field-value averages, the covariant four-potential operators are $A_i(\phi) = \hbar Q(\sqrt{H^{-1}}\phi \otimes \rho_i), i = 1, 2, 3$, and $\Phi(\phi) = \hbar Q(\sqrt{H^{-1}}\phi \otimes \rho_4)$. Averages over mutually space-like regions commute. The total energy of the field is $\Omega(H) - c(A_1(s_1) + A_2(s_2) + A_3(s_3) + \Phi(s_4))$, where s_1, s_2, s_3, s_4 is any real, square-integrable, contravariant four-current density. Expectation values of $(\vec{\nabla} \cdot \vec{A} - . c^{-1}(\partial/\partial t)\Phi)(\phi)$ are always zero (on photons), and $(\Box^2 A_i)(\phi) = -c^{-1}(s_i, \phi)I, i = 1, 2, 3, (\Box^2\Phi)(\phi) = c^{-1}(s_4, \phi)I$, so Maxwell's equations are satisfied. A photon is polarized parallel to its electric vector and perpendicular to its magnetic vector—both perpendicular to its momentum. Its energy satisfies Planck's relation $E = h\nu$, where ν is the frequency of the induced field.

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[1] Stone, M. H., *Linear Transformations in Hilbert Space*, Am. Math. Soc. Coll. Publ., XV, New York, 1932.

[2] Fock, V., *Zeits. f. Phys.*, **75**, 622–647 (1932).

[3] Wentzel, G., *Einführung in die Quantentheorie der Wellenfelder*, Franz Deuticke, Vienna, 1943.

ON A CONJECTURE OF MURRAY AND VON NEUMANN

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1. *Introduction.*—In this note the authors present a proof of a conjecture of F. J. Murray and J. v. Neumann[1] concerning normalcy of factors.

A ring of operators[2] $\mathfrak{R}$ is said to be *normal* if each subring $\mathfrak{S}$ of $\mathfrak{R}$ coincides with the set of operators in $\mathfrak{R}$ each of which commutes with every operator in $\mathfrak{S}'_{\mathfrak{R}}$, where $\mathfrak{S}'_{\mathfrak{R}}$ is the ring of operators in $\mathfrak{R}$ each of which commutes with every operator in $\mathfrak{S}$. In symbols, normalcy requires that $(\mathfrak{S}'_{\mathfrak{R}})'_{\mathfrak{R}} = \mathfrak{S}$ for each subring $\mathfrak{S}$ of $\mathfrak{R}$. The center of a normal ring $\mathfrak{R}$ consists of the operators α I, α complex (put $\mathfrak{S} = \{\alpha \text{ I}\}$); i.e., $\mathfrak{R}$ is a *factor*. J. v. Neumann proved[3] that the factor $\mathfrak{B}$ of all bounded operators is normal. The question of which factors are normal was raised by F. J. Murray and J. v. Neumann (R.O. I, p. 185). They showed that all factors in case I (the discrete case) are normal and exhibited examples of non-normal factors in case II (the continuous case). Their later results establish the non-normalcy of each member of a restricted class of factors in case II, viz., the approximately finite factors (cf. R.O. I, pp. 209, 229; and R.O. IV, Theorem XIV, p. 781, Lemma 5.2.3, p. 787). The presumption is that no factor in case II is normal (cf. R.O. I, p. 185). We shall show that this is actually the case, and, indeed, that one can choose a *subfactor* which violates the normalcy.

The proof of the non-normalcy of factors in case II proceeds in four stages: a lemma concerning operators "almost in the center" of a finite factor is proved; the existence of maximal approximately finite subfactors of a factor of type II_1 is established; it is then shown, for each non-approximately finite factor of type II_1, that every maximal approximately finite subfactor violates normalcy; and, finally, the case II_∞ is reduced to the case II_1.

2. *Almost Central Operators.*—We note first a simple relation valid for functions on a locally compact group and reprove, as a special consequence of it, a result in R.O. IV (Lemma 4.7.1). The purpose of this section is to establish the following extension of the cited result.

THEOREM 1. *For a given* $\epsilon > 0$, *let the operator* A *in a factor* $\mathfrak{M}$ *of type* II_1 (*or* I_n) *satisfy the following conditions*[4]

$$(i) \qquad [[AX - XA]] \leq \epsilon \|X\| \text{ for all } X \text{ in } \mathfrak{M},$$
$$(ii) \qquad T(A) = 0.$$

Then $[[A]] \leq \epsilon$.

LEMMA 1. *Let* G *be a locally compact group and* φ *an integrable and square integrable function on* G. *Define* $\|\varphi\|^2 = \int_G |\varphi(g)|^2 dg$, *where* dg *refers to left-invariant Haar measure on* G. *Introduce the translated functions* φ_s: $\varphi_s(g) = \varphi(s^{-1}g)$, s *in* G. *If* $\int_G \varphi(g)dg = 0$ *then*

$$\int_G (\|\varphi - \varphi_s\|^2 - 2\|\varphi\|^2)\, ds = 0.$$

Hence, for some a *and* b *in* G,

$$\|\varphi - \varphi_a\|^2 \leq 2\|\varphi\|^2 \leq \|\varphi - \varphi_b\|^2.$$

Proof. Since $\|\varphi_s\|^2 = \|\varphi\|^2$,

$$\|\varphi - \varphi_s\|^2 - 2\|\varphi\|^2 = -2\mathrm{Re} \int_G \varphi(g)\overline{\varphi(s^{-1}g)}\, dg.$$

Defining $\tilde{\varphi}(g)$ to be $\overline{\varphi(g^{-1})}$, the integral on the right side becomes the convolution $\varphi * \tilde{\varphi}$ at s; and

$$\int_G \varphi * \tilde{\varphi}(s)\, ds = \int_G \varphi(s)\, ds \cdot \int_G \tilde{\varphi}(s)\, ds = 0.$$

LEMMA 2 (*cf. R.O. IV, Lemma 4.7.1*). *For a given* $\epsilon > 0$, *let a self-adjoint operator* A *in a factor* $\mathfrak{M}$ *of type* I_n ($n = 1, 2, \ldots$) *satisfy the following conditions*

$$(i) \qquad [[AX - XA]] \leq \epsilon \|X\| \qquad \text{for all } X \text{ in } \mathfrak{M}$$

(*ii*) $T(A) = 0.$

Then $[[A]] \leq \epsilon/\sqrt{2}.$

Proof. We can assume that $\mathfrak{N}$ is the full ring $\mathcal{B}_n$ of linear transformations on n-dimensional unitary space, since algebraic $*$ isomorphisms between $\mathfrak{N}$ and $\mathcal{B}_n$ preserve trace[5] and bound. Choose a basis $e_1, e_2, \ldots, e_n$ consisting of mutually orthogonal unit eigenvectors for A. Then $A = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Introduce the permutation matrices $U, U^2, \ldots$ by $U^s e_g = e_{g-s}$ where subscripts are read modulo n. Then $U^{-s}AU^s = \text{diag}\ \{\lambda_{1-s}, \lambda_{2-s}, \ldots, \lambda_{n-s}\}$. Since $[[VB]] = [[B]]$ for arbitrary B, and unitary V, both in $\mathfrak{M}$, we infer from (*i*) that $[[AU^s - U^sA]]^2 = [[U^{-s}AU^s - A]]^2 = (1/n) \sum_{g=1}^{n} |\lambda_{g-s} - \lambda_g|^2 \leq \epsilon^2$. In Lemma 1, take G as the additive group of integers modulo n, and let $\varphi(g) = \lambda_g$. It follows, then, that

$$2[[A]]^2 = 2(1/n) \sum_{g=1}^{n} |\lambda_g|^2 \leq (1/n) \sum_{g=1}^{n} |\lambda_g - \lambda_{g-s}|^2 \leq \epsilon^2, \text{ for some } s.$$

Proof of Theorem 1. It suffices to prove that (*i*) and (*ii*) imply $[[A]] \leq \epsilon/\sqrt{2}$ for A self-adjoint. In fact, for arbitrary A, write $A_1 = (1/2)(A + A^*)$, $A_2 = (1/2i)(A - A^*)$, and observe that

$$[[A]]^2 = T(A^*A) = [[A_1]]^2 + [[A_2]]^2 \quad (\text{since } T(A_1A_2) = T(A_2A_1)).$$

Moreover, $[[A_jX - XA_j]] \leq \epsilon\|X\|$, so that $[[A_j]] \leq \epsilon/\sqrt{2}, j = 1, 2$; and thus $[[A]] \leq \epsilon$.

Given $\delta > 0$, one can determine real numbers λ_k and orthogonal spectral projections E_k, $k = 1, 2, \ldots, m$, for the self-adjoint operator A such that $\sum_{k=1}^{m} E_k = I$ and

$$\|A - \sum_{k=1}^{m} \lambda_k E_k\| \leq \delta.$$

Let $c = \sum_{k=1}^{m} |\lambda_k|$ and choose an integer n greater than $(c/\delta)^2$. Express each E_k as a sum of orthogonal projections $E_k^{(j)}$ in $\mathfrak{M}$, all of relative dimension $1/n$, and a residual projection R_k of dimension less than $1/n$. Let $R = \sum_{k=1}^{m} R_k$: then $D(R) = r/n$ for some integer r less than n. Decompose R into a sum of r orthogonal projections P_i in $\mathfrak{M}$, each of dimension $1/n$, and define $B = \sum_{k=1}^{m} \lambda_k(E_k - R_k)$. Then

$$[[B - A]] \leq \delta + \sum_{k=1}^{m} |\lambda_k| \cdot [[R_k]] \leq \delta + c/\sqrt{n} \leq 2\delta.$$

Since the n projections $E_k^{(j)}$ and P_i are orthogonal, equivalent and have sum I, they lie in some subfactor $\mathfrak{N}$ of $\mathfrak{M}$, $\mathfrak{N}$ of type I_n ($\mathfrak{N}$ consists of all linear combinations of the members of a system of $n \times n$ matrix units, in $\mathfrak{M}$, based on the above n projections; cf. R.O. IV, §2.6). Note that B is in $\mathfrak{N}$ and that

$$[[BX - XB]] \leq [[AX - XA]] + 2[[B - A]]\cdot\|X\| \leq (\epsilon + 4\delta)\|X\|$$

for arbitrary X in $\mathfrak{M}$ and, *a fortiori*, in $\mathfrak{N}$. This same inequality obtains if B is altered by subtracting from it the central operator $T(B)\cdot I$.

Since the trace and norm in $\mathfrak{M}$, when restricted to $\mathfrak{N}$, coincide with the trace and norm in $\mathfrak{N}$, it follows, from the preceding lemma, that

$$[[B - T(B)I]] \leq (\epsilon + 4\delta)/\sqrt{2}.$$

From the Cauchy-Schwarz inequality, $|T(X^*Y)| \leq [[X]]\cdot[[Y]]$, one obtains

$$|T(B)| = |T(B - A)| \leq [[B - A]]\cdot[[I]] \leq 2\delta$$

(recall that $T(A) = 0$). The above inequalities imply that

$$[[A]] \leq [[A - B]] + [[B - T(B)I]] + |T(B)| \leq 2\delta + (\epsilon + 4\delta)/\sqrt{2} + 2\delta$$

for arbitrarily small δ. Hence $[[A]] \leq \epsilon/\sqrt{2}$ as asserted.

3. *Maximal Approximately Finite Factors.*—With the aid of Theorem 1, we are in a position to prove the key result. The proof is carried out by methods similar to those employed in R.O. IV (Theorem XIII, p. 780) for the construction of approximately finite subfactors.

THEOREM 2. *In a factor $\mathfrak{M}$ of type II_1, each infinite family $\{\mathcal{P}_\alpha\}$ of subfactors which is simply ordered by inclusion, generates a subfactor $\mathcal{P}$ of $\mathfrak{M}$, $\mathcal{P}$ of type II_1. If, in addition, each $\mathcal{P}_\alpha$ is either approximately finite or in case I then $\mathcal{P}$ is approximately finite.*

Each factor of type II_1 contains a subfactor which is maximal with respect to the property of being approximately finite.

Proof. The set theoretical union $\mathcal{S}$ of all $\mathcal{P}_\alpha$ is a self-adjoint subalgebra of $\mathfrak{M}$. In R.O. IV (Theorem I, p. 728) it is shown that $\mathcal{P}$ is the closure of $\mathcal{S}$ in the metric topology on $\mathfrak{M}$ induced by the norm "[[ ]]." We show that $\mathcal{P}$ is a factor. In fact, let A be an operator in the center of $\mathcal{P}$. Choose ϵ positive and determine an operator B in some $\mathcal{P}_\beta$ such that $[[B - A]] < \epsilon$. For each X in $\mathfrak{M}$, in particular for X in $\mathcal{P}_\beta$, one has

$$[[(B - T(B)I)X - X(B - T(B)I)]] = [[(B - A)X - X(B - A)]] \leq 2\epsilon\|X\|.$$

Applying Theorem 1, we see that $[[B - T(B)I]] \leq 2\epsilon$. Now $|T(B) - T(A)| \leq [[B - A]] \leq \epsilon$ so that

$$[[A - T(A)I]] \leq [[A - B]] + [[B - T(B)I]] + |T(B) - T(A)| \leq 4\epsilon$$

for arbitrary small ϵ. Thus $A = T(A)I$, and $\mathcal{P}$ is a factor.

Since $\mathfrak{M}$ is finite, $\mathcal{P}$ is of type II_1 (type I_n is excluded, for $\mathcal{P}$ is infinite dimensional as a vector space). Under the additional assumption that each $\mathcal{P}_\alpha$ is approximately finite or in case I we can conclude that $\mathcal{P}$ is approximately finite. In fact, given $\epsilon > 0$ and operators $A_1, \ldots, A_m$ in $\mathcal{P}$, find $B_1, \ldots, B_m$ in some $\mathcal{P}_\gamma$ such that $[[B_i - A_i]] \leq \epsilon/2$. Applying the criterion (A) (cf. R.O. IV, Definition 4.3.1, and Theorem XII, p. 778) for approximate finiteness, we determine $C_1, \ldots, C_m$ in some case I subfactor of $\mathcal{P}_\gamma$, such that $[[C_i - B_i]] \leq \epsilon/2$. Then $[[A_i - C_i]] \leq \epsilon$, and $\mathcal{P}$ is approximately finite (now by criterion (B); R.O. IV, Definition 4.5.2).

The last assertion of the theorem follows, now, from the fact that each factor of type II_1 contains an approximately finite subfactor (cf. R.O. IV, Theorem XIII, p. 780) and an application of Zorn's Lemma.

4. Non-normalcy in Case II.—The following theorem contains the principal result of this note.

THEOREM 3. *No factor in case II is normal. If $\mathfrak{M}$ is of type II_1 and not approximately finite, and $\mathcal{A}$ is a maximal approximately finite subfactor of $\mathfrak{M}$, then $(\mathcal{A}'_{\mathfrak{M}})'_{\mathfrak{M}}$ contains $\mathcal{A}$ properly.*

Proof. Consider first the case where $\mathfrak{M}$ is of type II_1. The non-normalcy of approximately finite factors was noted by F. J. Murray and J. v. Neumann, who proved that all approximately finite factors are algebraically isomorphic (R.O. IV, Theorem XIV, p. 781) and exhibited specific approximately finite factors which are non-normal, and indeed have a subfactor which violates normalcy (R.O. I, p. 209; and R.O. IV, Lemma 5.2.3, p. 787). We may assume, therefore, that $\mathfrak{M}$ is not approximately finite and select, by Theorem 2, a maximal approximately finite (proper) subfactor $\mathcal{A}$.

The proof proceeds by contradiction. Indeed, suppose that $(\mathcal{A}'_{\mathfrak{M}})'_{\mathfrak{M}} = \mathcal{A}$. This implies, in the first place, that $\mathcal{A}'_{\mathfrak{M}}$ is a factor. In fact, if A is in the center of $A'_{\mathfrak{M}}$, i.e., $A \in (\mathcal{A}'_{\mathfrak{M}})'_{\mathfrak{M}}$ $(= \mathcal{A})$, then A belongs to the center, $\{\alpha I\}$, of $\mathcal{A}$. In the second place, the assumption that $(\mathcal{A}'_{\mathfrak{M}})'_{\mathfrak{M}} = \mathcal{A}$ implies that $\mathcal{A}'_{\mathfrak{M}} \neq \{\alpha I\}$. Hence the factor $\mathcal{A}'_{\mathfrak{M}}$ contains a subfactor $\mathfrak{N}$ of type I_n, for some $n \geq 2$ (cf. R.O. IV, Lemma 2.6.2). The ring $\mathcal{P}$ generated by $\mathcal{A}$ and $\mathfrak{N}$ is an approximately finite subfactor of $\mathfrak{M}$ which contains $\mathcal{A}$ properly. This follows (cf. R.O. I, §2; and R.O. IV, Lemma 4.8.2) from the fact that $\mathcal{P}$ is algebraically the Kronecker product of the approximately finite factor $\mathcal{A}$ and the total matrix ring $\mathcal{B}_n$ (note that $\mathfrak{N}$ commutes with $\mathcal{A}$ and is algebraically isomorphic to $\mathcal{B}_n$). An alternative proof is obtained by applying the criterion for approximate finiteness formulated as Definition 4.1.1. in R.O. IV. The existence of such a $\mathcal{P}$ violates the maximality of $\mathcal{A}$, and we conclude that $(\mathcal{A}'_{\mathfrak{M}})'_{\mathfrak{M}}$ contains $\mathcal{A}$ properly.

It remains to establish the non-normalcy of factors of type II_∞. Each such factor $\mathfrak{M}$ is representable as an infinite matrix ring over a factor $\mathcal{P}$ of type II_1 in the sense of a Kronecker product $\mathcal{P} \otimes \mathcal{B}$ (cf. R.O. IV, Theorem IX, p. 746). Let $\mathfrak{Q}$ be a subfactor of $\mathcal{P}$ such that $(\mathfrak{Q}'_{\mathcal{P}})'_{\mathcal{P}} \neq \mathfrak{Q}$. Then the subfactor, $\mathfrak{Q} \otimes \mathcal{B}$, of $\mathfrak{M}$, obtained by restricting the coefficients of the matrix representation of $\mathfrak{M}$ to $\mathfrak{Q}$, violates the normalcy of $\mathfrak{M}$, as verified by a simple computation. This completes the proof.

In conclusion we note that J. v. Neumann[6] has established the existence of non-normal factors in case III. The techniques applied in the present note do not, however, seem to yield further information in the case III situation.

* The second named author is a National Research Fellow.

[1] Murray, F. J., and Neumann, J. v., "On Rings of Operators," *Ann. Math.*, **37**, 116–229 (1936). We shall refer to this paper as R.O. I, and to the paper "On Rings of Operators IV," *Ibid.*, **44**, 716–808 (1943), by the same authors, as R.O. IV.

[2] "A ring of operators" is a weakly closed, self-adjoint algebra of bounded, linear transformations on a Hilbert space, which contains the identity operator I.

[3] Neumann, J. v., "Zur Algebra der Funktionaloperatoren," *Math. Ann.*, **102**, 370–427 (1929).

[4] We denote by "$T(A)$" the *trace* of the operator A and by "$[[A]]$" the norm, $(T(A^*A))^{1/2}$, of A. Cf., Murray, F. J., and Neumann, J. v., "On Rings of Operators II," *Trans. Am. Math. Soc.*, **41**, 208–248 (1937); see especially pp. 218, 219 and 241 for the properties of the trace and norm.

[5] For a matrix $A = (a_{ij}),\ i, j = 1, \ldots, n;\ T(A)$ is the normalized trace $(\Sigma a_{ii})/n$, since $T(I) = 1$. Similarly $[[A]]^2 = (\Sigma|a_{ij}|^2)/n$.

[6] Neumann, J. v., "On Rings of Operators III," *Ann. Math.*, **41**, 94–161 (1940). See especially pp. 159–161.

ON DETERMINANTS AND A PROPERTY OF THE TRACE IN FINITE FACTORS

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1. Introduction.—In this note the authors wish to outline a theory of determinants in a finite factor. This theory originated in an attempt to prove that the trace[1] of a generalized nilpotent operator is zero. The properties of the determinant, which we shall derive, will allow us to prove, more generally, that the trace of an arbitrary operator lies in the convex hull of its spectrum.

There is no difficulty in proving that the trace of a *proper* nilpotent is zero or that the trace of a *normal* operator lies in the convex hull of its spectrum. For arbitrary finite matrices, this latter result is proved by