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ISOMETRIES OF OPERATOR ALGEBRAS[1]

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1. Introduction

Well known results of Banach [1][2] and M. H. Stone [8] determine all linear isometric maps of one $C(X)$ onto another (where '$C(X)$' denotes, throughout this paper, the set of all real-valued, continuous functions on the compact Hausdorff space X). Such isometries are the maps induced by homeomorphisms of the spaces involved followed by possible changes of sign in the function values on the various closed and open sets. An internal characterization of these isometries would classify them as an algebra isomorphism of the $C(X)$'s followed by a real unitary multiplication, i.e., multiplication by a real continuous function whose absolute value is 1. The situation in the case of the ring of complex continuous functions (which we denote by '$C'(X)$' throughout) is exactly the same; the real unitary multiplication being replaced, of course, by a complex unitary multiplication.

It is the purpose of this paper to present the non-commutative extension of the results stated above. A comment as to why this noncommutative extension takes form in a statement about algebras of operators on a Hilbert space seems to be in order. The work of Gelfand-Neumark [2] has as a very particular consequence the fact that each $C'(X)$ is faithfully representable as a self-adjoint, uniformly closed algebra of operators (C^*- algebra) on a Hilbert space. The representing algebra of operators is, of course, commutative. A statement about the norm and algebraic structure of $C'(X)$ finds then its natural non-commutative extension in the corresponding statement about not necessarily commutative C^*- algebras.

A cursory examination shows that one cannot hope for a word for word transference of the $C'(X)$ result to the non-commutative situation. An isometry between operator algebras is as likely to be an anti-isomorphism as an isomorphism. The direct sum of two C^*- algebras, which is again a C^*- algebra, by [2], with an automorphism in one component and an anti-automorphism in the other shows that isomorphisms and anti-isomorphisms together do not encompass all isometries. It is slightly surprising, in view of these facts, that any orderly classification of the isometries of a C^*- algebra is at all possible. It turns out, in fact, that all isometric maps are composites of a unitary multiplication and a map preserving the C^*- or quantum mechanical structure (see Segal [7])—of the operator algebra in question. More specifically, such maps are linear isomorphisms which commute with the $*-$ operation and are multiplicative on powers, composed with a multiplication by a unitary operator in the algebra.

[1] This paper was written while the author was a National Research Fellow.

[2] Numbers in brackets refer to the bibliography at the end of this paper.

The methods previously developed for handling $C(X)$ are entirely abandoned. They rely on intimate use of the relations between X and $C(X)$. The analogous procedure in the non-commutative case would involve dealing with the state space of the operator algebra (see Segal [6]), i.e., the space of self-adjoint, positive, normalized, linear functionals, and the pure states (extreme points) therein. The sparseness of knowledge concerning the pure states makes this procedure seem very difficult. Our investigation is actually carried out along lines which develop an intrinsic proof. The $C'(X)$ result is not used, and our proof yields in that case an intrinsic proof which is perhaps simpler than the original one.

This paper falls naturally into two parts. In §2 an internal study of C^*- algebras is made to determine the nature of the extreme points on their unit spheres. In §3 these theorems are applied to determine the isometries of C^*- algebras, and the paper concludes with a result concerning the possibility of distinguishing between factors by their Banach space structure alone.

The problem of extending the isometry theorem to the non-commutative case was posed to the author by I. Kaplansky who conjectured a result close to that which is actually the case.

2. Extreme points on the unit sphere

Since the isometries of one Banach space onto another carry the extreme points of the unit sphere in the first space onto the extreme points of the unit sphere in the second space, a knowledge of the nature of these extreme points in each of the spaces would seem to provide a useful tool in classifying the possible isometries. The following theorem provides us with this knowledge in the case of a C^*- algebra.

We note, at this point, that all C^*- algebras to which we refer throughout this paper contain the identity operator. When we use the term "B^*- algebra", we refer to the algebras of [2] satisfying the conditions $1'$–$5'$ of that paper, i.e., Banach algebras with a $*$-operation such that $\| a^*a \| = \| a \|^2$.

THEOREM 1. *The set of extreme points of the unit sphere $\mathfrak{S}$ of a C^*- algebra $\mathfrak{A}$ is exactly the set of partially isometric operators of $\mathfrak{A}$ of the form U, where $U^*U = E$, $UU^* = F$ and $(I - F)\mathfrak{A}(I - E) = (0)$. The only normal extreme points are the unitary operators in $\mathfrak{A}$ and these operators are the only extreme points with inverses.*

LEMMA 2. *In any B^*- algebra with identity e, e is an extreme point of the unit sphere.*

PROOF. Suppose $e = \frac{1}{2}(a + b)$; then $e^* = e = \frac{1}{2}(a^* + b^*)$, so that $e = \frac{1}{2}[\frac{1}{2}(a + a^*) + \frac{1}{2}(b + b^*)] = \frac{1}{2}(c + d)$, where a, b, c, d are all in the unit sphere and c and d are self-adjoint. Since $d = 2e - c$, d and c commute. Representing the real B^*-algebra generated by e, c, d as a $C(X)$, we see that $c = d = e$, i.e., $\frac{1}{2}(a + a^*) = \frac{1}{2}(b + b^*) = e$ (the lemma is clear for function spaces). Again $a^* = 2e - a$, so that a is normal, and, passing to the function space, this time a $C'(X)$, shows us that $a = a^* = e$. It follows that e is an extreme point of the unit sphere.

PROOF OF THEOREM 1. We show first that a partially isometric operator U of $\mathfrak{A}$ with initial space $E\mathfrak{H}$ ($\mathfrak{H}$ the Hilbert space) and final space $F\mathfrak{H}$ (see Murray-von Neumann [4]) such that $(I - F)\mathfrak{A}(I - E) = (0)$ is an extreme point of $\mathfrak{S}$. In fact, suppose $U = \frac{1}{2}(A + B)$ with A and B in $\mathfrak{S}$. Then $E = U^*U = \frac{1}{2}(U^*A + U^*B)$ and $E = \frac{1}{2}(U^*AE + U^*BE)$. Since $EU^* = U^*$, both U^*AE and U^*BE lie in the B^*-algebra $E\mathfrak{A}E$ whose identity is E. By Lemma 2, $U^*AE = U^*BE = E$. With x in $E\mathfrak{H}$, $U^*AEx = U^*Ax = Ex = x$. Now, since A is in $\mathfrak{S}$, $\| Ax \| \leqq \| x \|$. But $\| U^* \| = 1$, so that $\| Ax \| = \| x \|$; moreover U^* is norm preserving on $F\mathfrak{H}$ alone, so that Ax lies in $F\mathfrak{H}$. However, $U^*Ux = Ex = x$; and, since U^* is 1–1 on $F\mathfrak{H}$, $Ax = Ux$. Thus $AE = U$, and, by symmetry, $BE = U$. Now $U^* = \frac{1}{2}(A^* + B^*)$, so that, by symmetry again, $A^*F = B^*F = U^*$ or $FA = FB = U$. Our hypothesis, $(0) = (I - F)\mathfrak{A}(I - E)$, tells us that $A = FA(I - E) + AE = FB(I - E) + BE = B$ so that U is extreme.

Suppose now that T is an extreme point of $\mathfrak{S}$. We show first that T is a partial isometry by showing T^*T to be a projection. Consider the commutative, real C^*- algebra generated by I and T^*T; and its representing $C(X)$. Denoting operators and their representing functions by the same symbol, T^*T is a positive function of norm 1 (T, being extreme on $\mathfrak{S}$, has norm 1). Suppose that at some point x of X, T^*T takes a non-zero value less than one. It is clear that one can construct a function C small in a small neighborhood of x, vanishing outside this neighborhood, and non-zero at x such that if $R = I + C$, $S = I - C$, then $1 = \| T^*TR^2 \| = \| T^*TS^2 \|$. Thus $\| TR \|^2 = \| (TR)^*(TR) \| = \| R^*T^*TR \| = \| T^*TR^2 \| = 1 = \| TS \|^2$, so that TR and TS are in $\mathfrak{S}$. But $T = \frac{1}{2}(TR + TS)$, and, since T is extreme on $\mathfrak{S}$, $T = TR = T + TC$. Hence $TC = 0$, so that $T^*TC = 0$ contrary to the fact that both T^*T and C are non-zero at x. Thus T^*T takes no values different from 0 and 1 and is therefore a projection. It is an algebraic consequence of the fact that T^*T is a projection that TT^* is a projection, however, it is appropriate to note at this point that the $*-$ operation is a real linear isometry of $\mathfrak{A}$ onto $\mathfrak{A}$ so that T^* as well as T is extreme on $\mathfrak{S}$. The above argument then shows that $T^{**}T^* = TT^*$ is a projection.

It remains now to show that T is a partial isometry of the described type. Let $E = T^*T$ and $F = TT^*$, and suppose A is in $(I - F)\mathfrak{A}(I - E)$. We may of course assume A to be of norm less than 1. Let $z = x + y$ be of norm 1 with x in the range of E and y in the range of $I - E$; then $(T \pm A)z = Tx \pm Ay = FTx \pm (I - F)Ay$ so that $\| (T \pm A)z \| \leqq 1$ and thus $\| T \pm A \| \leqq 1$. But $T = \frac{1}{2}(T + A + T - A)$ with $T + A$ and $T - A$ in $\mathfrak{S}$. Thus $T + A = T$ or $A = 0$ so that $(I - F)\mathfrak{A}(I - E) = (0)$ as asserted.

With either F or E the identity, we have $(I - F)\mathfrak{A}(I - E) = 0$, so that "semi-unitary" operators (i.e. partially isometries U such that U^*U or UU^* is I) are always extreme. For further reference, we note that multiplication by a unitary operator U is an isometry of $\mathfrak{A}$ onto itself, for $\| UA \|^2 = \| (UA)^*(UA) \| = \| A^*U^*UA \| = \| A^*A \| = \| A \|^2$. Since I is extreme, $UI = U$ is, and we have another proof that the unitary operators of $\mathfrak{A}$ are extreme. Moreover, if U is normal and extreme on $\mathfrak{S}$, then by passing to the $C'(X)$ representing the C^*- algebra generated by I, U, and U^*, we see that U is unitary. In fact, if the function U was in absolute value less than 1 at some point x of X, a small function C non-zero at x and vanishing outside a small neighborhood of x could be constructed so that $U + C$ and $U - C$ were both in $\mathfrak{S}$. But then $U = \frac{1}{2}(U + C + U - C)$ with $U + C \neq U - C$. We thus have U unitary as stated.

If we know that T is extreme on $\mathfrak{S}$ and has an inverse it follows, from the above results, that T is unitary. This can be seen directly, however, from the polar decomposition $T = U(T^*T)^{\frac{1}{2}}$ with U unitary. Indeed, U^*T is extreme since multiplication by U^* is an isometry and T is extreme. Thus $(T^*T)^{\frac{1}{2}}$ is a self-adjoint extreme point and hence unitary, so that $T = U(T^*T)^{\frac{1}{2}}$ is unitary. The proof is complete.

An application of Theorem 1 to particular situations yields some special results of interest.

Rickart [5] defines "quasi-transitivity" of an algebra $\mathfrak{A}$ to mean "$A\mathfrak{A}B = 0$ if and only if A or B is 0." Applying the results of [5] (slightly modified to remove denumerability restrictions) to the case of rings of operators in the sense of Murray-von Neumann [4], we have that quasi-transitive rings are factors (central) and conversely. As a result of this, we can state:

Corollary 3. *The only extreme points of the unit sphere in a factor are the semi-unitary operators. In finite factors only the unitary operators are extreme.*

It is rather surprising that the case most closely resembling the commutative case, where only unitary operators appear as extreme points, is the totally non-commutative or factor case, and, indeed, in the finite factor situation the result is exactly the same as the commutative case.

We close this section by proving a somewhat simpler auxiliary result.

Theorem 4. *The set of extreme points of the positive portion $\mathfrak{P}$ of the unit sphere $\mathfrak{S}$ in a C^*- algebra $\mathfrak{A}$ is the set of projections in $\mathfrak{A}$.*

Proof. If T is an extreme point of $\mathfrak{P}$ it is a fortiori self-adjoint and positive. Passing to the function situation, one sees readily that T is a projection.

Suppose now that E is a projection in $\mathfrak{A}$ and that $E = \frac{1}{2}(A + B)$ with A and B in $\mathfrak{P}$. Then $2E - A = B \geqq 0$ so that $E \geqq A/2$. If $Ex = 0$ then $0 = (Ex, x) \geqq \frac{1}{2}(Ax, x) \geqq 0$ so that $(Ax, x) = 0$. But then $(A^{\frac{1}{2}}x, A^{\frac{1}{2}}x) = 0$ so that $A^{\frac{1}{2}}x = 0$ and $A^{\frac{1}{2}}A^{\frac{1}{2}}x = Ax = 0$. Thus $A(I - E) = 0$ so that $A = AE + A(I - E) = AE = (AE)^* = EA$ and by symmetry $B = EBE$. By Lemma 2, E is extreme on the unit sphere of $E\mathfrak{A}E$ so that $A = B = E$, and E is extreme on $\mathfrak{P}$.

3. The isometries

To this point, we have referred to the full algebraic structure of the operator algebra $\mathfrak{A}$ when we used the term "C^*- algebra". What is relevant, however, for the quantum-mechanical applications is the linear structure and the power structure of the self-adjoint elements of $\mathfrak{A}$ (see [7]). From the identity $(A + B)^2 - A^2 - B^2 = AB + BA$, the Jordan product of self-adjoint elements becomes meaningful, and this, together with the decomposition of the aribitrary operator A as $A = B + iC$ with B and C self-adjoint, gives meaning to the expression A^2 thus to the Jordan product of arbitrary operators and hence to arbitrary powers. This leads us to define a linear isomorphism ρ of one C^*- algebra $\mathfrak{A}$ onto (into) another $\mathfrak{A}'$ which preserves self-adjoints, their power structure, and, hence, the full Jordan structure as a "quantum mechanical isomorphism of $\mathfrak{A}$ onto (into) $\mathfrak{A}'$"—or, for brevity, a "C^*- isomorphism". In addition to the Jordan structure of $\mathfrak{A}$, it is of importance to know when two operators of $\mathfrak{A}$ commute. We shall note, as a consequence of the results of Jacobson-Rickart [9], that a C^*- isomorphism between two C^*- algebras automatically preserves commutativity. Moreover, the results of [9] make it possible to more closely determine the form of a C^*- isomorphism.

We begin with a theorem which generalizes a result of Gelfand-Neumark [2] and I. Kaplanksy [3]. These papers prove the isometric character of actual $*-$ ring isomorphisms.

THEOREM 5. *A C^*- isomorphism ρ of a C^*- algebra (B^*- algebra) $\mathfrak{A}$ onto a C^*- algebra (B^*- algebra) $\mathfrak{A}'$ is isometric and preserves commutativity.*

LEMMA 6. *With A and B in $\mathfrak{A}$, $\rho(BAB) = \rho(B)\rho(A)\rho(B)$ and $\rho((AB)^n + (BA)^n) = (\rho(A)\rho(B))^n + (\rho(B)\rho(A))^n$*

PROOF. We have observed that $\rho(AB + BA) = \rho(A)\rho(B) + \rho(B)\rho(A)$. Moreover, $\rho((A + B)^3) = (\rho(A) + \rho(B))^3 = \rho(A)^3 + \rho(A)^2\rho(B) + \rho(B)\rho(A)^2 + \rho(A)\rho(B)^2 + \rho(B)^2\rho(A) + \rho(B)^3 + \rho(ABA + BAB)$, so that $\rho(ABA + BAB) = \rho(A)\rho(B)\rho(A) + \rho(B)\rho(A)\rho(B)$. Similarly, considering $\rho((A - B)^3)$, we find $-\rho(ABA) + \rho(BAB) = -\rho(A)\rho(B)\rho(A) + \rho(B)\rho(A)\rho(B)$. Adding the last two equalities, we find $\rho(BAB) = \rho(B)\rho(A)\rho(B)$. In consequence of this, we have $\rho(B(AB)^{n-1}) = \rho(B)(\rho(A)\rho(B))^{n-1} = \rho((BA)^{n-1}B)$ so that $\rho((AB)^n + (BA)^n) = \rho(AB(AB)^{n-1} + (BA)^{n-1}BA) = \rho(A)\rho(B(AB)^{n-1}) + \rho((BA)^{n-1}B)\rho(A) = \rho(A)\rho(B)(\rho(A)\rho(B))^{n-1} + (\rho(B)\rho(A))^{n-1}\rho(B)\rho(A) = (\rho(A)\rho(B))^n + (\rho(B)\rho(A))^n$.

PROOF OF THEOREM 5. We can show $\rho(I) = I'$, the identity of $\mathfrak{A}'$; for otherwise $\rho(I)$ is a projection and $\rho(B) = \rho(IBI) = \rho(I)\rho(B)\rho(I)$, by Lemma 5, for arbitrary B in $\mathfrak{A}$, so that ρ maps $\mathfrak{A}$ into $\rho(I)\mathfrak{A}'\rho(I)$, a C^*- algebra with identity $\rho(I)$.

We show now that ρ is an order isomorphism, i.e., that ρ takes positive operators and only positive operators into positive operators. In fact, if A is positive then $A = B^2$ with B self-adjoint so that $\rho(A) = \rho(B)^2$ is positive. Suppose now that $\rho(A)$ is positive; then $\rho(|A|)^2 = \rho(|A|^2) = \rho(A^2) = \rho(A)^2$ so that $\rho(|A|) = \rho(A)$, since a positive operator has only one positive square root. (In the B^*- algebra case we take "positive" to mean "self-adjoint with non-negative spectrum." The uniqueness of positive square roots is true here from the following considerations. If $a^2 = b^2 = c$ with a and b positive then $ac = a^3 = ca$. Passing to the representing function algebra, a is the uniform limit of polynomials in c. Similarly b is the uniform limit of the same polynomials in c, so that $a = b$.) Since ρ is an isomorphism $A = |A| \geqq 0$. It follows at once that ρ is isometric on self-adjoint elements A, for both $\| \rho(A) \| I' \pm \rho(A)$ and $\| A \| I \pm A$ are positive so that, in the first case $\| \rho(A) \| I \pm A \geqq 0$ and, in the second, $\| A \| I' \pm \rho(A) \geqq 0$.

Suppose now that A is an arbitrary operator of norm 1 in $\mathfrak{A}$. Since both

$(AA^*)^n$ and $(A^*A)^n$ are self-adjoint, we have $1 = \| A \|^{4n} = \| AA^* \|^{2n} = \| (AA^*)^{2n} \| = \| \frac{1}{2}((AA^*)^n + i(A^*A)^n + ((AA^*)^n + i(A^*A)^n)^*) \| \cdot \| \frac{1}{2}((AA^*)^n - i(A^*A)^n + ((AA^*)^n - i(A^*A)^n)^*) \| \leqq \| (AA^*)^n + i(A^*A)^n \| \cdot \| (AA^*)^n - i(A^*A)^n \| = \| (AA^*)^{2n} + (A^*A)^{2n} + i((A^*A)^n(AA^*)^n - (AA^*)^n (A^*A)^n) \| = \| \rho((AA^*)^{2n} + (A^*A)^{2n}) + i\rho((A^*A)^n(AA^*)^n - (AA^*)^n (A^*A)^n) \| = \| (\rho(A)\rho(A^*))^{2n} + (\rho(A)^*\rho(A))^{2n} + i((\rho(A)^*\rho(A))^n(\rho(A)\rho(A)^*)^n - (\rho(A)\rho(A)^*)^n(\rho(A)^*\rho(A))^n) \| \leqq \| (AA^*)^{2n} \| + \| (A^*A)^{2n} \| + \| (A^*A)^n (AA^*)^n \| + \| (AA^*)^n(A^*A)^n \| \leqq 4 \| A \|^{4n} = 4$ (see Lemma 6). Thus $1 \leqq \| (\rho(A)\rho(A)^*)^{2n} + (\rho(A)^*\rho(A))^{2n} + i((\rho(A)^*\rho(A))^n(\rho(A)\rho(A)^*)^n - (\rho(A)\rho (A)^*)^n(\rho(A)^*\rho(A))^n) \| \leqq 4 \| \rho(A) \|^{4n}$ for arbitrary positive integral n so that $\| \rho(A) \| \geqq 1$. Moreover, treating $\rho(A)$ as we did A, we find that $\| \rho(A) \|^{4n} \leqq \| (\rho(A)\rho(A)^*)^{2n} + (\rho(A)^*\rho(A))^{2n} + i((\rho(A)^*\rho(A))^n(\rho(A)\rho(A)^*)^n - (\rho(A)\rho(A)^*)^n(\rho(A)^*\rho(A))^n) \| \leqq 4$ for arbitrary positive integral n so that $\| \rho(A) \| \leqq 1$. Thus for A of norm 1 in $\mathfrak{A}$, $\| \rho(A) \| = 1$. It is immediate that ρ is an isometry.

In [9; Corollary 1], it is shown that a Jordan homomorphism onto a special Jordan algebra whose enveloping algebra has a center which contains no nilpotents, preserves commutativity. Our C^*- isomorphisms are, of course, Jordan homorphisms, the enveloping algebras of the images of which are the image C^*- algebras. Since the center of a C^*- algebra is representable as a $C'(X)$, it contains no nilpotents. Thus [9] applies, and we see that ρ preserves commutativity. For completeness, we sketch the short proof of the quoted result. A simple computation shows that ρ preserves the Lie triple product $[[A, B], C]$; and the square of the Lie product, $[A, B]^2$. Thus if A and B commute, then $[[A, B], C] = 0$ for arbitrary C in $\mathfrak{A}$, so that $[[\rho(A), \rho(B)], \rho(C)] = 0$ and $[\rho(A), \rho(B)]$ is in the center of $\mathfrak{A}'$. But $[A, B]^2 = 0$ so that $[\rho(A), \rho(B)]^2 = 0$, and $[\rho(A), \rho(B)]$ is a central nilpotent. Hence $[\rho(A), \rho(B)] = 0$, or $\rho(A)$ and $\rho(B)$ commute. The proof of Theorem 5 is complete.

When the term "isometry" is employed, in the following theorem, it refers solely to a linear, norm preserving map with no restrictive conditions as regards the $*-$ structure.

THEOREM 7. *A linear isomorphism ρ of one C^*- algebra $\mathfrak{A}$ onto another $\mathfrak{A}'$ which is isometric is a C^*- isomorphism followed by left multiplication by a fixed unitary operator, viz.,* $\rho(I)$.

LEMMA 8. *A linear map η of one B^*-algebra into another which carries the identity into the identity and is isometric on normal elements preserves adjoints,* i.e., $\eta(a^*) = \eta(a)^*$.

PROOF. Suppose a is self-adjoint of norm 1 and $\eta(a) = b + ic$ with b and c self-adjoint. If c is non-zero there is a non-zero real number β in the spectrum of c; say $\beta > 0$ (otherwise consider $-a$). Then $\| a + inI \| = (1 + n^2)^{\frac{1}{2}} < \beta + n \leqq \| c + nI \| \leqq \| \eta(a + inI) \|$ for large n. Since η is isometric on normal elements, $\beta = 0$. Thus c must have zero spectrum and, being self-adjoint, is itself 0, i.e., $\eta(a)$ is self-adjoint. Now for arbitrary a, with $a = a' + ia''$, a', a'' self-adjoint, we have $\eta(a)^* = \eta(a') - i\eta(a'')$ and $\eta(a^*) = \eta(a') - i\eta(a'')$ so that $\eta(a)^* = \eta(a^*)$.

PROOF OF THEOREM 7. We show first that $\rho(I)$ is a unitary operator. Since ρ is an isometry of $\mathfrak{A}$ onto $\mathfrak{A}'$, the extreme points of the unit sphere in $\mathfrak{A}$ are mapped by ρ into extreme points of the unit sphere of $\mathfrak{A}'$. By Lemma 2, I is such an extreme-point in $\mathfrak{A}$, so that $\rho(I)$ is extreme on the unit sphere of $\mathfrak{A}'$. By Theorem 1, $\rho(I)$ is a partially isometric operator U such that

$$(I - F)\mathfrak{A}'(I - E) = 0,$$

where $U^*U = E$ and $UU^* = F$. Our task is to show that $E = F = I$. Suppose that A is an operator of $\mathfrak{A}$ such that $\| A \| = \sup \{| \lambda | : \lambda \text{ in the spectrum of } A\}$ and that $| \gamma | = \| A \|$ with γ in the spectrum of A. Then $\gamma I + A$ has 2γ in its spectrum so that $2 \| A \| = | 2\gamma | \leqq \| \gamma I + A \| \leqq | \gamma | + \| A \| = 2 \| A \|$ and thus $2 \| A \| = \| \gamma I + A \| = \| \gamma\rho(I) + \rho(A) \| = \| \gamma U + \rho(A) \|$. One can find, therefore, a sequence of unit vectors $z_n = x_n + y_n$ with $x_n = Ez_n$, $y_n = (I - E)z_n$ such that $\| (\gamma U + \rho(A))z_n \| \to 2 \| A \|$. Since $\| (\gamma U + \rho(A))z_n \| \leqq \| \gamma U z_n \| + \| \rho(A)z_n \| \leqq \| A \| + \| A \|$, we have $\| \gamma U z_n \| \to \| A \|$ and $\| \rho(A)z_n \| \to \| A \| = \| \rho(A) \|$. It follows that $\| Uz_n \| = \| Ux_n \| = \| x_n \| \to 1$ so that $\| y_n \| \to 0$. Hence

$$\begin{aligned} \| A \| = \| \rho(A) \| &\geqq \| \rho(A)x_n \| \\ &= \| \rho(A)z_n - \rho(A)y_n \| \geqq \| \rho(A)z_n \| - \| \rho(A)y_n \| \to \| A \| \end{aligned}$$

so that $\| \rho(A)x_n \| \to \| A \|$. We assert that $\gamma Ux_n - \rho(A)x_n \to 0$. In fact, by the parallelogram law,

$$\begin{aligned} &\| \gamma Ux_n - \rho(A)x_n \|^2 \\ &\quad = 2(\| \gamma Ux_n \|^2 + \| \rho(A)x_n \|^2) - \| \gamma Ux_n + \rho(A)x_n \|^2 \\ &\quad \to 4 \| A \|^2 - (2 \| A \|)^2 = 0 \end{aligned}$$

It follows now that for A as above, and, in particular, for A normal, $\| U^*\rho(A)E \| = \| A \|$. In fact, $\| A \| = \| \rho(A) \| \geqq \| U^*\rho(A)E \| \geqq \| U^*\rho(A)Ex_n \| = \| U^*\rho(A)x_n \| = \| U^*\rho(A)x_n - \gamma U^*Ux_n + \gamma U^*Ux_n \| = \| U^*(\rho(A)x_n - \gamma Ux_n) + \gamma x_n \| \to | \gamma | = \| A \|$.

Let η be defined on $\mathfrak{A}$ as, $\eta(B) = U^*\rho(B)E$. Since $EU^* = U^*$, η maps $\mathfrak{A}$ into the B^*- algebra $E\mathfrak{A}'E$, and $\eta(I) = U^*UE = E^2 = E$, the identity of $E\mathfrak{A}'E$. The preceding computation shows η to be isometric on normal operators so that η is as in Lemma 8 and preserves adjoints. Suppose now that $A + iB$, A, B self-adjoint, is such that $\rho(A + iB)$ is either $I - F$ or $I - E$. In either of these cases $\eta(A + iB) = \eta(A) + i\eta(B) = U^*\rho(A + iB)E = 0$. Thus, since $\eta(A)$ and $\eta(B)$ are self-adjoint and of the same norm as A and B respectively, we have $\eta(A) = \eta(B) = A = B = 0$, and, thus, $I - F = I - E = 0$. It follows that $\rho(I) = U$ is unitary.

Therefore $\eta = U^*\rho$ is an isometry of $\mathfrak{A}$ onto $\mathfrak{A}'$ which takes I into I'. We show η to be a C^*- isomorphism. In fact, by Lemma 8, η preserves adjoints. Moreover, η preserves order, for suppose A is of norm 1 and $A \geqq 0$, then $\eta(A)$

is self-adjoint. Now $\| A - I \| \leqq 1$, so that $\| \eta(A) - I' \| \leqq 1$. Thus $\eta(A)$ can have no strictly negative spectrum, i.e., $\eta(A)$ is positive.

We digress briefly to give an alternative ending to the proof in the ring of operator (weakly closed) case. Since η is an isometry which preserves order it preserves the set of extreme points of the positive portion of the unit sphere, viz., by Theorem 4, the projections in $\mathfrak{A}$. Since η preserves order, it preserves the lattice structure of the projections in $\mathfrak{A}$ (assuming $\mathfrak{A}$ weakly closed). Suppose E and F are two orthogonal projections in $\mathfrak{A}$. Then $E + F$ is a projection as are $\eta(E)$, $\eta(F)$, and $\eta(E) + \eta(F)$. Hence

$$(\eta(E) + \eta(F))^2 = \eta(E) + \eta(E)\eta(F) + \eta(F)\eta(E) + \eta(F) = \eta(E) + \eta(F).$$

Thus $\eta(E)\eta(F) = -\eta(F)\eta(E)$ and $\eta(E)\eta(F) = \eta(E)\eta(F)^2 = -\eta(F)\eta(E)\eta(F)$. Taking adjoints we have $\eta(F)\eta(E) = -\eta(F)\eta(E)\eta(F) = \eta(E)\eta(F) = -\eta(F)\eta(E)$ so that $\eta(F)\eta(E) = 0$. Let E and F now be arbitrary commuting projections in $\mathfrak{A}$. Then $(E - EF)(F - EF) = 0$ so that $(\eta(E) - \eta(EF))(\eta(F) - \eta(EF)) = 0$ and $\eta(E)\eta(F) - \eta(EF) - \eta(EF) + \eta(EF) = 0$, i.e., $\eta(E)\eta(F) = \eta(EF)$ (recall that $\eta(EF)$ is less than both $\eta(E)$ and $\eta(F)$). Since each self-adjoint operator in $\mathfrak{A}$ is a uniform limit of linear combinations of projections in $\mathfrak{A}$, $\eta(AB) = \eta(A)\eta(B)$ when A and B are commuting self-adjoint operators. This settles the question for the ring case but does not contribute to the general proof. We return now to the arbitrary C*— algebra $\mathfrak{A}$.

As a consequence of this theorem, we shall see that isometries preserve unitary operators. For present purposes, however, we prove a partial result in this direction, viz., η sends unitary operators U whose real part $(\frac{1}{2}(U + U^*) = A)$ is positive and invertible into unitary operators. In fact, let α be the minimum of the spectrum of A. By assumption $\alpha > 0$. Choose n so large that $2n\alpha > 1$. Then $\| U - nI \| = \sup \{((n - \lambda)^2 + 1 - \lambda^2)^{\frac{1}{2}} : \lambda \text{ in the spectrum of } A\} = \sup \{(n^2 - 2\lambda n + 1)^{\frac{1}{2}} : \lambda \text{ in the spectrum of } A\} = (n^2 - 2\alpha n + 1)^{\frac{1}{2}} < n$. But if $\eta(U)$ were not a unitary operator it would, by Theorem 1, being an extreme point of the unit sphere in $\mathfrak{A}'$, be a partially isometric operator without an inverse. In this case, $\| \eta(U) - nI' \| \geqq n > \| U - nI \|$ contradicting the isometric character of η. Thus $\eta(U)$ is unitary.

Suppose now that A is an arbitrary positive invertible operator of $\mathfrak{A}$ of norm less than or equal to 1. The unitary operator $A + i(I - A^2)^{\frac{1}{2}}$ has a positive invertible real part so that $\eta(A) + i\eta((I - A^2)^{\frac{1}{2}})$ is unitary. Since $(I - A^2)^{\frac{1}{2}}$ is positive, $\eta((I - A^2)^{\frac{1}{2}})$ is positive so that $\eta((I - A^2)^{\frac{1}{2}}) = (I - \eta(A)^2)^{\frac{1}{2}}$ (by considering the functional representation). The same conclusion can be stated for αA with α small and positive. Now, by the binomial expansion, which is applicable in the Banach algebra situation,

$$(I - (\alpha A)^2)^{\frac{1}{2}} = I - \tfrac{1}{2}\alpha^2 A^2 - \alpha^4 A^4/8 - \cdots ,$$

and

$$(I - \eta(\alpha A)^2)^{\frac{1}{2}} = I - \tfrac{1}{2}\alpha^2 \eta(A)^2 - \tfrac{1}{8}\alpha^4 \eta(A)^4 - \cdots .$$

Since η is an isometry and a fortiori continuous in the uniform topology, we have

$$\eta((I - (\alpha A)^2)^{\frac{1}{2}}) = I - \tfrac{1}{2}\alpha^2\eta(A^2) - \alpha^4\eta(A^4)/8 - \cdots$$
$$= (I - \eta(\alpha A)^2)^{\frac{1}{2}} = I - \tfrac{1}{2}\alpha^2\eta(A)^2 - \tfrac{1}{8}\alpha^4\eta(A)^4 - \cdots .$$

Thus

$$-\tfrac{1}{2}\eta(A^2) - \alpha^2\eta(A^4)/8 - \cdots = -\tfrac{1}{2}\eta(A)^2 - \alpha^2\eta(A)^4/8 - \cdots ,$$

and, letting α tend to 0, we see that $-\frac{1}{2}\eta(A^2) = -\frac{1}{2}\eta(A)^2$ or $\eta(A^2) = \eta(A)^2$. To obtain this property for A positive, invertible with arbitrary norm, we apply the result just proved to $A/2 \| A \|$. For A and B positive, invertible, and commuting, $A + B$ is positive and invertible (functional representation). Thus $\eta((A + B)^2) = (\eta(A) + \eta(B))^2$ so that $2\eta(AB) = \eta(A)\eta(B) + \eta(B)\eta(A)$. For arbitrary self-adjoint in $\mathfrak{A}$, we write $A = A \vee 0 + I - (I - A \wedge 0)$ where $A \vee 0 + I = B$ and $I - A \wedge 0 = C$ are positive, invertible, and commute. Thus $\eta(A^2) = \eta(B^2) - 2\eta(BC) + \eta(C^2) = \eta(B)^2 - \eta(B)\eta(C) - \eta(C)\eta(B) + \eta(C)^2 = (\eta(B) - \eta(C))^2 = \eta(A)^2$. Thus η is a C^*- isomorphism, and the proof is complete.

The results of this section, when applied to a single C^*- algebra, find expression in more algebraic form through the following considerations. Suppose ρ, τ are two isometries of the C^*- algebra $\mathfrak{A}$ onto itself, with the group of all quantum automorphisms of $\mathfrak{A}$ denoted by $\mathfrak{Q}$, $\mathfrak{U}$ the group of all unitary operators in $\mathfrak{A}$, and $\mathfrak{I}$ the group of all isometries of $\mathfrak{A}$ onto itself. From our preceding results, $\rho = U \cdot \eta$, $\tau = V \cdot \zeta$ with U, V in $\mathfrak{U}$ and η, ζ in $\mathfrak{Q}$. (This decomposition is obviously unique). The group $\mathfrak{U}$ acts as an automorphism group on $\mathfrak{Q}$ as follows: to U in $\mathfrak{U}$ we assign the automorphism $\eta(\cdot) \rightarrow U^*\eta(\cdot)U = U(\eta)$ of $\mathfrak{Q}$ onto itself. If for a group G we denote by $G\#$ the anti-isomorphic group (i.e., the elements of G with the multiplication $a \cdot b = ba$), we can state

COROLLARY 9. *$\mathfrak{I}\#$ is the semi-direct product of $\mathfrak{U}\#$ and $\mathfrak{Q}$.*

PROOF. In fact, $\rho = (U, \eta)$, $\tau = (V, \zeta)$, and $\rho \cdot \tau = \tau\rho = V\zeta U\eta = VU(U^*\zeta U)\eta = U \cdot V(U^*\zeta U)\eta = (U \cdot V, U(\zeta)\eta)$; with the notation as established above.

It would seem appropriate, at this point, to investigate, in a more detailed fashion, the nature of C^*- isomorphisms. A complete examination would, of course, entail the determination of all isomorphisms and anti-isomorphisms of C^*- algebras, a program far beyond the scope of this paper. We propose, rather than such a complete investigation, the program of relating the C^*- isomorphisms to the more tractible $*-$ isomorphisms and $*-$ anti-isomorphisms. Again the author was fortunate in having the burden of the algebraic portions of this study carried by [9]. Since Jacobson-Rickart propose to consider this question in the case of rings with an involution (which, in particular, includes our situation), we confine our attention to those results which lie close to the work of [9] and Kaplansky [10].

We say that a C^*- isomorphism ρ of $\mathfrak{A}$ onto $\mathfrak{A}'$ is the sum of the $*-$ isomorphism η and $*-$ anti-isomorphism φ when $\rho = \eta + \varphi$ as linear maps and there exist C^*- subalgebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ of $\mathfrak{A}$ and $\mathfrak{A}_1'$, $\mathfrak{A}_2'$ of $\mathfrak{A}'$ such that $\mathfrak{A}$ is the direct sum of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, $\mathfrak{A}'$ is the direct sum of $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$, η is a $*-$ isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_1'$ and is 0 on $\mathfrak{A}_2$, and φ is a $*-$ anti-isomorphism of $\mathfrak{A}_2$ onto $\mathfrak{A}_2'$ and is 0 on $\mathfrak{A}_1$.

THEOREM 10. *A C^*- isomorphism ρ of a ring of operators $\mathfrak{A}$ onto a C^*- algebra $\mathfrak{A}'$ is the sum of a $*-$ isomorphism and a $*-$ anti-isomorphism.*

PROOF. Suppose first that we can decompose $\mathfrak{A}$ into a direct sum of $*-$ algebras such that the restriction of ρ to each of these algebras decomposes into a sum of a $*-$ isomorphism and a $*-$ anti-isomorphism. It is clear then, from simple algebra, that ρ is such a sum. Suppose again that in a ring of operators $\mathfrak{A}_1$ it is possible to halve the identity operator I, i.e., there exist orthogonal equivalent projections E, F in the ring such that $E + F = I$. Then $\mathfrak{A}_1$ is ring isomorphic to a 2×2 matrix ring. In fact, if U is a partial isometry such that $U^*U = E$ and $UU^* = F$, then $E = e_{11}$, $F = e_{22}$, $U = e_{21}$, $U^* = e_{12}$ are a set of matrix units, and $\mathfrak{A}_1$ is isomorphic to the 2×2 matrix ring over the ring of elements in $\mathfrak{A}_1$ which commute with E, F, U, and U^*. As a consequence of the results of [9], we shall see that a C^*- isomorphism of such a matrix ring is the sum of a $*-$ isomorphism and a $*-$ anti-isomorphism. Now the results of Kaplanksy [10], allow us to decompose an operator ring $\mathfrak{A}$ into the direct sum of the following five summands: two parts of type I and type II each, a finite and a purely infinite part, and a summand of type III, which is purely infinite. (A ring of operators is finite if the identity is equivalent to no properly smaller projection in the ring; infinite otherwise; and purely infinite if every non-zero central projection is infinite. A ring $\mathfrak{A}$ is of type I if every direct summand has an abelian projection, i.e., a projection E such that $E\mathfrak{A}E$ is abelian, type II if it has no abelian projections and every direct summand has a finite projection; type III if all projections are infinite). Kaplanksy shows that in rings with no abelian projections and in purely infinite rings the identity can be halved. He also shows that finite rings of type I are direct sums of finite matrix rings over a commutative algebra with identity. Thus rings fall into two easily managed portions, one in which the identity can be halved and so is a 2×2 matrix ring and the other a finite ring of type I, which is a direct sum of finite matrix rings. Our initial remarks apply and the proof is complete once we make the slight modification of the results of [9] necessary for our situation.

In [9] it is shown that a Jordan homomorphism of an nxn matrix ring $D_n (n \geqq 2)$ over an arbitrary ring D with identity is the sum of a homomorphism and an anti-homomorphism (say J_1 and J_2 respectively), i.e., the image ring is the direct sum of two rings R_1 and R_2 such that J_1 is a homomorphism of D_n into R_1, J_2 is an anti-homomorphism of D_n into R_2, and $J = J_1 + J_2$ as linear maps. If J is a Jordan isomorphism, it is clear, from linear space considerations alone that D_n is the direct sum of $D_n^{(1)} = J^{-1}(R_1)$ and $D_n^{(2)} = J^{-1}(R_2)$ as linear spaces, and that $J = J_1$ on $D_n^{(1)}$ and $J = J_2$ on $D_n^{(2)}$. From this it follows that $D_n^{(1)}$ and $D_n^{(2)}$ are indeed subrings of D_n, that J_1 is an isomorphism of $D_n^{(1)}$ onto R_1, J_2

an anti-isomorphism of $D_n^{(2)}$ onto R_2, and J is the sum of J_1 and J_2 in the sense which we defined before (applied to C^*- isomorphisms). It remains for us to prove that the $*-$ operation behaves properly under the isomorphisms and anti-isomorphisms exhibited. In the proof of the quoted result of [9] two central idempotents g and h are constructed in the envelope of the image ring of J, and J_1, J_2 are defined by $J_1(x) = J(x)g$, $J_2(x) = J(x)h$. In our situation we have the C^*- isomorphism ρ expressed as a sum of the isomorphism η and the anti-isomorphism φ where the image of ρ and the envelope of the image of ρ coincide and are the C^*- algebra $\mathfrak{A}'$. The central idempotents E, F, given by the proof in [9], such that $\eta(A) = \rho(A)E$, $\varphi(A) = \rho(A)F$, commute, in particular, with their adjoints so that they are necessarily self-adjoint (normal idempotents) and hence projections. Thus for arbitrary A, $\eta(A)^* = E\rho(A)^* = E\rho(A^*) = \rho(A^*)E = \eta(A^*)$ so that η and, similarly, φ are $*-$ preserving maps. The proof is complete.

Since factors (central rings of operators) are indecomposable in the sense of having no proper direct summands, we have as an immediate consequence of Theorem 10:

Corollary 11. *Any C^*- isomorphism of a factor is either a $*-$ isomorphism or a $*-$ anti-isomorphism.*

We turn our attention now to an examination of isometries in factors. First, however, we prove a preliminary lemma.

Lemma 12. *If ρ is an isometry of the C^*- algebra $\mathfrak{A}$ onto the C^*- algebra $\mathfrak{A}'$ and U is a unitary operator of $\mathfrak{A}$ then $\rho(U)$ is unitary.*

Proof. By Theorem 7, $\rho = V\cdot\eta$ with η a C^*- isomorphism and V unitary in $\mathfrak{A}'$. It suffices, therefore, to show that $\eta(U)$ is unitary. But $I = \eta(UU^*) = \frac{1}{2}(\eta(U)\eta(U)^* + \eta(U)^*\eta(U))$, and, by the extreme point property of I, $I = \eta(U)\eta(U)^* = \eta(U)^*\eta(U)$ so that $\eta(U)$ is unitary.

By a conjugate isometry we shall mean a conjugate linear map which is isometric. The relation "x is the image of y under some isometry or conjugate isometry of $\mathfrak{A}$" is clearly an equivalence relation on the set of extreme points of the unit sphere in the operator algebra $\mathfrak{A}$. The extreme points fall into disjoint sets under this equivalence relation. We shall refer to such an equivalence class of extreme points as "an extreme point class". We shall also say "A can be reached from B" when the operator A is the image of B under some isometry of $\mathfrak{A}$. With this terminology established, we state:

Theorem 13. *Factors of type I_n and II_1 have exactly one extreme point class, the set of all unitary operators they contain. Factors of type I_∞ (considered algebraically and isometrically, now, as all bounded operators of some Hilbert space) have an extreme point class corresponding to each cardinal number up to the dimension of the Hilbert space. Each such class, other than the unitary operators, consists of all those proper semi-unitary operators in the factor whose initial or final spaces (whichever is not the whole Hilbert space) have their complementary manifolds of dimension a fixed cardinal (the cardinal corresponding to the extreme point class). In the case of separable Hilbert space, factors of type III have exactly two extreme point classes;*

the unitary operators forming one class and the proper semi-unitary operators the other. Factors of type II_∞ have at least three extreme point classes: the unitary operators form one class, proper semi-unitary operators whose initial or final spaces (whichever is not the whole Hilbert space) have their complementary projections of finite relative dimension and those for which the complementary projections have infinite relative dimension do not lie in the same extreme point class (in the separable case, the latter set forms an extreme point class).

PROOF. It is clear that each unitary operator of a factor $\mathfrak{A}$ can be reached from every other unitary operator of $\mathfrak{A}$. Moreover, Lemma 12 shows that no operator other than a unitary operator can be reached from a unitary operator. It follows that the set of unitary operators in a factor always forms an extreme point class. Since the only extreme points of the unit sphere in finite factors are the unitary operators, we have the first assertion of this theorem.

Suppose U and V are semi-unitary operators in the factor $\mathfrak{A}$ with $UU^* = E$, $VV^* = F$, E and F different from I and $I - E$ equivalent to $I - F$. Then V can be reached from U. In fact, suppose W is a partial isometry in $\mathfrak{A}$ mapping the range of $I - E$ onto the range of $I - F$ and the range of E into (0). A simple computation shows that $W + VU^*$ is a unitary operator in $\mathfrak{A}$ left multiplication by which sends U into V. Now, for factors of type I_∞ (assuming that we have already made the faithful isometric, algebraic representation as all bounded operators on some Hilbert space), $I - E$ and $I - F$ are equivalent if and only if they project on manifolds of the same dimension. Thus, for $\mathfrak{A}$ of type I_∞, U and V are in the same extreme point class when $I - E$ and $I - F$ project on manifolds of the same dimension. Moreover, if $\rho(U) = V$ with ρ an isometry then $I - E$ and $I - F$ are equivalent (in the I_∞ case). In fact, $\rho = T \cdot \eta$ with T unitary and η either a * automorphism or a * anti-automorphism of $\mathfrak{A}$. Thus $\eta(U) = T^*V$ and $\eta(UU^*) = \eta(E) = T^*VV^*T = T^*FT$. From this we have $\eta(I - E) = T^*(I - F)T$. Since η preserves projections and their orthogonality relations, we see that the cardinal number of a maximal set of projections mutually orthogonal and smaller than $T^*(I - F)T$ is the same as that of a maximal set of orthogonal projections smaller than $I - E$. These cardinal numbers, however, are the dimensions of the ranges of $T^*(I - F)T$ and $I - E$ respectively. Clearly the ranges of $T^*(I - F)T$ and $I - F$ have the same dimension, so that $I - E$ and $I - F$ project on manifolds of the same dimension and are equivalent. Observe that, since all non-zero projections in case III are infinite and hence equivalent (separable space case), U and V are in the same extreme point class, for arbitrary semi-unitary operators, so long as U and V have $\mathfrak{H}$ for initial space. Since the * operation is a conjugate isometry all extreme point classes are self-adjoint (i.e. contain U^* along with U), and each conjugate isometry is the composition of an isometry and the * map. Thus the extreme point class of U consists of those extreme points which can be reached from U together with their adjoints. This gives the results for factors of type I_∞ and III.

Suppose now that $\mathfrak{A}$ is of type II_∞ with U and V proper semi-unitary operators in $\mathfrak{A}$ such that $UU^* = E$, $VV^* = F$ and $V = \rho(U)$ for some isometry ρ of $\mathfrak{A}$. With the notation as before, $\rho = T \cdot \eta$ and $\eta(I - E) = T^*(I - F)T$ (if $U^*U =$

E the same equality results). Now $I - E$ is infinite if and only if $T^*(I - F)T$ is infinite, since η preserves order and sends partially isometric operators into partially isometric operators. Moreover, $T^*(I - F)T$ is equivalent to $I - F$ under the partially isometry $T^*(I - F)$. Thus $I - E$ is infinite if and only if $I - F$ is. We conclude, therefore, that if $I - E$ is finite and $I - F$ infinite neither U nor U^* is in the same extreme point class as V. Thus factors of type II_∞ have at least three extreme point classes. In the separable case, all infinite projections are equivalent so that, from previous remarks, all semi-unitary operators V with associated $I - F$ infinite can be reached from one another and hence form an extreme point class. The proof is complete.

It is an immediate corollary of the above result that no two factors of different type are isometric ("equivalent" in the terminology of [1]). A stronger statement than this can be made.

THEOREM 14. *If $\mathfrak{A}$ and $\mathfrak{A}'$ are rings of operators with isometric Banach spaces then $\mathfrak{A}$ and $\mathfrak{A}'$ are * isomorphic as real algebras.*

PROOF. Suppose ρ is an isometry of $\mathfrak{A}$ onto $\mathfrak{A}'$. By Theorem 7, $\rho = U \cdot \eta$ with $U(= \rho(I))$ a unitary operator in $\mathfrak{A}'$ and η a quantum isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$. By Theorem 10, η is the sum of a * isomorphism φ and a * anti-isomorphism ψ. Now ψ maps a subring $\mathfrak{A}_1 = E\mathfrak{A}E$ of $\mathfrak{A}$ anti-isomorphically onto a subring (sub operator ring) $\mathfrak{A}_1' = E'\mathfrak{A}'E'$. Composing ψ with a* anti-automorphism of $\mathfrak{A}_1'$ yields ψ', an isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_1'$. The sum of φ and ψ' would then give the desired isomorphism between $\mathfrak{A}$ and $\mathfrak{A}'$. As (real) * anti-isomorphism we may use the * operation. Thus $\mathfrak{A}$ and $\mathfrak{A}'$ are * isomorphic as real algebras (the isomorphism, in fact, being complex linear on one direct summand and conjugate complex linear on the other).

Note that if ρ is a conjugate isometry, the map ρ *, which is the map ρ followed by the * map, is isometric. By Theorem 7, $\rho^* = U \cdot \eta$ with U unitary and η a C^*-isomorphism, so that $\rho(A) = \eta(A^*)U^* = U^*U\eta^*(A)U^* = U^*\eta_0^*(A)$. Hence conjugate isometries of C^*- algebras are the composition of C^*- isomorphisms, the * map, and left multiplication by unitary operators.

It results from Theorem 13, that factors of type I_n, II_1, and III are recognizable by their Banach space structure alone. The only possible confusion is that which can arise from factors of type I_∞ and II_∞, and this is due to our incomplete knowledge of the extreme point classes of type II_∞.

We conclude with the remark that the classification result for isometries of C^*- algebras goes over to the not necessarily closed $*-$ algebras. In fact, an isometry of such an algebra is uniquely extendable to its closure, and, now, our other results are applicable.

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