^* (cf. R.O. I, Lemma 6.2.1). Since $\mathfrak{M}$ is a finite factor, the closure of the range of A is $\mathfrak{K}$. Observe, moreover, that for each element x in $\mathfrak{K}$, $\| B \| \, \| Ax \| \geqq \| BAx \| = \| x \|$. Thus A is regular.

LEMMA 5. *For the determinant* Δ, *extended to singular operators by application of the definition in* §2, *we have the following continuity properties*:

(1°) $\lim_{\varepsilon \to 0+} \Delta(H + \varepsilon I) = \Delta(H)$, *for* $H \geqq 0$.

(2°) $\Delta(H_1) \geqq \Delta(H_2)$, *when* $H_1 \geqq H_2 \geqq 0$.

(3°) $\overline{\lim}_{n\to\infty} \Delta(X_n) \leqq \Delta(X)$, *when* X_n *tends to* X *uniformly.*

(4°) $\lim_{n\to\infty} \Delta(H_n) = \Delta(H)$ *if* $H_n \geqq H \geqq 0$ *and* H_n *tends to* H *uniformly.*

PROOF. Statement (1°) is proved by applying Lemma 1, (4°) to $f(\lambda) = \lambda + \varepsilon$ and observing that $\log(\lambda + \varepsilon)$ tends to $\log \lambda$, monotonically as $\varepsilon \to 0+$, $\lambda \geqq 0$. If $H_1 \geqq H_2 \geqq 0$ and $\varepsilon > 0$, then $H_1 + \varepsilon I \geqq H_2 + \varepsilon I$ and $H_2 + \varepsilon I$ is regular, so that, by Theorem 1, (4°), $\Delta(H_1 + \varepsilon I) \geqq \Delta(H_2 + \varepsilon I)$. Letting $\varepsilon \to 0+$ we arrive at statement (2°), by means of (1°). When X_n tends to X uniformly then $X_n^* X_n$ tends to X^*X uniformly. If statement (3°) holds for positive operators, we see that

$$\overline{\lim_{n\to\infty}} \, \Delta(X_n) = \overline{\lim_{n\to\infty}} \, [\Delta(X_n^* X_n)]^{\frac{1}{2}} \leqq [\Delta(X^*X)]^{\frac{1}{2}} = \Delta(X).$$

If, now, $X_n \geqq 0$ and $\varepsilon > 0$, then $X_n + \varepsilon I$ is regular and tends uniformly to the regular operator $X + \varepsilon I$, so that $\Delta(X_n + \varepsilon I)$ tends to $\Delta(X + \varepsilon I)$. In view of statement (2°), this implies that $\overline{\lim}_n \Delta(X_n) \leqq \overline{\lim}_n \Delta(X_n + \varepsilon I) = \Delta(X + \varepsilon I)$

[6] In an infinite factor, there are singular operators whose product is regular; for example, one can find U such that $U^*U = I$ but $UU^* = E$, a projection different from I.

for every positive ε. Now let ε tend to $0+$ and apply statement (1°). Finally, statement (4°) is an immediate consequence of statements (2°) and (3°).

With the aid of the above lemma, we can prove that $\Delta(XY) = \Delta(X)\Delta(Y)$ unrestrictedly (for the analytic extension of Δ). By an argument employed in the proof[7] of Theorem 1, it suffices to prove $\Delta(HKH) = \Delta(H)\Delta(K)\Delta(H)$ for positive H and K. For this purpose, note that, for $\varepsilon > 0$,

$$A(\varepsilon) = (H + \varepsilon^2(K + \varepsilon I)^{-1})(K + \varepsilon I)(H + \varepsilon^2(K + EI)^{-1})$$
$$= HKH + \varepsilon H^2 + 2\varepsilon^2 H + \varepsilon^4(K + \varepsilon I)^{-1} \geqq HKH.$$

Now $A(\varepsilon)$ tends to HKH uniformly as ε tends to 0. Thus, by (4°) of the preceding lemma and the regularity of $H + \varepsilon^2(K + \varepsilon I)^{-1}$, $K + \varepsilon I$;

$$\Delta[A(\varepsilon)] = \Delta(H + \varepsilon^2(K + \varepsilon I)^{-1})\Delta(K + \varepsilon I)\Delta(H + \varepsilon^2(K + \varepsilon I)^{-1}) \to \Delta(HKH)$$

as $\varepsilon \to 0+$. Again, by (4°), $\Delta(H + \varepsilon^2(K + \varepsilon I)^{-1}) \to \Delta(H)$ and $\Delta(K + \varepsilon I) \to \Delta(K)$ as $\varepsilon \to 0+$, since $H + \varepsilon^2(K + \varepsilon I)^{-1} \geqq H$ and $K + \varepsilon I \geqq K$. Thus $\Delta(HKH) = \Delta(H)\Delta(K)\Delta(H)$ as we wished to prove.

We should remark, at this point, that the two extensions introduced above actually differ from one another. In fact, let $H = \int_0^1 \lambda \, dE_\lambda$ where $D(E_\lambda) = \lambda$ (cf. R.O. II, Lemma 3.1.3). Then, for the analytic extension, $\Delta(H) = \exp \int_0^1 \log \lambda \, d\lambda = 1/e$, whereas $\Delta(H) = 0$ for the algebraic extension, since H is singular.

Although, for extensions other than the algebraic extension, the determinant is non-zero on some singular operators, we shall show:

LEMMA 6. *If Δ_1 is an arbitrary extension of the determinant from regular operators to all operators in $\mathfrak{M}$, and X is an arbitrary operator with a nullspace, then $\Delta_1(X) = 0$ (in fact, we shall only use that $\Delta_1(XY) = \Delta_1(X)\Delta_1(Y)$, and that $\Delta_1 \not\equiv 1$).*

PROOF. Let E be the projection on the orthogonal complement of the nullspace of X. Then E is in $\mathfrak{M}$ and $E \neq I$. Moreover, $X = XE$, so that $\Delta_1(X) = \Delta_1(X)\Delta_1(E)$. Thus it suffices to prove $\Delta_1(E) = 0$ for all projections E in $\mathfrak{M}$ which differ from I. Since $\Delta_1(E) = \Delta_1(E^2) = [\Delta_1(E)]^2$, we see that $\Delta_1(E)$ equals 0 or 1. Observe that $\Delta_1(E) = \Delta_1(F)$ if $E \sim F$ (i.e., $E = U^{-1}FU$ for some unitary operator U in $\mathfrak{M}$). Thus Δ_1 has the same value $d(\alpha)$ for all projections of dimension α. Moreover, $d(\alpha) = 0$ if $\alpha \leqq \frac{1}{2}$, since, in this case, we can choose two orthogonal projections E and F, each of dimension α, so that $d(\alpha)^2 = \Delta_1(E)\Delta_1(F) = \Delta_1(EF) = \Delta_1(0)$. But $\Delta_1(0) = 0$; for we can choose X in $\mathfrak{M}$ such that $\Delta_1(X) \neq 1$, whence $\Delta_1(0) = \Delta_1(0X) = \Delta_1(0)\Delta_1(X)$. We show, now, that $d(1 - \beta) = d(1 - 2\beta)$ when $0 < \beta < \frac{1}{2}$, by repeated application of which, we obtain $d(1 - \beta) = d(\nu)$ for some ν in the interval $(0, \frac{1}{2})$, so that $d(\alpha) = 0$ for *all* α less than 1. In fact, choose E and F such that $F \leqq E$, $D(E) = 1 - \beta$.

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[7] In a finite factor every operator X has a decomposition $X = UH$ with $H = (X^*X)^{\frac{1}{2}}$ and U unitary (recall that, in a finite factor, any partially isometric operator has a unitary extension).

$D(F) = 1 - 2\beta$. Let $G = (I - E) + F$, and observe that

$$D(G) = \beta + (1 - 2\beta) = 1 - \beta,$$

and that $F = EG$, so that $d(1 - 2\beta) = [d(1 - \beta)]^2 = d(1 - \beta)$ (recall that $d(\alpha)$ is 0 or 1).

With the aid of the above lemma we can prove:

THEOREM 6. *No extension of the determinant Δ from the regular operators to all operators in $\mathfrak{M}$, is continuous in the uniform topology.*

PROOF. Among all singular operators in $\mathfrak{M}$, those with a nullspace lie dense, in the uniform topology.[8] For positive singular operators, the necessary approximation follows from the spectral theorem. For arbitrary singular X in $\mathfrak{M}$, write $X = UH$, where U is unitary and $H \geqq 0$. Since X is singular and U is regular, H is singular. Determine an operator K in $\mathfrak{M}$, with a nullspace, such that $\| K - H \| < \varepsilon$. Then $\| UK - X \| < \varepsilon$, and UK has the same nullspace as K.

If Δ_1 is a uniformly continuous extension of the determinant then Δ_1 is zero on all singular operators by the above remark in conjunction with the preceding lemma. In other words, Δ_1 would have to be the algebraic extension of Δ. This extension is, however, not continuous. Indeed, let H be positive and singular and such that the analytically extended determinant is $\alpha(\neq 0)$ on H (cf. the example preceding Lemma 6). The regular operators $H + (1/n)I$ tend to H in the uniform topology. Lemma 5, (1°), shows that $\Delta(H + (1/n)I) \rightarrow \alpha$. Hence the algebraic extension of Δ is not continuous, from which the theorem follows.

We may remark, in conclusion, that the proof of Theorem 3 contains a proof of the fact that no determinant with properties (1°)–(4°) of Theorem 3 exists in an infinite factor; since only finite factors admit a trace, and, by the proof, the existence of such a determinant implies the existence of a trace.

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[8] The corresponding statement for infinite factors is false. In fact, choose U such that $U^*U = I \neq UU^*$. If $Ax = 0$ ($\| x \| = 1$), then $\| U - A \| \geqq \| Ux - Ax \| = \| Ux \| = 1$.