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Inequalities for Jacobi polynomials

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Abstract A Bernstein-type inequality is obtained for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which is uniform for all degrees $n \geq 0$, all real $\alpha, \beta \geq 0$, and all values $x \in [-1, 1]$. It provides uniform bounds on a complete set of matrix coefficients for the irreducible representations of SU(2) with a decay of $d^{-1/4}$ in the dimension d of the representation. Moreover, it complements previous results of Krasikov on a conjecture of Erdélyi, Magnus, and Nevai.

Keywords Jacobi polynomial · Bernstein inequality · Matrix coefficient

Mathematics Subject Classification 22E46 · 33C45

1 Introduction

For $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > -1$, and a nonnegative integer n, we denote by $P_n^{(\alpha,\beta)}$ the Jacobi polynomial with the standard normalization. Recall that in terms of the Gauss hypergeometric function,

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right).$$

Recall also that for a fixed pair (α, β), these functions are orthogonal polynomials on $[-1, 1]$ for the weight function

$$w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$$

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with the explicit values

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)^2 w^{(\alpha,\beta)}(x)\,dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$$

(see [15], Eq. (4.3.3)).

For $x \in [-1,1]$ and $\alpha, \beta \geq 0$, let

$$g_n^{(\alpha,\beta)}(x) = \left(\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right)^{1/2}\left(\frac{1-x}{2}\right)^{\alpha/2}\left(\frac{1+x}{2}\right)^{\beta/2} P_n^{(\alpha,\beta)}(x).$$

Then these functions are orthogonal on $[-1,1]$ for the constant weight. Moreover,

$$\frac{1}{2}\int_{-1}^{1} g_n^{(\alpha,\beta)}(x)^2\,dx = \frac{1}{2n+\alpha+\beta+1}. \tag{1}$$

In suitable coordinates the functions $g_n^{(\alpha,\beta)}$ with arbitrary nonnegative integers α, β and n comprise a natural and complete set of matrix coefficients for the irreducible representations of SU(2) (see Sect. 2 below). The value $2n+\alpha+\beta+1$ in (1) is exactly the dimension of the corresponding irreducible representation.

We shall prove the following uniform upper bound.

Theorem 1.1 *There exists a constant $C > 0$ such that*

$$\left|\left(1-x^2\right)^{\frac{1}{4}} g_n^{(\alpha,\beta)}(x)\right| \leq C(2n+\alpha+\beta+1)^{-\frac{1}{4}}$$

for all $x \in [-1,1]$, all $\alpha, \beta \geq 0$, and all nonnegative integers n.

We have not made a serious effort to find the best value of C, but at least our proof shows that $C < 12$.

With standard normalization, the inequality in Theorem 1.1 amounts to the following uniform bound for the Jacobi polynomials:

$$\begin{aligned}(\sin\theta)^{\alpha+\frac{1}{2}}(\cos\theta)^{\beta+\frac{1}{2}}\left|P_n^{(\alpha,\beta)}(\cos 2\theta)\right| \\ \leq \frac{C}{\sqrt{2}}(2n+\alpha+\beta+1)^{-\frac{1}{4}}\left(\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}\right)^{1/2}\end{aligned} \tag{2}$$

for $0 \leq \theta \leq \pi/2$. The decay rate of $1/4$ in Theorem 1.1 is optimal as α and β tend to infinity, see Remark 4.4. However, if the pair (α,β) is fixed, then $P_n^{(\alpha,\beta)}(x)$ is $O(n^{-1/2})$ for each $x \neq \pm 1$, cf. [15], Theorem 7.32.2. In particular, in Legendre's case $\alpha = \beta = 0$ where $P_n^{(\alpha,\beta)}(x)$ specializes to the Legendre polynomial $P_n(x)$, the Bernstein inequality (refined by Antonov and Kholshevnikov)

$$\left(1-x^2\right)^{1/4}\left|P_n(x)\right| \leq (4/\pi)^{1/2}(2n+1)^{-1/2}, \quad x \in [-1,1], \tag{3}$$

is known to be sharp, see [15], Theorem 7.3.3, and [13]. We refer to [5] for a further discussion of the sharpest constant in (2), with a subset of the current parameter range.

It is of interest also to express our inequality in terms of the orthonormal polynomials defined by

$$\hat{P}_n^{(\alpha,\beta)}(x) = \left(\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \right)^{1/2} P_n^{(\alpha,\beta)}(x),$$

for which

$$\int_{-1}^{1} \hat{P}_n^{(\alpha,\beta)}(x)^2 w^{(\alpha,\beta)}(x)\, dx = 1.$$

Here our estimate reads

$$\left(1-x^2\right)^{\frac{1}{4}} \sqrt{w^{\alpha,\beta}(x)} \left| \hat{P}_n^{(\alpha,\beta)}(x) \right| \leq \frac{C}{\sqrt{2}} (2n+\alpha+\beta+1)^{\frac{1}{4}}$$

for all $\alpha, \beta \geq 0$ and all integers $n \geq 0$, with the same constant C as before. The following generalization of Bernstein's inequality (3) was conjectured by Erdélyi, Magnus, and Nevai [4]:

$$\left(1-x^2\right)^{\frac{1}{4}} \sqrt{w^{\alpha,\beta}(x)} \left| \hat{P}_n^{(\alpha,\beta)}(x) \right| \leq C'(\alpha+\beta+2)^{1/4} \tag{4}$$

for all $\alpha, \beta \geq -\frac{1}{2}$ and all integers $n \geq 0$, with a uniform constant $C' > 0$. A stronger form of the conjecture, in which the right-hand side of (4) is replaced by

$$C''(\alpha+\beta+2)^{1/6} \left(1 + \frac{\alpha+\beta}{n}\right)^{1/12},$$

was recently established by Krasikov [10], but only in the parameter range $\alpha, \beta \geq \frac{1+\sqrt{2}}{4}$, $n \geq 6$. Our estimate is valid for a more general range, but it does not have the stronger form suggested by Krasikov, and it involves $2n+\alpha+\beta$ rather than $\alpha+\beta$. Note however that by combining our results with those of [10], one can remove Krasikov's restriction $n \geq 6$ in the parameter range for the validity of (4). In a range disjoint from that of [10], but overlapping with the range of the current paper, inequality (4) was established for $-\frac{1}{2} \leq \alpha, \beta \leq \frac{1}{2}$ in [1] (see also [3] and [5]).

Estimate (2) implies a similar estimate for the ultrasperical (Gegenbauer) polynomials $C_n^{(\lambda)}(x)$, as these are directly related to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = \beta = \lambda - \frac{1}{2}$. Previous to [10], this case had been considered in [11] and, as above, (2) allows the removal of a restriction on the degree.

The proof of Theorem 1.1 is based on an expression for $P_n^{(\alpha,\beta)}(x)$ as a contour integral, for which we can estimate the integrand by elementary analysis. The proof is simpler when α and β are integers. In this case, which is treated in Sect. 3, the contour is just a circle. The general case is the discussed in Sect. 5.

2 Motivation from representation theory

It is well known that the irreducible representations of SU(2) can be expressed by Jacobi polynomials. In the physics literature it is customary to denote the corresponding matrix representations as *Wigner's d-matrices*. We recall a few details (see [17], Sect. 38, [16], Chap. 3, or [9]). The irreducible representations π_l of SU(2) are parameterized by the nonnegative integers or half-integers $l = 0, \frac{1}{2}, 1, \dots$, where $2l + 1$ is the corresponding dimension. The standard representation space for π_l is the space $\mathcal{P}_l$ of polynomials in two complex variables z_1, z_2, homogeneous of degree $2l$, on which the representation is given by

$$\left[\pi_l \begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right](z_1, z_2) = f(az_1 + cz_2, bz_1 + dz_2).$$

Let

$$k_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \quad \text{and} \quad t_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for $\phi, \theta \in \mathbb{R}$. Then every element $A \in \mathrm{SU}(2)$ allows a decomposition of the form $A = k_\phi t_\theta k_{-\psi}$. The monomials $z_1^j z_2^k$ with $j + k = 2l$ form a basis for $\mathcal{P}_l$, and it is convenient to use the notation

$$h_p^l(z_1, z_2) = z_1^{l-p} z_2^{l+p},$$

where $p = -l, -l + 1, \dots, l$. Notice that these are weight vectors

$$\pi_l(k_\phi) h_p^l = e^{-i2p\phi} h_p^l \quad (p = -1, \dots, l).$$

Choosing the inner product on $\mathcal{P}_l$ so that π_l is unitary, the functions h_p^l form an orthogonal basis. We denote by $\hat{h}_p^l$ the corresponding normalized basis vectors. For $A \in \mathrm{SU}(2)$, the matrix elements

$$m_{pq}^l(A) = \langle \pi_l(A) \hat{h}_q^l, \hat{h}_p^l \rangle$$

with $p, q = -l, \dots, l$ form the so-called Wigner's d-matrix. Our result for the Jacobi polynomials implies the following.

Theorem 2.1 *Let C be the constant from Theorem 1.1. Then*

$$|\sin 2\theta|^{1/2} \left| m_{pq}^l(k_\phi t_\theta k_{-\psi}) \right| \le C(2l + 1)^{-1/4} \tag{5}$$

for all $\phi, \theta, \psi \in \mathbb{R}$, all $l = 0, \frac{1}{2}, 1, \dots$ and all $p, q = -l, \dots, l$. Moreover, the exponent $1/4$ on the right-hand side is best possible.

Proof Explicitly the matrix elements are given as follows (see [9, 16, 17]). For $p, q = -l, \dots, l$ such that $|q| \le p$,

$$m_{pq}^l(k_\phi t_\theta k_{-\psi}) = e^{-i2p\phi} e^{i2q\psi} g_n^{(\alpha,\beta)}(\cos 2\theta),$$

where

$$\alpha = p - q, \quad \beta = p + q, \quad n = l - p.$$

For other values of p and q, there are similar expressions, and in all cases one has

$$\left| m_{pq}^l(k_\phi t_\theta k_{-\psi}) \right| = \left| g_n^{(\alpha,\beta)}(\cos 2\theta) \right|,$$

where $\alpha = |p-q|$, $\beta = |p+q|$, and $n = l - \max\{|p|, |q|\}$. Moreover,

$$\dim \pi_l = 2l + 1 = 2n + \alpha + \beta + 1.$$

Thus (5) follows directly from Theorem 1.1. For the last statement of Theorem 2.1, see Remark 4.4. □

Remark 2.2 For l integral, π_l descends to a representation of SO(3), and the matrix elements m^l_{p0} with $q = 0$ descend to spherical harmonic functions on $S^2 \simeq \mathrm{SO}(3)/\mathrm{SO}(2)$. With the common normalization from quantum mechanics, the spherical harmonics Y_l^m with $-l \le m \le l$ satisfy

$$Y_l^m(\theta, \phi) = \pm \frac{(2l+1)^{1/2}}{(4\pi)^{1/2}} g_{l-\alpha}^{(\alpha,\alpha)}(\cos\theta) e^{im\phi},$$

where $\alpha = |m|$. From Theorem 1.1 we obtain the uniform estimate

$$|\sin\theta|^{1/2} \left| Y_l^m(\theta, \phi) \right| \le \frac{C}{(4\pi)^{1/2}} (2l+1)^{1/4}$$

for all θ, ϕ and all integers l, m with $|m| \le l$.

The Jacobi polynomials are also related to the harmonic analysis on the complex spheres with respect to the action of the unitary group. The spherical functions for the pair $(U(q), U(q-1))$ are functions on the unit sphere in $\mathbb{C}^q$, and in suitable coordinates they can be expressed by means of the Jacobi functions $P_n^{(\alpha,\beta)}$ with $\alpha = q - 2$ (see [8, 14]). The direct motivation for the present paper was an application of this observation for $q = 2$ to a study of $\mathrm{Sp}(2, \mathbb{R})$. In [7] the first author and de Laat apply the uniform estimates of the present paper for the case $\alpha = 0$, to show that $\mathrm{Sp}(2, \mathbb{R})$ does not have the approximation property (AP) introduced by the first author and Kraus in [6]. Earlier, Bernstein's inequality (3) had been used in [12] to prove that the group $\mathrm{SL}(3, \mathbb{R})$ does not have property (AP).

3 Integral parameters

The proof is based on the following integral expression, which is obtained by applying Cauchy's formula to Rodrigues' formula for $P_n^{(\alpha,\beta)}(x)$ (see [15], Eq. (4.3.1)):

$$(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) = \left(-\frac{1}{2}\right)^n I_n^{(\alpha,\beta)}(x) \tag{6}$$

for $x \in (-1, 1)$, where

$$I_n^{(\alpha,\beta)}(x) = \frac{1}{2\pi i} \int_{\gamma(x)} \frac{(1-z)^{n+\alpha}(1+z)^{n+\beta}}{(z-x)^n} \frac{dz}{z-x}. \tag{7}$$

Here $\gamma(x)$ is any closed contour encircling x in the positive direction. We assume in this section that α and β are integers ≥ 0. Without this assumption one would have to request also that $\gamma(x)$ does not enclose the points $z = \pm 1$. We shall take $\gamma(x) = C(x, r)$, the circle centered at x and with a radius $r > 0$ to be specified later.

The case $n = 0$ will be treated separately in Lemma 4.3 below. Here we assume that $n \geq 1$ and let $a = \alpha/n$ and $b = \beta/n$. Then

$$\begin{aligned} I_n^{(\alpha,\beta)}(x) &= \frac{1}{2\pi i} \int_{C(x,r)} \left(\frac{(1-z)^{a+1}(1+z)^{b+1}}{z-x} \right)^n \frac{dz}{z-x} \\ &= \frac{1}{2\pi i} \int_{C(0,r)} \left(\frac{(1-x-s)^{a+1}(1+x+s)^{b+1}}{s} \right)^n \frac{ds}{s}. \end{aligned}$$

In order to select a suitable radius r, we look for the stationary points of the expression inside the parentheses, as a function of s. We let

$$\psi(s) = (a+1)\log(1-x-s) + (b+1)\log(1+x+s) - \log s$$

for $s \in \mathbb{C}$ and analyze the derivative

$$\psi'(s) = \frac{a+1}{s+x-1} + \frac{b+1}{s+x+1} - \frac{1}{s},$$

which is independent of the branch cut used for the complex logarithm. Now

$$\psi'(s) = \frac{As^2 + B(x)s + C(x)}{(s+x-1)(x+s+1)s},$$

where

$$A = a+b+1, \qquad B(x) = (a+b)x + a - b, \qquad C(x) = 1 - x^2.$$

The numerator is a second-order polynomial in s with the discriminant

$$\begin{aligned} \Delta(x) &= B(x)^2 - 4AC(x) \\ &= (a+b+2)^2 x^2 + 2\big(a^2 - b^2\big)x + (a-b)^2 - 4(a+b+1), \end{aligned}$$

which coincides with the polynomial Δ defined in [2]. The polynomial $\Delta(x)$ has two real roots

$$\left.\begin{matrix} x^+ \\ x^- \end{matrix}\right\} = \frac{b^2 - a^2 \pm 4\sqrt{(a+1)(b+1)(a+b+1)}}{(a+b+2)^2},$$

for which $-1 \leq x^- < x^+ \leq 1$. For $x^- < x < x^+$, we have $\Delta(x) < 0$, and thus there are two conjugate solutions $s = s_1, s_2$ to the equation $As^2 + B(x)s + C(x) = 0$. They are

$$s_1, s_2 = \frac{-B(x) \pm i\sqrt{-\Delta(x)}}{2A}.$$

Note that

$$|s_1|^2 = |s_2|^2 = s_1 s_2 = \frac{C(x)}{A} = \frac{1-x^2}{a+b+1}.$$

Hence, if we choose the radius

$$r = \sqrt{\frac{1-x^2}{a+b+1}}, \tag{8}$$

then our contour $C(0,r)$ will pass through the stationary points of ψ. We define r by (8) for all $x \in (-1,1)$ (also when $\Delta(x) \geq 0$).

We now find

$$\left|I_n^{(\alpha,\beta)}(x)\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|\left(1-x-re^{i\theta}\right)^{1+a}\left(1+x+re^{i\theta}\right)^{1+b} r^{-1}\right|^n d\theta$$

and write

$$\left|\left(1-x-re^{i\theta}\right)^{1+a}\left(1+x+re^{i\theta}\right)^{1+b} r^{-1}\right| = e^{f(\cos\theta)},$$

where

$$f(t) = \frac{a+1}{2}\ln\left(r^2+(1-x)^2-2r(1-x)t\right) + \frac{b+1}{2}\ln\left(r^2+(1+x)^2+2r(1+x)t\right) - \ln(r) \tag{9}$$

for $t \in [-1,1]$. Notice that we allow the possible value $f(t) = -\infty$ at the end points $t = \pm 1$. Let

$$t_2 = \frac{r^2+(1-x)^2}{2r(1-x)}, \qquad t_1 = -\frac{r^2+(1+x)^2}{2r(1+x)}. \tag{10}$$

Then $t_1 \leq -1$ and $1 \leq t_2$. It follows that

$$f(t) = \frac{a+1}{2}\ln(t_2-t) + \frac{b+1}{2}\ln(t-t_1) + K, \tag{11}$$

where

$$K = \frac{a+1}{2}\ln(1-x) + \frac{b+1}{2}\ln(1+x) + \frac{a+b}{2}\ln r + \frac{a+b+2}{2}\ln 2 \tag{12}$$

is independent of t. With (11) we can extend the domain of definition for f to $[t_1,t_2] \supset [-1,1]$. For later reference, we note that from (10) and (8) it follows that

$$t_1 = \frac{-(a+b+2)-(a+b)x}{2\sqrt{a+b+1}\sqrt{1-x^2}}, \qquad t_2 = \frac{(a+b+2)-(a+b)x}{2\sqrt{a+b+1}\sqrt{1-x^2}}, \tag{13}$$

and

$$t_2 - t_1 = \frac{a+b+2}{\sqrt{a+b+1}\sqrt{1-x^2}}. \tag{14}$$

We have

$$\left|I_n^{(\alpha,\beta)}(x)\right| \leq \frac{1}{2\pi}\int_0^{2\pi} e^{nf(\cos\theta)}\, d\theta.$$

From (11) we find

$$f'(t) = -\frac{a+1}{2(t_2-t)} + \frac{b+1}{2(t-t_1)} = \frac{(a+b+2)(t_0-t)}{2(t_2-t)(t-t_1)}, \tag{15}$$

where t_0 is the convex combination

$$t_0 = \frac{(a+1)t_1+(b+1)t_2}{a+b+2} = \frac{-a+b-(a+b)x}{2\sqrt{a+b+1}\sqrt{1-x^2}} \in (t_1,t_2). \tag{16}$$

Moreover,

$$f''(t) = -\frac{a+1}{2(t_2-t)^2} - \frac{b+1}{2(t-t_1)^2} < 0.$$

Hence, the function $f(t)$ is concave and has a global maximum at t_0. We thus obtain the initial estimate

$$\left|I_n^{(\alpha,\beta)}(x)\right| \le \frac{1}{\pi}\int_0^\pi e^{nf(\cos\theta)}\,d\theta \le e^{nf(t_0)}. \tag{17}$$

Since

$$t_2 - t_0 = \frac{(a+1)(t_2-t_1)}{a+b+2}, \qquad t_0 - t_1 = \frac{(b+1)(t_2-t_1)}{a+b+2}, \tag{18}$$

we find

$$f(t_0) = \frac{a+1}{2}\ln\frac{(a+1)(t_2-t_1)}{a+b+2} + \frac{b+1}{2}\ln\frac{(b+1)(t_2-t_1)}{a+b+2} + K,$$

and from (12) and (14) it then follows that

$$f(t_0) = \frac{1}{2}\ln\left(\frac{2^{a+b+2}(a+1)^{a+1}(b+1)^{b+1}}{(a+b+1)^{a+b+1}}(1-x)^a(1+x)^b\right).$$

Thus,

$$\begin{aligned} e^{nf(t_0)} &\le \left(\frac{2^{a+b+2}(a+1)^{a+1}(b+1)^{b+1}}{(a+b+1)^{a+b+1}}(1-x)^a(1+x)^b\right)^{n/2} \\ &= \left(\frac{2^{a+b+2}(a+1)^{a+1}(b+1)^{b+1}}{(a+b+1)^{a+b+1}}\right)^{n/2}(1-x)^{\alpha/2}(1+x)^{\beta/2}. \end{aligned}$$

The inequality

$$\begin{aligned} &\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\left(\frac{(a+1)^{a+1}(b+1)^{b+1}}{(a+b+1)^{a+b+1}}\right)^n \\ &\quad \le \left(\frac{(n+1)(n+\alpha+\beta+1)}{(n+\alpha+1)(n+\beta+1)}\right)^{1/2} \end{aligned} \tag{19}$$

will be shown in Lemma 4.1. Inserting (17) and (19) into our definition of $g_n^{(\alpha,\beta)}$, we obtain the initial bound

$$\left|g_n^{(\alpha,\beta)}(x)\right| \le \left(\frac{(n+1)(n+\alpha+\beta+1)}{(n+\alpha+1)(n+\beta+1)}\right)^{1/4}. \tag{20}$$

In particular, since $(n+1)(n+\alpha+\beta+1) \le (n+\alpha+1)(n+\beta+1)$, it follows that $|g_n^{(\alpha,\beta)}(x)| \le 1$ (which could also be seen directly from the fact that $g_n^{(\alpha,\beta)}$ is a unitary matrix coefficient of orthonormal vectors).

In order to improve the estimate, we need to replace the inequality $f(t) \le f(t_0)$ by a stronger inequality. In Proposition 3.1 below we shall establish the inequality

$$f(t) \le f(t_0) + \frac{D}{1+t_0^2}f''(t_0)(t-t_0)^2 \tag{21}$$

for $t \in [-1, 1]$, with a suitable constant $D > 0$. Following the argument from before and taking into account the second term in (21), we can then improve (17) with the extra factor

$$\frac{1}{\pi}\int_0^\pi \exp\left(\frac{nD}{1+t_0^2} f''(t_0)(\cos\theta - t_0)^2\right) d\theta$$

on the right-hand side.

For the estimation of the exponential integral, we use Lemma 3.6 below, which is applicable since $f''(t_0) < 0$. We let

$$u = t_0\sqrt{\frac{nD}{1+t_0^2}\left|f''(t_0)\right|}, \qquad v = \sqrt{\frac{nD}{1+t_0^2}\left|f''(t_0)\right|},$$

and observe that $u^2 + v^2 = nD|f''(t_0)|$. We thus obtain

$$\left|I_n^{(\alpha,\beta)}(x)\right| \leq 2e^{nf(t_0)}\left(nD\left|f''(t_0)\right|\right)^{-1/4}, \tag{22}$$

and hence (20) has been improved to

$$\left|g_n^{(\alpha,\beta)}(x)\right| \leq \left(\frac{(n+1)(n+\alpha+\beta+1)}{(n+\alpha+1)(n+\beta+1)}\right)^{1/4} 2\left(nD\left|f''(t_0)\right|\right)^{-1/4}.$$

From (15), (18), and (14) it follows that

$$f''(t_0) = -\frac{a+b+2}{2(t_0-t_1)(t_2-t_0)} = -\frac{(a+b+1)(a+b+2)}{2(a+1)(b+1)}\left(1-x^2\right), \tag{23}$$

and hence,

$$|f''(t_0)| = \frac{(\alpha+\beta+n)(\alpha+\beta+2n)}{2(\alpha+n)(\beta+n)}\left(1-x^2\right).$$

Since

$$\frac{n+\alpha+\beta+1}{(n+\alpha+1)(n+\beta+1)} \leq \frac{n+\alpha+\beta}{(n+\alpha)(n+\beta)}$$

and

$$\frac{n+1}{n(2n+\alpha+\beta)} \leq \frac{3}{2n+\alpha+\beta+1}$$

for all $n \geq 1$ and $\alpha, \beta \geq 0$, it finally follows that

$$\left|g_n^{(\alpha,\beta)}(x)\right| \leq C'(\alpha+\beta+2n+1)^{-1/4}\left(1-x^2\right)^{-1/4},$$

where $C' = 2\sqrt[4]{6/D} = 2\sqrt[4]{168} < 8$ with the value $D = 1/28$ from below. This completes the proof of Theorem 1.1 in the integral case (up to the cited results from below).

Proposition 3.1 *Fix $x \in [-1, 1]$ and let $f(t)$ and t_0 be as above. Then*

$$f(t) \le f(t_0) + \frac{1}{28(1+t_0^2)} f''(t_0)(t-t_0)^2$$

for all $t \in [-1, 1]$.

Proof We begin the proof by a sequence of lemmas.

Lemma 3.2 *The following relation holds*:

$$(a+b)^2 + 4(a+b+1)t_0^2 = \frac{2a^2}{1-x} + \frac{2b^2}{1+x}. \tag{24}$$

Proof Using (16), we obtain

$$4(a+b+1)t_0^2 = \frac{(a-b+(a+b)x)^2}{1-x^2}.$$

On the other hand,

$$\frac{2a^2}{1-x} + \frac{2b^2}{1+x} = \frac{2(a^2+b^2+(a^2-b^2)x)}{1-x^2}.$$

Hence, (24) follows from the identity

$$(a+b)^2(1-x^2) + \big(a-b+(a+b)x\big)^2 = 2\big(a^2+b^2+\big(a^2-b^2\big)x\big),$$

which is straightforward. □

Lemma 3.3 *We have*

$$1-x^2 \le 16\frac{(a+1)(b+1)}{(a+b+2)^2}\big(1+t_0^2\big)$$

for all $x \in [-1, 1]$.

Proof Note first that if we replace the triple (a, b, x) by $(b, a, -x)$, then t_1, t_0, t_2 are replaced by $-t_2, -t_0, -t_1$, and hence the asserted inequality is unchanged. We may thus assume that $a \le b$.

It follows from Lemma 3.2 that

$$(a+b)^2 + 4(a+b+1)t_0^2 \ge \frac{2b^2}{1+x}$$

and therefore

$$1+x \ge \frac{2b^2}{(a+b)^2 + 4(a+b+1)t_0^2}.$$

Hence,

$$1-x \le 2-\frac{2b^2}{(a+b)^2+4(a+b+1)t_0^2} = 2\frac{a^2+2ab+4(a+b+1)t_0^2}{(a+b)^2+4(a+b+1)t_0^2}$$

and

$$1-x^2 \le 2(1-x) \le 4\frac{a^2+2ab+4(a+b+1)t_0^2}{(a+b)^2+4(a+b+1)t_0^2}.$$

Since the right-hand side is an increasing function of t_0^2, we have for $t_0^2 \le 1$ that

$$1-x^2 \le 4\frac{a^2+2ab+4(a+b+1)}{(a+b)^2+4(a+b+1)} \le 16\frac{(a+1)(b+1)}{(a+b+2)^2},$$

where in the last step we used that $a \le b$ implies $a^2+2ab \le 4ab$. For $t_0^2 \ge 1$, we obtain similarly

$$1-x^2 \le 4\frac{(a^2+2ab)t_0^2+4(a+b+1)t_0^2}{(a+b)^2+4(a+b+1)} \le 16\frac{(a+1)(b+1)}{(a+b+2)^2}t_0^2.$$

This completes the proof of Lemma 3.3. □

Lemma 3.4 *We have*

$$t_2-t_0 \ge \frac{1}{4(1+t_0^2)^{1/2}} \quad \text{and} \quad t_0-t_1 \ge \frac{1}{4(1+t_0^2)^{1/2}}. \tag{25}$$

Proof It follows from (14) and Lemma 3.3 that

$$t_2-t_1 \ge \frac{(a+b+2)^2}{4\sqrt{(a+1)(b+1)(a+b+1)}}\bigl(1+t_0^2\bigr)^{-1/2},$$

and hence, by (18),

$$t_2-t_0 \ge \frac{\sqrt{a+1}(a+b+2)}{4\sqrt{(b+1)(a+b+1)}}\bigl(1+t_0^2\bigr)^{-1/2}.$$

Using $(b+1)(a+b+1) \le (a+b+2)^2$ and $\sqrt{a+1} \ge 1$, we obtain the first inequality in (25). The second one is analogous. □

Lemma 3.5 *We have*

$$(u-t_1)(t_2-u) \le 14\bigl(1+t_0^2\bigr)(t_0-t_1)(t_2-t_0) \tag{26}$$

for all $u \in [t_1, t_2]$ *for which* $-1 \le u \le t_0$ *or* $t_0 \le u \le 1$.

Proof We first assume that $a \leq b$. Then by (18)

$$u - t_1 \leq t_2 - t_1 = \frac{a+b+2}{b+1}(t_0 - t_1) \leq 2(t_0 - t_1). \tag{27}$$

In order to estimate $t_2 - u$, we first note that $|u - t_0| \leq 1 + |t_0|$ and hence,

$$t_2 - u \leq t_2 - t_0 + |t_0 - u| \leq t_2 - t_0 + 1 + |t_0|.$$

By Lemma 3.4,

$$1 + |t_0| \leq \sqrt{2}\big(1 + t_0^2\big)^{1/2} \leq 4\sqrt{2}\big(1 + t_0^2\big)(t_2 - t_0),$$

and hence,

$$t_2 - u \leq (1 + 4\sqrt{2})\big(1 + t_0^2\big)(t_2 - t_0) \leq 7\big(1 + t_0^2\big)(t_2 - t_0). \tag{28}$$

Now (27) and (28) together imply (26). The proof for $a \geq b$ is analogous. □

We can now prove Proposition 3.1. Let $t \in [-1, 1]$. It follows from (15), (26), and (23) that

$$\begin{aligned}\frac{f'(u)}{u - t_0} &= -\frac{a+b+2}{2(u - t_1)(t_2 - u)}\\ &\leq -\frac{a+b+2}{28(1 + t_0^2)(t_0 - t_1)(t_2 - t_0)} = \frac{f''(t_0)}{14(1 + t_0^2)}\end{aligned}$$

for all $u \in \mathbb{R}$ between t and t_0. Hence,

$$\begin{aligned}f(t) &= f(t_0) + \int_{t_0}^{t} f'(u)\,du\\ &\leq f(t_0) + \frac{f''(t_0)}{14(1 + t_0^2)} \int_{t_0}^{t} (u - t_0)\,du = f(t_0) + \frac{f''(t_0)}{28(1 + t_0^2)}(t - t_0)^2.\end{aligned}$$

□

Lemma 3.6 *Let $u, v \in \mathbb{R}$ with $u^2 + v^2 > 0$. Then*

$$\frac{1}{\pi}\int_0^{\pi} e^{-(u + v\cos s)^2}\,ds \leq \frac{2}{\big(u^2 + v^2\big)^{1/4}}. \tag{29}$$

Proof We will show (29) with the slightly stronger bound

$$\frac{\sqrt{2}}{\sqrt{\max\{|u|, |v|\}}}.$$

The statement is invariant under the map $(u, v) \mapsto (-u, -v)$ and, using the substitution $s \mapsto \pi - s$, also under $v \mapsto -v$. Hence, it is sufficient to show that

$$\frac{1}{\pi}\int_0^{\pi} e^{-(u - v\cos s)^2}\,ds \leq \frac{\sqrt{2}}{\sqrt{\max\{u, v\}}}$$

for $u \geq 0$, $v \geq 0$.

Suppose first $0 \le u \le v$; then $v \ne 0$. Let $\sigma \in [0, \frac{\pi}{2}]$ be such that $\cos\sigma = \frac{u}{v}$. Then

$$u - v\cos s = v(\cos\sigma - \cos s) = 2v \sin\left(\frac{s+\sigma}{2}\right)\sin\left(\frac{s-\sigma}{2}\right).$$

Note that $\sin(\frac{s+\sigma}{2}) \ge |\sin(\frac{s-\sigma}{2})|$ because $\sin^2(\frac{s+\sigma}{2}) - \sin^2(\frac{s-\sigma}{2}) = \sin s \sin\sigma \ge 0$ for $s \in [0,\pi]$ and $\sigma \in [0, \frac{\pi}{2}]$. Using also that $|\sin t| \ge \frac{2}{\pi}|t|$ for $|t| \le \frac{\pi}{2}$, we have that

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi e^{-(u-v\cos s)^2}\,ds &= \frac{1}{\pi}\int_0^\pi e^{-4v^2\sin^2(\frac{s+\sigma}{2})\sin^2(\frac{s-\sigma}{2})}\,ds\\ &\le \frac{1}{\pi}\int_0^\pi e^{-4v^2\pi^{-4}(s-\sigma)^4}\,ds\\ &\le \frac{1}{\pi}\int_{-\infty}^\infty e^{-4v^2\pi^{-4}s^4}\,ds \le \frac{2}{\sqrt{2v}},\end{aligned}$$

where we used that $\int_0^\infty e^{-t^4}\,dt = \Gamma(\frac{5}{4}) \le 1$.

Suppose next that $0 \le v \le u \le 2v$. Then $u - v\cos s \ge v(1-\cos s) = 2v\sin^2(\frac{s}{2})$. Hence,

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi e^{-(u-v\cos s)^2}\,ds &\le \frac{1}{\pi}\int_0^\pi e^{-4v^2\sin^4(\frac{s}{2})}\,ds\\ &\le \frac{1}{\pi}\int_0^\pi e^{-4v^2\pi^{-4}s^4}\,ds \le \frac{1}{\sqrt{2v}} \le \frac{1}{\sqrt{u}},\end{aligned}$$

using again $\int_0^\infty e^{-t^4}\,dt \le 1$.

Suppose finally that $0 \le 2v \le u$. Then $u - v\cos s \ge \frac{u}{2}$, and hence,

$$\frac{1}{\pi}\int_0^\pi e^{-(u-v\cos s)^2}\,ds \le e^{-\frac{u^2}{4}} \le \frac{1}{\sqrt{u}},$$

where we used that $xe^{-x^4} \le \frac{1}{\sqrt{2}}$ for all $x \ge 0$. □

4 Some inequalities with gamma functions

In this section we prove some inequalities that were used in the preceding section. We assume that α, β are real and nonnegative.

Lemma 4.1 *Let $n, \alpha, \beta \ge 0$. Then*

$$\begin{aligned}&\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\\ &\quad\le \frac{n^n(\alpha+\beta+n)^{\alpha+\beta+n}}{(\alpha+n)^{\alpha+n}(\beta+n)^{\beta+n}}\left(\frac{(n+1)(n+\alpha+\beta+1)}{(n+\alpha+1)(n+\beta+1)}\right)^{1/2}.\end{aligned} \tag{30}$$

Proof We have, for $x, y, z \geq 0$,

$$\ln \frac{\Gamma(x+1)\Gamma(x+y+z+1)}{\Gamma(x+y+1)\Gamma(x+z+1)} = \int_0^y \int_0^z (\ln \Gamma)''(x+s+t+1)\, dt\, ds. \tag{31}$$

We claim that

$$(\ln \Gamma)''(u+1) \leq \frac{1}{u} - \frac{1}{2(u+1)^2} \tag{32}$$

for all $u > 0$. The asserted inequality (30) follows easily from (31) and (32).

In order to prove (32), we recall that

$$(\ln \Gamma)''(u+1) = \sum_{k=1}^{\infty} \frac{1}{(u+k)^2} = \sum_{k=0}^{\infty} A(u+k),$$

where

$$A(u) = \frac{1}{(u+1)^2}.$$

For the other side of (32), we use the telescoping series

$$\frac{1}{u} = \sum_{k=0}^{\infty} B(u+k), \qquad \frac{1}{2(u+1)^2} = \sum_{k=0}^{\infty} C(u+k),$$

where

$$B(u) = \frac{1}{u} - \frac{1}{u+1} = \frac{1}{u(u+1)}$$

and

$$C(u) = \frac{1}{2(u+1)^2} - \frac{1}{2(u+2)^2} = \frac{2u+3}{2(u+1)^2(u+2)^2}.$$

We observe that

$$C(u) \leq \frac{1}{(u+1)^2(u+2)}$$

and, hence,

$$B(u) - C(u) \geq \frac{1}{u(u+1)} - \frac{1}{(u+1)^2(u+2)} = \frac{u^2+2u+2}{u(u+1)^2(u+2)} \geq A(u).$$

We obtain (32) by termwise application of this inequality to the series. □

Lemma 4.2 *For* $\alpha, \beta \geq 0$,

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \leq \frac{(\alpha+\beta+\frac{1}{2})^{\alpha+\beta+\frac{1}{2}}(\frac{1}{2})^{\frac{1}{2}}}{(\alpha+\frac{1}{2})^{\alpha+\frac{1}{2}}(\beta+\frac{1}{2})^{\beta+\frac{1}{2}}}.$$

Proof Following the preceding proof, we deduce this inequality from

$$(\ln \Gamma)''(u+1) \le \frac{1}{u+\frac{1}{2}}.$$

The latter inequality is also seen as in the preceding proof, by using the telescoping series

$$\frac{1}{u+\frac{1}{2}} = \sum_{k=0}^{\infty} D(u+k),$$

where

$$D(u) = \frac{1}{u+\frac{1}{2}} - \frac{1}{u+\frac{3}{2}} = \frac{1}{(u+\frac{1}{2})(u+\frac{3}{2})} \ge \frac{1}{(u+1)^2} = A(u).$$ □

Lemma 4.3 *Let $\alpha, \beta \ge 0$ and $-1 \le x \le 1$. Then*

$$0 \le \left(1-x^2\right)^{1/4} g_0^{(\alpha,\beta)}(x) \le (\alpha+\beta+1)^{-1/4}.$$

Proof Since $P_0^{(\alpha,\beta)}(x) = 1$, we have $g_0^{(\alpha,\beta)}(x) \ge 0$ and

$$\left(1-x^2\right)^{\frac{1}{2}} g_0^{(\alpha,\beta)}(x)^2 = \frac{2\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left(\frac{1-x}{2}\right)^{\alpha+\frac{1}{2}} \left(\frac{1+x}{2}\right)^{\beta+\frac{1}{2}}.$$

For $\mu, \nu \ge 0$, the function $\varphi(x) = (1-x)^{\mu}(1+x)^{\nu}$ on $[-1, 1]$ satisfies

$$\max_{x\in[-1,1]} \varphi(x) = \varphi\left(\frac{\nu-\mu}{\nu+\mu}\right) = \frac{2^{\mu+\nu}\mu^{\mu}\nu^{\nu}}{(\mu+\nu)^{\mu+\nu}}.$$

Hence, by Lemma 4.2,

$$\begin{aligned} \max_{x\in[-1,1]} \left(1-x^2\right)^{\frac{1}{2}} g_0^{(\alpha,\beta)}(x)^2 &= \frac{2\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{(\alpha+\frac{1}{2})^{\alpha+\frac{1}{2}}(\beta+\frac{1}{2})^{\beta+\frac{1}{2}}}{(\alpha+\beta+1)^{\alpha+\beta+1}} \\ &\le h(\alpha+\beta)(\alpha+\beta+1)^{-1/2}, \end{aligned} \tag{33}$$

where

$$h(t) = \sqrt{2}\left(\frac{t+\frac{1}{2}}{t+1}\right)^{t+\frac{1}{2}}.$$

Since

$$(\log h)'(t) = \frac{1}{2(t+1)} + \log\left(\frac{t+\frac{1}{2}}{t+1}\right) = \int_{t+\frac{1}{2}}^{t+1} \left(\frac{1}{t+1} - \frac{1}{u}\right) du \le 0,$$

it follows that $h(t) \le h(0) = 1$ for all $t \ge 0$. This proves Lemma 4.3. □

Remark 4.4 It follows from (33) and Stirling's formula that

$$\max\big(1-x^2\big)^{1/4}|g_0^{(\alpha,\beta)}(x)| \sim (2/\pi)^{1/4}(\alpha+\beta+1)^{-1/4}$$

as $\alpha \to \infty$ and $\beta \to \infty$. Hence, the decay rate $1/4$ in Theorem 1.1 cannot be improved. This was observed already in [4], p. 604.

In this connection it can be noted that for each $l = 0, \frac{1}{2}, 1, \dots,$ the irreducible representation π_l of SU(2) will exhibit matrix coefficients in which the functions $g_0^{(\alpha,\beta)}$ for $\alpha+\beta = 2l$ occur (see Sect. 2). In particular, it follows that a positive solution to the EMN-conjecture mentioned in the introduction will not significantly improve the representation theoretic content of Theorem 1.1, discussed in Sect. 2.

5 The general case

In this section, $n \in \mathbb{N}_0$, and α, β are nonnegative real numbers. We have already proved in Lemma 4.3 that

$$\big|g_0^{(\alpha,\beta)}(x)\big| \le (\alpha+\beta+1)^{-1/4}, \quad x \in [-1,1], \quad \alpha,\beta \ge 0,$$

so we can assume that $n > 0$. As in Sect. 3, we put $a = \alpha/n$ and $b = \beta/n$ and use the integral representation (6)–(7) of $P_n^{(\alpha,\beta)}(x)$, with a closed contour $\gamma(x)$ encircling x in the positive direction. In addition, we assume now that $\gamma(x)$ does not intersect the branch cuts $]-\infty,-1]$ and $[1,\infty[$. As before, we define $r > 0$ by (8) and consider the circle $C(x,r)$. For $|x| < 1$, we find

$$1 < x + r \quad \Leftrightarrow \quad x > \frac{a+b}{a+b+2},$$

and, consequently,

$$-1 > x - r \quad \Leftrightarrow \quad x < -\frac{a+b}{a+b+2}.$$

Hence, we can distinguish the following cases:

Case 1 $\frac{a+b}{a+b+2} < x < 1$. Then 1 is inside, and -1 is outside $C(x,r)$.
Case 2 $|x| < \frac{a+b}{a+b+2}$. Both 1 and -1 are outside $C(x,r)$.
Case 3 $-1 < x < -\frac{a+b}{a+b+2}$. Here 1 is outside, and -1 is inside $C(x,r)$.

By continuity it suffices to prove Theorem 1.1 in each of these three cases. As the proof given in Sect. 3 is valid without modification in Case 2, we need only consider the other two cases. Note that the integral

$$J_n^{(\alpha,\beta)}(x) := \frac{1}{2\pi i}\int_{C(x,r)} \frac{(1-z)^{n+\alpha}(1+z)^{n+\beta}}{(z-x)^{n+1}}\,dz$$

makes sense for all $\alpha, \beta \ge 0$, although the argument of the integrand may become discontinuous at $z = x + r$ or at $z = x - r$ when these points belong to the branch cuts. As in Sect. 3, see (17),

$$\left|J_n^{(\alpha,\beta)}(x)\right| \le \frac{1}{\pi}\int_0^\pi e^{nf(\cos\theta)}\,d\theta,$$

where f is the function defined by (9). Note that f depends on a, b, and x. When necessary, we denote it by $f = f_{a,b,x}$.

Lemma 5.1 *The integral* (7) *satisfies*

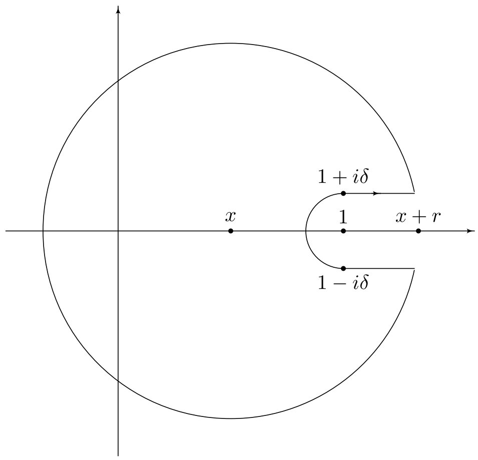
$$I_n^{(\alpha,\beta)}(x) = J_n^{(\alpha,\beta)}(x) + R_n^{(\alpha,\beta)}(x), \tag{34}$$

where $|R_n^{(\alpha,\beta)}(x)| \le e^{nf(1)}$ *in Case* 1, $R_n^{(\alpha,\beta)}(x) = 0$ *in Case* 2, *and* $|R_n^{(\alpha,\beta)}(x)| \le e^{nf(-1)}$ *in Case* 3.

Proof Consider first Case 1 and note that

$$f(1) = \ln\bigl((r-1+x)^{a+1}(r+1+x)^{b+1}r^{-1}\bigr).$$

We let the closed contour $\gamma(x)$ follow $C(x,r)$ except for a small arc around the possible locus of discontinuity at $x+r$. Let $\delta > 0$ be such that the removed arc consist of points $z_1 + iz_2$ in the strip $|z_2| < \delta$. The end points below and above $x+r$ are joined to $1 \pm i\delta$ by line segments along the axis. Finally, $1 - i\delta$ and $1 + i\delta$ are connected by a half circle crossing the axis to the left of $z = 1$.



In the limit $\delta \to 0^+$, we obtain (34) with

$$\begin{aligned} R_n^{(\alpha,\beta)}(x) &= -\frac{\sin(\pi(n+\alpha))}{\pi}\int_1^{x+r}\frac{(z-1)^{n+\alpha}(1+z)^{n+\beta}}{(z-x)^{n+1}}\,dz \\ &= (-1)^{n-1}\frac{\sin(\pi\alpha)}{\pi}\int_{1-x}^{r}\frac{(s+x-1)^{n+\alpha}(1+s+x)^{n+\beta}}{s^{n+1}}\,ds. \end{aligned}$$

In particular, $R_n^{(\alpha,\beta)}(x) = 0$ if $\alpha = 0$, so that we may assume that $\alpha > 0$. For $x < 1$ and $0 < s < r$, we have $\frac{s}{r}(1-x) \le 1-x$, and hence $s + x - 1 \le \frac{s}{r}(r + x - 1)$. It follows that

$$\frac{(s+x-1)^{n+\alpha}(1+s+x)^{n+\beta}}{s^{n+1}} \le \frac{(r+x-1)^{n+\alpha}(1+r+x)^{n+\beta}s^{\alpha-1}}{r^{n+\alpha}}$$

for $0 < 1 - x < s < r$. Thus,

$$\begin{aligned}\left|R_n^{(\alpha,\beta)}(x)\right| &\le \frac{|\sin(\pi\alpha)|}{\pi}\frac{(r+x-1)^{n+\alpha}(1+r+x)^{n+\beta}}{r^{n+\alpha}}\int_0^r s^{\alpha-1}\,ds\\ &= \frac{|\sin(\pi\alpha)|}{\pi\alpha}\frac{(r+x-1)^{n+\alpha}(1+r+x)^{n+\beta}}{r^{n}} = \frac{|\sin(\pi\alpha)|}{\pi\alpha}e^{nf(1)},\end{aligned}$$

completing the proof for Case 1.

Case 2 is trivial since 1 and -1 are both outside $C(x,r)$. For the last case, we observe that

$$I_n^{(\alpha,\beta)}(x) = (-1)^n I_n^{(\beta,\alpha)}(-x)$$

and likewise

$$J_n^{(\alpha,\beta)}(x) = (-1)^n J_n^{(\beta,\alpha)}(-x).$$

Moreover, from (9) we see that $f_{b,a,-x}(t) = f_{a,b,x}(-t)$. Now Case 3 follows easily from Case 1. □

Lemma 5.2 *Let* $t_0 \in (t_1, t_2)$ *be given by* (16). *Then*

$$f(1) \le f(t_0) + \frac{1}{140}f''(t_0)$$

in Case 1, *and likewise*, *in Case* 3,

$$f(-1) \le f(t_0) + \frac{1}{140}f''(t_0).$$

Proof It follows from (16) that the derivative of $t_0 = t_0(x)$ as a function of x is

$$\frac{-(a+b)+(b-a)x}{2(a+b+1)^{1/2}(1-x^2)^{3/2}}.$$

Since $|b-a| \le a+b$, it follows that t_0 is a decreasing function of $x \in (-1,1)$. Hence, in Case 1,

$$t_0(x) < t_0\left(\frac{a+b}{a+b+2}\right) = \frac{(b-a)(a+b+2)-(a+b)^2}{4(a+b+1)} \le \frac{1}{2},$$

where the last inequality follows from

$$(b-a)(a+b+2)-(a+b)^2 = -2a(a+b+1)+2b \le 2(a+b+1).$$

From Proposition 3.1 and (23) we have

$$f(1) \le f(t_0) + \frac{(1-t_0)^2}{28(1+t_0)^2} f''(t_0)$$

with $f''(t_0) < 0$. Since $t_0 \le \frac{1}{2}$, we find

$$4t_0^2 - 10t_0 + 4 = 4\left(t_0 - \frac{1}{2}\right)(t_0 - 2) \ge 0$$

and

$$\frac{(1-t_0)^2}{1+t_0^2} - \frac{1}{5} = \frac{4t_0^2 - 10t_0 + 4}{5(1+t_0^2)} \ge 0.$$

Hence,

$$f(1) \le f(t_0) + \frac{1}{140} f''(t_0),$$

as claimed. The proof in Case 3 follows by the observation at the end of the proof of Lemma 5.1 since the t_0 associated with the data $b, a, -x$ is the negative of the t_0 associated with a, b, x. □

We can now complete the proof of Theorem 1.1. As in (22), we find

$$\left|J_n^{(\alpha,\beta)}(x)\right| \le \frac{1}{\pi}\int_0^{\pi} e^{nf(\cos\theta)}\, d\theta \le C_1 e^{nf(t_0)} \left(n\left|f''(t_0)\right|\right)^{-1/4},$$

where $C_1 = 2D^{-1/4} = 2\sqrt[4]{28}$. Since $e^{-t} \le \frac{1}{\sqrt{2}} t^{-1/4}$ for all $t > 0$, we obtain from Lemmas 5.1 and 5.2 that

$$\left|R_n^{(\alpha,\beta)}(x)\right| \le C_2 e^{nf(t_0)} \left(n\left|f''(t_0)\right|\right)^{-1/4}$$

with $C_2 = \frac{1}{\sqrt{2}}\sqrt[4]{140} = \sqrt[4]{35}$. All together,

$$\left|I_n^{(\alpha,\beta)}(x)\right| \le C_3 e^{nf(t_0)} \left(n\left|f''(t_0)\right|\right)^{-1/4}$$

with $C_3 = C_1 + C_2$. Still proceeding as in Sect. 3 and using Lemma 4.1, we finally get

$$\begin{aligned}\left|g_n^{(\alpha,\beta)}(x)\right| &\le C_3 \left(\frac{(n+1)(n+\alpha+\beta+1)}{(n+\alpha+1)(n+\beta+1)}\right)^{1/4} \left(n\left|f''(t_0)\right|\right)^{-1/4} \\ &\le C(1+\alpha+\beta+2n)^{-1/4}\left(1-x^2\right)^{-1/4}\end{aligned}$$

for $C = \sqrt[4]{6}C_3$. In particular, we find $C < 12$.

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