On the Free Gamma Distributions

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ABSTRACT. For each positive number α, we study the analog ν_α in free probability of the classical Gamma distribution with parameter α. We prove that ν_α is absolutely continuous, and establish the main properties of the density, including analyticity and unimodality. We study further the asymptotic behavior of ν_α as $\alpha \downarrow 0$.

1. INTRODUCTION

In this paper, we study the free Gamma distributions. These are the images of the classical Gamma distributions under the bijection, introduced by Bercovici and Pata, between the classes of infinitely divisible measures in classical and free probability (cf. [BP99] and [BNT02a]). More precisely, for any positive number α, the free gamma distribution ν_α with parameter α is defined as $\Lambda(\mu_\alpha)$, where Λ is the Bercovici-Pata bijection (see Section 2) and μ_α is the classical Gamma distribution with parameter α, that is,

$$\mu_\alpha(B) = \frac{1}{\Gamma(\alpha)} \int_{B\cap[0,\infty)} t^{\alpha-1} \mathrm{e}^{-t}\,\mathrm{d}t \tag{1.1}$$

for any Borel set B in $\mathbb{R}$.

The classical Gamma distributions form perhaps the simplest class of self-decomposable measures on $\mathbb{R}$ that are not stable (see Section 2). Since Λ preserves the notions of stability and self-decomposability (see [BP99] and [BNT02a]), the measures ν_α are thus of interest as (the simplest?) examples of non-stable self-decomposable measures with respect to free (additive) convolution. Of particular interest is the free χ^2-distribution $\Lambda(\chi_1^2)$, which (up to scaling by 2) equals the measure $\nu_{1/2}$. Apart from the general importance of the χ^2-distribution in classical probability, this is mainly due to the fact that the square of the semi-circle distribution (the analog of the Gaussian distribution in free probability) equals the free Poisson distribution (the image of the classical Poisson law by Λ) as observed

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in, for example, [VDN92]. Since Λ is injective, the relationship between the Gaussian and the χ^2-distribution thus breaks down in free probability, and from that point of view, it is of some interest to identify further the measure $\Lambda(\chi_1^2)$.

In an appendix to the paper [BP99], P. Biane studied the freely stable distributions and established their absolute continuity (with respect to Lebesgue measure) as well as the main features of their densities, in particular, analyticity and unimodality. Applying the same method as Biane (based on Stieltjes inversion), we establish in the present paper that the free Gamma distributions ν_α are absolutely continuous with analytic densities, and that they have supports in the form $[s_\alpha, \infty)$ for some strictly positive number s_α, which increases (strictly) with α and tends to 0 and ∞ as α goes to 0 and ∞, respectively. We derive an (implicit) expression for the density f_α of ν_α in the form

$$f_\alpha(P_\alpha(x)) = \frac{1}{\pi}\frac{v_\alpha(x)}{x^2 + v_\alpha(x)^2} \quad (x \in [-c_\alpha, \infty)), \tag{1.2}$$

where P_α is a strictly increasing function given by

$$P_\alpha(x) = 2x + \alpha - \alpha\int_0^\infty \frac{t^2 \mathrm{e}^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,\mathrm{d}t, \quad (x \in [-c_\alpha, \infty)),$$

and c_α is a positive constant such that $P_\alpha(-c_\alpha) = s_\alpha$. Moreover, $v_\alpha \colon \mathbb{R} \to \mathbb{R}$ is a function essentially defined by the condition

$$(x + \mathrm{i}v_\alpha(x))[1 + \alpha G_{\mu_1}(x + \mathrm{i}v_\alpha(x))] \in \mathbb{R},$$

where G_μ denotes the Cauchy (or Stieltjes) transform of a probability measure μ on $\mathbb{R}$ (see formula (2.3) below). This condition emerges naturally from the method of Stieltjes inversion in combination with the key formula

$$G_{\nu_\alpha}(z(1 + \alpha G_{\mu_1}(z))) = \frac{1}{z}, \tag{1.3}$$

which holds for all z in $\mathbb{C}^+$ satisfying that $z(1+\alpha G_{\mu_1}(z)) \in \mathbb{C}^+$. The passage from (1.3) to (1.2) via Stieltjes inversion depends heavily on an application of Lindelöf's Theorem, which we specify in Lemma 2.1 below for the reader's convenience.

By careful studies of the functions v_α, P_α and the right-hand side of (1.2), we derive some main features of the density f_α, such as analyticity, unimodality, and asymptotic behavior:

$$\lim_{\xi\to\infty} \frac{f_\alpha(\xi)}{\xi^{-1}\mathrm{e}^{-\xi}} = \alpha\mathrm{e}^\alpha, \quad \text{and} \quad \lim_{\xi\downarrow s_\alpha}\left(\frac{f_\alpha(\xi)}{\sqrt{\xi - s_\alpha}}\right) = \frac{\sqrt{2}}{\pi c_\alpha\sqrt{s_\alpha - c_\alpha^2}}. \tag{1.4}$$

In particular, it follows that ν_α has moments of all orders, which is in accordance with the results of Benaych-George in [BG06].

We study also the asymptotic behavior of ν_α as $\alpha \downarrow 0$, and we prove that the measures $(1/\alpha)\nu_\alpha$ converge to the measure $x^{-1}\mathrm{e}^{-x}1_{(0.\infty)}(x)\,\mathrm{d}x$ in moments and in the sense of point-wise convergence of the densities:

$$\lim_{\alpha\downarrow 0}\frac{f_\alpha(\xi)}{\alpha} = \xi^{-1}\mathrm{e}^{-\xi} \quad (\xi \in (0,\infty)).$$

The remainder of the paper is organized as follows. In Section 2, we collect background material on infinite divisibility, the Bercovici-Pata bijection, and Stieltjes inversion. In Section 3, we establish absolute continuity of ν_α and prove the expression (1.2) for the density. In Section 4, we establish the asymptotic behavior (1.4) and study how the quantities c_α and s_α vary as functions of α. In Section 5, we prove that ν_α is unimodal, and in the final Section 6, we study the asymptotic behavior of ν_α as $\alpha \downarrow 0$. The main results in Sections 3–6 depend in part on some basic properties of the functions v_α and P_α, the proofs of which are (not surprisingly) rather technical. To maintain the flow of the paper, these proofs are deferred to Appendix A at the end of the paper.

2. Background

2.1. Classical and free infinite divisibility. A (Borel-) probability measure μ on $\mathbb{R}$ is called infinitely divisible if there exists, for each positive integer n, a probability measure μ_n on $\mathbb{R}$ such that

$$\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ terms}}, \tag{2.1}$$

where $*$ denotes the usual convolution of probability measures (based on classical independence). We denote by $\mathcal{ID}(*)$ the class of all such measures on $\mathbb{R}$.

We recall that a probability measure μ on $\mathbb{R}$ is infinitely divisible if and only if its characteristic function (or Fourier transform) $\hat{\mu}$ has the Lévy-Khintchine representation

$$\hat{\mu}(u) = \exp\left[\mathrm{i}\eta u - \frac{1}{2}au^2 + \int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t))\rho(\mathrm{d}t)\right], \quad (u \in \mathbb{R}), \tag{2.2}$$

where η is a real constant, a is a non-negative constant, and ρ is a Lévy measure on $\mathbb{R}$, meaning that

$$\rho(\{0\}) = 0, \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, t^2\}\rho(\mathrm{d}t) < \infty.$$

The parameters a, ρ and η are uniquely determined by μ, and the triplet (a, ρ, η) is called the *characteristic triplet* for μ.

For two probability measures μ and ν on $\mathbb{R}$, the free convolution $\mu \boxplus \nu$ is defined as the distribution of $x + y$, where x and y are *freely independent* (possibly unbounded) self-adjoint operators on a Hilbert space with spectral distribution μ

and ν, respectively (see [BV93] for further details). The class $\mathcal{ID}(\boxplus)$ of infinitely divisible probability measures with respect to free convolution $\boxplus$ is defined by replacing classical convolution $*$ by free convolution $\boxplus$ in (2.1).

For a (Borel-) probability measure μ on $\mathbb{R}$ with support $\mathrm{supp}(\mu)$, the Cauchy (or Stieltjes) transform is the mapping $G_\mu\colon \mathbb{C}\setminus\mathrm{supp}(\mu)\to\mathbb{C}$ defined by

$$G_\mu(z)=\int_{\mathbb{R}}\frac{1}{z-t}\,\mu(\mathrm{d}t),\qquad (z\in\mathbb{C}\setminus\mathrm{supp}(\mu)). \tag{2.3}$$

The *free cumulant transform* C_μ of μ is then given by

$$C_\mu(z)=zG_\mu^{\langle -1\rangle}(z)-1 \tag{2.4}$$

for all z in a certain region R of $\mathbb{C}^-$ (the lower half complex plane), where the (right) inverse $G_\mu^{\langle -1\rangle}$ of G_μ is well defined. Specifically, R may be chosen in the form

$$R=\left\{z\in\mathbb{C}^- : \frac{1}{z}\in\Delta_{\eta,M}\right\},$$
$$\text{where } \Delta_{\eta,M}=\{z\in\mathbb{C}^+ : |\operatorname{Re}(z)|<\eta\operatorname{Im}(z),\ \operatorname{Im}(z)>M\}$$

for suitable positive numbers η and M. It was proved in [BV93] (see also [Ma92] and [Vo86]) that C_μ constitutes the free analog of $\log\hat{\mu}$ in the sense that it linearizes free convolution:

$$C_{\mu\boxplus\nu}(z)=C_\mu(z)+C_\nu(z),$$

for all probability measures μ and ν on $\mathbb{R}$ and all z in a region where all three transforms are defined. The results in [BV93] are presented in terms of a variant, φ_μ, of C_μ, which is often referred to as the Voiculescu transform, and which is again a variant of the R-transform $\mathcal{R}_\mu$ introduced in [Vo86]. The relationship is the following:

$$\varphi_\mu(z)=\mathcal{R}_\mu\left(\frac{1}{z}\right)=zC_\mu\left(\frac{1}{z}\right) \tag{2.5}$$

for all z in a region $\Delta_{\eta,M}$ as above. In [BV93], it was proved additionally that $\mu\in\mathcal{ID}(\boxplus)$ if and only if there exists a in $[0,\infty)$, η in $\mathbb{R}$, and a Lévy measure ρ, such that C_μ has the *free Lévy-Khintchine representation*:

$$C_\mu(z)=\eta z+az^2+\int_{\mathbb{R}}\left(\frac{1}{1-tz}-1-tz1_{[-1,1]}(t)\right)\rho(\mathrm{d}t) \tag{2.6}$$

(cf. also [BNT02b]). In particular, for μ in $\mathcal{ID}(\boxplus)$, it follows that C_μ can be extended to an analytic map (also denoted C_μ) defined on all of $\mathbb{C}^-$. The triplet (a,ρ,η) is uniquely determined and is called the *free characteristic triplet* for μ.

It follows from the paper [Be08] that any measure in $\mathcal{ID}(\boxplus)$ has continuous singular part (with respect to Lebesgue measure) equal to zero. Moreover, it was proved already in [BV93, Proposition 5.12] that any measure ν in $\mathcal{ID}(\boxplus)$ has at most one atom. In fact, the proof of that proposition reveals that an atom a for ν is necessarily equal to the non-tangential limit of $\varphi_\nu(z)$ as $z \to 0$, $z \in \mathbb{C}^+$. We say that a function $u\colon \mathbb{C}^+ \to \mathbb{C}$ has a non-tangential limit ℓ at 0, if for any positive number δ, we have that

$$\ell = \lim_{z\to 0,\, z\in\triangle_\delta} u(z), \quad \text{where } \triangle_\delta = \{z \in \mathbb{C}^+ \mid \mathrm{Im}(z) > \delta|\,\mathrm{Re}(z)|\}. \tag{2.7}$$

The following lemma is a consequence of Lindelöf's theorem (for this, see [Li15], [CoLo66, Theorem 2.2]). It is extremely useful in connection with the method of Stieltjes Inversion (see Subsection 2.3 below), a fact from which we profit strongly in Section 3 below.

Lemma 2.1. *Let $G\colon \mathbb{C}^+ \to \mathbb{C}^-$ be an analytic function, and assume there exists a curve $(z_t)_{t\in[0,1)}$ in $\mathbb{C}^+$, such that $\lim_{t\to 1} z_t = 0$, and such that $\alpha := \lim_{t\to 1} G(z_t)$ exists in $\mathbb{C}$. Then, for any positive number δ, we also have $\lim_{z\to 0,\, z\in\triangle_\delta} G(z) = \alpha$; that is, G has non-tangential limit α at* 0.

Theorem 2.2 in [CoLo66] provides a result similar to Lemma 2.1, which holds in particular for any *bounded* analytic function

$$f\colon \{x + \mathrm{i}y \mid x > 0,\ y \in \mathbb{R}\} \to \mathbb{C}.$$

Recalling that the mapping $\zeta \mapsto (\zeta - 1)/(\zeta + 1)$ is a conformal bijection of $\{x + \mathrm{i}y \mid x > 0,\ y \in \mathbb{R}\}$ onto the open unit disc in $\mathbb{C}$, Lemma 2.1 may then be verified by applying [CoLo66, Theorem 2.2] to the bounded function

$$f(z) = \frac{\mathrm{i}G(\mathrm{e}^{\mathrm{i}\pi/2}z) - 1}{\mathrm{i}G(\mathrm{e}^{\mathrm{i}\pi/2}z) + 1}, \quad (z \in \{x + \mathrm{i}y \mid x > 0,\ y \in \mathbb{R}\}).$$

2.2. The Bercovici-Pata bijection. In [BP99], Bercovici and Pata introduced a bijection Λ between the two classes $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$, which may formally be defined as the mapping sending a measure μ from $\mathcal{ID}(*)$ with characteristic triplet (a, ρ, η) onto the measure $\Lambda(\mu)$ in $\mathcal{ID}(\boxplus)$ with *free* characteristic triplet (a, ρ, η). It is then obvious that Λ is a bijection, and it turns out that Λ further enjoys the following properties (see [BP99] and [BNT02a]):

(a) If $\mu_1, \mu_2 \in \mathcal{ID}(*)$, then $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$;
(b) If $\mu \in \mathcal{ID}(*)$ and $c \in \mathbb{R}$, then $\Lambda(D_c\mu) = D_c\Lambda(\mu)$, where, for example, $D_c\mu$ is the transformation of μ by the mapping $x \mapsto cx\colon \mathbb{R} \to \mathbb{R}$;
(c) For any constant c in $\mathbb{R}$, we have $\Lambda(\delta_c) = \delta_c$, where δ_c denotes Dirac measure at c;
(d) Λ is a homeomorphism with respect to weak convergence.

Most of these properties can be established rather easily from the following convenient formula:

$$C_{\Lambda(\mu)}(\mathrm{i}z) = \int_0^\infty \log \hat{\mu}(zx)\mathrm{e}^{-x}\,\mathrm{d}x, \quad (z \in (-\infty, 0),\ \mu \in \mathcal{ID}(*)), \tag{2.8}$$

which was derived in [BNT04]. The properties (a)–(c) imply that Λ preserves, for example, the classes of stable and self-decomposable measures. Specifically, let $\mathcal{P}$ denote the class of all (Borel-) probability measures on $\mathbb{R}$, and recall then that a measure μ from $\mathcal{P}$ is called *stable*, if it satisfies the condition

$$\forall\ \alpha, \alpha' > 0 \quad \exists\ \alpha'' > 0,\ \exists\ \beta \in \mathbb{R} \colon D_\alpha\mu * D_{\alpha'}\mu = D_{\alpha''}\mu * \delta_\beta. \tag{2.9}$$

Recall also that μ is *self-decomposable* if

$$\forall\ c \in (0,1) \quad \exists\ \mu_c \in \mathcal{P} \colon \mu = D_c\mu * \mu_c. \tag{2.10}$$

Denoting by $\mathcal{S}(*)$ and $\mathcal{L}(*)$ the classes of stable and self-decomposable measures, respectively, it is well known (see, e.g., [Sat99]) that $\mathcal{S}(*) \subseteq \mathcal{L}(*) \subseteq \mathcal{ID}(*)$. The classes $\mathcal{S}(\boxplus)$ and $\mathcal{L}(\boxplus)$ are defined by replacing classical convolution $*$ by free convolution $\boxplus$ in (2.9)–(2.10) above. It was shown in [BV93] and [BNT02a] that $\mathcal{S}(\boxplus) \subseteq \mathcal{L}(\boxplus) \subseteq \mathcal{ID}(\boxplus)$. By application of properties (a)–(c) above, it follows then easily that

$$\Lambda(\mathcal{S}(*)) = \mathcal{S}(\boxplus) \quad \text{and} \quad \Lambda(\mathcal{L}(*)) = \mathcal{L}(\boxplus) \tag{2.11}$$

(see [BV93] and [BNT02a]). The measures in $\mathcal{S}(*)$ may alternatively by characterized as those measures in $\mathcal{ID}(*)$ whose Lévy measure has the form

$$\rho(\mathrm{d}t) = \left(c_-|t|^{-1-a_-}1_{(-\infty,0)}(t) + c_+t^{-1-a_+}1_{(0,\infty)}(t)\right)\mathrm{d}t$$

for suitable numbers c_+, c_- in $[0,\infty)$ and a_+, a_- in $(0,2)$. Similarly, $\mathcal{L}(*)$ may be characterized as the class of measures in $\mathcal{ID}(*)$ with Lévy measures in the form $\rho(\mathrm{d}t) = |t|^{-1}k(t)\,\mathrm{d}t$, where $k \colon \mathbb{R} \setminus \{0\} \to [0,\infty)$ is increasing on $(-\infty, 0)$ and decreasing on $(0,\infty)$. By the definition of Λ and (2.11), we have the exact same characterizations of the measures in $\mathcal{S}(\boxplus)$ and $\mathcal{L}(\boxplus)$, respectively, if we let the term "Lévy measure" refer to the free Lévy-Khintchine representation (2.6) rather than the classical one (2.2).

For any positive number α, the classical Gamma distribution μ_α with parameter α (cf. (1.1)) has Lévy measure

$$\rho_\alpha(\mathrm{d}t) = \alpha t^{-1}\mathrm{e}^{-t}1_{(0,\infty)}(t)\,\mathrm{d}t,$$

and thus $\mu_\alpha \in \mathcal{L}(*) \setminus \mathcal{S}(*)$. The corresponding free Gamma distribution, namely $\nu_\alpha = \Lambda(\mu_\alpha)$, satisfies accordingly that $\nu_\alpha \in \mathcal{L}(\boxplus) \setminus \mathcal{S}(\boxplus)$. As mentioned in the Introduction, the purpose of the present paper is to disclose the main features of ν_α for any α in $(0,\infty)$.

2.3. Stieltjes inversion. Let μ be a (Borel-) probability measure on $\mathbb{R}$, and consider its cumulative distribution function

$$F_\mu(t) = \mu((-\infty, t]), \quad (t \in \mathbb{R}),$$

as well as its Lebesgue decomposition $\mu = \rho + \sigma$, where the measures ρ and σ are, respectively, absolutely continuous and singular with respect to Lebesgue measure λ on $\mathbb{R}$. It follows from De la Vallé Poussin's Theorem (see [Sak37, Theorem IV.9.6]) that ρ and σ may be identified with the restrictions of μ to the sets

$$D_1 = \left\{x \in \mathbb{R} \mid \lim_{h\to 0} \frac{F_\mu(x+h) - F_\mu(x)}{h} \text{ exists in } \mathbb{R}\right\}$$

and

$$D_\infty = \left\{x \in \mathbb{R} \mid \lim_{h\to 0} \frac{F_\mu(x+h) - F_\mu(x)}{h} = \infty\right\},$$

respectively. In addition, we have that (see, e.g., Theorem 3.23 and Proposition 3.31 in [Fo84])

$$\lambda(\mathbb{R} \setminus D_1) = 0 \quad \text{and} \quad \rho(\mathrm{d}t) = F'_\mu(t) 1_{D_1}(t)\,\mathrm{d}t,$$

where, for any t in D_1, $F'_\mu(t)$ denotes the derivative of F_μ at t.

Consider now additionally the Cauchy (or Stieltjes) transform G_μ defined in (2.3). It follows then from general theory of Poisson-Stieltjes integrals (see [Do62]) that

$$F'_\mu(x) = -\frac{1}{\pi} \lim_{y\downarrow 0} \operatorname{Im}(G_\mu(x+\mathrm{i}y)) \quad \text{for all } x \text{ in } D_1,$$

and that

$$\lim_{y\downarrow 0} |(G_\mu(x+\mathrm{i}y))| = \infty \qquad \text{for all } x \text{ in } D_\infty.$$

In particular, we may conclude that the singular part σ of μ is concentrated on the set

$$\{x \in \mathbb{R} \mid \lim_{y\downarrow 0} |G_\mu(x+\mathrm{i}y)| = \infty\}$$

(see also Chapter XIII in [RS78]).

3. Absolute Continuity of ν_α

In this section, we establish absolute continuity of the free Gamma distributions ν_α, $\alpha > 0$, and prove the formula (1.2) for the densities. Our starting point is the derivation of the formula (1.3), and we introduce for that purpose the function $H_\alpha \colon \mathbb{C} \setminus [0,\infty) \to \mathbb{C}$ given by

$$H_\alpha(z) = z + z\alpha G_{\mu_1}(z) = z + z\alpha \int_0^\infty \frac{\mathrm{e}^{-t}}{z-t}\,\mathrm{d}t = z + \alpha + \alpha \int_0^\infty \frac{t\mathrm{e}^{-t}}{z-t}\,\mathrm{d}t \tag{3.1}$$

for z in $\mathbb{C} \setminus [0, \infty)$. By differentiation under the integral sign, note that H_α is analytic on $\mathbb{C} \setminus [0, \infty)$ with derivatives given by

$$\begin{aligned} H'_\alpha(z) &= 1 - \alpha \int_0^\infty \frac{t\mathrm{e}^{-t}}{(z-t)^2}\,\mathrm{d}t, && (z \in \mathbb{C} \setminus [0,\infty)), \\ H^{(k)}_\alpha(z) &= (-1)^k \alpha k! \int_0^\infty \frac{t\mathrm{e}^{-t}}{(z-t)^{k+1}}\,\mathrm{d}t, && (z \in \mathbb{C} \setminus [0,\infty),\ k \geq 2). \end{aligned} \tag{3.2}$$

In the following, we also consider the function $F \colon \mathbb{C} \setminus [0, \infty) \to (0, \infty)$ given by

$$F(x + \mathrm{i}y) = \int_0^\infty \frac{t\mathrm{e}^{-t}}{|x + \mathrm{i}y - t|^2}\,\mathrm{d}t = \int_0^\infty \frac{t\mathrm{e}^{-t}}{(x-t)^2 + y^2}\,\mathrm{d}t \tag{3.3}$$

for all $x, y \in \mathbb{R}$ such that $x + \mathrm{i}y \in \mathbb{C} \setminus [0, \infty)$.

Lemma 3.1. *Let α be a positive number. Then, the following hold:*

(i) *There exists a unique positive real number c_α such that*

$$\frac{1}{\alpha} = F(-c_\alpha) = \int_0^\infty \frac{t\mathrm{e}^{-t}}{(c_\alpha + t)^2}\,\mathrm{d}t. \tag{3.4}$$

The number c_α increases with α, and satisfies that

$$\lim_{\alpha \to 0} c_\alpha = 0, \quad \text{and} \quad \lim_{\alpha \to \infty} c_\alpha = \infty.$$

(ii) *There is a function $v_\alpha \colon \mathbb{R} \to [0, \infty)$, such that*

$$\{z \in \mathbb{C}^+ \mid H_\alpha(z) \in \mathbb{C}^+\} = \{x + \mathrm{i}y \mid x, y \in \mathbb{R},\ y > v_\alpha(x)\}. \tag{3.5}$$

The function v_α is given by

$$v_\alpha(x) = 0, \qquad \text{if } x \in (-\infty, -c_\alpha], \tag{3.6}$$

$$F(x + \mathrm{i}v_\alpha(x)) = \frac{1}{\alpha}, \qquad \text{if } x \in (-c_\alpha, \infty). \tag{3.7}$$

(iii) *For all x in $\mathbb{R}$ we have that $H_\alpha(x + \mathrm{i}v_\alpha(x)) \in \mathbb{R}$.*

(iv) *The function v_α satisfies that $v_\alpha(x) > 0$ for all x in $(-c_\alpha, \infty)$.*

Proof. (i) The function $x \mapsto F(-x) = \int_0^\infty t\mathrm{e}^{-t}/(x+t)^2\,\mathrm{d}t$ is clearly strictly decreasing and (by dominated convergence) continuous on $(0, \infty)$. Moreover, by monotone convergence, $\lim_{x \downarrow 0} F(-x) = \infty$ and $\lim_{x \to \infty} F(-x) = 0$. Hence, there exists a unique number c_α in $(0, \infty)$ such that $F(-c_\alpha) = 1/\alpha$. The last assertions in (i) are immediate from the last equality in (3.4).

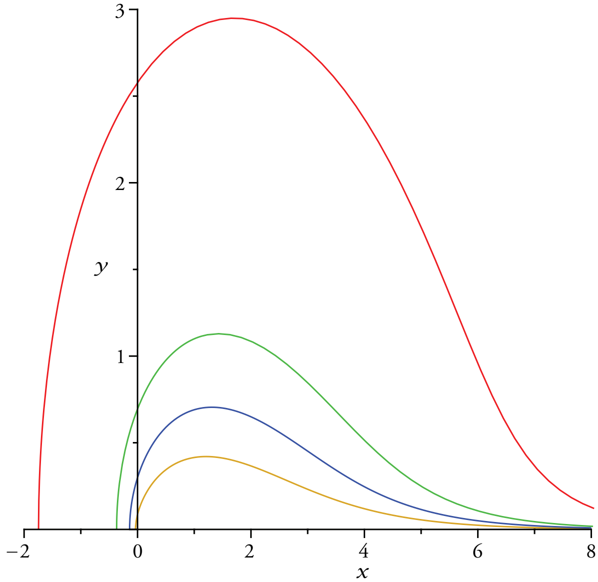

FIGURE 3.1. The graphs of the functions $v_{1/2}$, v_1, v_2, and v_{10}

(ii) For x, y in $\mathbb{R}$ such that $x + \mathrm{i}y \in \mathbb{C} \setminus [0, \infty)$, we find from formula (3.1) that

$$\begin{aligned}(3.8)\qquad \operatorname{Im}(H_\alpha(x+\mathrm{i}y)) &= y + \alpha \operatorname{Im}\left(\int_0^\infty \frac{t\mathrm{e}^{-t}}{x+\mathrm{i}y-t}\,\mathrm{d}t\right) \\ &= y - \alpha y \int_0^\infty \frac{t\mathrm{e}^{-t}}{(x-t)^2+y^2}\,\mathrm{d}t = y(1-\alpha F(x+\mathrm{i}y)).\end{aligned}$$

For fixed x in $\mathbb{R}$, the function $y \mapsto F(x+\mathrm{i}y)$ is clearly strictly decreasing on $(0, \infty)$, and $F(x+\mathrm{i}y) \to 0$ as $y \to \infty$. Moreover, by monotone convergence,

$$\lim_{y \downarrow 0} F(x+\mathrm{i}y) = \begin{cases} \infty, & \text{if } x \geq 0, \\ F(x), & \text{if } x < 0. \end{cases}$$

Thus, if $x > -c_\alpha$, then $\lim_{y\downarrow 0} F(x+\mathrm{i}y) > F(-c_\alpha) = 1/\alpha$, and there exists a unique y_x in $(0, \infty)$ such that $F(x+\mathrm{i}y_x) = 1/\alpha$. Thus, if we put $v_\alpha(x) = y_x$, then $\alpha F(x+\mathrm{i}y) < 1$, and hence $\operatorname{Im}(H_\alpha(x+\mathrm{i}y)) > 0$ for all y in $(v_\alpha(x), \infty)$. Similarly, $\operatorname{Im}(H_\alpha(x+\mathrm{i}y)) < 0$ for y in $(0, v_\alpha(x))$.

If $x \leq -c_\alpha$, then for all y in $(0, \infty)$, we have that $F(x+\mathrm{i}y) < F(x) \leq F(-c_\alpha) = 1/\alpha$, and hence that $\operatorname{Im}(H_\alpha(x+\mathrm{i}y)) > 0$. Thus, if we put $v_\alpha(x) = 0$ for x in $(-\infty, -c_\alpha]$, it follows altogether that v_α satisfies (3.5), and that v_α is given by (3.6)–(3.7).

Finally, we note that (iii) follows immediately from (3.8) in combination with (3.6)–(3.7), and that (iv) is a consequence of the way v_α was defined in the proof of (ii). ◻

In continuation of Lemma 3.1, we introduce next the following notation:

$$\mathcal{G}_\alpha = \{x + \mathrm{i}v_\alpha(x) \mid x \in \mathbb{R}\}, \tag{3.9}$$

$$\mathcal{G}'_\alpha = \{x + \mathrm{i}v_\alpha(x) \mid x \in [-c_\alpha, \infty)\} = \mathcal{G}_\alpha \setminus (-\infty, -c_\alpha), \tag{3.10}$$

$$\mathcal{G}^+_\alpha = \{x + \mathrm{i}y \mid x, y \in \mathbb{R}, \text{ and } y > v_\alpha(x)\}. \tag{3.11}$$

Note, in particular, that $0 \notin \mathcal{G}_\alpha$ (since $v_\alpha(0) > 0$), that $\mathcal{G}_\alpha = \partial\mathcal{G}^+_\alpha$, and that $\mathcal{G}_\alpha, \mathcal{G}^+_\alpha \subseteq \mathbb{C} \setminus [0, \infty)$.

Proposition 3.2. *Let α be a positive number, and consider the free Gamma distribution ν_α with parameter α. Consider further the classical Gamma distribution μ_1 with parameter 1 (cf. (1.1)). We then have (cf. formulae (2.3) and (2.4))*

(i) *$\mathcal{C}_{\nu_\alpha}(1/z) = \alpha G_{\mu_1}(z)$ for all z in $\mathbb{C}^+$;*
(ii) *$G_{\nu_\alpha}(H_\alpha(z)) = 1/z$ for all z in $\mathcal{G}^+_\alpha$.*

Proof. (i) The classical Gamma distribution μ_α has characteristic function

$$\hat{\mu}_\alpha(u) = \exp\left(\alpha \int_0^\infty (\mathrm{e}^{\mathrm{i}ut} - 1)\frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t\right), \quad (u \in \mathbb{R}), \tag{3.12}$$

(see, e.g., [Sat99, Example 8.10]). By formula (2.8) and Fubini's theorem, it follows then for any u in $(-\infty, 0)$ that

$$\begin{aligned} \mathcal{C}_{\nu_\alpha}(\mathrm{i}u) &= \int_0^\infty \log \hat{\mu}_\alpha(ux)\mathrm{e}^{-x}\,\mathrm{d}x = \alpha \int_0^\infty \left(\int_0^\infty (\mathrm{e}^{\mathrm{i}uxt} - 1)\frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t\right)\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \alpha \int_0^\infty \frac{\mathrm{e}^{-t}}{t}\left(\frac{1}{1 - \mathrm{i}ut} - 1\right)\mathrm{d}t = \alpha\mathrm{i}u \int_0^\infty \frac{\mathrm{e}^{-t}}{1 - \mathrm{i}ut}\,\mathrm{d}t. \end{aligned}$$

Setting $u = -1/y$, we find for any y in $(0, \infty)$ that

$$\mathcal{C}_{\nu_\alpha}\left(\frac{1}{\mathrm{i}y}\right) = \alpha \int_0^\infty \frac{\mathrm{e}^{-t}}{\mathrm{i}y(1 - t/(\mathrm{i}y))}\,\mathrm{d}t = \alpha \int_0^\infty \frac{\mathrm{e}^{-t}}{\mathrm{i}y - t}\,\mathrm{d}t = \alpha G_{\mu_1}(\mathrm{i}y).$$

By analytic continuation, we conclude that $\mathcal{C}_{\nu_\alpha}(1/z) = \alpha G_{\mu_1}(z)$ for all z in $\mathbb{C}^+$.
(ii) Recall from the definition of $\mathcal{C}_{\nu_\alpha}$ (see Subsection 2.1) that

$$\mathcal{C}_{\nu_\alpha}\left(\frac{1}{z}\right) = \frac{1}{z}G^{\langle -1 \rangle}_{\nu_\alpha}\left(\frac{1}{z}\right) - 1$$

for all z in a region of the form

$$\Delta_{\eta,M} = \{z \in \mathbb{C}^+ : |\operatorname{Re}(z)| < \eta \operatorname{Im}(z),\ \operatorname{Im}(z) > M\}$$

for suitable positive numbers η and M. Taking (i) into account, we find that

$$(3.13)\quad G_{\nu_\alpha}^{\langle -1\rangle}\left(\frac{1}{z}\right) = z + z\alpha G_{\mu_1}(z) = H_\alpha(z), \quad \text{and hence } \frac{1}{z} = G_{\nu_\alpha}(H_\alpha(z))$$

for all z in $\Delta_{\eta,M}$. Since G_{ν_α} and H_α are analytic on $\mathbb{C}^+$, it follows from Lemma 3.1(ii) and analytic continuation that the latter equation in (3.13) holds for all z in $\mathcal{G}_\alpha^+$. This completes the proof. □

In order to combine Proposition 3.2(ii) with the method of Stieltjes inversion (and Lemma 2.1), we need some further preparations, which are presented in the series of lemmas to follow.

Lemma 3.3. *For any positive number α, we have that $H_\alpha'(-c_\alpha) = 0$ and $H_\alpha'(z) \neq 0$, for all z in $\mathcal{G}_\alpha' \setminus \{-c_\alpha\}$. In fact,*

$$\mathrm{Re}(H_\alpha'(x + \mathrm{i}v_\alpha(x))) = -\alpha v_\alpha(x)\frac{\partial}{\partial y}F(x + \mathrm{i}v_\alpha(x)) > 0$$

for all x in $(-c_\alpha, \infty)$.

Proof. Note first that, by (3.2)–(3.4), we have that

$$H_\alpha'(-c_\alpha) = 1 - \alpha\int_0^\infty \frac{t\mathrm{e}^{-t}}{(c_\alpha + t)^2}\,\mathrm{d}t = 1 - \alpha F(-c_\alpha) = 0.$$

For $z = x + \mathrm{i}y$ in $\mathbb{C}\setminus[0,\infty)$, we find next, by application of the Cauchy-Riemann equations and (3.8), that

$$\begin{aligned}\mathrm{Re}(H_\alpha'(z)) &= \frac{\partial}{\partial x}\mathrm{Re}(H_\alpha(z)) = \frac{\partial}{\partial y}\mathrm{Im}(H_\alpha(z))\\ &= \frac{\partial}{\partial y}(y(1 - \alpha F(z))) = (1 - \alpha F(z)) - \alpha y\frac{\partial}{\partial y}F(z).\end{aligned}$$

For any x in $(-c_\alpha, \infty)$, it thus follows from (3.7) that

$$\mathrm{Re}(H_\alpha'(x + \mathrm{i}v_\alpha(x))) = 0 - \alpha v_\alpha(x)\frac{\partial}{\partial y}F(x + \mathrm{i}v_\alpha(x)).$$

The proof is concluded by noting that differentiation with respect to y in (3.3) leads to

$$\frac{\partial}{\partial y}F(x + \mathrm{i}y) = -2y\int_0^\infty \frac{t\mathrm{e}^{-t}}{((x-t)^2 + y^2)^2}\,\mathrm{d}t,$$

where the right hand side is strictly negative whenever $y > 0$. □

In the following lemma, we collect some further properties of the function v_α that will be needed in various parts of the remainder of the paper. We defer the rather technical proof to Appendix A.

Lemma 3.4. *Let α be a positive number, and consider $v_\alpha\colon \mathbb{R} \to [0,\infty)$, the function given by (3.6)–(3.7). Then, v_α has the following properties:*

(i) *v_α is continuous on $\mathbb{R}$ and analytic on $\mathbb{R} \setminus \{-c_\alpha\}$.*
(ii) $\lim_{x\to\infty}(v_\alpha(x)/x\mathrm{e}^{-x}) = \alpha\pi$.
(iii) *For any positive numbers δ, γ, there exists a positive number α_0 such that*

$$\sup_{x\in[\delta,\infty)} \left| \frac{v_\alpha(x)}{\alpha} - \pi x \mathrm{e}^{-x} \right| \le \gamma, \quad \text{whenever } \alpha \in (0,\alpha_0].$$

Lemma 3.5. *Let α be a strictly positive number, let z be a point in $\mathcal{G}_\alpha$, and set $\xi = H_\alpha(z) = z + \alpha z G_{\mu_1}(z) \in \mathbb{R}$ (cf. Lemma 3.1(iii)).*

Then, the Cauchy transform G_{ν_α} of ν_α has the non-tangential limit $1/z$ at ξ. More precisely, for any positive number δ, we have that $\lim_{w\to 0,\, w\in\triangle_\delta} G_{\nu_\alpha}(\xi + w) = \frac{1}{z}$, where $\triangle_\delta$ is given by (2.7).

Proof. For any t in $(0,\infty)$ we have that $z + \mathrm{i}t \in \mathcal{G}_\alpha^+$, and hence also that $H_\alpha(z+\mathrm{i}t) \in \mathbb{C}^+$ (cf. formula (3.5)). Since $\mathcal{G}_\alpha \subseteq \mathbb{C} \setminus [0,\infty)$, the function H_α is continuous at z, and thus $H_\alpha(z+\mathrm{i}t) \to H_\alpha(z) = \xi$ as $t \downarrow 0$. At the same time, it follows from Proposition 3.2 that

$$G_{\nu_\alpha}(H_\alpha(z+\mathrm{i}t)) = \frac{1}{z+\mathrm{i}t} \longrightarrow \frac{1}{z}, \quad \text{as } t \downarrow 0$$

(recall that $z \neq 0$, since $0 \notin \mathcal{G}_\alpha$). Applying then Lemma 2.1 (to the function $w \mapsto G_{\nu_\alpha}(\xi + w)$), we may conclude that, actually, $\lim_{w\to 0,\, w\in\triangle_\delta} G_{\nu_\alpha}(\xi+w) = \frac{1}{z}$ for any positive number δ, as desired. □

For any α in $(0,\infty)$, we introduce next the function $P_\alpha\colon \mathbb{R} \to \mathbb{R}$ (cf. Lemma 3.1(iii)) given by

$$P_\alpha(x) = H_\alpha(x + \mathrm{i}v_\alpha(x)), \quad (x \in \mathbb{R}). \tag{3.14}$$

In particular, we set

$$s_\alpha = P_\alpha(-c_\alpha) = H_\alpha(-c_\alpha). \tag{3.15}$$

In the following lemma, we collect some properties of P_α that will be needed in the sequel. We defer the rather technical proof to Appendix A.

Lemma 3.6. *For any positive number α, the function $P_\alpha\colon \mathbb{R} \to \mathbb{R}$ has the following properties:*

(i) *P_α is continuous on $\mathbb{R}$ and analytic on $\mathbb{R} \setminus \{-c_\alpha\}$.*
(ii) *P_α satisfies that*

$$P_\alpha(x) = \begin{cases} x + \alpha + \alpha \displaystyle\int_0^\infty \frac{t\mathrm{e}^{-t}}{x-t}\,\mathrm{d}t, & \text{if } x < -c_\alpha, \\[2ex] 2x + \alpha - \alpha \displaystyle\int_0^\infty \frac{t^2\mathrm{e}^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,\mathrm{d}t, & \text{if } x \ge -c_\alpha. \end{cases} \tag{3.16}$$

(iii) *The number* $s_\alpha := P_\alpha(-c_\alpha)$ *is strictly positive.*
(iv) $\lim_{x\to\infty}(x + \alpha - P_\alpha(x)) = 0$.
(v) P_α *is a strictly increasing bijection of* $\mathbb{R}$ *onto* $\mathbb{R}$, *and* $P'_\alpha(x) > 0$ *for all* x *in* $\mathbb{R} \setminus \{-c_\alpha\}$.

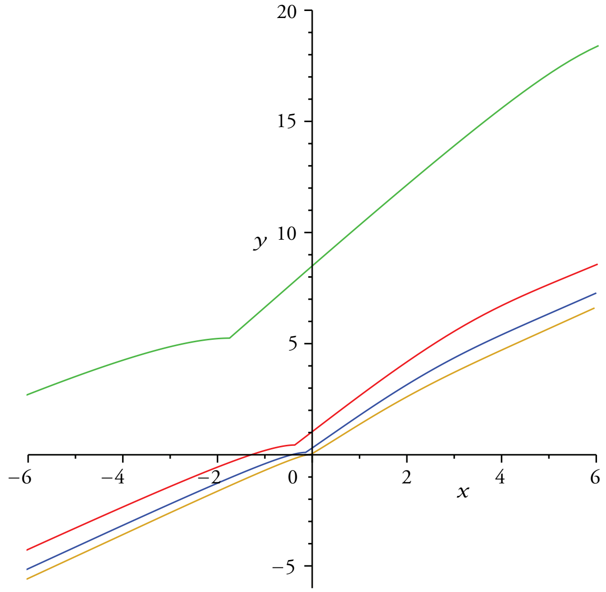

FIGURE 3.2. The graphs of the functions $P_{1/2}$, P_1, P_2 and P_{10}

Theorem 3.7. *Let* α *be a positive number, and consider* $Q_\alpha\colon \mathbb{R} \to [0,\infty)$, *the function defined by*

$$Q_\alpha(x) = \frac{v_\alpha(x)}{x^2 + v_\alpha(x)^2}, \qquad (x \in \mathbb{R}). \tag{3.17}$$

We then have the following:
(i) *The free Gamma distribution* ν_α *with parameter* α *is absolutely continuous with respect to Lebesgue measure.*
(ii) *The density* f_α *of* ν_α *is given by*

$$f_\alpha(\xi) = \begin{cases} 0, & \text{if } \xi \in (-\infty, s_\alpha], \\ \dfrac{1}{\pi} Q_\alpha(P_\alpha^{\langle -1\rangle}(\xi)), & \text{if } \xi \in (s_\alpha, \infty), \end{cases} \tag{3.18}$$

where $P_\alpha^{\langle -1\rangle}$ *denotes the inverse of* P_α (*cf.* (v) *in Lemma* 3.6).
(iii) *The support of* ν_α *is* $[s_\alpha, \infty)$.
(iv) f_α *is analytic on* (s_α, ∞).

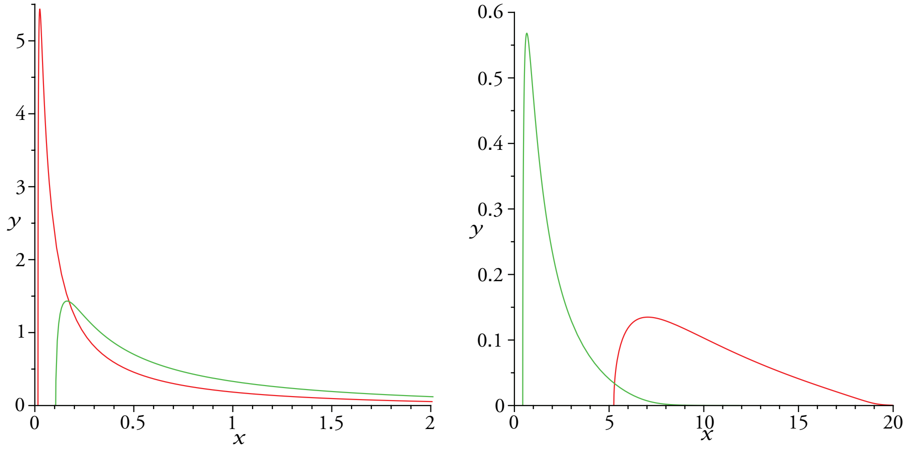

FIGURE 3.3. The graphs of the densities of the free gamma distributions $\nu_{1/2}, \nu_1$ and, respectively, ν_2, ν_{10}

Proof of Theorem 3.7. (i) From Lemma 3.6, it follows that P_α is a continuous, increasing bijection of $\mathbb{R}$ onto $\mathbb{R}$. By formula (3.14), this implies that H_α maps $\mathcal{G}_\alpha$ bijectively onto $\mathbb{R}$. Thus, for any ξ in $\mathbb{R}$, there is a unique z in $\mathcal{G}_\alpha$ such that $\xi = H_\alpha(z)$. Lemma 3.5 then yields in particular that

$$G_{\nu_\alpha}(\xi + \mathrm{i}y) \longrightarrow \frac{1}{z} \quad \text{as } y \downarrow 0. \tag{3.19}$$

Since the singular part of ν_α is concentrated (cf. Subsection 2.3) on the set

$$\{\xi \in \mathbb{R} \mid \lim_{y \downarrow 0} |G_{\nu_\alpha}(\xi + \mathrm{i}y)| = \infty\},$$

we may thus conclude that ν_α has no singular part, as desired.

(ii) Combining (i), formula (3.19), and the method of Stieltjes inversion (cf. Subsection 2.3), it follows that ν_α has a density f_α with respect to Lebesgue measure given by

$$f_\alpha(\xi) = -\frac{1}{\pi} \lim_{y \downarrow 0} \operatorname{Im}(G_{\nu_\alpha}(\xi + \mathrm{i}y)) = -\frac{1}{\pi} \operatorname{Im}\left(\frac{1}{z}\right), \quad (\xi \in \mathbb{R}),$$

where z is the unique point in $\mathcal{G}_\alpha$, such that $H_\alpha(z) = \xi$. Writing $z = x + \mathrm{i}v_\alpha(x)$ for some (unique) x in $\mathbb{R}$, we have that

$$\xi = H_\alpha(x + \mathrm{i}v_\alpha(x)) = P_\alpha(x), \quad \text{so that } x = P_\alpha^{\langle -1 \rangle}(\xi),$$

and therefore

$$
\begin{aligned}
f_\alpha(\xi) &= -\frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{x+\mathrm{i}v_\alpha(x)}\right) = \frac{1}{\pi}\frac{v_\alpha(x)}{x^2+v_\alpha(x)^2} \\
&= \frac{1}{\pi}Q_\alpha(x) = \frac{1}{\pi}Q_\alpha(P_\alpha^{\langle -1\rangle}(\xi)).
\end{aligned}
$$

The proof of (ii) is completed by noting that if $\xi < s_\alpha$, then $x < -c_\alpha$, so that $v_\alpha(x) = 0$, and therefore $f_\alpha(\xi) = 0$ by the previous calculation.

For (iii), we note that it is an immediate consequence of (i), (ii), and the fact that v_α (and hence, Q_α) is strictly positive on $(-c_\alpha, \infty)$ (cf. Lemma 3.1(iv)).

(iv) Since v_α is analytic on $(-c_\alpha, \infty)$ (cf. Lemma 3.4(i)), it follows immediately from (3.17) that so is Q_α. By (i) and (v) of Lemma 3.6, the function P_α is analytic on $(-c_\alpha, \infty)$ with strictly positive derivative. This implies that $P_\alpha^{\langle -1\rangle}$ is analytic on (s_α, ∞), and altogether we thus conclude that $f_\alpha = (1/\pi)Q_\alpha \circ P_\alpha^{\langle -1\rangle}$ is analytic on (s_α, ∞). □

4. Behavior at the Limits of the Support

In this section, we study the behavior of the density f_α of ν_α around the lower bound s_α of the support and at infinity. We start with the latter aspect.

Proposition 4.1. *Let α be a positive number, and consider the density f_α of ν_α (cf. (3.18)). We then have*

$$
\lim_{\xi\to\infty}\frac{f_\alpha(\xi)}{\xi^{-1}\mathrm{e}^{-\xi}} = \alpha\mathrm{e}^{\alpha}.
$$

Proof. Consider the function P_α introduced in (3.14). Since P_α is a strictly increasing bijection of $\mathbb{R}$ onto $\mathbb{R}$, it suffices to prove that

$$
\lim_{x\to\infty} f_\alpha(P_\alpha(x))P_\alpha(x)\mathrm{e}^{P_\alpha(x)} = \alpha\mathrm{e}^{\alpha}.
$$

Using Theorem 3.7(ii), Lemma 3.4(ii), and Lemma 3.6(iv), we find that

$$
\begin{aligned}
\lim_{x\to\infty} f_\alpha(P_\alpha(x))P_\alpha(x)\mathrm{e}^{P_\alpha(x)} &= \lim_{x\to\infty}\frac{1}{\pi}\frac{v_\alpha(x)}{x^2+v_\alpha(x)^2}P_\alpha(x)\mathrm{e}^{P_\alpha(x)} \\
&= \lim_{x\to\infty}\frac{1}{\pi}\left(\frac{x^2}{x^2+v_\alpha(x)^2}\right)\left(\frac{v_\alpha(x)}{x\mathrm{e}^{-x}}\right)\left(\frac{P_\alpha(x)}{x}\right)\mathrm{e}^{-x+P_\alpha(x)} \\
&= \frac{1}{\pi}\cdot 1\cdot\alpha\pi\cdot 1\cdot \mathrm{e}^{\alpha} = \alpha\mathrm{e}^{\alpha},
\end{aligned}
$$

as desired. □

We turn next to the behavior of $f_\alpha(\xi)$ as $\xi \downarrow s_\alpha$. We study initially how s_α varies as a function of α.

Proposition 4.2. *For any positive number α, consider the function H_α and the quantities c_α and s_α defined by* (3.1), (3.4), *and* (3.15), *respectively. We then have the following conclusions:*

(i) *H_α satisfies the differential equation*

$$H'_\alpha(z) = \alpha + z + (z^{-1} - 1)H_\alpha(z), \quad (z \in \mathbb{C} \setminus [0, \infty)).$$

(ii) $s_\alpha = (c_\alpha/(1 + c_\alpha))(\alpha - c_\alpha)$.

(iii) $H''_\alpha(-c_\alpha) = 1 - s_\alpha/c_\alpha^2 < 0$.

(iv) $\lim_{\alpha\to 0} s_\alpha = 0$, *and* $\lim_{\alpha\to\infty} s_\alpha = \infty$.

(v) *c_α is an analytic function of α, and*

$$\frac{\mathrm{d}c_\alpha}{\mathrm{d}\alpha} = \frac{c_\alpha(1 + c_\alpha)}{\alpha(\alpha - 2c_\alpha - c_\alpha^2)}.$$

(vi) *s_α is an analytic function of α, and*

$$\frac{\mathrm{d}s_\alpha}{\mathrm{d}\alpha} = \frac{c_\alpha(\alpha + 1)}{\alpha(1 + c_\alpha)}.$$

In particular, s_α is a strictly increasing function of α.

Proof. (i) Differentiation in the first equality in (3.1), along with partial integration, leads to

$$\begin{aligned} H'_\alpha(z) &= 1 + \alpha \int_0^\infty \frac{\mathrm{e}^{-t}}{z - t}\,\mathrm{d}t - \alpha z \int_0^\infty \frac{\mathrm{e}^{-t}}{(z - t)^2}\,\mathrm{d}t \\ &= z^{-1}H_\alpha(z) - \alpha z\left(\left[\frac{\mathrm{e}^{-t}}{z - t}\right]_0^\infty + \int_0^\infty \frac{\mathrm{e}^{-t}}{z - t}\,\mathrm{d}t\right) \\ &= z^{-1}H_\alpha(z) + \alpha - (H_\alpha(z) - z) = \alpha + z + (z^{-1} - 1)H_\alpha(z) \end{aligned}$$

for all z in $\mathbb{C} \setminus [0, \infty)$.

(ii) From Lemma 3.3, (i), and (3.15), it follows that

$$0 = H'_\alpha(-c_\alpha) = \alpha - c_\alpha + (-c_\alpha^{-1} - 1)H_\alpha(-c_\alpha) = \alpha - c_\alpha - \frac{1 + c_\alpha}{c_\alpha}s_\alpha,$$

from which (ii) follows immediately.

(iii) Differentiation in (i) leads to the formula

$$H''_\alpha(z) = 1 - z^{-2}H_\alpha(z) + (z^{-1} - 1)H'_\alpha(z), \quad (z \in \mathbb{C} \setminus [0, \infty)).$$

Combining this with Lemma 3.3, we find that

$$H''_\alpha(-c_\alpha) = 1 - c_\alpha^{-2}H_\alpha(-c_\alpha) + 0 = 1 - c_\alpha^{-2}s_\alpha.$$

At the same time, it follows from (3.2) that

$$H_\alpha''(-c_\alpha) = 2\alpha \int_0^\infty \frac{te^{-t}}{(-c_\alpha - t)^3}\, dt = -2\alpha \int_0^\infty \frac{te^{-t}}{(c_\alpha + t)^3}\, dt < 0,$$

and thus (iii) is established.

(iv) From Lemma 3.1(i), we know that $c_\alpha \to 0$ as $\alpha \to 0$, and that $c_\alpha \to \infty$ as $\alpha \to \infty$. In combination with (ii) and (iii), respectively, it follows that s_α has the same properties.

(v) For any x in $(0, \infty)$, it follows from (3.2)–(3.3) that

$$H_\alpha'(-x) = 1 - \alpha \int_0^\infty \frac{te^{-t}}{(x + t)^2}\, dt = 1 - \alpha F(-x),$$

so that

$$F(-x) = \frac{1}{\alpha}(1 - H_\alpha'(-x)) \text{ and } F'(-x) = -\frac{1}{\alpha} H_\alpha''(-x), \quad (x \in (0, \infty)). \tag{4.1}$$

Using (iii) and (ii), we thus find that

$$\begin{aligned} F'(-c_\alpha) &= -\frac{1}{\alpha} H_\alpha''(-c_\alpha) = \frac{1}{\alpha}(c_\alpha^{-2} s_\alpha - 1) \\ &= \frac{1}{\alpha}\left(\frac{\alpha - c_\alpha}{c_\alpha(1 + c_\alpha)} - 1\right) = \frac{\alpha - 2c_\alpha - c_\alpha^2}{\alpha c_\alpha(1 + c_\alpha)}. \end{aligned}$$

In particular, we see from (iii) that $F'(-c_\alpha) > 0$, and (4.1) shows that F is analytic on $(-\infty, 0)$. From the defining formula $F(-c_\alpha) = 1/\alpha$ and the implicit function theorem (for analytic functions; see [FG02, Theorem 7.6]), it thus follows that c_α is an analytic function of α with derivative given by

$$\frac{dc_\alpha}{d\alpha} = \frac{1}{\alpha^2 F'(-c_\alpha)} = \frac{c_\alpha(1 + c_\alpha)}{\alpha(\alpha - 2c_\alpha - c_\alpha^2)}, \quad (\alpha \in (0, \infty)).$$

(vi) From (ii) and (v), it is clear that s_α is an analytic function of α. Consider now the function $S\colon (0, \infty) \times (0, \infty) \to \mathbb{R}$ given by

$$S(c, \alpha) = \frac{c}{1 + c}(\alpha - c), \quad (c, \alpha \in (0, \infty)),$$

and note that $s_\alpha = S(c_\alpha, \alpha)$ for all positive α, and that

$$\frac{\partial S}{\partial c}(c, \alpha) = \frac{\alpha - 2c - c^2}{(1 + c)^2} \text{ and } \frac{\partial S}{\partial \alpha}(c, \alpha) = \frac{c}{1 + c}, \quad (c, \alpha \in (0, \infty)).$$

Using the chain rule and (v), it thus follows that

$$\begin{aligned}\frac{\mathrm{d}s_\alpha}{\mathrm{d}\alpha} &= \frac{\partial S}{\partial c}(c_\alpha,\alpha)\frac{\mathrm{d}c_\alpha}{\mathrm{d}\alpha} + \frac{\partial S}{\partial \alpha}(c_\alpha,\alpha)\\ &= \frac{\alpha - 2c_\alpha - c_\alpha^2}{(1+c_\alpha)^2}\cdot\frac{c_\alpha(1+c_\alpha)}{\alpha(\alpha-2c_\alpha-c_\alpha^2)} + \frac{c_\alpha}{1+c_\alpha}\\ &= \frac{c_\alpha}{\alpha(1+c_\alpha)} + \frac{c_\alpha}{1+c_\alpha} = \frac{c_\alpha(\alpha+1)}{\alpha(1+c_\alpha)},\end{aligned}$$

and this completes the proof. ◻

Let a be a real number contained in an interval I. For functions $g,h\colon I\to\mathbb{C}$, such that $0\notin h(I)$, we use in the following proposition the notation $g(x)\sim h(x)$ as $x\to a$ to express that $\lim_{x\to a}(g(x)/h(x)) = 1$.

Proposition 4.3. *For any positive number α, we set $\gamma_\alpha = \dfrac{6H_\alpha''(-c_\alpha)}{H_\alpha'''(-c_\alpha)}$. Then, $\gamma_\alpha > 0$, and we have that*

(i) $$v_\alpha(x) \sim \gamma_\alpha^{1/2}(x+c_\alpha)^{1/2}, \quad \text{as } x\downarrow -c_\alpha,$$

(ii) $$\lim_{x\downarrow -c_\alpha}\frac{P_\alpha(x)-s_\alpha}{x+c_\alpha} = -\frac{1}{2}\gamma_\alpha H_\alpha''(-c_\alpha) > 0, \text{ and}$$

(iii) $$f_\alpha(\xi) \sim \frac{\sqrt{2}}{\pi c_\alpha\sqrt{s_\alpha - c_\alpha^2}}(\xi - s_\alpha)^{1/2}, \quad \text{as } \xi\downarrow s_\alpha.$$

Proof. Using formula (3.2), we note first, for any k in $\{2,3,4,\dots\}$, that

$$H_\alpha^{(k)}(-c_\alpha) = -\alpha k!\int_0^\infty \frac{t\mathrm{e}^{-t}}{(t+c_\alpha)^{k+1}}\,\mathrm{d}t < 0, \tag{4.2}$$

and in particular this verifies that $\gamma_\alpha > 0$.

(i) Using formula (A.4) (in Appendix A), we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}(v_\alpha(x)^2) = 2v_\alpha(x)v_\alpha'(x) \xrightarrow[x\downarrow -c_\alpha]{} \frac{2\displaystyle\int_0^\infty \frac{t\mathrm{e}^{-t}}{(t+c_\alpha)^3}\,\mathrm{d}t}{\displaystyle\int_0^\infty \frac{t\mathrm{e}^{-t}}{(t+c_\alpha)^4}\,\mathrm{d}t} = \gamma_\alpha,$$

where the last equality results from (4.2). Thus, l'Hôpital's rule yields that

$$\lim_{x\downarrow -c_\alpha}\frac{v_\alpha(x)^2}{\gamma_\alpha(x+c_\alpha)} = \frac{\gamma_\alpha}{\gamma_\alpha} = 1.$$

(ii) Using that $H_\alpha(-c_\alpha) = s_\alpha$ and $H'_\alpha(-c_\alpha) = 0$ (cf. Lemma 3.3), we find by Taylor expansion that

$$\begin{aligned} P_\alpha(x) &= H_\alpha(x + \mathrm{i}v_\alpha(x)) \\ &= s_\alpha + \frac{1}{2}H''_\alpha(-c_\alpha)(x + c_\alpha + \mathrm{i}v_\alpha(x))^2 + o(|x + c_\alpha + \mathrm{i}v_\alpha(x)|^2), \end{aligned}$$

and hence by application of (i), we have

$$\begin{aligned} \frac{P_\alpha(x) - s_\alpha}{x + c_\alpha} &= \frac{1}{2}H''_\alpha(-c_\alpha)\left(x + c_\alpha - \frac{v_\alpha(x)^2}{x + c_\alpha}\right) + \frac{o((x + c_\alpha)^2 + v_\alpha(x)^2)}{x + c_\alpha} \\ &\longrightarrow -\frac{1}{2}H''_\alpha(-c_\alpha)\gamma_\alpha \end{aligned}$$

as $x \downarrow -c_\alpha$. Since $\gamma_\alpha > 0$, formula (4.2) shows that the resulting expression above is positive, and hence (ii) is established.

(iii) Recall from Theorem 3.7 that

$$f_\alpha(P_\alpha(x)) = \frac{1}{\pi}\frac{v_\alpha(x)}{x^2 + v_\alpha(x)^2}, \quad (x \in [-c_\alpha, \infty)).$$

By application of (i), it thus follows that

$$f_\alpha(P_\alpha(x)) \sim \frac{\gamma_\alpha^{1/2}}{\pi c_\alpha^2}(x + c_\alpha)^{1/2}, \quad \text{as } x \downarrow -c_\alpha. \tag{4.3}$$

Using (ii), we have also that

$$\lim_{\xi \downarrow s_\alpha}\left(\frac{\xi - s_\alpha}{P_\alpha^{\langle -1\rangle}(\xi) + c_\alpha}\right) = \lim_{\xi \downarrow s_\alpha}\left(\frac{P_\alpha(P_\alpha^{\langle -1\rangle}(\xi)) - s_\alpha}{P_\alpha^{\langle -1\rangle}(\xi) + c_\alpha}\right) = -\frac{1}{2}\gamma_\alpha H''_\alpha(-c_\alpha),$$

and hence

$$P_\alpha^{\langle -1\rangle}(\xi) + c_\alpha \sim -\frac{2}{\gamma_\alpha H''_\alpha(-c_\alpha)}(\xi - s_\alpha), \quad \text{as } \xi \downarrow s_\alpha.$$

Combining this with (4.3), we find that

$$f_\alpha(\xi) \sim \frac{\gamma_\alpha^{1/2}(P_\alpha^{\langle -1\rangle}(\xi) + c_\alpha)^{1/2}}{\pi c_\alpha^2} \sim \frac{\sqrt{2}}{\pi c_\alpha^2(-H''_\alpha(-c_\alpha))^{1/2}}(\xi - s_\alpha)^{1/2},$$

as $\xi \downarrow s_\alpha$. Applying finally Proposition 4.2(iii), we obtain (iii). $\square$

5. UNIMODALITY

In this section, we establish unimodality of the densities f_α. We start with a few preparatory results.

Lemma 5.1. *For each positive number R, let $\Phi_R\colon (0,\pi) \to (0,\infty)$ be the function given by*

$$\Phi_R(\theta) = F(R\sin(\theta)\mathrm{e}^{\mathrm{i}\theta}), \quad (\theta \in (0,\pi)),$$

where F is the function introduced in (3.3). Then, for any R in $(0,\infty)$, there exists a unique number θ_R in $(0,\pi)$ such that Φ_R is strictly decreasing on $(0,\theta_R]$ and strictly increasing on $[\theta_R,\pi)$.

Proof. We note first that, for any r in $(0,\infty)$ and θ in $(-\pi,\pi]$, we have, using the change of variables $t = ru$, that

$$\begin{aligned} F(r\mathrm{e}^{\mathrm{i}\theta}) &= \int_0^\infty \frac{t\mathrm{e}^{-t}}{(r\cos(\theta)-t)^2 + r^2\sin^2(\theta)}\,\mathrm{d}t \\ &= \int_0^\infty \frac{ru\mathrm{e}^{-ru}}{r^2(\cos(\theta)-u)^2 + r^2\sin^2(\theta)} r\,\mathrm{d}u \\ &= \int_0^\infty \frac{u\mathrm{e}^{-ru}}{1-2u\cos(\theta)+u^2}\,\mathrm{d}u. \end{aligned}$$

Hence, for a fixed positive number R, we have that

$$\Phi_R(\theta) = F(R\sin(\theta)\mathrm{e}^{\mathrm{i}\theta}) = \int_0^\infty \frac{u\mathrm{e}^{-uR\sin(\theta)}}{1-2u\cos(\theta)+u^2}\,\mathrm{d}u, \quad (\theta \in (0,\pi)).$$

Then, define the function $\Psi_R\colon (-1,1) \to (0,\infty)$ by

$$\Psi_R(s) = \int_0^\infty \frac{u\mathrm{e}^{-uR\sqrt{1-s^2}}}{1-2us+u^2}\,\mathrm{d}u, \quad (s \in (-1,1)),$$

so that $\Phi_R(\theta) = \Psi_R(\cos(\theta))$ for θ in $(0,\pi)$. Since the function $\theta \mapsto \cos(\theta)$ is strictly decreasing on $(0,\pi)$, it suffices then to show that Ψ_R is strictly decreasing on $(-1,\eta_R]$ and strictly increasing on $[\eta_R,1)$ for some number η_R in $(-1,1)$. For this, we consider for any u in $(0,\infty)$ the function $\psi_{R,u}\colon (-1,1) \to (0,\infty)$ given by

$$\psi_{R,u}(s) = \frac{u\mathrm{e}^{-uR\sqrt{1-s^2}}}{1-2us+u^2}, \quad (s \in (-1,1)).$$

By a standard application of the theorem on differentiation under the integral sign, it follows that Ψ_R is differentiable on $(-1,1)$ with derivative

$$\Psi_R'(s) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\psi_{R,u}(s)\,\mathrm{d}u, \quad (s \in (-1,1)). \tag{5.1}$$

For any u in $(0,\infty)$ and s in $(-1,1)$, we note further that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\ln(\psi_{R,u}(s)) &= \frac{\mathrm{d}}{\mathrm{d}s}\big(\ln(u) - uR\sqrt{1-s^2} - \ln(1-2us+u^2)\big)\\ &= uRs(1-s^2)^{-1/2} + 2u(1-2us+u^2)^{-1},\end{aligned}$$

so that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ln(\psi_{R,u}(s)) &= uR(1-s^2)^{-1/2} + uRs^2(1-s^2)^{-3/2}\\ &\qquad + 4u^2(1-2us+u^2)^{-2}\\ &= uR(1-s^2)^{-3/2} + 4u^2(1-2us+u^2)^{-2} > 0.\end{aligned}$$

Since

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ln(\psi_{R,u}(s)) = \frac{\psi''_{R,u}(s)}{\psi_{R,u}(s)} - \frac{\psi'_{R,u}(s)^2}{\psi_{R,u}(s)^2},$$

we may thus conclude that $\psi''_{R,u} > 0$, and hence that $\psi'_{R,u}$ is strictly increasing on $(-1,1)$. Since this holds for all u in $(0,\infty)$, it follows further from (5.1) that Ψ'_R is *strictly* increasing on $(-1,1)$. Thus, Ψ_R is either strictly increasing, strictly decreasing, or of the form asserted above. However, by Fatou's lemma,

$$\liminf_{s\uparrow 1}\Psi_R(s) \geq \int_0^\infty \frac{u}{(1-u)^2}\,\mathrm{d}u = \infty,$$

and

$$\liminf_{s\downarrow -1}\Psi_R(s) \geq \int_0^\infty \frac{u}{(1+u)^2}\,\mathrm{d}u = \infty,$$

and hence Ψ_R must have the claimed form. $\square$

Lemma 5.2. *Let α be a strictly positive number, and consider the functions Q_α, P_α, and f_α given in* (3.17), (3.16), *and* (3.18). *We then have the following:*

(i) *For any ρ in $(0,\infty)$ the equation: $Q_\alpha(x) = \rho$ has at most two solutions in $(-c_\alpha,\infty)$.*
(ii) *For any ρ in $(0,\infty)$ the equation $f_\alpha(\xi) = \rho$ has at most two solutions in (s_α,∞).*

Proof. (i) Let ρ be a strictly positive number, and assume that there exist three distinct points x_1, x_2, x_3 in $(-c_\alpha,\infty)$ such that

$$\rho = Q_\alpha(x_j) = -\operatorname{Im}\left(\frac{1}{x_j + \mathrm{i}v_\alpha(x_j)}\right), \qquad (j = 1,2,3).$$

It is elementary to check that the points z in $\mathbb{C}\setminus\{0\}$, for which $-\operatorname{Im}(1/z) = \rho$, constitute the circle C_ρ in $\mathbb{C}$ with center $(1/(2\rho))\mathrm{i}$ and radius $1/(2\rho)$ (except for the origin). Thus, our assumption is that C_ρ intersects the set $\mathcal{G}'_\alpha$ (given in (3.10)) at three distinct points. Note that

$$\begin{aligned} C_\rho &= \left\{ \frac{1}{2\rho}(\mathrm{i} + \mathrm{e}^{\mathrm{i}\beta}) \mid \beta \in (-\pi, \pi] \right\} \\ &= \left\{ \frac{1}{2\rho}(\cos(\beta) + \mathrm{i}(1 + \sin(\beta))) \mid \beta \in (-\pi, \pi] \right\}. \end{aligned}$$

Writing a point $(1/(2\rho))(\cos(\beta)+\mathrm{i}(1+\sin(\beta))$ from $C_\rho \setminus \{0\}$ in polar coordinates $r\mathrm{e}^{\mathrm{i}\theta}$ ($r > 0$, $\theta \in (0, \pi)$), it follows that

$$r\sin(\theta) = \frac{1}{2\rho}(1 + \sin(\beta))$$

and

$$r^2 = \frac{1}{4\rho^2}(\cos^2(\beta) + 1 + \sin^2(\beta) + 2\sin(\beta)) = \frac{1}{2\rho^2}(1 + \sin(\beta)),$$

so that

$$r = \frac{1 + \sin(\beta)}{2\rho^2 r} = \frac{r\sin(\theta)}{\rho r} = \frac{1}{\rho}\sin(\theta).$$

Hence,

$$C_\rho = \left\{ \frac{1}{\rho}\sin(\theta)\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in (0, \pi] \right\},$$

and our assumption thus implies that there are three distinct points $\theta_1, \theta_2, \theta_3$ in $(0, \pi)$, such that $(1/\rho)\sin(\theta_j)\mathrm{e}^{\mathrm{i}\theta_j} \in \mathcal{G}'_\alpha$, $j = 1, 2, 3$. According to (3.10) and (3.7), this means that the equation

$$F\left(\frac{1}{\rho}\sin(\theta)\mathrm{e}^{\mathrm{i}\theta}\right) = \frac{1}{\alpha} \tag{5.2}$$

has (at least) three distinct solutions in $(0, \pi)$. However, Lemma 5.1 asserts that the function

$$\Phi_{1/\rho}(\theta) = F\left(\frac{1}{\rho}\sin(\theta)\mathrm{e}^{\mathrm{i}\theta}\right), \quad (\theta \in (0, \pi)),$$

is strictly decreasing on $(0, \theta_{1/\rho}]$ and strictly increasing on $[\theta_{1/\rho}, \pi)$ for some $\theta_{1/\rho}$ in $(0, \pi)$. Hence, the equation (5.2) has at most two solutions in $(0, \pi)$, and we have reached the desired contradiction.

(ii) Let ρ be a strictly positive number, and assume that there exist three distinct ξ_1, ξ_2, ξ_3 in (s_α, ∞) such that $f_\alpha(\xi_j) = \rho$, $j = 1, 2, 3$. Then, there exist three distinct points x_1, x_2, x_3 in $(-c_\alpha, \infty)$, such that $P_\alpha(x_j) = \xi_j$, $j = 1, 2, 3$, and it follows from formula (3.18) that

$$\rho = f_\alpha(P_\alpha(x_j)) = \frac{1}{\pi} Q_\alpha(x_j), \quad (j = 1, 2, 3).$$

This contradicts (i), and the proof is completed. ◻

Theorem 5.3. *For each α in $(0, \infty)$, the density f_α of the free Gamma distribution ν_α is unimodal. In fact, there exists a number ω_α in (s_α, ∞) such that f_α is strictly increasing on $[s_\alpha, \omega_\alpha]$ and strictly decreasing on $[\omega_\alpha, \infty)$.*

Proof. The proof is an elementary consequence of Lemma 5.2(ii); for completeness, we provide the details. We know that f_α is continuous, that $f_\alpha(\xi) > 0$ whenever $\xi > s_\alpha$, and that

$$f_\alpha(s_\alpha) = 0 = \lim_{\xi \to \infty} f_\alpha(\xi)$$

(cf. Lemma 3.1(iv), Theorem 3.7 and Proposition 4.1). In particular, it follows that f_α attains a strictly positive maximum at some point ω_α in (s_α, ∞). We show next that f_α is non-decreasing on $[s_\alpha, \omega_\alpha]$. Indeed, if this were not the case, we could choose ξ_1, ξ_2 in $(s_\alpha, \omega_\alpha)$ such that

$$\xi_1 < \xi_2, \quad \text{and} \quad f_\alpha(\xi_1) > f_\alpha(\xi_2).$$

Choosing an arbitrary number ρ in $(f_\alpha(\xi_2), f_\alpha(\xi_1))$, it follows then by continuity of f_α that there must exist s_1 in (s_α, ξ_1), s_2 in (ξ_1, ξ_2) and s_3 in (ξ_2, ω_α) such that

$$f_\alpha(s_i) = \rho, \quad (i = 1, 2, 3).$$

Since this contradicts Lemma 5.2(ii), we conclude that f_α is non-decreasing on $[s_\alpha, \omega_\alpha]$. This further implies that f_α is strictly increasing on that same interval, since otherwise f_α would be constant on a non-degenerate sub-interval, which is precluded by Lemma 5.2(ii).

Similar (symmetric) arguments show that f_α is strictly decreasing on $[\omega_\alpha, \infty)$, and this completes the proof. ◻

6. Asymptotic Behavior as $\alpha \to 0$

In this section, we study the asymptotic behavior of the free Gamma distributions ν_α, as $\alpha \downarrow 0$. We start by considering convergence in moments.

Proposition 6.1. *The measures $(1/\alpha)\nu_\alpha$ converge in moments to the measure $t^{-1}\mathrm{e}^{-t}1_{(0,\infty)}(t)\,\mathrm{d}t$ as $\alpha \downarrow 0$. More precisely, we have for any p in $\mathbb{N}$ that*

$$\frac{1}{\alpha}\int_0^\infty t^p \nu_\alpha(\mathrm{d}t) \longrightarrow \int_0^\infty t^{p-1}\mathrm{e}^{-t}\,\mathrm{d}t, \quad \text{as } \alpha \downarrow 0.$$

Proof. It follows from Proposition 4.1 that ν_α has moments of all orders (cf. also [BG06]). It follows, moreover, from [An01, Lemma 6.5] that, for all p in $\mathbb{N}$, the free cumulant $r_p(\alpha)$ of ν_α equals the classical cumulant $c_p(\alpha)$ of μ_α (the classical Gamma distribution with parameter α). The latter cumulants may be identified by considering the Taylor expansion at 0 of $\log(\hat{\mu}_\alpha(u))$. By using dominated convergence, it follows that, for any u in $(-1,1)$, we have that (cf. (3.12))

$$\begin{aligned}\log(\hat{\mu}_\alpha(u)) &= \alpha\int_0^\infty (\mathrm{e}^{\mathrm{i}ut}-1)\frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t = \alpha\int_0^\infty \Big(\sum_{p=1}^\infty \frac{\mathrm{i}^p u^p t^{p-1}}{p!}\Big)\mathrm{e}^{-t}\,\mathrm{d}t\\ &= \alpha\sum_{p=1}^\infty \frac{\mathrm{i}^p(p-1)!}{p!}u^p,\end{aligned}$$

from which we may deduce that $r_p(\alpha) = c_p(\alpha) = \alpha(p-1)!$ for all p in $\mathbb{N}$. Using the Moment-Cumulant Formula (cf. [NiSp06]), it follows further that the p'th moment $m_p(\alpha)$ of ν_α is given by

$$m_p(\alpha) = r_p(\alpha) + \sum_{k=2}^p \frac{1}{k}\binom{p}{k-1}\sum_{\substack{q_1,\dots,q_k\ge 1\\ q_1+\cdots+q_k=p}} r_{q_1}(\alpha)r_{q_2}(\alpha)\cdots r_{q_k}(\alpha)$$

for any p in $\mathbb{N}$. In particular, we see that $m_p(\alpha)$ is a polynomial in α of degree p with no constant term and linear term $\alpha(p-1)!$. For any p in $\mathbb{N}$, we may thus conclude that

$$\frac{1}{\alpha}\int_0^\infty t^p\nu_\alpha(\mathrm{d}t) = \frac{1}{\alpha}m_p(\alpha)\xrightarrow[\alpha\to 0]{} (p-1)! = \int_0^\infty t^{p-1}\mathrm{e}^{-t}\,\mathrm{d}t,$$

as desired. □

We show next that the densities of $(1/\alpha)\nu_\alpha$ actually converge pointwise to $t^{-1}\mathrm{e}^{-t}1_{(0,\infty)}(t)$ as $\alpha\downarrow 0$.

Lemma 6.2. *Consider the functions P_α defined in (3.14). We have the following:*

(i) *For any x in $(0,\infty)$, we have that $P_\alpha(x)\to x$, as $\alpha\downarrow 0$.*

(ii) *For any y in $(0,\infty)$, we have that $P_\alpha^{\langle -1\rangle}(y)\to y$, as $\alpha\downarrow 0$.*

Proof. (i) Let x be a fixed number in $(0,\infty)$. From (3.1) and (3.14), it follows that

$$P_\alpha(x) = x + \mathrm{i}v_\alpha(x) + \alpha + \alpha\int_0^\infty \frac{t\mathrm{e}^{-t}}{x-t+\mathrm{i}v_\alpha(x)}\,\mathrm{d}t, \qquad (\alpha\in(0,\infty)).$$

Lemma 3.4(iii) clearly implies that $v_\alpha(x)\to 0$, as $\alpha\to 0$, and hence it suffices to show that

$$\alpha\int_0^\infty \frac{t\mathrm{e}^{-t}}{x-t+\mathrm{i}v_\alpha(x)}\,\mathrm{d}t \longrightarrow 0, \quad \text{as } \alpha\to 0. \tag{6.1}$$

From Lemma 3.4(iii), it follows furthermore that we may choose α_1 in $(0,\infty)$, such that $v_\alpha(x)/\alpha \geq (\pi/2)x\mathrm{e}^{-x}$, whenever $\alpha \in (0,\alpha_1]$. Then, for all t in $(0,\infty)$ and α in $(0,\alpha_1]$, we have that

$$\alpha\left|\frac{t\mathrm{e}^{-t}}{x-t+\mathrm{i}v_\alpha(x)}\right| \leq \frac{t\mathrm{e}^{-t}}{v_\alpha(x)/\alpha} \leq \frac{t\mathrm{e}^{-t}}{(\pi/2)x\mathrm{e}^{-x}} = \frac{2}{\pi}x^{-1}\mathrm{e}^{x}t\mathrm{e}^{-t}. \tag{6.2}$$

For any t in $(0,\infty)\setminus\{x\}$, we note further that

$$\alpha\left|\frac{t\mathrm{e}^{-t}}{x-t+\mathrm{i}v_\alpha(x)}\right| \leq \alpha\frac{t\mathrm{e}^{-t}}{|x-t|} \longrightarrow 0, \quad \text{as } \alpha \to 0. \tag{6.3}$$

If we combine (6.2) and (6.3), it follows by dominated convergence that (6.1) holds, as desired.

(ii) Let y in $(0,\infty)$ and ε in $(0,y)$ be given. From (i), we know that $P_\alpha(y-\varepsilon) \to y-\varepsilon$, and $P_\alpha(y+\varepsilon) \to y+\varepsilon$, as $\alpha \to 0$. Hence, we may choose α_2 in $(0,\infty)$ such that

$$P_\alpha(y-\varepsilon) < y \text{ and } P_\alpha(y+\varepsilon) > y, \quad \text{whenever } \alpha \in (0,\alpha_2].$$

Then, for any α in $(0,\alpha_2]$, we have that

$$y \in [P_\alpha(y-\varepsilon), P_\alpha(y+\varepsilon)] = P_\alpha([y-\varepsilon, y+\varepsilon]), \tag{6.4}$$

since P_α is increasing and continuous. It follows from (6.4) that

$$P_\alpha^{\langle -1\rangle}(y) \in [y-\varepsilon, y+\varepsilon], \quad \text{whenever } \alpha \in (0,\alpha_2],$$

and since ε was chosen arbitrarily in $(0,y)$, this establishes (ii). $\square$

Proposition 6.3. *For any x in $(0,\infty)$, we have that $\frac{1}{\alpha}f_\alpha(x) \to x^{-1}\mathrm{e}^{-x}$, as $\alpha \to 0$.*

Proof. Let x be a fixed number in $(0,\infty)$, and note that Lemma 3.4(iii) implies that $v_\alpha(x)/\alpha \to \pi x\mathrm{e}^{-x}$, as $\alpha \to 0$. Using then formula (3.18), we find that

$$\frac{1}{\alpha}f_\alpha(P_\alpha(x)) = \frac{v_\alpha(x)/\alpha}{\pi(x^2+v_\alpha(x)^2)} \longrightarrow \frac{x\mathrm{e}^{-x}}{x^2+0} = x^{-1}\mathrm{e}^{-x}, \quad \text{as } \alpha \to 0.$$

It suffices thus to show that

$$\frac{1}{\alpha}|f_\alpha(P_\alpha(x)) - f_\alpha(x)| \longrightarrow 0, \quad \text{as } \alpha \to 0.$$

For all positive α, we set $y_\alpha := P_\alpha^{\langle -1\rangle}(x)$, and Lemma 6.2(ii) then asserts that $y_\alpha \to x$ as $\alpha \to 0$. Given any number δ in $(0,x)$, we may thus choose α_1 in $(0,\infty)$ such that

$$y_\alpha \in [\delta,\infty), \quad \text{whenever } \alpha \in (0,\alpha_1]. \tag{6.5}$$

For any α in $(0,\alpha_1]$, we find then by application of (3.18) that

$$\begin{aligned}
\frac{1}{\alpha}\,|f_\alpha(P_\alpha(x)) - f_\alpha(x)| &= \frac{1}{\alpha}\,|f_\alpha(P_\alpha(x)) - f_\alpha(P_\alpha(y_\alpha))| \\
&= \frac{1}{\pi}\left|\frac{v_\alpha(x)/\alpha}{x^2+v_\alpha(x)^2} - \frac{v_\alpha(y_\alpha)/\alpha}{y_\alpha^2+v_\alpha(y_\alpha)^2}\right| \\
&\le \frac{1}{\pi}\left|\frac{v_\alpha(x)/\alpha - v_\alpha(y_\alpha)/\alpha}{x^2+v_\alpha(x)^2}\right| + \frac{v_\alpha(y_\alpha)}{\pi\alpha}\left|\frac{1}{x^2+v_\alpha(x)^2} - \frac{1}{y_\alpha^2+v_\alpha(y_\alpha)^2}\right|.
\end{aligned} \tag{6.6}$$

Consider, in addition, an arbitrary number γ in $(0,1)$. By Lemma 3.4(iii), we may then choose α_2 in $(0,\alpha_1]$, such that

$$\sup_{u\in[\delta,\infty)}\left|\frac{v_\alpha(u)}{\alpha} - \pi u \mathrm{e}^{-u}\right| \le \gamma, \quad \text{whenever } \alpha \in (0,\alpha_2]. \tag{6.7}$$

Using (6.5) and (6.7), we find that

$$\frac{v_\alpha(y_\alpha)}{\alpha} \le \pi y_\alpha \mathrm{e}^{-y_\alpha} + \gamma \le \pi \sup_{u\in(0,\infty)} u\mathrm{e}^{-u} + 1 < \infty, \tag{6.8}$$

whenever $\alpha \in (0,\alpha_2]$. Together with the fact that $y_\alpha \to x$ as $\alpha \to 0$, this implies that

$$\frac{1}{y_\alpha^2+v_\alpha(y_\alpha)^2} = \frac{1}{y_\alpha^2+\alpha^2(v_\alpha(y_\alpha)/\alpha)^2} \longrightarrow \frac{1}{x^2}, \quad \text{as } \alpha \to 0.$$

Since also $1/(x^2+v_\alpha(x)^2) \to 1/x^2$, as $\alpha \to 0$, another application of (6.8) then yields that

$$\frac{v_\alpha(y_\alpha)}{\alpha}\left|\frac{1}{x^2+v_\alpha(x)^2} - \frac{1}{y_\alpha^2+v_\alpha(y_\alpha)^2}\right| \longrightarrow 0, \quad \text{as } \alpha \to 0.$$

In view of (6.6), it remains thus to show that

$$\left|\frac{v_\alpha(x)/\alpha - v_\alpha(y_\alpha)/\alpha}{x^2+v_\alpha(x)^2}\right| \longrightarrow 0, \quad \text{as } \alpha \to 0. \tag{6.9}$$

For this, note that, for any α in $(0,\alpha_2]$, we have by new applications of (6.5) and (6.7) that

$$\left|\frac{v_\alpha(x)/\alpha - v_\alpha(y_\alpha)/\alpha}{x^2+v_\alpha(x)^2}\right| \le \frac{2\gamma + \pi|x\mathrm{e}^{-x} - y_\alpha \mathrm{e}^{-y_\alpha}|}{x^2}.$$

Since $u \mapsto ue^{-u}$ is continuous at x, we may choose α_3 in $(0, \alpha_2]$ such that $|x\mathrm{e}^{-x} - y_\alpha \mathrm{e}^{-y_\alpha}| \le \pi^{-1}\gamma$, whenever $\alpha \in (0, \alpha_3]$, and then

$$\left| \frac{v_\alpha(x)/\alpha - v_\alpha(y_\alpha)/\alpha}{x^2 + v_\alpha(x)^2} \right| \le \frac{3\gamma}{x^2}, \quad \text{whenever } \alpha \in (0, \alpha_3].$$

As γ was chosen arbitrarily in $(0, 1)$, this verifies (6.9) and completes the proof. □

Appendix A. Proofs of Lemmas 3.4 and 3.6

In this appendix, we provide detailed (but rather technical) proofs of Lemma 3.4 and Lemma 3.6. We start with the following preparatory result:

Lemma A.1. *Let α be a positive number, and consider $v_\alpha \colon \mathbb{R} \to [0, \infty)$, the function given by (3.6)–(3.7). Then, we have the following:*

(i) *If $0 < \varepsilon < x$, then*

$$v_\alpha(x) \ge 2\alpha(x - \varepsilon)\mathrm{e}^{-x-\varepsilon} \arctan \frac{\varepsilon}{v_\alpha(x)}.$$

(ii) *For any ε in $(0, 1)$ we have for all sufficiently large x that*

$$v_\alpha(x) \le \frac{2\alpha(x + \varepsilon)}{1 - \varepsilon} \mathrm{e}^{-x+\varepsilon} \arctan \frac{\varepsilon}{v_\alpha(x)}.$$

Proof. (i) Recall first (cf. (3.7)) that

$$\text{(A.1)} \qquad \frac{1}{\alpha} = \int_0^\infty \frac{t\mathrm{e}^{-t}}{(x - t)^2 + v_\alpha(x)^2} \,\mathrm{d}t, \quad (x \in (-c_\alpha, \infty)).$$

Assume next that $0 < \varepsilon < x$. Using (A.1) as well as the change of variables $u = (t - x)/v_\alpha(x)$, we find that

$$\text{(A.2)} \qquad \begin{aligned} \frac{1}{\alpha} &\ge \int_{x-\varepsilon}^{x+\varepsilon} \frac{t\mathrm{e}^{-t}}{(t - x)^2 + v_\alpha(x)^2} \,\mathrm{d}t \\ &\ge \frac{(x - \varepsilon)\mathrm{e}^{-x-\varepsilon}}{v_\alpha(x)^2} \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{1 + ((t - x)/v_\alpha(x))^2} \,\mathrm{d}t \\ &= \frac{(x - \varepsilon)\mathrm{e}^{-x-\varepsilon}}{v_\alpha(x)^2} \int_{-\varepsilon/v_\alpha(x)}^{\varepsilon/v_\alpha(x)} \frac{1}{1 + u^2} v_\alpha(x) \,\mathrm{d}u \\ &= \frac{2(x - \varepsilon)\mathrm{e}^{-x-\varepsilon}}{v_\alpha(x)} \arctan \frac{\varepsilon}{v_\alpha(x)}, \end{aligned}$$

from which the desired estimate follows immediately.

(ii) Let ε be a given number in $(0, 1)$, and note that, for any t in $(0, \infty)$ and x in (ε, ∞),

$$\frac{t\mathrm{e}^{-t}}{(t - x)^2 + v_\alpha(x)^2} 1_{[0,x-\varepsilon]}(t), \ \frac{t\mathrm{e}^{-t}}{(t - x)^2 + v_\alpha(x)^2} 1_{[x+\varepsilon,\infty)}(t) \le \varepsilon^{-2} t\mathrm{e}^{-t}.$$

Hence, by dominated convergence,

$$\int_0^{x-\varepsilon} \frac{t\mathrm{e}^{-t}}{(t-x)^2+v_\alpha(x)^2}\,\mathrm{d}t, \int_{x+\varepsilon}^{\infty} \frac{t\mathrm{e}^{-t}}{(t-x)^2+v_\alpha(x)^2}\,\mathrm{d}t \longrightarrow 0, \quad \text{as } x \to \infty.$$

Thus, for all sufficiently large x, we have by (A.1) that

$$\begin{aligned}(1-\varepsilon)\frac{1}{\alpha} &\le \int_{x-\varepsilon}^{x+\varepsilon} \frac{t\mathrm{e}^{-t}}{(t-x)^2+v_\alpha(x)^2}\,\mathrm{d}t \\ &\le \frac{(x+\varepsilon)\mathrm{e}^{-x+\varepsilon}}{v_\alpha(x)^2}\int_{x-\varepsilon}^{x+\varepsilon}\frac{1}{1+((t-x)/v_\alpha(x))^2}\,\mathrm{d}t \\ &= \frac{2(x+\varepsilon)\mathrm{e}^{-x+\varepsilon}}{v_\alpha(x)}\arctan\frac{\varepsilon}{v_\alpha(x)},\end{aligned} \tag{A.3}$$

which yields the desired estimate. ◻

Proof of Lemma 3.4. (i) Consider the function $\tilde{F}\colon \mathbb{R}\times(0,\infty)\to\mathbb{R}$ given by

$$\tilde{F}(x,y) = F(x+\mathrm{i}y) = \int_0^\infty \frac{t\mathrm{e}^{-t}}{(x-t)^2+y^2}\,\mathrm{d}t, \qquad (x\in\mathbb{R},\ y>0).$$

Using formula (3.8) in the case $\alpha = 1$, it follows that

$$\tilde{F}(x,y) = 1 - y^{-1}\operatorname{Im}(H_1(x+\mathrm{i}y)), \quad ((x,y)\in\mathbb{R}\times(0,\infty)),$$

and since the imaginary part of an analytic function is analytic (as a function of two real variables), we may conclude from this that $\tilde{F}$ is analytic on $\mathbb{R}\times(0,\infty)$. By differentiation under the integral sign, we find in particular that

$$\frac{\partial\tilde{F}}{\partial x}(x,y) = -2\int_0^\infty \frac{(x-t)t\mathrm{e}^{-t}}{((x-t)^2+y^2)^2}\,\mathrm{d}t$$

and

$$\frac{\partial\tilde{F}}{\partial y}(x,y) = -2y\int_0^\infty \frac{t\mathrm{e}^{-t}}{((x-t)^2+y^2)^2}\,\mathrm{d}t.$$

Since $v_\alpha(x) > 0$ and $\tilde{F}(x, v_\alpha(x)) = 1/\alpha$ for all x in $(-c_\alpha,\infty)$, and since

$$\frac{\partial\tilde{F}}{\partial y}(x,y) < 0 \quad \text{for all } (x,y)\in\mathbb{R}\times(0,\infty),$$

it follows then from the implicit function theorem (for analytic functions; see [FG02, Theorem 7.6]) that v_α is analytic on $(-c_\alpha,\infty)$ with derivative given by

$$\text{(A.4)}\quad v_\alpha'(x) = \frac{-\dfrac{\partial \tilde{F}}{\partial x}(x, v_\alpha(x))}{\dfrac{\partial \tilde{F}}{\partial y}(x, v_\alpha(x))} = \frac{-\displaystyle\int_0^\infty \frac{(x-t)t\mathrm{e}^{-t}}{((x-t)^2 + v_\alpha(x)^2)^2}\,\mathrm{d}t}{v_\alpha(x)\displaystyle\int_0^\infty \frac{t\mathrm{e}^{-t}}{((x-t)^2 + v_\alpha(x)^2)^2}\,\mathrm{d}t}, \qquad (x \in (-c_\alpha, \infty)).$$

In particular, v_α is continuous on $(-c_\alpha, \infty)$. From the defining relations (A.1) and (3.4), it is standard to check that $v_\alpha(x) \to 0$ as $x \downarrow -c_\alpha$. Thus, v_α is continuous at $-c_\alpha$ as well, and hence on all of $\mathbb{R}$.

(ii) Using Lemma A.1(i), we find, for any positive ε, that

$$\liminf_{x\to\infty} \frac{v_\alpha(x)}{x\mathrm{e}^{-x}} \geq \liminf_{x\to\infty} \frac{2\alpha(x-\varepsilon)\mathrm{e}^{-x-\varepsilon}\arctan(\varepsilon/v_\alpha(x))}{x\mathrm{e}^{-x}} = 2\alpha\mathrm{e}^{-\varepsilon}\frac{\pi}{2} = \mathrm{e}^{-\varepsilon}\alpha\pi,$$

where we have used that $v_\alpha(x) \to 0$ as $x \to \infty$ (cf. Lemma A.1(ii)). Letting then $\varepsilon \to 0$, it follows that

$$\text{(A.5)}\qquad \liminf_{x\to\infty} \frac{v_\alpha(x)}{x\mathrm{e}^{-x}} \geq \alpha\pi.$$

Using Lemma A.1(ii), we find similarly for ε in $(0, 1)$ that

$$\limsup_{x\to\infty} \frac{v_\alpha(x)}{x\mathrm{e}^{-x}} \leq \limsup_{x\to\infty} \frac{2\alpha(x+\varepsilon)\mathrm{e}^{-x+\varepsilon}\arctan(\varepsilon/v_\alpha(x))}{(1-\varepsilon)x\mathrm{e}^{-x}} = \frac{\mathrm{e}^{\varepsilon}}{1-\varepsilon}\alpha\pi,$$

and letting then $\varepsilon \to 0$, we conclude that

$$\text{(A.6)}\qquad \limsup_{x\to\infty} \frac{v_\alpha(x)}{x\mathrm{e}^{-x}} \leq \alpha\pi.$$

Combining (A.5) and (A.6) completes the proof of (ii).

(iii) Let ε be a fixed number in $(0, \frac{1}{2}]$. For any x in $[\varepsilon, \infty)$, we note then that

$$\begin{aligned}\alpha\Big(\int_0^{x-\varepsilon} &\frac{t\mathrm{e}^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,\mathrm{d}t + \int_{x+\varepsilon}^\infty \frac{t\mathrm{e}^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,\mathrm{d}t\Big) \\ &\leq \alpha\varepsilon^{-2}\Big(\int_0^{x-\varepsilon} t\mathrm{e}^{-t}\,\mathrm{d}t + \int_{x+\varepsilon}^\infty t\mathrm{e}^{-t}\,\mathrm{d}t\Big) \leq \alpha\varepsilon^{-2}\int_0^\infty t\mathrm{e}^{-t}\,\mathrm{d}t = \alpha\varepsilon^{-2},\end{aligned}$$

and thus

$$\sup_{x\in[\varepsilon,\infty)} \alpha\Big(\int_0^{x-\varepsilon} \frac{t\mathrm{e}^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,\mathrm{d}t + \int_{x+\varepsilon}^\infty \frac{t\mathrm{e}^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,\mathrm{d}t\Big) \longrightarrow 0, \quad \text{as } \alpha \to 0.$$

In combination with (A.1), this implies that we may choose α_1 in $(0,\infty)$ such that, for all α in $(0,\alpha_1]$ and all x in $[\varepsilon,\infty)$, we have that

$$\begin{aligned}1-\varepsilon &\le \alpha\int_{x-\varepsilon}^{x+\varepsilon}\frac{t\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t\\ &\le 2\alpha(x+\varepsilon)\mathrm{e}^{-x+\varepsilon}v_\alpha(x)^{-1}\arctan\frac{\varepsilon}{v_\alpha(x)}\le \pi\alpha(x+\varepsilon)\mathrm{e}^{-x+\varepsilon}v_\alpha(x)^{-1},\end{aligned}$$

where we have reused the calculation (A.3). Hence, it follows that

$$\text{(A.7)}\qquad \frac{v_\alpha(x)}{\alpha}\le\frac{\pi(x+\varepsilon)\mathrm{e}^{-x+\varepsilon}}{1-\varepsilon}\quad\text{for all } x \text{ in } [\varepsilon,\infty) \text{ and } \alpha \text{ in } (0,\alpha_1].$$

Since $\varepsilon\le\frac{1}{2}$, we find, in particular, for all α in $(0,\alpha_1]$, that

$$\text{(A.8)}\qquad \sup_{x\in[\varepsilon,\infty)} v_\alpha(x)\le K_\varepsilon\alpha,\quad\text{where}\quad K_\varepsilon := 2\pi\sqrt{\mathrm{e}}\sup_{x\in[\varepsilon,\infty)}\left(x+\frac{1}{2}\right)\mathrm{e}^{-x}<\infty.$$

For any x in $[\varepsilon,\infty)$ and α in $(0,\infty)$, we note next that

$$\begin{aligned}1 &\ge \alpha\int_{x-\varepsilon}^{x+\varepsilon}\frac{t\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t\\ &\ge \alpha(x-\varepsilon)\mathrm{e}^{-x-\varepsilon}v_\alpha(x)^{-2}\int_{x-\varepsilon}^{x+\varepsilon}\frac{1}{1+((t-x)/v_\alpha(x))^2}\,\mathrm{d}t\\ &= 2\alpha(x-\varepsilon)\mathrm{e}^{-x-\varepsilon}v_\alpha(x)^{-1}\arctan\frac{\varepsilon}{v_\alpha(x)}.\end{aligned}$$

In combination with (A.8), this shows that, for all x in $[\varepsilon,\infty)$ and α in $(0,\alpha_1]$, we have that

$$\frac{v_\alpha(x)}{\alpha}\ge 2(x-\varepsilon)\mathrm{e}^{-x-\varepsilon}\arctan(\varepsilon K_\varepsilon^{-1}\alpha^{-1}).$$

Hence, we may choose α_2 in $(0,\alpha_1]$, such that

$$\text{(A.9)}\qquad \frac{v_\alpha(x)}{\alpha}\ge 2(x-\varepsilon)\mathrm{e}^{-x-\varepsilon}(1-\varepsilon)\frac{\pi}{2}=(1-\varepsilon)\pi(x-\varepsilon)\mathrm{e}^{-x-\varepsilon}$$

for all x in $[\varepsilon,\infty)$ and α in $(0,\alpha_2]$. Combining now (A.7) and (A.9), it follows for any α in $(0,\alpha_2]$ that

$$\begin{aligned}\text{(A.10)}\qquad &\sup_{x\in[\varepsilon,\infty)}\left|\frac{v_\alpha(x)}{\alpha}-\pi x\mathrm{e}^{-x}\right|\\ &\quad\le\pi\sup_{x\in[\varepsilon,\infty)}\Big[\big(x\mathrm{e}^{-x}-(1-\varepsilon)(x-\varepsilon)\mathrm{e}^{-x-\varepsilon}\big)\\ &\qquad\qquad\vee\big((1-\varepsilon)^{-1}(x+\varepsilon)\mathrm{e}^{-x+\varepsilon}-x\mathrm{e}^{-x}\big)\Big].\end{aligned}$$

Using the fact that the function $x \mapsto x\mathrm{e}^{-x}$ is bounded on $(0,\infty)$, it is standard to check that

$$\text{(A.11)} \qquad \begin{aligned} &\sup_{x\in(0,\infty)} \big(x\mathrm{e}^{-x} - (1-\varepsilon)(x-\varepsilon)\mathrm{e}^{-x-\varepsilon}\big) \xrightarrow[\varepsilon\to 0]{} 0, \\ &\sup_{x\in(0,\infty)} \big((1-\varepsilon)^{-1}(x+\varepsilon)\mathrm{e}^{-x+\varepsilon} - x\mathrm{e}^{-x}\big) \xrightarrow[\varepsilon\to 0]{} 0. \end{aligned}$$

To complete the proof, assume that positive numbers δ and γ are given. By (A.11), we may then choose ε in $(0,\delta\wedge\frac{1}{2}]$ such that the right-hand side of (A.10) is smaller than γ. Applying the above considerations to this ε, it follows that we may choose α_2 in $(0,\infty)$, such that

$$\sup_{x\in[\delta,\infty)} \left|\frac{v_\alpha(x)}{\alpha} - \pi x\mathrm{e}^{-x}\right| \le \sup_{x\in[\varepsilon,\infty)} \left|\frac{v_\alpha(x)}{\alpha} - \pi x\mathrm{e}^{-x}\right| \le \gamma,$$

whenever $\alpha \in (0,\alpha_2]$. $\square$

For the proof of Lemma 3.6, we need the following auxiliary result.

Lemma A.2. *Let α, r, ε be positive numbers such that $\varepsilon < 1$. We then have the following:*

(i) *As $x\to\infty$, we have* $x\displaystyle\int_{(0,\infty)\setminus(x-\varepsilon/x,\,x+\varepsilon/x)} \frac{t^r\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t \longrightarrow 0.$

(ii) *If $0<\varepsilon<x$, then*

$$\begin{aligned}\frac{2(x-\varepsilon)^r\mathrm{e}^{-x-\varepsilon}}{v_\alpha(x)}\arctan\frac{\varepsilon}{v_\alpha(x)} &\le \int_{x-\varepsilon}^{x+\varepsilon} \frac{t^r\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t \\ &\le \frac{2(x+\varepsilon)^r\mathrm{e}^{-x+\varepsilon}}{v_\alpha(x)}\arctan\frac{\varepsilon}{v_\alpha(x)}.\end{aligned}$$

(iii) *For all sufficiently large positive x, we have that*

$$\frac{(x-\varepsilon)(x^2-\varepsilon)^2\mathrm{e}^{-2\varepsilon/x}}{\alpha x^2(x^2+\varepsilon)} \le \int_{x-\varepsilon/x}^{x+\varepsilon/x} \frac{t^2\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t \le \frac{(x^2+\varepsilon)^2\mathrm{e}^{2\varepsilon/x}}{\alpha x(x^2-\varepsilon)}.$$

Proof. (i) We note first that, for x in, say, $(2,\infty)$, we have that

$$\begin{aligned}\int_{x+\varepsilon/x}^{2x} \frac{t^r\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t &\le (2x)^r\mathrm{e}^{-x-\varepsilon/x}\int_{x+\varepsilon/x}^{2x}\frac{1}{(\varepsilon/x)^2}\,\mathrm{d}t \\ &= 2^r\varepsilon^{-2}x^r(x^3-\varepsilon x)\mathrm{e}^{-x-\varepsilon/x},\end{aligned}$$

and similarly that

$$\begin{aligned}\int_{x/2}^{x-\varepsilon/x} \frac{t^r\mathrm{e}^{-r}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t &\le \left(x-\frac{\varepsilon}{x}\right)^r\mathrm{e}^{-x/2}\int_{x/2}^{x-\varepsilon/x}\frac{1}{(\varepsilon/x)^2}\,\mathrm{d}t \\ &= \varepsilon^{-2}\left(x-\frac{\varepsilon}{x}\right)^r\left(\frac{x^3}{2}-\varepsilon x\right)\mathrm{e}^{-x/2}.\end{aligned}$$

Moreover,

$$\int_{2x}^{\infty} \frac{t^r e^{-r}}{(x-t)^2 + v_\alpha(x)^2}\,dt \le \int_{2x}^{\infty} \frac{t^r e^{-r}}{x^2}\,dt = \frac{1}{x^2}\int_{2x}^{\infty} t^r e^{-r}\,dt,$$

and

$$\int_{0}^{x/2} \frac{t^r e^{-r}}{(x-t)^2 + v_\alpha(x)^2}\,dt \le \int_{0}^{x/2} \frac{t^r e^{-r}}{(x/2)^2}\,dt = \frac{4}{x^2}\int_{0}^{x/2} t^r e^{-r}\,dt.$$

Now, the sum of the left-hand sides of the four estimates above is equal to the integral

$$\int_{(0,\infty)\setminus(x-\varepsilon/x,x+\varepsilon/x)} \frac{t^r e^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,dt,$$

and the sum of the right-hand sides is clearly of size $o(x^{-1})$ as $x \to \infty$. Thus, this shows (i).

(ii) Assume $0 < \varepsilon < x$. Arguing as in the proof of Lemma 3.4(ii), we find that

$$\begin{aligned}\int_{x-\varepsilon}^{x+\varepsilon} \frac{t^r e^{-t}}{(t-x)^2 + v_\alpha(x)^2}\,dt &\le \frac{(x+\varepsilon)^r e^{-x+\varepsilon}}{v_\alpha(x)^2}\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{1+((t-x)/v_\alpha(x))^2}\,dt \\ &= \frac{2(x+\varepsilon)^r e^{-x+\varepsilon}}{v_\alpha(x)} \arctan\frac{\varepsilon}{v_\alpha(x)},\end{aligned}$$

which proves the second estimate in (ii). The first estimate follows similarly.

(iii) Considering x in $(1,\infty)$, we find by application of (ii) and Lemma A.1(i) (with ε replaced by ε/x) that

$$\begin{aligned}\int_{x-\varepsilon/x}^{x+\varepsilon/x} \frac{t^2 e^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,dt &\le \frac{2(x+\varepsilon/x)^2 e^{-x+\varepsilon/x}}{v_\alpha(x)} \arctan\frac{\varepsilon}{x v_\alpha(x)} \\ &\le \frac{2(x+\varepsilon/x)^2 e^{-x+\varepsilon/x} \arctan(\varepsilon/(x v_\alpha(x)))}{2\alpha(x-\varepsilon/x)e^{-x-\varepsilon/x} \arctan(\varepsilon/(x v_\alpha(x)))} \\ &= \frac{(x^2+\varepsilon)^2 e^{2\varepsilon/x}}{\alpha x(x^2-\varepsilon)},\end{aligned}$$

which proves the second estimate in (iii). Regarding the first estimate, we note first that it follows from (i) that

$$\int_{(0,\infty)\setminus(x-\varepsilon/x,x+\varepsilon/x)} \frac{t e^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,dt \le \frac{\varepsilon}{\alpha x}$$

for all sufficiently large x, and hence by (A.1) and (ii), we have

$$\begin{aligned}\frac{(1-\varepsilon/x)}{\alpha} &\le \int_{x-\varepsilon/x}^{x+\varepsilon/x} \frac{t e^{-t}}{(x-t)^2 + v_\alpha(x)^2}\,dt \\ &\le \frac{2(x+\varepsilon/x)e^{-x+\varepsilon/x}}{v_\alpha(x)} \arctan\frac{\varepsilon}{x v_\alpha(x)}\end{aligned}$$

for all sufficiently large x. For such x, we may thus conclude that

$$v_\alpha(x) \le \frac{2\alpha(x+\varepsilon/x)\mathrm{e}^{-x+\varepsilon/x}}{1-\varepsilon/x}\arctan\frac{\varepsilon}{xv_\alpha(x)},$$

which in combination with (ii) yields that

$$\begin{aligned}\int_{x-\varepsilon/x}^{x+\varepsilon/x}\frac{t^2\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t &\ge \frac{2(x-\varepsilon/x)^2\mathrm{e}^{-x-\varepsilon/x}}{v_\alpha(x)}\arctan\frac{\varepsilon}{xv_\alpha(x)}\\ &\ge \frac{2(1-\varepsilon/x)(x-\varepsilon/x)^2\mathrm{e}^{-x-\varepsilon/x}\arctan(\varepsilon/(xv_\alpha(x)))}{2\alpha(x+\varepsilon/x)\mathrm{e}^{-x+\varepsilon/x}\arctan(\varepsilon/(xv_\alpha(x)))}\\ &= \frac{(x-\varepsilon)(x^2-\varepsilon)^2\mathrm{e}^{-2\varepsilon/x}}{\alpha x^2(x^2+\varepsilon)},\end{aligned}$$

for all sufficiently large x. This completes the proof. $\square$

Proof of Lemma 3.6. (i) Since H_α is analytic on $\mathbb{C}\setminus[0,\infty)$ and v_α is analytic on $\mathbb{R}\setminus\{-c_\alpha\}$, it follows immediately from (3.14) that P_α is analytic on $\mathbb{R}\setminus\{-c_\alpha\}$. Since v_α is continuous on $\mathbb{R}$, it follows also that so is P_α.

(ii) For x in $(-\infty,-c_\alpha)$, formula (3.16) follows immediately from (3.1), since $v_\alpha(x)=0$. For x in $[-c_\alpha,\infty)$ we find, using Lemma 3.1(iii), (3.1), and (A.1), that

$$\begin{aligned}P_\alpha(x) &= H_\alpha(x+\mathrm{i}v_\alpha(x)) = \mathrm{Re}\left(H_\alpha(x+\mathrm{i}v_\alpha(x))\right)\\ &= x+\alpha+\alpha\int_0^\infty \mathrm{Re}\left(\frac{t\mathrm{e}^{-t}}{x+\mathrm{i}v_\alpha(x)-t}\right)\mathrm{d}t\\ &= x+\alpha+\alpha\int_0^\infty\frac{(x-t)t\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t\\ &= x+\alpha+\alpha x\int_0^\infty\frac{t\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t-\alpha\int_0^\infty\frac{t^2\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t\\ &= 2x+\alpha-\alpha\int_0^\infty\frac{t^2\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t,\end{aligned}$$

as desired.

(iii) If we consider the function H_α restricted to $(-\infty,0)$, it follows from (3.1) and dominated convergence that

$$\lim_{z\to-\infty,\, z\in\mathbb{R}} H_\alpha(z) = -\infty \quad\text{and}\quad \lim_{z\to 0,\, z\in(-\infty,0)} H_\alpha(z) = 0.$$

From (3.2), it follows further that

$$H_\alpha'(z)>0,\ \text{if } z\in(-\infty,-c_\alpha) \quad\text{and}\quad H_\alpha'(z)<0,\ \text{if } z\in(-c_\alpha,0).$$

Since $P_\alpha = H_\alpha$ on $(-\infty, -c_\alpha]$, we deduce from these observations that P_α is strictly increasing on $(-\infty, -c_\alpha]$, and that $s_\alpha > \inf_{z\in(-c_\alpha,0)} H_\alpha(z) = 0$.

(iv) Using formula (3.16) as well as (i) and (iii) of Lemma A.2, we find, for any ε in $(0,1)$, that

$$\begin{aligned}
\limsup_{x\to\infty}(x+\alpha-P_\alpha(x)) &= \limsup_{x\to\infty}\Big(-x+\alpha\int_0^\infty \frac{t^2\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t\Big)\\
&= \limsup_{x\to\infty}\Big(-x+\alpha\int_{x-\varepsilon/x}^{x+\varepsilon/x} \frac{t^2\mathrm{e}^{-t}}{(x-t)^2+v_\alpha(x)^2}\,\mathrm{d}t\Big)\\
&\le \limsup_{x\to\infty}\left(-x+\alpha\frac{(x^2+\varepsilon)^2\mathrm{e}^{2\varepsilon/x}}{\alpha x(x^2-\varepsilon)}\right)\\
&= \limsup_{x\to\infty}\left(-x+\frac{(x^2+\varepsilon)}{x}\left(1+\frac{2\varepsilon}{x^2-\varepsilon}\right)\mathrm{e}^{2\varepsilon/x}\right)\\
&= \limsup_{x\to\infty}\left(x(\mathrm{e}^{2\varepsilon/x}-1)+\left(\frac{2\varepsilon x}{x^2-\varepsilon}+\frac{\varepsilon}{x}+\frac{2\varepsilon^2}{x(x^2-\varepsilon)}\right)\mathrm{e}^{2\varepsilon/x}\right)\\
&= \lim_{x\to\infty}\frac{\mathrm{e}^{2\varepsilon/x}-1}{1/x-0} = 2\varepsilon.
\end{aligned}$$

Arguing similarly, we find next that

$$\begin{aligned}
\liminf_{x\to\infty}(x+\alpha-P_\alpha(x)) &\ge \liminf_{x\to\infty}\left(-x+\alpha\frac{(x-\varepsilon)(x^2-\varepsilon)^2\mathrm{e}^{-2\varepsilon/x}}{\alpha x^2(x^2+\varepsilon)}\right),\\
&= \liminf_{x\to\infty}\left(-x+\frac{(x-\varepsilon)(x^2-\varepsilon)}{x^2}\left(1-\frac{2\varepsilon}{x^2+\varepsilon}\right)\mathrm{e}^{-2\varepsilon/x}\right)\\
&= \liminf_{x\to\infty}\Bigg(x(\mathrm{e}^{-2\varepsilon/x}-1)-\frac{2\varepsilon x}{x^2+\varepsilon}\mathrm{e}^{-2\varepsilon/x}\\
&\qquad+\frac{\varepsilon^2-x\varepsilon-x^2\varepsilon}{x^2}\left(1-\frac{2\varepsilon}{x^2+\varepsilon}\right)\mathrm{e}^{-2\varepsilon/x}\Bigg) = -3\varepsilon.
\end{aligned}$$

(v) From (3.16) and (iv), it is clear that $P_\alpha(x)\to\pm\infty$ as $x\to\pm\infty$, and since P_α is continuous, it suffices thus to prove that $P_\alpha'(x)>0$ for all x in $\mathbb{R}\setminus\{-c_\alpha\}$. In the proof of (iii), we already noted that $P_\alpha'(x)>0$ for all x in $(-\infty,-c_\alpha)$. For x in $(-c_\alpha,\infty)$, we find by differentiation in (3.14) that

$$\begin{aligned}
P_\alpha'(x) &= H_\alpha'(x+\mathrm{i}v_\alpha(x))(1+\mathrm{i}v_\alpha'(x))\\
&= \mathrm{Re}(H_\alpha'(x+\mathrm{i}v_\alpha(x))) - \mathrm{Im}(H_\alpha'(x+\mathrm{i}v_\alpha(x)))v_\alpha'(x),
\end{aligned}\tag{A.12}$$

where we have used that $P_\alpha'(x)\in\mathbb{R}$, so that

$$0 = \mathrm{Im}(P_\alpha'(x)) = \mathrm{Re}\,(H_\alpha'(x+\mathrm{i}v_\alpha(x)))v_\alpha'(x) + \mathrm{Im}\,(H_\alpha'(x+\mathrm{i}v_\alpha(x))).\tag{A.13}$$

According to Lemma 3.3, $\mathrm{Re}(H'_\alpha(x + \mathrm{i}v_\alpha(x))) > 0$, and hence (A.13) implies that

$$v'_\alpha(x) = \frac{-\mathrm{Im}(H'_\alpha(x + \mathrm{i}v_\alpha(x)))}{\mathrm{Re}(H'_\alpha(x + \mathrm{i}v_\alpha(x)))},$$

which, inserted into (A.12), yields that

$$\begin{aligned} P'_\alpha(x) &= \mathrm{Re}(H'_\alpha(x + \mathrm{i}v_\alpha(x))) + \frac{\mathrm{Im}(H'_\alpha(x + \mathrm{i}v_\alpha(x)))^2}{\mathrm{Re}(H'_\alpha(x + \mathrm{i}v_\alpha(x)))} \\ &= \frac{|H'_\alpha(x + \mathrm{i}v_\alpha(x))|^2}{\mathrm{Re}(H'_\alpha(x + \mathrm{i}v_\alpha(x)))} > 0, \end{aligned}$$

as desired. $\square$

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