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ASYMPTOTIC EXPANSIONS FOR THE GAUSSIAN UNITARY ENSEMBLE

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Let $g:\mathbb{R} \to \mathbb{C}$ be a C^∞-function with all derivatives bounded and let tr_n denote the normalized trace on the $n \times n$ matrices. In Ref. 3 Ercolani and McLaughlin established asymptotic expansions of the mean value $\mathbb{E}\{\mathrm{tr}_n(g(X_n))\}$ for a rather general class of random matrices X_n, including the Gaussian Unitary Ensemble (GUE). Using an analytical approach, we provide in the present paper an alternative proof of this asymptotic expansion in the GUE case. Specifically we derive for a $\mathrm{GUE}(n, \frac{1}{n})$ random matrix X_n that

$$\mathbb{E}\{\mathrm{tr}_n(g(X_n))\} = \frac{1}{2\pi}\int_{-2}^{2} g(x)\sqrt{4-x^2}\,\mathrm{d}x + \sum_{j=1}^{k} \frac{\alpha_j(g)}{n^{2j}} + O(n^{-2k-2}),$$

where k is an arbitrary positive integer. Considered as mappings of g, we determine the coefficients $\alpha_j(g)$, $j \in \mathbb{N}$, as distributions (in the sense of L. Schwarts). We derive a similar asymptotic expansion for the covariance $\mathrm{Cov}\{\mathrm{Tr}_n[f(X_n)], \mathrm{Tr}_n[g(X_n)]\}$, where f is a function of the same kind as g, and $\mathrm{Tr}_n = n\,\mathrm{tr}_n$. Special focus is drawn to the case where $g(x) = \frac{1}{\lambda - x}$ and $f(x) = \frac{1}{\mu - x}$ for λ, μ in $\mathbb{C}\backslash\mathbb{R}$. In this case the mean and covariance considered above correspond to, respectively, the one- and two-dimensional Cauchy (or Stieltjes) transform of the $\mathrm{GUE}(n, \frac{1}{n})$.

Keywords: Gaussian Unitary Ensemble; expectation and covariance of traces; asymptotic expansion; Cauchy transform.

AMS Subject Classification: 60B20

1. Introduction

Since the groundbreaking Ref. 17 by Voiculescu, the asymptotics for families of large, independent GUE random matrices has become an important tool in the theory of operator algebras. In the paper [8] it was established that if $X_1^{(n)}, \ldots, X_r^{(n)}$ are independent GUE$(n, \frac{1}{n})$ random matrices (see Definition 2.1 below), then with probability one we have for any polynomial p in r non-commuting variables that

$$\lim_{n\to\infty} \|p(X_1^{(n)}, \ldots, X_r^{(n)})\| = \|p(x_1, \ldots, x_r)\|, \tag{1.1}$$

where $\{x_1, \ldots, x_r\}$ is a free semi-circular family of self-adjoint operators in a C^*-probability space $(\mathcal{A}, \tau)$ (see Ref. 18 for definitions), and where $\|\cdot\|$ denotes the operator norm. This result leads in particular to the fact that there are non-invertible elements in the extension semi-group of the reduced C^*-algebra associated to the free group on r generators (see Ref. 8).

A key step in the proof of (1.1) was to establish precise estimates of the expectation and variance of $\mathrm{tr}_n[g(p(X_1^{(n)}, \ldots, X_r^{(n)}))]$, where tr_n denotes the normalized trace, g is a C^∞-function with compact support, and where we assume now that p is a self-adjoint polynomial. In fact it was established in Refs. 8 and 6 that in this setup we have the estimates:

$$\mathbb{E}\{\mathrm{tr}_n[g(p(X_1^{(n)}, \ldots, X_r^{(n)}))]\} = \tau[g(p(x_1, \ldots, x_r))] + O(n^{-2}), \tag{1.2}$$

$$\mathbb{V}\{\mathrm{tr}_n[g(p(X_1^{(n)}, \ldots, X_r^{(n)}))]\} = O(n^{-2}). \tag{1.3}$$

Furthermore, if the derivative g' vanishes on the spectrum of the operator $p(x_1, \ldots, x_r)$, then we actually have that

$$\mathbb{V}\{\mathrm{tr}_n[g(p(X_1^{(n)}, \ldots, X_r^{(n)}))]\} = O(n^{-4}).$$

If we assume instead that g is a polynomial, then the left-hand sides of (1.2) and (1.3) may actually be expanded as polynomials in n^{-2}. More precisely it was proved in Ref. 16 that for any function $w\colon \{1, 2, \ldots, p\} \to \{1, 2, \ldots, r\}$ we have that[a]

$$\mathbb{E}\big\{\mathrm{tr}_n\big[X_{w(1)}^{(n)} X_{w(2)}^{(n)} \cdots X_{w(p)}^{(n)}\big]\big\} = \sum_{\gamma\in T(w)} n^{-2\sigma(\gamma)}, \tag{1.4}$$

where $T(w)$ is a certain class of permutations of $\{1, 2, \ldots, p\}$, and $\sigma(\gamma) \in \mathbb{N}_0$ for all γ in $T(w)$ (see Refs. 16 or 12 for details). It was established furthermore in Ref. 12 that for two functions $w\colon \{1, 2, \ldots, p\} \to \{1, 2, \ldots, r\}$ and $v\colon \{1, 2, \ldots, q\} \to \{1, 2, \ldots, r\}$ we have that

$$\mathbb{E}\big\{\mathrm{tr}_n\big[X_{w(1)}^{(n)} X_{w(2)}^{(n)} \cdots X_{w(p)}^{(n)}\big]\mathrm{tr}_n\big[X_{v(1)}^{(n)} X_{v(2)}^{(n)} \cdots X_{v(q)}^{(n)}\big]\big\} = \sum_{\gamma\in T(w,v)} n^{-2\sigma(\gamma)}, \tag{1.5}$$

where now $T(w, v)$ is a certain class of permutations of $\{1, 2, \ldots, p + q\}$ and again $\sigma(\gamma) \in \mathbb{N}_0$ for all γ in $T(w, v)$ (see Ref. 12 for details).

[a]When $r = 1$, formula (1.4) corresponds to the Harer–Zagier recursion formulas (see Ref. 7).

In view of (1.4) and (1.5) it is natural to ask whether the left-hand sides of (1.2) and (1.3) may in general be expanded as "power series" in n^{-2}, when g is, say, a compactly supported C^∞-function. In the case $r = 1$, this question was answered affirmatively by Ercolani and McLaughlin (see Theorem 1.4 in Ref. 3) for a more general class of random matrices than the GUE. More precisely, Ercolani and McLaughlin established for a single matrix X_n (from the considered class of random matrices) and any C^∞-function g with at most polynomial growth the existence of a sequence $(\alpha_j(g))_{j\in\mathbb{N}_0}$ of complex numbers, such that for any n in $\mathbb{N}$ and k in $\mathbb{N}_0$,

$$\mathbb{E}\{\mathrm{tr}_n(g(X_n))\} = \sum_{j=0}^{k} \frac{\alpha_j(g)}{n^{2j}} + O(n^{-2k-2}). \tag{1.6}$$

Their proof is rather involved and is based on Riemann–Hilbert techniques developed by Deift, McLaughlin and co-authors. In this paper we provide an alternative proof for (1.6) in the case where X_n is a GUE$(n, \frac{1}{n})$ random matrix. For technical ease, we only establish (1.6) for functions in the class $C_b^\infty(\mathbb{R})$ consisting of all C^∞-functions $g : \mathbb{R} \to \mathbb{C}$, such that all derivatives $g^{(k)}$, $k \in \mathbb{N}_0$, are bounded on $\mathbb{R}$. However, all (relevant) results of the present paper can easily be extended to all C^∞-functions with at most polynomial growth. For each j in $\mathbb{N}$ we show that the coefficient $\alpha_j(g)$ is explicitly given in the form:

$$\alpha_j(g) = \frac{1}{2\pi} \int_{-2}^{2} [T^j g](x)\sqrt{4 - x^2}\,\mathrm{d}x$$

for a certain linear operator $T : C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ (see Theorem 3.5 and Corollary 3.6), and we describe α_j explicitly as a distribution (in the sense of L. Schwarts) in terms of Chebychev polynomials (cf. Corollary 4.6). The proof of (1.6) is based on the fact, proved by Götze and Tikhomirov in Ref. 4, that the spectral density h_n of a GUE$(n, \frac{1}{n})$ random matrix satisfies the following third-order differential equation:

$$\frac{1}{n^2} h_n'''(x) + (4 - x^2)h_n'(x) + x h_n(x) = 0, \quad (x \in \mathbb{R}). \tag{1.7}$$

In the special case where $g(x) = \frac{1}{\lambda - x}$ for some non-real complex number λ, the integral $\int_{\mathbb{R}} g(x)h_n(x)\mathrm{d}x$ is the Cauchy (or Stieltjes) transform $G_n(\lambda)$ for the measure $h_n(x)\mathrm{d}x$, and asymptotic expansions like (1.6) appeared already in Ref. 1 for a rather general class of random matrices (including the GUE). In the GUE case, our analytical approach leads to the following explicit expansion (see Sec. 4):

$$G_n(\lambda) = \eta_0(\lambda) + \frac{\eta_1(\lambda)}{n^2} + \frac{\eta_2(\lambda)}{n^4} + \cdots + \frac{\eta_k(\lambda)}{n^{2k}} + O(n^{-2k-2}), \tag{1.8}$$

where

$$\eta_0(\lambda) = \frac{\lambda}{2} - \frac{1}{2}(\lambda^2 - 4)^{\frac{1}{2}}, \quad \text{and} \quad \eta_j(\lambda) = \sum_{r=2j}^{3j-1} C_{j,r}(\lambda^2 - 4)^{-r-\frac{1}{2}} \quad (j \in \mathbb{N}).$$

The constants $C_{j,r}$, $2j \leq r \leq 3j-1$, appearing above are positive numbers for which we provide recursion formulas (see Proposition 4.5).

As for the "power series expansion" of (1.3), consider again for each n a single $\mathrm{GUE}(n, \frac{1}{n})$ random matrix. For any functions f, g from $C_b^\infty(\mathbb{R})$, we establish in Sec. 5 the existence of a sequence $(\beta_j(f,g))_{j\in\mathbb{N}_0}$ of complex numbers, such that for any k in $\mathbb{N}_0$ and n in $\mathbb{N}$,

$$\mathrm{Cov}\{\mathrm{Tr}_n[f(X_n)], \mathrm{Tr}_n[g(X_n)]\} = \sum_{j=0}^{k} \frac{\beta_j(f,g)}{n^{2j}} + O(n^{-2k-2}), \tag{1.9}$$

where Tr_n denotes the un-normalized trace on $M_n(\mathbb{C})$, and where the covariance $\mathrm{Cov}[Y,Z]$ of two complex-valued square integrable random variables is defined by

$$\mathrm{Cov}\{Y,Z\} = \mathbb{E}\{(Y-\mathbb{E}\{Y\})(Z-\mathbb{E}\{Z\})\}.$$

The proof of (1.9) is based on the following formula, essentially due to Pastur and Scherbina (see Ref. 13):

$$\begin{aligned}&\mathrm{Cov}\{\mathrm{Tr}_n[f(X_n)], \mathrm{Tr}_n[g(X_n)]\}\\ &\quad= \int_{\mathbb{R}^2} \left(\frac{f(x)-f(y)}{x-y}\right)\left(\frac{g(x)-g(y)}{x-y}\right)\rho_n(x,y)\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{1.10}$$

where the kernel ρ_n is given by

$$\rho_n(x,y) = \frac{n}{4}\left[\varphi_n\left(\sqrt{\frac{n}{2}}x\right)\varphi_{n-1}\left(\sqrt{\frac{n}{2}}y\right) - \varphi_{n-1}\left(\sqrt{\frac{n}{2}}x\right)\varphi_n\left(\sqrt{\frac{n}{2}}y\right)\right]^2,$$

with φ_n the nth Hermite function (see formula (2.1) below). The essential step then is to establish the formula (see Theorem 5.4):

$$\rho_n(x,y) = \frac{1}{4}\left[\tilde{h}_n(x)\tilde{h}_n(y) - 4h_n'(x)h_n'(y) - \frac{1}{n^2}h_n''(x)h_n''(y)\right], \quad ((x,y)\in\mathbb{R}^2), \tag{1.11}$$

where $\tilde{h}_n(x) = h_n(x) - xh_n'(x)$, and h_n is as before the spectral density of $\mathrm{GUE}(n, \frac{1}{n})$. Using (1.10)–(1.11) and Fubini's theorem, the expansion (1.9) may be derived from (1.6).

In the particular case where

$$f(x) = \frac{1}{\lambda - x}, \quad \text{and} \quad g(x) = \frac{1}{\mu - x}, \quad (x\in\mathbb{R}),$$

we obtain in Sec. 6 the following specific expansion for the two-dimensional Cauchy-transform:

$$\begin{aligned}&\mathrm{Cov}\{\mathrm{Tr}_n[(\lambda\mathbf{1} - X_n)^{-1}], \mathrm{Tr}_n[(\mu\mathbf{1} - X_n)^{-1}]\}\\ &\quad= \frac{1}{2(\lambda-\mu)^2}\sum_{j=0}^{k}\frac{\Gamma_j(\lambda,\mu)}{n^{2j}} + O(n^{-2k-2}),\end{aligned} \tag{1.12}$$

where the coefficients $\Gamma_j(\lambda,\mu)$ are given explicitly in terms of the functions η_l appearing in (1.8) (see Corollary 6.3). The leading term $\frac{\Gamma_0(\lambda,\mu)}{2(\lambda-\mu)^2}$ may also be identified as the integral

$$\int_{\mathbb{R}^2}\left(\frac{(\lambda-x)^{-1}-(\lambda-y)^{-1}}{x-y}\right)\left(\frac{(\mu-x)^{-1}-(\mu-y)^{-1}}{x-y}\right)\rho(x,y)\mathrm{d}x\mathrm{d}y,$$

where $\rho(x,y)\mathrm{d}x\mathrm{d}y$ is the weak limit of the measures $\rho_n(x,y)\mathrm{d}x\mathrm{d}y$ as $n\to\infty$ (cf. (1.10)). The limiting density ρ is explicitly given by

$$\rho(x,y)=\frac{1}{4\pi^2}\frac{4-xy}{\sqrt{4-x^2}\sqrt{4-y^2}}1_{(-2,2)}(x)1_{(-2,2)}(y),\quad (x,y\in\mathbb{R}),\tag{1.13}$$

and we provide a proof of this fact at the end of Sec. 5.

In Ref. 1 the authors derive for a rather general class of random matrices an expansion for the two-dimensional Cauchy transform in the form:

$$\mathrm{Cov}\{\mathrm{Tr}_n[(\lambda\mathbf{1}-X_n)^{-1}],\mathrm{Tr}_n[(\mu\mathbf{1}-X_n)^{-1}]\}=\sum_{j=0}^{k}d_{j,n}(\lambda,\mu)n^{-j}+o(n^{-k}),$$

where the leading coefficient $d_{0,n}$ is given explicitly by

$$d_{0,n}(\lambda,\mu)=\frac{1}{2(\lambda-\mu)^2}\left(\frac{\lambda\mu-a^2}{\sqrt{\lambda^2-a^2}\sqrt{\mu^2-a^2}}-1\right),\tag{1.14}$$

with a the variance of the relevant limiting semi-circle distribution. In the GUE setup considered in the present paper, $a=2$, and in this case it is easily checked that $d_{0,n}$ is identical to the leading coefficient $\frac{\Gamma_0(\lambda,\mu)}{2(\lambda-\mu)^2}$ in (1.12).

The density ρ given by (1.13) has previously appeared in Ref. 2. There the author proves that if X_n is a GUE$(n,\frac{1}{n})$ random matrix, then for any polynomial f, such that $\int_{-2}^{2}f(t)\sqrt{4-t^2}\mathrm{d}t=0$, the random variable $\mathrm{Tr}_n(f(X_n))$ converges, as $n\to\infty$, in distribution to the Gaussian distribution $N(0,\sigma^2)$, where the limiting variance σ^2 is given by

$$\sigma^2=\int_{[-2,2]\times[-2,2]}\left(\frac{f(x)-f(y)}{x-y}\right)^2\rho(x,y)\mathrm{d}x\mathrm{d}y.$$

The density ρ has also been identified in the physics literature as the (leading term for the) correlation function of the formal level density for the GUE (see Ref. 11 and references therein).

In a forthcoming paper (under preparation) we establish results similar to those obtained in the present paper for random matrices of Wishart type.

2. Auxiliary Differential Equations

In this section we consider two differential equations, both of which play a crucial role in the definition of the operator T introduced in Sec. 3. The former is a third-order differential equation for the spectral density of the GUE. We start thus by reviewing the GUE and its spectral distribution.

Consider a random $n \times n$ matrix $Y = (y_{ij})_{1\le i,j\le n}$ defined on some probability space $(\Omega, \mathcal{F}, P)$. The distribution of Y is then the probability measure P_Y on the set $M_n(\mathbb{C})$ of $n \times n$-matrices (equipped with Borel-σ-algebra) given by

$$P_Y(B) = P(Y \in B)$$

for any Borel-subset B of $M_n(\mathbb{C})$.

Throughout the paper we focus exclusively on the *Gaussian Unitary Ensemble* (GUE), which is the class of random matrices defined as follows:

Definition 2.1. Let n be a positive integer and σ^2 a positive real number. By $\mathrm{GUE}(n, \sigma^2)$ we then denote the distribution of a random $n \times n$ matrix $X = (x_{ij})_{1\le i,j\le n}$ (defined on some probability space) satisfying the following four conditions:

(i) For any i, j in $\{1, 2, \ldots, n\}$, $x_{ij} = \overline{x_{ji}}$.
(ii) The random variables x_{ij}, $1 \le i \le j \le n$, are independent.
(iii) If $1 \le i < j \le n$, then $\mathrm{Re}(x_{ij}), \mathrm{Im}(x_{ij})$ are i.i.d. with distribution $N(0, \frac{1}{2}\sigma^2)$.
(iv) For any i in $\{1, \ldots, n\}$, x_{ii} is a real-valued random variable with distribution $N(0, \sigma^2)$.

We recall now the specific form of the spectral distribution of a GUE random matrix. Let $\varphi_0, \varphi_1, \varphi_2, \ldots,$ be the sequence of Hermite functions, i.e.

$$\varphi_k(t) = (2^k k! \sqrt{\pi})^{-1/2} H_k(t) \exp(-t^2/2), \tag{2.1}$$

where $H_0, H_1, H_2, \ldots,$ is the sequence of Hermite polynomials, i.e.

$$H_k(t) = (-1)^k \exp(t^2) \left[\frac{\mathrm{d}^k}{\mathrm{d}t^k} \exp(-t^2) \right]. \tag{2.2}$$

Recall then (see e.g., Corollary 1.6 in Ref. 7) that the spectral distribution of a random matrix X from $\mathrm{GUE}(n, \frac{1}{n})$ has density

$$h_n(t) = \frac{1}{\sqrt{2n}} \sum_{k=1}^{n-1} \varphi_k\left(\sqrt{\frac{n}{2}}t\right)^2, \tag{2.3}$$

with respect to Lebesgue measure. More precisely,

$$\mathbb{E}\{\mathrm{tr}_n(g(X))\} = \int_{\mathbb{R}} g(t) h_n(t) \mathrm{d}t,$$

for any Borel function $g : \mathbb{R} \to \mathbb{R}$, for which the integral on the right-hand side is well-defined.

Götze and Tikhomirov established the following third-order differential equation for h_n:

Proposition 2.2 ([4]). *For each n in $\mathbb{N}$, h_n is a solution to the differential equation*:

$$\frac{1}{n^2}h_n'''(t) + (4 - t^2)h_n'(t) + th_n(t) = 0, \quad (t \in \mathbb{R}).$$

Proof. See Lemma 2.1 in Ref. 4. □

Via the differential equation in Proposition 2.2 and integration by parts, we are led (see the proof of Theorem 3.5 below) to consider the following differential equation:

$$(t^2 - 4)f'(t) + 3tf(t) = g(t), \tag{2.4}$$

for suitable given C^∞-functions g. The same differential equation was studied by Götze and Tikhomorov in Lemma 3.1 in Ref. 5 for a different class of functions g in connection with their Stein's Method approach to Wigner's semi-circle law.

Proposition 2.3. *For any C^∞-function $g: \mathbb{R} \to \mathbb{C}$, the differential equation*

$$(t^2 - 4)f'(t) + 3tf(t) = g(t), \quad (t \in \mathbb{R}), \tag{2.5}$$

has unique C^∞-solutions on $(-\infty, 2)$ and on $(-2, \infty)$. Furthermore, there is a C^∞-solution to (2.5) *on all of $\mathbb{R}$, if and only if g satisfies*

$$\int_{-2}^{2} g(t)\sqrt{4 - t^2}\mathrm{d}t = 0. \tag{2.6}$$

Proof. We note first that by splitting f and g in their real and imaginary parts, we may assume that they are both real-valued functions.

Uniqueness. By linearity it suffices to prove uniqueness in the case $g = 0$, i.e. that 0 is the only solution to the homogeneous equation:

$$(t^2 - 4)f'(t) + 3tf(t) = 0 \tag{2.7}$$

on $(-\infty, 2)$ and on $(-2, \infty)$. By standard methods we can solve (2.7) on each of the intervals $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$. We find thus that any solution to (2.7) must satisfy that

$$f(t) = \begin{cases} c_1(t^2 - 4)^{-\frac{3}{2}}, & \text{if } t < -2, \\ c_2(4 - t^2)^{-\frac{3}{2}}, & \text{if } t \in (-2, 2), \\ c_3(t^2 - 4)^{-\frac{3}{2}}, & \text{if } t > 2, \end{cases}$$

for suitable constants c_1, c_2, c_3 in $\mathbb{R}$. Since a solution to (2.7) on $(-\infty, 2)$ is continuous at $t = -2$, it follows that for such a solution we must have $c_1 = c_2 = 0$. Similarly, 0 is the only solution to (2.7) on $(-2, \infty)$.

Existence. The existence part is divided into three steps.

Step 1. We start by finding the solution to (2.5) on $(-2,\infty)$. By standard methods, it follows that the solution to (2.5) on $(2,\infty)$ is given by

$$f_c(t)=(t^2-4)^{-\frac{3}{2}}\int_2^t(s^2-4)^{\frac{1}{2}}g(s)\mathrm{d}s+c(t^2-4)^{-\frac{3}{2}},\quad (t\in(2,\infty),\ c\in\mathbb{C}),$$

whereas the solution to (2.5) on $(-2,2)$ is given by

$$f_c(t)=(4-t^2)^{-\frac{3}{2}}\int_t^2(4-s^2)^{\frac{1}{2}}g(s)\mathrm{d}s+c(4-t^2)^{-\frac{3}{2}},\quad (t\in(-2,2),\ c\in\mathbb{C}).$$

Now consider the function $f:(-2,\infty)\to\mathbb{R}$ given by

$$f(t)=\begin{cases}(4-t^2)^{-\frac{3}{2}}\displaystyle\int_t^2(4-s^2)^{\frac{1}{2}}g(s)\mathrm{d}s, & \text{if } t\in(-2,2),\\ \dfrac{1}{6}g(2), & \text{if } t=2,\\ (t^2-4)^{-\frac{3}{2}}\displaystyle\int_2^t(s^2-4)^{\frac{1}{2}}g(s)\mathrm{d}s, & \text{if } t\in(2,\infty).\end{cases}\tag{2.8}$$

We claim that f is a C^∞-function on $(-2,\infty)$. Once this has been verified, f is automatically a solution to (2.5) on all of $(-2,\infty)$ (by continuity at $t=2$). To see that f is a C^∞-function on $(-2,\infty)$, it suffices to show that f is C^∞ on $(0,\infty)$, and for this we use the following change of variables:

$$y=t^2-4,\quad \text{i.e. } t=\sqrt{y+4},\quad (t>2,\ y>0).$$

For y in $(0,\infty)$, we have

$$f(\sqrt{4+y})=y^{-\frac{3}{2}}\int_2^{\sqrt{4+y}}(s^2-4)^{\frac{1}{2}}g(s)\mathrm{d}s=y^{-\frac{3}{2}}\int_0^y u^{\frac{1}{2}}\cdot\frac{g(\sqrt{u+4})}{2\sqrt{u+4}}\mathrm{d}u.$$

Using then the change of variables

$$u=vy,\quad v\in[0,1],$$

we find that

$$f(\sqrt{4+y})=y^{-\frac{3}{2}}\int_0^1 v^{\frac{1}{2}}y^{\frac{1}{2}}\cdot\frac{g(\sqrt{4+vy})}{2\sqrt{4+vy}}y\mathrm{d}v=\int_0^1 v^{\frac{1}{2}}\cdot\frac{g(\sqrt{4+vy})}{2\sqrt{4+vy}}\mathrm{d}v,$$

for any y in $(0,\infty)$. Now, consider the function

$$l(y)=\int_0^1 v^{\frac{1}{2}}\cdot\frac{g(\sqrt{4+vy})}{2\sqrt{4+vy}}\mathrm{d}v,$$

which is well-defined on $(-4,\infty)$. By the usual theorem on differentiation under the integral sign (see e.g., Theorem 11.5 in Ref. 15), it follows that l is a C^∞-function on $(-4+\epsilon,K)$, for any positive numbers ϵ and K such that $0<\epsilon<K$. Hence l is

a C^∞-function on all of $(-4,\infty)$. Note also that

$$l(0) = \frac{1}{6}g(2) = f(2).$$

Furthermore, by performing change of variables as above in the reversed order, we find for any y in $(-4,0)$ that

$$l(y) = (-y)^{-\frac{3}{2}} \int_{\sqrt{4+y}}^{2} (4-s^2)^{\frac{1}{2}} g(s)\mathrm{d}s = f(\sqrt{4+y}).$$

Hence, we have established that $f(\sqrt{4+y}) = l(y)$ for any y in $(-4,\infty)$. Since l is a C^∞-function on $(-4,\infty)$, and since $f(t) = l(t^2-4)$ for all t in $(0,\infty)$, it follows that $f \in C^\infty((0,\infty))$, as desired.

Step 2. Next, we find the solution to (2.5) on $(-\infty,2)$. For this, consider the differential equation:

$$(t^2-4)\psi'(t) + 3t\psi(t) = g(-t), \quad (t \in (-2,\infty)). \tag{2.9}$$

From what we established in Step 1, it follows that (2.9) has a unique solution ψ in $C^\infty((-2,\infty))$. Then put

$$f_1(t) = -\psi(-t), \quad (t \in (-\infty,2)), \tag{2.10}$$

and note that $f_1 \in C^\infty((-\infty,2))$, which satisfies (2.5) on $(-\infty,2)$.

Step 3. It remains to verify that the solutions f and f_1, found in Steps 1 and 2 above, coincide on $(-2,2)$, if and only if Eq. (2.6) holds. With ψ as in Step 2, note that ψ is given by the right-hand side of (2.8), if $g(s)$ is replaced by $g(-s)$. Thus, for any t in $(-2,2)$, we have that

$$\begin{aligned}
f(t) - f_1(t) &= f(t) + \psi(-t) \\
&= (4-t^2)^{-\frac{3}{2}} \int_t^2 (4-s^2)^{\frac{1}{2}} g(s)\mathrm{d}s \\
&\quad + (4-t^2)^{-\frac{3}{2}} \int_{-t}^2 (4-s^2)^{\frac{1}{2}} g(-s)\mathrm{d}s \\
&= (4-t^2)^{-\frac{3}{2}} \int_t^2 (4-s^2)^{\frac{1}{2}} g(s)\mathrm{d}s \\
&\quad + (4-t^2)^{-\frac{3}{2}} \int_{-2}^t (4-s^2)^{\frac{1}{2}} g(s)\mathrm{d}s \\
&= (4-t^2)^{-\frac{3}{2}} \int_{-2}^2 (4-s^2)^{\frac{1}{2}} g(s)\mathrm{d}s,
\end{aligned}$$

from which the assertion follow readily. □

Proposition 2.4. *For any C^∞-function $g:\mathbb{R}\to\mathbb{C}$, there is a unique C^∞-function $f:\mathbb{R}\to\mathbb{C}$, such that*

$$g(t)=\frac{1}{2\pi}\int_{-2}^{2}g(s)\sqrt{4-s^2}\mathrm{d}s+(t^2-4)f'(t)+3tf(t),\quad (t\in\mathbb{R}). \tag{2.11}$$

If $g\in C_b^\infty(\mathbb{R})$, then $f\in C_b^\infty(\mathbb{R})$ too.

Proof. Let g be a function from $C^\infty(\mathbb{R})$, and consider the function

$$g_c=g-\frac{1}{2\pi}\int_{-2}^{2}g(s)\sqrt{4-s^2}\,\mathrm{d}s.$$

Since $\int_{-2}^{2}g_c(s)\sqrt{4-s^2}\mathrm{d}s=0$, it follows immediately from Proposition 2.3 that there is a unique C^∞-solution f to (2.11). Moreover (cf. the proof of Proposition 2.3), f satisfies that

$$f(t)=\begin{cases}(t^2-4)^{-\frac{3}{2}}\displaystyle\int_2^t(s^2-4)^{\frac{1}{2}}g_c(s)\mathrm{d}s, & \text{if } t\in(2,\infty),\\ -(t^2-4)^{-\frac{3}{2}}\displaystyle\int_2^{|t|}(s^2-4)^{\frac{1}{2}}g_c(-s)\mathrm{d}s, & \text{if } t\in(-\infty,-2).\end{cases}$$

Assume now that g (and hence g_c) is in $C_b^\infty(\mathbb{R})$, and choose a number R in $(0,\infty)$, such that $|g_c(t)|\le R$ for all t in $\mathbb{R}$. Then, for any t in $(2,\infty)$, we find that

$$|f(t)|\le(t^2-4)^{-\frac{3}{2}}R\int_2^t(s^2)^{\frac{1}{2}}\mathrm{d}s=\frac{1}{2}R(t^2-4)^{-\frac{1}{2}},$$

and thus f is bounded on, say, $(3,\infty)$. It follows similarly that f is bounded on, say, $(-\infty,-3)$. Hence, since f is continuous, f is bounded on all of $\mathbb{R}$.

Taking first derivatives in (2.11), we note next that

$$(t^2-4)f''(t)+5tf'(t)+3f(t)=g'(t),\quad (t\in\mathbb{R}),$$

and by induction we find that in general

$$(t^2-4)f^{(k+1)}(t)+(2k+3)tf^{(k)}(t)+k(k+2)f^{(k-1)}(t)=g^{(k)}(t),\quad (k\in\mathbb{N},\,t\in\mathbb{R}).$$

Thus, for t in $\mathbb{R}\setminus\{-2,2\}$,

$$f'(t)=\frac{-3tf(t)}{t^2-4}+\frac{g_c(t)}{t^2-4}, \tag{2.12}$$

and

$$f^{(k+1)}(t)=\frac{-(2k+3)tf^{(k)}(t)}{t^2-4}-\frac{k(k+2)f^{(k-1)}(t)}{t^2-4}+\frac{g^{(k)}(t)}{t^2-4},\quad (k\in\mathbb{N}). \tag{2.13}$$

Since f and g_c are bounded, it follows from (2.12) that f' is bounded on, say, $\mathbb{R}\backslash[-3,3]$ and hence on all of $\mathbb{R}$. Continuing by induction, it follows similarly from (2.13) that $f^{(k)}$ is bounded for all k in $\mathbb{N}$. □

3. Asymptotic Expansion for Expectations of Traces

In this section we establish the asymptotic expansion (1.6). We start by equipping $C_b^\infty(\mathbb{R})$ with a sequence of norms, which make it into a Fréchét space.

Definition 3.1. Consider the vector space $C_b^\infty(\mathbb{R})$ of C^∞-functions $f:\mathbb{R}\to\mathbb{C}$, satisfying that

$$\forall\, k\in\mathbb{N}_0 : \sup_{t\in\mathbb{R}}\left|\frac{\mathrm{d}^k}{\mathrm{d}t^k}f(t)\right|<\infty.$$

We introduce then a sequence $\|\cdot\|_{(k)}$ of norms on $C_b^\infty(\mathbb{R})$ as follows:

$$\|g\|_\infty=\sup_{x\in\mathbb{R}}|g(x)|,\quad (g\in C_b^\infty(\mathbb{R})),$$

and for any k in $\mathbb{N}_0$:

$$\|g\|_{(k)}=\max_{j=0,\dots,k}\|g^{(j)}\|_\infty,\quad (g\in C_b^\infty(\mathbb{R})),$$

where $g^{(j)}$ denotes the jth derivative of g. Equipped with the sequence $(\|\cdot\|_{(k)})_{k\in\mathbb{N}}$ of norms, $C_b^\infty(\mathbb{R})$ becomes a Fréchét space (see e.g., Theorem 1.37 and Remark 1.38(c) in Ref. 14).

The following lemma is well-known, but for the reader's convenience we include a proof.

Lemma 3.2. *Consider $C_b^\infty(\mathbb{R})$ as a Fréchét space as described in Definition* 3.1. *Then a linear mapping $L:C_b^\infty(\mathbb{R})\to C_b^\infty(\mathbb{R})$ is continuous, if and only if the following condition is satisfied*:

$$\forall\, k\in\mathbb{N}\,\exists\, m_k\in\mathbb{N}\,\exists\, C_k>0\;\forall\, g\in C_b^\infty(\mathbb{R}):\|Lg\|_{(k)}\le C_k\|g\|_{(m_k)}. \tag{3.1}$$

Proof. A sequence (g_n) from $C_b^\infty(\mathbb{R})$ converges to a function g in $C_b^\infty(\mathbb{R})$ in the described Fréchét topology, if and only if $\|g_n-g\|_{(k)}\to 0$ as $n\to\infty$ for any k in $\mathbb{N}$. Therefore condition (3.1) clearly implies continuity of L.

To establish the converse implication, note that by Theorem 1.37 in Ref. 11, a neighborhood basis at 0 for $C_b^\infty(\mathbb{R})$ is given by

$$U_{k,\epsilon}=\{h\in C_b^\infty(\mathbb{R})\mid \|h\|_{(k)}<\epsilon\},\quad (k\in\mathbb{N},\,\epsilon>0).$$

Thus, if L is continuous, there exists for any k in $\mathbb{N}$ an m in $\mathbb{N}$ and a positive δ, such that $L(U_{m,\delta})\subseteq U_{k,1}$. For any non-zero function g in $C_b^\infty(\mathbb{R})$, we have that

$\frac{1}{2}\delta\|g\|_{(m)}^{-1}g \in U_{m,\delta}$, and therefore

$$\frac{1}{2}\delta\|g\|_{(m)}^{-1}\|Lg\|_{(k)} < 1, \quad \text{i.e. } \|Lg\|_{(k)} < \frac{2}{\delta}\|g\|_{(m)},$$

which establishes (3.1). □

Remark 3.3. Appealing to Proposition 2.4, we may define a mapping $S\colon C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ by setting, for g in $C_b^\infty(\mathbb{R})$, $Sg = f$, where f is the unique solution to (2.11). By uniqueness, S is automatically a linear mapping. We define next the linear mapping $T\colon C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ by the formula:

$$Tg = (Sg)''', \quad (g \in C_b^\infty(\mathbb{R})).$$

Proposition 3.4. *The linear mappings $S, T\colon C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ introduced in Remark* 3.3 *are continuous when $C_b^\infty(\mathbb{R})$ is viewed as a Fréchét space as described in Definition* 3.1.

Proof. Since differentiation is clearly a continuous mapping from $C_b^\infty(\mathbb{R})$ into itself, it follows immediately that T is continuous, if S is.

To prove that S is continuous, it suffices to show that the graph of S is closed in $C_b^\infty(\mathbb{R}) \times C_b^\infty(\mathbb{R})$ equipped with the product topology (cf. Theorem 2.15 in Ref. 14). So let (g_n) be a sequence of functions in $C_b^\infty(\mathbb{R})$, such that $(g_n, Sg_n) \to (g, f)$ in $C_b^\infty(\mathbb{R}) \times C_b^\infty(\mathbb{R})$ for some functions f, g in $C_b^\infty(\mathbb{R})$. In particular then,

$$g_n \to g, \quad Sg_n \to f, \quad \text{and} \quad (Sg_n)' \to f' \quad \text{uniformly on } \mathbb{R} \text{ as } n \to \infty.$$

It follows that for any t in $\mathbb{R}$,

$$\begin{aligned} g(t) = \lim_{n\to\infty} g_n(t) &= \lim_{n\to\infty} \left(\frac{1}{2\pi}\int_{-2}^{2} g_n(s)\sqrt{4-s^2}\mathrm{d}s \right. \\ &\qquad \left. + (t^2-4)(Sg_n)'(t) + 3t(Sg_n)(t) \right) \\ &= \frac{1}{2\pi}\int_{-2}^{2} g(s)\sqrt{4-s^2}\mathrm{d}s + (t^2-4)f'(t) + 3tf(t). \end{aligned}$$

Therefore, by uniqueness of solutions to (2.11), $Sg = f$, and the graph of S is closed. □

Theorem 3.5. *Consider the spectral density h_n for* $\mathrm{GUE}(n, \frac{1}{n})$ *and the linear operator $T\colon C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ introduced in Remark* 3.3. *Then for any function g in $C_b^\infty(\mathbb{R})$ we have that*

$$\int_{\mathbb{R}} g(t)h_n(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-2}^{2} g(t)\sqrt{4-t^2}\mathrm{d}t + \frac{1}{n^2}\int_{\mathbb{R}} Tg(t)\cdot h_n(t)\mathrm{d}t.$$

Proof. Consider a fixed function g from $C_b^\infty(\mathbb{R})$, and then put $f = Sg$, where S is the linear operator introduced in Remark 3.3. Recall that

$$g(t) = \frac{1}{2\pi}\int_{-2}^{2} g(s)\sqrt{4-s^2}\mathrm{d}s + (t^2-4)f'(t) + 3tf(t), \quad (t \in \mathbb{R}). \tag{3.2}$$

By Proposition 2.2 and partial integration it follows that

$$\begin{aligned} 0 &= \int_{\mathbb{R}} f(t)[n^{-2}h_n'''(t) + (4-t^2)h_n'(t) + th_n(t)]\mathrm{d}t \\ &= -n^{-2}\int_{\mathbb{R}} f'''(t)h_n(t)\mathrm{d}t - \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}[f(t)(4-t^2)]h_n(t)\mathrm{d}t + \int_{\mathbb{R}} tf(t)h_n(t)\mathrm{d}t \\ &= \int_{\mathbb{R}}[-n^{-2}f'''(t) - (4-t^2)f'(t) + 3tf(t)]h_n(t)\mathrm{d}t, \end{aligned}$$

so that

$$\int_{\mathbb{R}}[(t^2-4)f'(t) + 3tf(t)]h_n(t)\mathrm{d}t = \frac{1}{n^2}\int_{\mathbb{R}} f'''(t)h_n(t)\mathrm{d}t = \frac{1}{n^2}\int_{\mathbb{R}} Tg(t)\cdot h_n(t)\mathrm{d}t.$$

Using (3.2) and the fact that $h_n(t)\mathrm{d}t$ is a probability measure, we conclude that

$$\begin{aligned} \int_{\mathbb{R}} g(t)\cdot h_n(t)\mathrm{d}t &= \frac{1}{2\pi}\int_{-2}^{2} g(t)\sqrt{4-t^2}\mathrm{d}t + \int_{\mathbb{R}}[(t^2-4)f'(t) + 3tf(t)]h_n(t)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-2}^{2} g(t)\sqrt{4-t^2}\mathrm{d}t + \frac{1}{n^2}\int_{\mathbb{R}} Tg(t)\cdot h_n(t)\mathrm{d}t, \end{aligned}$$

which is the desired expression. □

As an easy corollary of Proposition 3.5, we may now derive (in the GUE case) Ercolani's and McLaughlin's asymptotic expansion (see Theorem 1.4 in Ref. 3).

Corollary 3.6. *Let* $T : C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ *be the linear mapping introduced in Remark* 3.3. *Then for any* k *in* $\mathbb{N}$ *and* g *in* $C_b^\infty(\mathbb{R})$, *we have*:

$$\begin{aligned} \int_{\mathbb{R}} g(t)h_n(t)\mathrm{d}t &= \frac{1}{2\pi}\sum_{j=0}^{k-1}\frac{1}{n^{2j}}\int_{-2}^{2}[T^jg](t)\sqrt{4-t^2}\mathrm{d}t + \frac{1}{n^{2k}}\int_{\mathbb{R}}[T^kg](t)\cdot h_n(t)\mathrm{d}t. \\ &= \frac{1}{2\pi}\sum_{j=0}^{k-1}\frac{1}{n^{2j}}\int_{-2}^{2}[T^jg](t)\sqrt{4-t^2}\mathrm{d}t + O(n^{-2k}). \end{aligned}$$

Proof. The first equality in the corollary follows immediately by successive applications of Theorem 3.5. To show the second one, it remains to establish that for any k in $\mathbb{N}$

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}} |[T^kg](t)|\cdot h_n(t)\mathrm{d}t < \infty.$$

But this follows immediately from the fact that T^kg is bounded, and the fact that $h_n(t)\mathrm{d}t$ is a probability measure for each n. □

4. Asymptotic Expansion for the Cauchy Transform

For a GUE$(n, \frac{1}{n})$ random matrix X_n, we consider now the Cauchy transform given by

$$G_n(\lambda) = \mathbb{E}\{\mathrm{tr}_n[(\lambda \mathbf{1}_n - X_n)^{-1}]\} = \int_{\mathbb{R}} \frac{1}{\lambda - t} h_n(t)\mathrm{d}t, \quad (\lambda \in \mathbb{C}\backslash\mathbb{R}).$$

Setting

$$g_\lambda(t) = g(\lambda, t) = \frac{1}{\lambda - t}, \quad (t \in \mathbb{R},\, \lambda \in \mathbb{C}\backslash\mathbb{R}),$$

we have by the usual theorem on differentiation under the integral sign (for analytical functions) that G_n is analytical on $\mathbb{C}\backslash\mathbb{R}$ with derivatives

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} G_n(\lambda) = \int_{\mathbb{R}} \frac{(-1)^k k!}{(\lambda - t)^{k+1}} h_n(t)\mathrm{d}t = (-1)^k \int_{\mathbb{R}} \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} g_\lambda(t)\right) h_n(t)\mathrm{d}t, \tag{4.1}$$

for any k in $\mathbb{N}$ and λ in $\mathbb{C}\backslash\mathbb{R}$.

Lemma 4.1. *The Cauchy transform G_n of a* GUE$(n, \frac{1}{n})$ *random matrix X_n satisfies the following differential equation*:

$$n^{-2} \frac{\mathrm{d}^3}{\mathrm{d}\lambda^3} G_n(\lambda) + (4 - \lambda^2) \frac{\mathrm{d}}{\mathrm{d}\lambda} G_n(\lambda) + \lambda G_n(\lambda) = 2, \tag{4.2}$$

for all λ in $\mathbb{C}\backslash\mathbb{R}$.

Proof. From Proposition 2.2 and partial integration we obtain for fixed λ in $\mathbb{C}\backslash\mathbb{R}$ that

$$\begin{aligned} 0 &= \int_{\mathbb{R}} g_\lambda(t)[n^{-2} h_n'''(t) + (4 - t^2) h_n'(t) + t h_n(t)]\mathrm{d}t \\ &= \int_{\mathbb{R}} [-n^{-2} g_\lambda'''(t) - (4 - t^2) g_\lambda'(t) + 3t g_\lambda(t)] h_n(t)\mathrm{d}t. \end{aligned} \tag{4.3}$$

Note here that

$$(4 - t^2) g_\lambda'(t) = \frac{4 - t^2}{(\lambda - t)^2} = \frac{4 - \lambda^2}{(\lambda - t)^2} + \frac{2\lambda}{\lambda - t} - 1,$$

and that

$$3t g_\lambda(t) = \frac{3t}{\lambda - t} = \frac{3\lambda}{\lambda - t} - 3.$$

Inserting this into (4.3) and using (4.1) and the fact that h_n is a probability density, we find that

$$0 = n^{-2} \frac{\mathrm{d}^3}{\mathrm{d}\lambda^3} G_n(\lambda) + (4 - \lambda^2) \frac{\mathrm{d}}{\mathrm{d}\lambda} G_n(\lambda) + \lambda G_n(\lambda) - 2,$$

as desired. $\square$

For each fixed λ in $\mathbb{C}\backslash\mathbb{R}$, we apply next Corollary 3.6 to the function g_λ and obtain for any k in $\mathbb{N}_0$ the expansion:

$$G_n(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - t} h_n(t)\mathrm{d}t = \eta_0(\lambda) + \frac{\eta_1(\lambda)}{n^2} + \frac{\eta_2(\lambda)}{n^4} + \cdots + \frac{\eta_k(\lambda)}{n^{2k}} + O(n^{-2k-2}), \tag{4.4}$$

where $\eta_j(\lambda) = \frac{1}{2\pi}\int_{-2}^{2}[T^j g_\lambda](t)\sqrt{4-t^2}\mathrm{d}t$ for all j. To determine these coefficients we shall insert the expansion (4.4) into the differential equation (4.2) in order to obtain differential equations for the η_j's. To make this rigorous, we need first to establish analyticity of the η_j's as functions of λ.

Lemma 4.2. (i) *For any k in $\mathbb{N}_0$ the mapping $\lambda \mapsto T^k g_\lambda$ is analytical as a mapping from $\mathbb{C}\backslash\mathbb{R}$ into the Fréchét space $C_b^\infty(\mathbb{R})$, and*

$$\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} T^k g_\lambda = T^k\left(\frac{\partial^j}{\partial\lambda^j} g(\lambda, \cdot)\right) \quad \textit{for any } j \textit{ in } \mathbb{N}.$$

(ii) *For any k, n in $\mathbb{N}$, consider the mappings $\eta_k, R_{k,n} : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ given by*

$$\eta_k(\lambda) = \int_{-2}^{2} [T^k g_\lambda](s)\sqrt{4-s^2}\mathrm{d}s, \quad (\lambda \in \mathbb{C}\backslash\mathbb{R}), \tag{4.5}$$

$$R_{k,n}(\lambda) = \int_{\mathbb{R}} [T^{k+1} g_\lambda](s) h_n(s)\mathrm{d}s, \qquad (\lambda \in \mathbb{C}\backslash\mathbb{R}). \tag{4.6}$$

These mappings are analytical on $\mathbb{C}\backslash\mathbb{R}$ with derivatives:

$$\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\eta_k(\lambda) = \int_{-2}^{2}\left[T^k\left(\frac{\partial^j}{\partial\lambda^j} g(\lambda, \cdot)\right)\right](s)\sqrt{4-s^2}\mathrm{d}s, \quad (\lambda \in \mathbb{C}\backslash\mathbb{R},\ j \in \mathbb{N}),$$

$$\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} R_{k,n}(\lambda) = \int_{\mathbb{R}}\left[T^{k+1}\left(\frac{\partial^j}{\partial\lambda^j} g(\lambda, \cdot)\right)\right](s) h_n(s)\mathrm{d}s, \qquad (\lambda \in \mathbb{C}\backslash\mathbb{R},\ j \in \mathbb{N}).$$

Proof. (i) By standard methods it follows that for any λ in $\mathbb{C}\backslash\mathbb{R}$ and l, j in $\mathbb{N}_0$,

$$\lim_{h\to 0}\left(\sup_{t\in\mathbb{R}}\left|\frac{1}{h}\left(\frac{\partial^l}{\partial t^l}\frac{\partial^j}{\partial\lambda^j} g(\lambda+h, t) - \frac{\partial^l}{\partial t^l}\frac{\partial^j}{\partial\lambda^j} g(\lambda, t)\right) - \frac{\partial^l}{\partial t^l}\frac{\partial^{j+1}}{\partial\lambda^{j+1}} g(\lambda, t)\right|\right) = 0. \tag{4.7}$$

When $j = 0$, formula (4.7) shows that the mapping $F : \mathbb{C}\backslash\mathbb{R} \to C_b^\infty(\mathbb{R})$ given by

$$F(\lambda) = g(\lambda, \cdot), \quad (\lambda \in \mathbb{C}\backslash\mathbb{R}),$$

is analytical on $\mathbb{C}\backslash\mathbb{R}$ with derivative $\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda) = \frac{\partial}{\partial\lambda} g(\lambda, \cdot)$ (cf. Definition 3.30 in Ref. 14). Using then (4.7) and induction on j, it follows that moreover

$$\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} F(\lambda) = \frac{\partial^j}{\partial\lambda^j} g(\lambda, \cdot), \quad (\lambda \in \mathbb{C}\backslash\mathbb{R})$$

for all j in $\mathbb{N}$. For each k in $\mathbb{N}$ the mapping $T^k : C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ is linear and continuous (cf. Proposition 3.4), and it follows therefore immediately that the composed mapping $T^k \circ F : \mathbb{C}\backslash\mathbb{R} \to C_b^\infty(\mathbb{R})$ is again analytical on $\mathbb{C}\backslash\mathbb{R}$ with derivatives

$$\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} T^k g(\lambda, \cdot) = \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} T^k \circ F(\lambda) = T^k \left(\frac{\partial^j}{\partial \lambda^j} g(\lambda, \cdot) \right) \quad \text{for all } j \text{ in } \mathbb{N}.$$

This establishes (i).

(ii) As an immediate consequence of (i), for each fixed s in $\mathbb{R}$ the mapping $\lambda \mapsto [T^k g(\lambda, \cdot)](s)$ is analytical with derivatives

$$\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} [T^k g(\lambda, \cdot)](s) = \left[T^k \frac{\partial^j}{\partial \lambda^j} g(\lambda, \cdot) \right](s), \quad (j \in \mathbb{N}).$$

Note here that by Lemma 3.2

$$\left\| T^k \frac{\partial^j}{\partial \lambda^j} g(\lambda, \cdot) \right\|_\infty \le C(k, 0) \left\| \frac{\partial^j}{\partial \lambda^j} g(\lambda, \cdot) \right\|_{(m(k,0))}$$

for suitable constants $C(k, 0)$ in $(0, \infty)$ and $m(k, 0)$ in $\mathbb{N}$. Hence, for any closed ball B inside $\mathbb{C}\backslash\mathbb{R}$ and any j in $\mathbb{N}$ we have that

$$\sup_{\lambda \in B} \left\| T^k \left(\frac{\partial^j}{\partial \lambda^j} g(\lambda, \cdot) \right) \right\|_\infty \le C(k, 0) \sup_{\lambda \in B} \left\| \frac{\partial^j}{\partial \lambda^j} g(\lambda, \cdot) \right\|_{(m(k,0))} < \infty.$$

It follows now by application of the usual theorem on differentiation under the integral sign, that for any finite Borel-measure μ on $\mathbb{R}$, the mapping $\lambda \mapsto \int_{\mathbb{R}} [T^k g(\lambda, \cdot)](s) \mu(\mathrm{d}s)$ is analytical on $\mathbb{C}\backslash\mathbb{R}$ with derivatives

$$\begin{aligned} \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} \int_{\mathbb{R}} [T^k g(\lambda, \cdot)](s) \mu(\mathrm{d}s) &= \int_{\mathbb{R}} \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} [T^k g(\lambda, \cdot)](s) \mu(\mathrm{d}s) \\ &= \int_{\mathbb{R}} \left[T^k \left(\frac{\partial^j}{\partial \lambda^j} g(\lambda, \cdot) \right) \right](s) \mu(\mathrm{d}s). \end{aligned}$$

In particular this implies (ii). □

Lemma 4.3. *Let G_n denote the Cauchy-transform of $h_n(x)\mathrm{d}x$, and consider for each λ in $\mathbb{C}\backslash\mathbb{R}$ and k in $\mathbb{N}_0$ the asymptotic expansion*:

$$G_n(\lambda) = \eta_0(\lambda) + \frac{\eta_1(\lambda)}{n^2} + \frac{\eta_2(\lambda)}{n^4} + \cdots + \frac{\eta_k(\lambda)}{n^{2k}} + O(n^{-2k-2}) \tag{4.8}$$

given by Corollary 3.6. *Then the coefficients $\eta_j(\lambda)$ are analytical as functions of λ, and they satisfy the following recursive system of differential equations*:

$$\begin{aligned} &(4 - \lambda^2)\eta_0'(\lambda) + \lambda\eta_0(\lambda) = 2, \\ &(\lambda^2 - 4)\eta_j'(\lambda) - \lambda\eta_j(\lambda) = \eta_{j-1}'''(\lambda), \quad (j \in \mathbb{N}). \end{aligned} \tag{4.9}$$

Proof. For each j in $\mathbb{N}_0$ the coefficient $\eta_j(\lambda)$ is given by (4.5) (cf. Corollary 3.6), and hence Lemma 4.2 asserts that η_j is analytical on $\mathbb{C}\backslash\mathbb{R}$. Recall also from Corollary 3.6 that the $O(n^{-2k-2})$ term in (4.8) has the form $n^{-2k-2}R_{k,n}(\lambda)$, where $R_{k,n}(\lambda)$ is given by (4.6) and is again an analytical function on $\mathbb{C}\backslash\mathbb{R}$ according to Lemma 4.2. Inserting now (4.8) into the differential equation (4.2), we obtain for λ in $\mathbb{C}\backslash\mathbb{R}$ that

$$\begin{aligned}
2 &= n^{-2}G_n'''(\lambda) + (4-\lambda^2)G_n'(\lambda) + \lambda G_n(\lambda)\\
&= n^{-2}\left(\sum_{j=0}^{k} n^{-2j}\eta_j'''(\lambda) + n^{-2k-2}R_{k,n}'''(\lambda)\right) + (4-\lambda^2)\\
&\quad\cdot\left(\sum_{j=0}^{k} n^{-2j}\eta_j'(\lambda) + n^{-2k-2}R_{k,n}'(\lambda)\right) + \lambda\left(\sum_{j=0}^{k} n^{-2j}\eta_j(\lambda) + n^{-2k-2}R_{k,n}(\lambda)\right)\\
&= [(4-\lambda^2)\eta_0'(\lambda) + \lambda\eta_0(\lambda)] + \sum_{j=1}^{k} n^{-2j}[\eta_{j-1}'''(\lambda) + (4-\lambda^2)\eta_j'(\lambda) + \lambda\eta_j(\lambda)]\\
&\quad + n^{-2k-2}[\eta_k'''(\lambda) + (4-\lambda^2)R_{k,n}'(\lambda) + \lambda R_{k,n}(\lambda)] + n^{-2k-4}R_{k,n}'''(\lambda). \qquad (4.10)
\end{aligned}$$

Using Lemma 4.2, we note here that for fixed k and λ we have for any l in $\mathbb{N}_0$ that

$$\begin{aligned}
\sup_{n\in\mathbb{N}}\left|\frac{\mathrm{d}^l}{\mathrm{d}\lambda^l}R_{k,n}(\lambda)\right| &= \sup_{n\in\mathbb{N}}\left|\int_{\mathbb{R}}\left[T^{k+1}\left(\frac{\partial^l}{\partial\lambda^l}g(\lambda,\cdot)\right)\right](s)h_n(s)\mathrm{d}s\right|\\
&\leq \left\|T^{k+1}\left(\frac{\partial^l}{\partial\lambda^l}g(\lambda,\cdot)\right)\right\|_\infty < \infty,
\end{aligned}$$

since $T^{k+1}(\frac{\partial^l}{\partial\lambda^l}g(\lambda,\cdot)) \in C_b^\infty(\mathbb{R})$. Thus, letting $n\to\infty$ in (4.10), it follows that

$$(4-\lambda^2)\eta_0'(\lambda) + \lambda\eta_0(\lambda) = 2, \quad (\lambda\in\mathbb{C}\backslash\mathbb{R}).$$

and subsequently by multiplication with n^2 that

$$\begin{aligned}
0 &= \sum_{j=1}^{k} n^{-2j+2}[\eta_{j-1}'''(\lambda) + (4-\lambda^2)\eta_j'(\lambda) + \lambda\eta_j(\lambda)]\\
&\quad + n^{-2k}[\eta_k'''(\lambda) + (4-\lambda^2)R_{k,n}'(\lambda) + \lambda R_{k,n}(\lambda)] + n^{-2k-2}R_{k,n}'''(\lambda). \qquad (4.11)
\end{aligned}$$

Letting then $n\to\infty$ in (4.11), we find similarly (assuming $k\geq 1$) that

$$\eta_0'''(\lambda) + (4-\lambda^2)\eta_1'(\lambda) + \lambda\eta_1(\lambda) = 0,$$

and subsequently that

$$\begin{aligned}
0 &= \sum_{j=2}^{k} n^{-2j+4}[\eta_{j-1}'''(\lambda) + (4-\lambda^2)\eta_j'(\lambda) + \lambda\eta_j(\lambda)]\\
&\quad + n^{-2k+2}[\eta_k'''(\lambda) + (4-\lambda^2)R_{k,n}'(\lambda) + \lambda R_{k,n}(\lambda)] + n^{-2k}R_{k,n}'''(\lambda).
\end{aligned}$$

Continuing like this (induction), we obtain (4.9) for any j in $\{1,2,\ldots,k\}$. Since k can be chosen arbitrarily in $\mathbb{N}$, we obtain the desired conclusion. □

For any odd integer k we shall in the following use the conventions:

$$(\lambda^2-4)^{\frac{1}{2}} = \lambda\sqrt{1-\frac{4}{\lambda^2}}, \quad \text{and} \quad (\lambda^2-4)^{\frac{k}{2}} = ((\lambda^2-4)^{\frac{1}{2}})^k \tag{4.12}$$

for any λ in the region

$$\Omega := \mathbb{C}\backslash[-2,2],$$

and where $\sqrt{\cdot}$ denotes the usual main branch of the square root on $\mathbb{C}\backslash(-\infty,0]$. We note in particular that

$$|(\lambda^2-4)^{\frac{1}{2}}| \to \infty, \quad \text{as } |\lambda| \to \infty. \tag{4.13}$$

Lemma 4.4. *For any r in $\mathbb{Z}\backslash\{-3,-4\}$ the complete solution to the differential equation*:

$$(\lambda^2-4)\frac{\mathrm{d}}{d\lambda}f(\lambda) - \lambda f(\lambda) = \frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}(\lambda^2-4)^{-r-\frac{1}{2}}, \quad (\lambda\in\Omega) \tag{4.14}$$

is given by

$$f(\lambda) = \frac{(r+1)(2r+1)(2r+3)}{(r+3)(\lambda^2-4)^{r+5/2}} + \frac{2(2r+1)(2r+3)(2r+5)}{(r+4)(\lambda^2-4)^{r+7/2}} + C(\lambda^2-4)^{\frac{1}{2}}, \tag{4.15}$$

for all λ in Ω, and where C is an arbitrary complex constant.

Proof. By standard methods the complete solution to (4.14) is given by

$$f(\lambda) = (\lambda^2-4)^{\frac{1}{2}}\int(\lambda^2-4)^{-\frac{3}{2}}\frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}(\lambda^2-4)^{-r-\frac{1}{2}}\mathrm{d}\lambda, \quad (\lambda\in\Omega), \tag{4.16}$$

where $\int(\lambda^2-4)^{-\frac{3}{2}}\frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}(\lambda^2-4)^{-r-\frac{1}{2}}\mathrm{d}\lambda$ denotes the class of anti-derivatives (on Ω) to the function $(\lambda^2-4)^{-\frac{3}{2}}\frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}(\lambda^2-4)^{-r-\frac{1}{2}}$. Note here that by a standard calculation,

$$\begin{aligned}(\lambda^2-4)^{-\frac{3}{2}}\frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}(\lambda^2-4)^{-r-\frac{1}{2}} &= \frac{-(2r+1)(2r+2)(2r+3)\lambda}{(\lambda^2-4)^{r+4}} \\ &\quad - \frac{4(2r+1)(2r+3)(2r+5)\lambda}{(\lambda^2-4)^{r+5}}.\end{aligned}$$

Assuming that $r\notin\{-3,-4\}$, we have (since Ω is connected) for k in $\{4,5\}$ that

$$\int\lambda(\lambda^2-4)^{-r-k}\mathrm{d}\lambda = \frac{-1}{2(r+k-1)}(\lambda^2-4)^{-r-k+1} + C, \quad (C\in\mathbb{C}).$$

We obtain thus that

$$\begin{aligned}
&\int (\lambda^2-4)^{-\frac{3}{2}} \frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}(\lambda^2-4)^{-r-\frac{1}{2}}\mathrm{d}\lambda \\
&\quad = \frac{(2r+1)(2r+2)(2r+3)}{2(r+3)(\lambda^2-4)^{r+3}} + \frac{4(2r+1)(2r+3)(2r+5)}{2(r+4)(\lambda^2-4)^{r+4}} + C \\
&\quad = \frac{(r+1)(2r+1)(2r+3)}{(r+3)(\lambda^2-4)^{r+3}} + \frac{2(2r+1)(2r+3)(2r+5)}{(r+4)(\lambda^2-4)^{r+4}} + C,
\end{aligned}$$

where C is an arbitrary constant. Inserting this expression into (4.16), formula (4.15) follows readily. □

Proposition 4.5. *Let G_n denote the Cauchy-transform of $h_n(x)\mathrm{d}x$, and consider for each λ in $\mathbb{C}\backslash\mathbb{R}$ and k in $\mathbb{N}_0$ the asymptotic expansion*:

$$G_n(\lambda) = \eta_0(\lambda) + \frac{\eta_1(\lambda)}{n^2} + \frac{\eta_2(\lambda)}{n^4} + \cdots + \frac{\eta_k(\lambda)}{n^{2k}} + O(n^{-2k-2})$$

given by Corollary 3.6. *Then for λ in $\mathbb{C}\backslash\mathbb{R}$ we have that*

$$\eta_0(\lambda) = \frac{\lambda}{2} - \frac{1}{2}(\lambda^2-4)^{\frac{1}{2}}, \tag{4.17}$$

$$\eta_1(\lambda) = (\lambda^2-4)^{-\frac{5}{2}}, \tag{4.18}$$

and generally for j in $\mathbb{N}$, η_j takes the form:

$$\eta_j(\lambda) = \sum_{r=2j}^{3j-1} C_{j,r}(\lambda^2-4)^{-r-\frac{1}{2}}$$

for constants $C_{j,r}$, $2j \le r \le 3j-1$. Whenever $j \ge 1$, these constants satisfy the recursion formula:

$$\begin{aligned}
C_{j+1,r} = \frac{(2r-3)(2r-1)}{r+1}&((r-1)C_{j,r-2} \\
&+ (4r-10)C_{j,r-3}), \quad (2j+2 \le r \le 3j+2),
\end{aligned} \tag{4.19}$$

where for r in $\{2j+2, 3j+2\}$ we adopt the conventions: $C_{j,2j-1} = 0 = C_{j,3j}$.

Before proceeding to the proof of Proposition 4.5, we note that for any j in $\mathbb{N}_0$ and λ in $\mathbb{C}\backslash\mathbb{R}$ we have by Lemma 3.2 that

$$|\eta_j(\lambda)| \le \|T^j g_\lambda\|_\infty \le C(j,0)\|g_\lambda\|_{m(j,0)},$$

for suitable constants $C(j,0)$ in $(0,\infty)$ and $m(j,0)$ in $\mathbb{N}$ (not depending on λ). In particular it follows that

$$|\eta_j(\mathrm{i}x)| \to 0, \quad \text{as } x \to \infty, \quad x \in \mathbb{R}. \tag{4.20}$$

Proof of Proposition 4.5. The function η_0 is the Cauchy transform of the standard semi-circle distribution, which is well-known to equal the right-hand side of (4.17) (see e.g., Ref. 18). Now, $\eta_0'''(\lambda) = -\frac{1}{2}\frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}(\lambda^2-4)^{\frac{1}{2}}$, so by (4.9) and Lemma 4.4 (with $r=-1$), it follows that

$$\eta_1(\lambda) = -\frac{1}{2}(-2(\lambda^2-4)^{1-\frac{7}{2}}) + C(\lambda^2-4)^{\frac{1}{2}} = (\lambda^2-4)^{-\frac{5}{2}} + C(\lambda^2-4)^{\frac{1}{2}},$$

for a suitable constant C in $\mathbb{C}$. Comparing (4.20) and (4.13), it follows that we must have $C=0$, which establishes (4.18).

Proceeding by induction, assume that for some j in $\mathbb{N}$ we have established that

$$\eta_j(\lambda) = \sum_{r=2j}^{3j-1} C_{j,r}(\lambda^2-4)^{-r-\frac{1}{2}}$$

for suitable constants $C(j,r)$, $r=2j,2j+1,\dots,3j-1$. Then by (4.9), Lemma 4.4 and linearity it follows that modulo a term of the form $C(\lambda^2-4)^{1/2}$ we have that

$$\begin{aligned}
\eta_{j+1}(\lambda) = {} & \sum_{r=2j}^{3j-1} C_{j,r}\frac{(r+1)(2r+1)(2r+3)}{r+3}(\lambda^2-4)^{-r-\frac{5}{2}} \\
& + \sum_{r=2j}^{3j-1} C_{j,r}\frac{2(2r+1)(2r+3)(2r+5)}{r+4}(\lambda^2-4)^{-r-\frac{7}{2}} \\
= {} & \sum_{s=2j+2}^{3j+1} C_{j,s-2}\frac{(s-1)(2s-3)(2s-1)}{s+1}(\lambda^2-4)^{-s-\frac{1}{2}} \\
& + \sum_{s=2j+3}^{3j+2} C_{j,s-3}\frac{2(2s-5)(2s-3)(2s-1)}{s+1}(\lambda^2-4)^{-s-\frac{1}{2}} \\
= {} & C_{j,2j}\frac{(2j+1)(4j+1)(4j+3)}{2j+3}(\lambda^2-4)^{-2j-2-\frac{1}{2}} \\
& + C_{j,3j-1}\frac{2(6j-1)(6j+1)(6j+3)}{3j+3}(\lambda^2-4)^{-3j-2-\frac{1}{2}} \\
& + \sum_{s=2j+3}^{3j+1} \frac{(2s-3)(2s-1)}{s+1}[(s-1)C_{j,s-2} \\
& + (4s-10)C_{j,s-3}](\lambda^2-4)^{-s-\frac{1}{2}}.
\end{aligned}$$

As before (4.20) and (4.13) imply that the neglected term $C(\lambda^2-4)$ must vanish anyway. The resulting expression in the calculation above has the form

$$\sum_{s=2(j+1)}^{3(j+1)-1} C_{j+1,s}(\lambda^2-4)^{-s-\frac{1}{2}},$$

where the constants $C_{j+1,s}$ are immediately given by (4.19), whenever $2j+3 \leq s \leq 3j+1$. Recalling the convention that $C_{j,2j-1} = 0 = C_{j,3j}$, it is easy to check that also when $s = 2j+2$ or $s = 3j+2$, formula (4.19) produces, respectively, the coefficients to $(\lambda^2-4)^{-2j-\frac{5}{2}}$ and $(\lambda^2-4)^{-3j-\frac{5}{2}}$ appearing in the resulting expression above. $\square$

Using the recursion formula (4.19), it follows easily that

$$\eta_2(\lambda) = 21(\lambda^2-4)^{-\frac{9}{2}} + 105(\lambda^2-4)^{-\frac{11}{2}},$$
$$\eta_3(\lambda) = 1485(\lambda^2-4)^{-\frac{13}{2}} + 18018(\lambda^2-4)^{-\frac{15}{2}} + 50050(\lambda^2-4)^{-\frac{17}{2}}.$$

We close this section by identifying the functionals $g \mapsto \frac{1}{2\pi}\int_{-2}^{2}[T^j g](t)\sqrt{4-t^2}\mathrm{d}t$ as distributions (in the sense of L. Schwarts). Before stating the result, we recall that the Chebychev polynomials $T_0, T_1, T_2, \dots$ of the first kind are the polynomials on $\mathbb{R}$ determined by the relation:

$$T_k(\cos\theta) = \cos(k\theta), \quad (\theta \in [0,\pi],\, k \in \mathbb{N}_0). \tag{4.21}$$

Corollary 4.6. *For each j in $\mathbb{N}_0$ consider the mapping $\alpha_j : C_b^\infty(\mathbb{R}) \to \mathbb{C}$ given by*

$$\alpha_j(g) = \frac{1}{2\pi}\int_{-2}^{2}[T^j g](t)\sqrt{4-t^2}\mathrm{d}t, \quad (g \in C_b^\infty(\mathbb{R})),$$

where $T : C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ is the linear mapping introduced in Theorem 3.5. *Consider in addition for each k in $\mathbb{N}_0$ the mapping $E_k : C_b^\infty(\mathbb{R}) \to \mathbb{C}$ given by*

$$E_k(g) = \frac{1}{\pi}\int_{-2}^{2} g^{(k)}(x)\frac{T_k(\frac{x}{2})}{\sqrt{4-x^2}}\mathrm{d}x,$$

where $T_0, T_1, T_2, \dots$ are the Chebychev polynomials given by (4.21). *Then for any j in $\mathbb{N}$,*

$$\alpha_j = \sum_{k=2j}^{3j-1} C_{j,k}\frac{k!}{(2k)!}E_k, \tag{4.22}$$

where $C_{j,2j}, C_{j,2j+1}, \dots, C_{j,3j-1}$ are the constants described in Proposition 4.5.

From Corollary 4.6 it follows in particular that α_j (restricted to $C_c^\infty(\mathbb{R})$) is a distribution supported on $[-2,2]$ (i.e. $\alpha_j(\varphi) = 0$ for any function φ from $C_c^\infty(\mathbb{R})$ such that $\mathrm{supp}(\varphi) \cap [-2,2] = \emptyset$). In addition it follows from (4.22) that α_j is a distribution of order at most $3j-1$ (cf. Ref. 14, p. 156), and it is not hard to show that in fact the order of α_j equals $3j-1$.

Proof of Corollary 4.6. Let j in $\mathbb{N}$ be given and let Λ_j denote the right-hand side of (4.22). Since both α_j and Λ_j are supported on $[-2,2]$, it suffices to show that their Stieltjes transforms coincide, i.e. that

$$\alpha_j(g_\lambda) = \Lambda_j(g_\lambda), \quad (\lambda \in \mathbb{C}\backslash\mathbb{R}), \tag{4.23}$$

where as before $g_\lambda(x) = \frac{1}{\lambda - x}$ for all x in $\mathbb{R}$. Since the mapping $\lambda \mapsto g_\lambda$ is analytical from $\mathbb{C}\backslash\mathbb{R}$ into $C_b^\infty(\mathbb{R})$ (cf. Lemma 4.2), and since the linear functionals $\alpha_j, \Lambda_j : C_b^\infty(\mathbb{R}) \to \mathbb{C}$ are continuous, the functions $\lambda \mapsto \alpha_j(g_\lambda)$ and $\lambda \mapsto \Lambda_j(g_\lambda)$ are analytical on $\mathbb{C}\backslash\mathbb{R}$. It suffices thus to establish (4.23) for λ in $\mathbb{C}\backslash\mathbb{R}$ such that $|\lambda| > 2$. So consider in the following a fixed such λ. We know from Proposition 4.5 that

$$\alpha_j(g_\lambda) = \eta_j(\lambda) = \sum_{k=2j}^{3j-1} C_{j,k}(\lambda^2 - 4)^{-k-\frac{1}{2}},$$

with $(\lambda^2 - 4)^{-k-\frac{1}{2}}$ defined as in (4.12). It suffices thus to show that

$$E_k(g_\lambda) = \frac{(2k)!}{k!}(\lambda^2 - 4)^{-k-\frac{1}{2}}$$

for all k in $\mathbb{N}$. So let k from $\mathbb{N}$ be given, and recall that $g_\lambda(x) = \frac{1}{\lambda}\sum_{\ell=0}^\infty (\frac{x}{\lambda})^\ell$ for all x in $[-2,2]$. Since $\int_{-2}^2 \frac{|T_k(\frac{x}{2})|}{\sqrt{4-x^2}}\mathrm{d}x < \infty$, and since the power series

$$\sum_{\ell=r}^\infty \ell(\ell-1)\cdots(\ell-r+1)z^{\ell-r}$$

converges uniformly on $\{z \in \mathbb{C} \,|\, |z| \le \frac{2}{|\lambda|}\}$ for any r in $\mathbb{N}_0$, it follows that we may change the order of differentiation, summation and integration in the following calculation:

$$\begin{aligned}
E_k(g_\lambda) &= \frac{1}{\pi\lambda}\int_{-2}^2 \left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sum_{\ell=0}^\infty \Big(\frac{x}{\lambda}\Big)^\ell\right] \frac{T_k(\frac{x}{2})}{\sqrt{4-x^2}}\mathrm{d}x \\
&= \frac{1}{\pi\lambda}\int_{-2}^2 \left[\lambda^{-k}\sum_{\ell=k}^\infty \ell(\ell-1)\cdots(\ell-k+1)\Big(\frac{x}{\lambda}\Big)^{\ell-k}\right] \frac{T_k(\frac{x}{2})}{\sqrt{4-x^2}}\mathrm{d}x \\
&= \sum_{\ell=k}^\infty \frac{\ell!}{(\ell-k)!}\left[\frac{1}{\pi}\int_{-2}^2 x^{\ell-k}\frac{T_k(\frac{x}{2})}{\sqrt{4-x^2}}\mathrm{d}x\right]\lambda^{-\ell-1}.
\end{aligned}$$

Using the substitution $x = 2\cos\theta, \theta \in (0,\pi)$, as well as (4.21) and Euler's formula for $\cos\theta$, it follows by a standard calculation that

$$\frac{1}{\pi}\int_{-2}^2 x^p \frac{T_k(\frac{x}{2})}{\sqrt{4-x^2}}\mathrm{d}x = \begin{cases} \binom{p}{(p-k)/2}, & \text{if } p \in \{k+2m \,|\, m \in \mathbb{N}_0\}, \\ 0, & \text{otherwise.}\end{cases}$$

We thus find that

$$\begin{aligned}
E_k(g_\lambda) &= \sum_{m=0}^{\infty} \frac{(2k+2m)!}{(k+2m)!}\binom{k+2m}{m}\lambda^{-2m-2k-1} = \sum_{m=0}^{\infty}\frac{(2k+2m)!}{m!(k+m)!}\lambda^{-2m-2k-1}\\
&= \frac{(2k)!}{k!}\sum_{m=0}^{\infty}4^m\binom{k+m-\frac{1}{2}}{m}\lambda^{-2m-2k-1}\\
&= \lambda^{-2k-1}\frac{(2k)!}{k!}\sum_{m=0}^{\infty}\binom{k+m-\frac{1}{2}}{m}\left(\frac{4}{\lambda^2}\right)^m,
\end{aligned}$$

where the third equality results from a standard calculation on binomial coefficients. Recall now that

$$(1-z)^{-k-\frac{1}{2}} = \sum_{m=0}^{\infty}\binom{k+m-\frac{1}{2}}{m}z^m, \quad (z\in\mathbb{C},\ |z|<1),$$

where the left-hand side is formally defined as $(\sqrt{1-z})^{-2k-1}$, with $\sqrt{\cdot}$ the usual holomorphic branch of the square root on $\mathbb{C}\backslash(-\infty,0]$. We may thus conclude that

$$\begin{aligned}
E_k(g_\lambda) &= \lambda^{-2k-1}\frac{(2k)!}{k!}\left(1-\frac{4}{\lambda^2}\right)^{-k-\frac{1}{2}} = \frac{(2k)!}{k!}\left(\lambda\sqrt{1-\frac{4}{\lambda^2}}\right)^{-2k-1}\\
&= \frac{(2k)!}{k!}(\lambda^2-4)^{-k-\frac{1}{2}},
\end{aligned}$$

where the last equality follows from (4.12). This completes the proof. □

5. Asymptotic Expansion for Second-Order Statistics

In this section we shall establish asymptotic expansions, similar to Corollary 3.6, for covariances in the form $\mathrm{Cov}\{\mathrm{Tr}_n[f(X_n)], \mathrm{Tr}_n[g(Y_n)]\}$, where $f, g \in C_b^\infty(\mathbb{R})$, X_n is a $\mathrm{GUE}(n, \frac{1}{n})$ random matrix and Tr_n denotes the (un-normalized) trace on $M_n(\mathbb{C})$.

For complex-valued random variables Y, Z with second moments (and defined on the same probability space), we use the notation:

$$\mathbb{V}\{Y\} = \mathbb{E}\{(Y-\mathbb{E}\{Y\})^2\}, \quad \text{and} \quad \mathrm{Cov}\{Y,Z\} = \mathbb{E}\{(Y-\mathbb{E}\{Y\})(Z-\mathbb{E}\{Z\})\}.$$

Note in particular that $\mathbb{V}\{Y\}$ is generally not a positive number, and that $\mathrm{Cov}\{Y,Z\}$ is truly linear in both Y and Z.

Lemma 5.1. *Let σ be a positive number, and let X_N be a* $\mathrm{GUE}(n,\sigma^2)$ *random matrix. For any function f from $C_b^\infty(\mathbb{R})$ we then have that*

$$\mathbb{V}\{\mathrm{Tr}_n[f(X_n)]\} = \frac{1}{4\sigma^2}\int_{\mathbb{R}^2}(f(x)-f(y))^2\psi_n\left(\frac{x}{\sqrt{2\sigma^2}},\frac{y}{\sqrt{2\sigma^2}}\right)^2 \mathrm{d}x\mathrm{d}y, \tag{5.1}$$

where the kernel ψ_n *is given by*

$$\psi_n(x,y) = \sum_{j=0}^{n-1} \varphi_j(x)\varphi_j(y) = \sqrt{\frac{n}{2}} \frac{\varphi_n(x)\varphi_{n-1}(y) - \varphi_{n-1}(x)\varphi_n(y)}{x-y}, \tag{5.2}$$

and the φ_j*'s are the Hermite functions introduced in* (2.1).

Proof. Formula (5.1) appears in the proof of [13, Lemma 3] with ψ given by the first equality in (5.2). The second equality in (5.2) is equivalent to the Christoffel–Darboux formula for the Hermite polynomials (see Ref. 9, p. 193 formula (11)). $\square$

Corollary 5.2. *Let* X_n *be a* $\mathrm{GUE}(n, \frac{1}{n})$ *random matrix.*

(i) *For any function* f *from* $C_b^\infty(\mathbb{R})$ *we have that*

$$\mathbb{V}\{\mathrm{Tr}_n[f(X)]\} = \int_{\mathbb{R}^2} \left(\frac{f(x)-f(y)}{x-y}\right)^2 \rho_n(x,y)\mathrm{d}x\mathrm{d}y,$$

where the kernel ρ_n *is given by*

$$\rho_n(x,y) = \frac{n}{4}\left[\varphi_n\left(\sqrt{\frac{n}{2}}x\right)\varphi_{n-1}\left(\sqrt{\frac{n}{2}}y\right) - \varphi_{n-1}\left(\sqrt{\frac{n}{2}}x\right)\varphi_n\left(\sqrt{\frac{n}{2}}y\right)\right]^2. \tag{5.3}$$

(ii) *For any functions* f *and* g *from* $C_b^\infty(\mathbb{R})$ *we have that*

$$\begin{aligned}&\mathrm{Cov}\{\mathrm{Tr}_n[f(X_n)], \mathrm{Tr}_n[g(X_n)]\} \\ &\quad = \int_{\mathbb{R}^2} \left(\frac{f(x)-f(y)}{x-y}\right)\left(\frac{g(x)-g(y)}{x-y}\right) \rho_n(x,y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Proof. (i) This follows from Lemma 5.1 by a straightforward calculation, setting $\sigma^2 = \frac{1}{n}$ in (5.1).

(ii) Using (i) on the functions $f+g$ and $f-g$ we find that

$$\begin{aligned}&\mathrm{Cov}\{\mathrm{Tr}_n[f(X_n)], \mathrm{Tr}_n[g(X_n)]\} \\ &\quad = \frac{1}{4}(\mathbb{V}\{\mathrm{Tr}_n[f(X_n)+g(X_n)]\} - \mathbb{V}\{\mathrm{Tr}_n[f(X_n)-g(X_n)]\}) \\ &\quad = \frac{1}{4}\int_{\mathbb{R}^2} \frac{((f+g)(x)-(f+g)(y))^2 - ((f-g)(x)-(f-g)(y))^2}{(x-y)^2} \\ &\qquad \cdot \rho_n(x,y)\mathrm{d}x\mathrm{d}y \\ &\quad = \frac{1}{4}\int_{\mathbb{R}^2} \frac{4f(x)g(x)+4f(y)g(y)-4f(x)g(y)-4f(y)g(x)}{(x-y)^2}\rho_n(x,y)\mathrm{d}x\mathrm{d}y \\ &\quad = \int_{\mathbb{R}^2} \left(\frac{f(x)-f(y)}{x-y}\right)\left(\frac{g(x)-g(y)}{x-y}\right)\rho_n(x,y)\mathrm{d}x\mathrm{d}y,\end{aligned}$$

as desired. $\square$

In order to establish the desired asymptotic expansion of $\mathrm{Cov}\{\mathrm{Tr}_n[f(X_n)], \mathrm{Tr}_n[g(X_n)]\}$, we are led by Corollary 5.2(ii) to study the asymptotic behavior, as $n \to \infty$, of the probability measures $\rho_n(x,y)\mathrm{d}x\mathrm{d}y$. As a first step, it is instructive to note that $\rho_n(x,y)\mathrm{d}x\mathrm{d}y$ converges weakly, as $n \to \infty$, to the probability measure $\rho(x,y)\mathrm{d}x\mathrm{d}y$, where

$$\rho(x,y) = \frac{1}{4\pi^2}\frac{4-xy}{\sqrt{4-x^2}\sqrt{4-y^2}}1_{(-2,2)}(x)1_{(-2,2)}(y). \tag{5.4}$$

We shall give a short proof of this fact in Proposition 5.11 below. It implies in particular that if (X_n) is a sequence of random matrices, such that $X_n \sim \mathrm{GUE}(n, \frac{1}{n})$ for all n, then

$$\lim_{n\to\infty} \mathrm{Cov}\{\mathrm{Tr}_n[f(X_n)], \mathrm{Tr}_n[g(X_n)]\} = \int_{\mathbb{R}^2}\left(\frac{f(x)-f(y)}{x-y}\right)\left(\frac{g(x)-g(y)}{x-y}\right)\rho(x,y)\mathrm{d}x\mathrm{d}y,$$

for all $f, g \in C_b^\infty(\mathbb{R})$.

The key point in the approach given below is to express the density ρ_n in terms of the spectral density h_n of $\mathrm{GUE}(n, \frac{1}{n})$ (see Proposition 5.4 below).

Lemma 5.3. *Consider the functions* $\zeta_n : \mathbb{R}^2 \to \mathbb{R}$ *and* $\beta_n : \mathbb{R} \to \mathbb{R}$ *given by*

$$\zeta_n(x,y) = \frac{1}{2}[\varphi_n(x)\varphi_{n-1}(y) - \varphi_{n-1}(x)\varphi_n(y)]^2, \quad ((x,y) \in \mathbb{R}^2),$$

and

$$\beta_n(x) = \sum_{j=0}^{n-1}\varphi_j(x)^2, \quad (x \in \mathbb{R}),$$

with $\varphi_0, \varphi_1, \varphi_2, \ldots$ *the Hermite functions given in* (2.1). *We then have*

$$\zeta_n(x,y) = f_n(x)f_n(y) - g_n(x)g_n(y) - k_n(x)k_n(y), \quad ((x,y) \in \mathbb{R}^2),$$

where

$$f_n(x) = \frac{1}{2}(\varphi_n(x)^2 + \varphi_{n-1}(x)^2) = \frac{1}{2n}(\beta_n(x) - x\beta_n'(x)), \tag{5.5}$$

$$g_n(x) = \frac{1}{2}(\varphi_n(x)^2 - \varphi_{n-1}(x)^2) = \frac{1}{4n}\beta_n''(x), \tag{5.6}$$

$$k_n(x) = \varphi_{n-1}(x)\varphi_n(x) = \frac{-1}{\sqrt{2n}}\beta_n'(x), \tag{5.7}$$

for all x *in* $\mathbb{R}$.

Proof. Note first that with f_n, g_n and k_n defined by the leftmost equalities in (5.5)–(5.7) we have that

$$f_n(x) + g_n(x) = \varphi_n(x)^2 \quad \text{and} \quad f_n(x) - g_n(x) = \varphi_{n-1}(x)^2,$$

for all x in $\mathbb{R}$. Therefore,

$$\begin{aligned}\zeta_n(x,y) &= \frac{1}{2}[\varphi_n(x)\varphi_{n-1}(y) - \varphi_{n-1}(x)\varphi_n(y)]^2 \\ &= \frac{1}{2}[(f_n(x)+g_n(x))(f_n(y)-g_n(y)) + (f_n(x) \\ &\quad - g_n(x))(f_n(y)+g_n(y)) - 2k_n(x)k_n(y)] \\ &= f_n(x)f_n(y) - g_n(x)g_n(y) - k_n(x)k_n(y),\end{aligned}$$

for any (x,y) in $\mathbb{R}^2$. It remains thus to establish the three rightmost equalities in (5.5)–(5.7). For this we use the well-known formulas (cf. e.g., formulas (2.3)–(2.6) in Ref. 7):

$$\varphi_n'(x) = \sqrt{\frac{n}{2}}\varphi_{n-1}(x) - \sqrt{\frac{n+1}{2}}\varphi_{n+1}(x), \tag{5.8}$$

$$x\varphi_n(x) = \sqrt{\frac{n+1}{2}}\varphi_{n+1}(x) + \sqrt{\frac{n}{2}}\varphi_{n-1}(x), \tag{5.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{n-1}\varphi_k(x)^2\right) = -\sqrt{2n}\varphi_n(x)\varphi_{n-1}(x), \tag{5.10}$$

which hold for all n in $\mathbb{N}_0$, when we adopt the convention: $\varphi_{-1} \equiv 0$.

The second equality in (5.7) is an immediate consequence of (5.10). Combining (5.8) with (5.9), we note next that

$$\varphi_n'(x) = -x\varphi_n(x) + \sqrt{2n}\varphi_{n-1}(x) \quad \text{and} \quad \varphi_{n-1}'(x) = x\varphi_{n-1}(x) - \sqrt{2n}\varphi_n(x),$$

and therefore by (5.10)

$$\begin{aligned}\beta_n''(x) &= -\sqrt{2n}(\varphi_{n-1}'(x)\varphi_n(x) + \varphi_{n-1}(x)\varphi_n'(x)) \\ &= -\sqrt{2n}(x\varphi_{n-1}(x)\varphi_n(x) - \sqrt{2n}\varphi_n(x)^2 - x\varphi_{n-1}(x)\varphi_n(x) + \sqrt{2n}\varphi_{n-1}(x)^2) \\ &= 2n(\varphi_n(x)^2 - \varphi_{n-1}(x)^2),\end{aligned}$$

from which the second equality in (5.6) follows readily. Using once more (5.8) and (5.9), we note finally that

$$\begin{aligned}x\beta_n'(x) &= 2\sum_{j=0}^{n-1} x\varphi_j(x)\varphi_j'(x) \\ &= 2\sum_{j=0}^{n-1}\left(\sqrt{\frac{j+1}{2}}\varphi_{j+1}(x) + \sqrt{\frac{j}{2}}\varphi_{j-1}(x)\right)\left(\sqrt{\frac{j}{2}}\varphi_{j-1}(x) - \sqrt{\frac{j+1}{2}}\varphi_{j+1}(x)\right)\end{aligned}$$

$$= \sum_{j=0}^{n-1} (\varphi_{j-1}(x)^2 + (j-1)\varphi_{j-1}(x)^2 - (j+1)\varphi_{j+1}(x)^2)$$
$$= \left(\sum_{j=0}^{n-2} \varphi_j(x)^2\right) - (n-1)\varphi_{n-1}(x)^2 - n\varphi_n(x)^2,$$

and therefore

$$\begin{aligned}\beta_n(x) - x\beta_n'(x) &= \varphi_{n-1}(x)^2 + (n-1)\varphi_{n-1}(x)^2 + n\varphi_n(x)^2 \\ &= n(\varphi_{n-1}(x)^2 + \varphi_n(x)^2),\end{aligned}$$

which establishes the second equality in (5.5). □

Proposition 5.4. *Let ρ_n be the kernel given by* (5.3) *and let h_n be the spectral density of a* GUE$(n, \frac{1}{n})$ *random matrix* (*cf.* (2.3)). *We then have*

$$\rho_n(x,y) = \frac{1}{4}\left[\tilde{h}_n(x)\tilde{h}_n(y) - 4h_n'(x)h_n'(y) - \frac{1}{n^2}h_n''(x)h_n''(y)\right], \quad ((x,y) \in \mathbb{R}^2), \tag{5.11}$$

where

$$\tilde{h}_n(x) = h_n(x) - xh_n'(x), \quad (x \in \mathbb{R}).$$

Proof. With ζ_n, f_n, g_n, k_n and β_n as in Lemma 5.3 we have that

$$\begin{aligned}\rho_n(x,y) &= \frac{n}{2}\zeta_n\left(\sqrt{\frac{n}{2}}x, \sqrt{\frac{n}{2}}y\right) \\ &= \frac{n}{2}\left(f_n\left(\sqrt{\frac{n}{2}}x\right) f_n\left(\sqrt{\frac{n}{2}}y\right) - g_n\left(\sqrt{\frac{n}{2}}x\right) g_n\left(\sqrt{\frac{n}{2}}y\right)\right. \\ &\qquad \left. - k_n\left(\sqrt{\frac{n}{2}}x\right) k_n\left(\sqrt{\frac{n}{2}}y\right)\right),\end{aligned} \tag{5.12}$$

and (cf. formula (2.3))

$$h_n(x) = \frac{1}{\sqrt{2n}}\beta_n\left(\sqrt{\frac{n}{2}}x\right). \tag{5.13}$$

Combining (5.13) with the rightmost equalities in (5.5)–(5.7), we find that

$$f_n\left(\sqrt{\frac{n}{2}}x\right) = \frac{1}{\sqrt{2n}}\tilde{h}_n(x), \quad g_n\left(\sqrt{\frac{n}{2}}x\right) = \frac{1}{\sqrt{2}n^{\frac{3}{2}}}h_n''(x), \quad \text{and}$$

$$k_n\left(\sqrt{\frac{n}{2}}x\right) = -\sqrt{\frac{2}{n}}h_n'(x),$$

and inserting these expressions into (5.12), formula (5.11) follows readily. □

By $C_b^\infty(\mathbb{R}^2)$ we denote the vector space of infinitely often differentiable functions $f:\mathbb{R}^2 \to \mathbb{C}$ satisfying that

$$\|D_1^k D_2^l f\|_\infty := \sup_{(x,y)\in\mathbb{R}^2} |D_1^k D_2^l f(x,y)| < \infty,$$

for any k, l in $\mathbb{N}_0$. Here D_1 and D_2 denote, respectively, the partial derivatives of f with respect to the first and the second variable.

Lemma 5.5. *Assume that $f \in C_b^\infty(\mathbb{R}^2)$ and consider the mapping $\varphi_f:\mathbb{R} \to C_b^\infty(\mathbb{R})$ given by*

$$\varphi_f(x) = f(x,\cdot), \quad (x \in \mathbb{R}).$$

Then φ_f is infinitely often differentiable from $\mathbb{R}$ into $C_b^\infty(\mathbb{R})$, and for any k in $\mathbb{N}$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\varphi_f(x) = [D_1^k f](x,\cdot), \quad (x \in \mathbb{R}). \tag{5.14}$$

Proof. By splitting f in its real and imaginary parts, we may assume that f is real-valued. For any k in $\mathbb{N}$ the function $D_1^k f$ is again an element of $C_b^\infty(\mathbb{R}^2)$. Therefore, by induction, it suffices to prove that φ_f is differentiable with derivative given by (5.14) (in the case $k = 1$). For this we need to establish that

$$\left\| \frac{\varphi_f(x+h) - \varphi_f(x)}{h} - [D_1 f](x,\cdot) \right\|_{(m)} \to 0, \quad \text{as } h \to 0,$$

for any m in $\mathbb{N}$ and any x in $\mathbb{R}$. This amounts to showing that for fixed x in $\mathbb{R}$ and l in $\mathbb{N}$ we have that

$$\sup_{y\in\mathbb{R}} \left| \frac{D_2^l f(x+h,y) - D_2^l f(x,y)}{h} - D_2^l D_1 f(x,y) \right| \to 0, \quad \text{as } h \to 0.$$

For fixed y in $\mathbb{R}$ second-order Taylor expansion for the function $[D_2^l f](\cdot, y)$ yields that

$$D_2^l f(x+h,y) - D_2^l f(x,y) = D_1 D_2^l f(x,y)h + \frac{1}{2} D_1^2 D_2^l f(\xi,y)h^2,$$

for some real number $\xi = \xi(x,y,h)$ between $x+h$ and x. Consequently,

$$\begin{aligned} \sup_{y\in\mathbb{R}} &\left| \frac{D_2^l f(x+h,y) - D_2^l f(x,y)}{h} - D_2^l D_1 f(x,y) \right| \\ &\le \frac{h}{2}\|D_1^2 D_2^l f\|_\infty \to 0, \quad \text{as } h \to 0, \end{aligned}$$

as desired. ◻

Corollary 5.6. *Let T be the linear mapping introduced in Remark* 3.3, *and let f be a function from $C_b^\infty(\mathbb{R}^2)$. We then have*

(i) *For any j in $\mathbb{N}_0$ the mapping*

$$\psi_f : x \mapsto T^j f(x,\cdot) : \mathbb{R} \to C_b^\infty(\mathbb{R})$$

is infinitely often differentiable with derivatives given by

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\psi_f(x) = T^j([D_1^k f](x,\cdot)). \tag{5.15}$$

(ii) *For any j in $\mathbb{N}_0$ the mapping $\upsilon_j : \mathbb{R} \to \mathbb{C}$ given by*

$$\upsilon_j(x) = \frac{1}{2\pi}\int_{-2}^{2}[T^j f(x,\cdot)](t)\sqrt{4-t^2}\mathrm{d}t, \quad (x \in \mathbb{R}),$$

is a $C_b^\infty(\mathbb{R})$-function. Moreover, for any k in $\mathbb{N}$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\upsilon_j(x) = \frac{1}{2\pi}\int_{-2}^{2}[T^j([D_1^k]f(x,\cdot))](t)\sqrt{4-t^2}\mathrm{d}t. \tag{5.16}$$

Proof. (i) As in the proof of Lemma 5.5 it suffices to prove that ψ_f is differentiable with derivative given by (5.15) (in the case $k = 1$). But this follows immediately from Lemma 5.5, since $\psi_f = T^j \circ \varphi_f$, where $T : C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ is a linear, continuous mapping (cf. Proposition 3.4).

(ii) It suffices to prove that υ_j is bounded and differentiable with derivative given by (5.16) (in the case $k = 1$). To prove that υ_j is differentiable with the prescribed derivative, it suffices, in view of (i), to establish that the mapping

$$g \mapsto \frac{1}{2\pi}\int_{-2}^{2} g(t)\sqrt{4-t^2}\mathrm{d}t : C_b^\infty(\mathbb{R}) \to \mathbb{R}$$

is linear and continuous. It is clearly linear, and since

$$\left|\frac{1}{2\pi}\int_{-2}^{2} g(t)\sqrt{4-t^2}\mathrm{d}t\right| \le \|g\|_\infty, \quad (g \in C_b^\infty(\mathbb{R})),$$

it is also continuous. To see finally that υ_j is a bounded mapping, we note that since $T^j : C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$ is continuous, there are (cf. Lemma 3.2) constants C from $(0,\infty)$ and m in $\mathbb{N}$, such that

$$\|T^j f(x,\cdot)\|_\infty \le C \max_{l=1,\dots,m}\|D_2^l f(x,\cdot)\|_\infty \le C \max_{l=1,\dots,m}\|D_2^l f\|_\infty,$$

for any x in $\mathbb{R}$. Therefore,

$$\sup_{x\in\mathbb{R}}|\upsilon_j(x)| \le \sup_{x\in\mathbb{R}}\|T^j f(x,\cdot)\|_\infty \le C \max_{l=1,\dots,m}\|D_2^l f\|_\infty < \infty,$$

since $f \in C_b^\infty(\mathbb{R}^2)$. □

Proposition 5.7. *For any function f in $C_b^\infty(\mathbb{R}^2)$ there exists a sequence $(\beta_j(f))_{j\in\mathbb{N}_0}$ of complex numbers such that*

$$\int_{\mathbb{R}^2} f(x,y)h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y = \sum_{j=0}^{k}\frac{\beta_j(f)}{n^{2j}} + O(n^{-2k-2}),$$

for any k in $\mathbb{N}_0$.

Proof. Let k in $\mathbb{N}_0$ be given. For fixed x in $\mathbb{R}$ the function $f(x,\cdot)$ belongs to $C_b^\infty(\mathbb{R})$ and hence Corollary 3.6 asserts that

$$\int_{\mathbb{R}} f(x,y)h_n(y)\mathrm{d}y = \sum_{j=1}^{k} \frac{\upsilon_j(x)}{n^{2j}} + \frac{1}{n^{2k+2}} \int_{\mathbb{R}} [T^{k+1}f(x,\cdot)](t)h_n(t)\mathrm{d}t, \tag{5.17}$$

where the functions $\upsilon_j : \mathbb{R} \to \mathbb{R}$ are given by

$$\upsilon_j(x) = \frac{1}{2\pi}\int_{-2}^{2} [T^j f(x,\cdot)](t)\sqrt{4-t^2}\mathrm{d}t, \quad (x \in \mathbb{R},\ j = 1,\dots,k).$$

As noted in the proof of Corollary 5.6, there exist constants C from $(0,\infty)$ and m in $\mathbb{N}$, such that

$$\|T^{k+1}f(x,\cdot)\|_\infty \le C \max_{l=1,\dots,m} \|D_2^l f\|_\infty, \quad (x \in \mathbb{R}).$$

Hence, since h_n is a probability density,

$$C_k^f := \sup_{x\in\mathbb{R}} \left| \int_{\mathbb{R}} [T^{k+1}f(x,\cdot)](t)h_n(t)\mathrm{d}t \right| \le C \max_{l=1,\dots,m} \|D_2^l f\|_\infty < \infty.$$

Using now Fubini's theorem and (5.17) we find that

$$\begin{aligned}\int_{\mathbb{R}^2} f(x,y)h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)h_n(y)\mathrm{d}y\right) h_n(x)\mathrm{d}x \\ &= \sum_{j=0}^{k} n^{-2j} \int_{\mathbb{R}} \upsilon_j(x)h_n(x)\mathrm{d}x + O(n^{-2k-2}),\end{aligned} \tag{5.18}$$

where the $O(n^{-2k-2})$-term is bounded by $C_k^f n^{-2k-2}$. According to Corollary 5.6(ii), $\upsilon_j \in C_b^\infty(\mathbb{R})$ for each j in $\{0,1,\dots,k\}$, and hence another application of Corollary 3.6 yields that

$$\int_{\mathbb{R}} \upsilon_j(x)h_n(x)\mathrm{d}x = \sum_{l=0}^{k-j} \frac{\xi_l^j(f)}{n^{2l}} + O(n^{-2k+2j-2}),$$

for suitable complex numbers $\xi_0^j(f),\dots,\xi_{k-j}^j(f)$. Inserting these expressions into (5.18) we find that

$$\begin{aligned}\int_{\mathbb{R}^2} f(x,y)h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y &= \sum_{j=0}^{k}\left(\sum_{l=0}^{k-j} \frac{\xi_l^j(f)}{n^{2(l+j)}} + O(n^{-2k-2})\right) + O(n^{-2k-2}) \\ &= \sum_{r=0}^{k} n^{-2r}\left(\sum_{j=0}^{r} \xi_{r-j}^j(f)\right) + O(n^{-2k-2}).\end{aligned}$$

Thus, setting $\beta_r(f) = \sum_{j=0}^{r} \xi_{r-j}^j(f)$, $r = 0,1,\dots,k$, we have obtained the desired expansion. $\square$

For the proof of Theorem 5.9 below we need to extend the asymptotic expansion in Proposition 5.7 to a larger class of functions than $C_b^\infty(\mathbb{R}^2)$.

Proposition 5.8. *Assume that $f:\mathbb{R}^2 \to \mathbb{C}$ is infinitely often differentiable, and polynomially bounded in the sense that*

$$|f(x,y)| \leq C(1+x^2+y^2)^m, \quad ((x,y)\in\mathbb{R}^2),$$

for suitable constants C from $(0,\infty)$ and m in $\mathbb{N}_0$. Then there exists a sequence $(\beta_j(f))_{j\in\mathbb{N}_0}$ of complex numbers, such that

$$\int_{\mathbb{R}^2} f(x,y)h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y = \sum_{j=0}^{k}\frac{\beta_j(f)}{n^{2j}} + O(n^{-2k-2}),$$

for any k in $\mathbb{N}_0$.

Proof. We start by choosing a function φ from $C_c^\infty(\mathbb{R}^2)$, satisfying that

- $\varphi(x,y)\in[0,1]$ for all (x,y) in $\mathbb{R}^2$.
- $\mathrm{supp}(f)\subseteq[-4,4]\times[-4,4]$.
- $\varphi\equiv 1$ on $[-3,3]\times[-3,3]$.

We then write $f = f\varphi + f(1-\varphi)$. Since $f\varphi\in C_c^\infty(\mathbb{R}^2)\subseteq C_b^\infty(\mathbb{R}^2)$, it follows from Proposition 5.7 that there exists a sequence $(\beta_j(f))_{j\in\mathbb{N}_0}$ of complex numbers, such that

$$\int_{\mathbb{R}^2} f(x,y)\varphi(x,y)h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y = \sum_{j=0}^{k}\frac{\beta_j(f)}{n^{2j}} + O(n^{-2k-2}),$$

for any k in $\mathbb{N}_0$. Therefore, it suffices to establish that

$$\int_{\mathbb{R}^2} f(x,y)(1-\varphi(x,y))h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y = O(n^{-2k-2}),$$

for any k in $\mathbb{N}_0$. Note here that $(1-\varphi)\equiv 0$ on $[-3,3]\times[-3,3]$, and that for some positive constant C' we have that

$$|f(x,y)(1-\varphi(x,y))| \leq C(1+x^2+y^2)^m \leq C'(x^{2m}+y^{2m}) \leq C'x^{2m}y^{2m},$$

for all (x,y) outside $[-3,3]\times[-3,3]$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(x,y)(1-\varphi(x,y))h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y \\
&\qquad \leq C'\int_{\mathbb{R}^2\setminus[-3,3]\times[-3,3]} x^{2m}y^{2m}h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y \\
&\qquad \leq 4C'\int_{\mathbb{R}}\int_3^\infty x^{2m}y^{2m}h_n(x)h_n(y)\mathrm{d}y\mathrm{d}x \\
&\qquad = 4C'\left(\int_{\mathbb{R}} x^{2m}h_n(x)\mathrm{d}x\right)\left(\int_3^\infty y^{2m}h_n(y)\mathrm{d}y\right),
\end{aligned}$$

where the second estimate uses symmetry of the function $(x,y) \mapsto x^{2m}y^{2m}h_n(x)h_n(y)$. By Wigner's semi-circle law (for moments)

$$\lim_{n\to\infty}\int_{\mathbb{R}} x^{2m}h_n(x)\mathrm{d}x = \frac{1}{2\pi}\int_{-2}^{2} x^{2m}\sqrt{4-x^2}\mathrm{d}x,$$

and therefore it now suffices to show that

$$\int_3^\infty y^{2m}h_n(y)\mathrm{d}y = O(n^{-2k-2}) \quad \text{for any } k \text{ in } \mathbb{N}_0. \tag{5.19}$$

Recall here that h_n is the spectral density of a $\mathrm{GUE}(n,\frac{1}{n})$ random matrix X_n, so that

$$\begin{aligned}\int_3^\infty y^{2m}h_n(y)\mathrm{d}y &= \mathbb{E}\{\mathrm{tr}_n[(X_n)^{2m}1_{(3,\infty)}(X_n)]\}\\ &= \frac{1}{n}\mathbb{E}\left\{\sum_{j=1}^n \left(\lambda_j^{(n)}\right)^{2m}1_{(3,\infty)}\left(\lambda_j^{(n)}\right)\right\},\end{aligned}$$

where $\lambda_1^{(n)} \le \lambda_2^{(n)} \le \cdots \le \lambda_n^{(n)}$ are the ordered (random) eigenvalues of X_n. Since the function $y \mapsto y^{2m}1_{(3,\infty)}(y)$ is non-decreasing on $\mathbb{R}$, it follows that

$$\frac{1}{n}\sum_{j=1}^n(\lambda_j^{(n)})^{2m}1_{(3,\infty)}(\lambda_j^{(n)}) \le (\lambda_n^{(n)})^{2m}1_{(3,\infty)}(\lambda_n^{(n)}) \le \|X_n\|^{2m}1_{(3,\infty)}(\|X_n\|).$$

Using (Ref. 6, Proposition 6.4) it thus follows that

$$\int_3^\infty y^{2m}h_n(y)\mathrm{d}y \le \mathbb{E}\{\|X_n\|^{2m}1_{(3,\infty)}(\|X_n\|)\} \le \gamma(2m)n\mathrm{e}^{-\frac{n}{2}},$$

for a suitable positive constant $\gamma(2m)$ (not depending on n). This clearly implies (5.19), and the proof is completed. □

Theorem 5.9. *Let ρ_n be the kernel given by* (5.3). *Then for any function f in $C_b^\infty(\mathbb{R}^2)$ there exists a sequence $(\beta_j(f))_{j\in\mathbb{N}_0}$ of complex numbers such that*

$$\int_{\mathbb{R}^2} f(x,y)\rho_n(x,y)\mathrm{d}x\mathrm{d}y = \sum_{j=0}^k \frac{\beta_j(f)}{n^{2j}} + O(n^{-2k-2}),$$

for any k in $\mathbb{N}_0$.

Proof. Using Proposition 5.4 we have that

$$\begin{aligned}\int_{\mathbb{R}^2} f(x,y)\rho_n(x,y)\mathrm{d}x\mathrm{d}y &= \frac{1}{4}\int_{\mathbb{R}^2} f(x,y)\tilde{h}_n(x)\tilde{h}_n(y)\mathrm{d}x\mathrm{d}y\\ &\quad - \int_{\mathbb{R}^2} f(x,y)h_n'(x)h_n'(y)\mathrm{d}x\mathrm{d}y\\ &\quad - \frac{1}{4n^2}\int_{\mathbb{R}^2} f(x,y)h_n''(x)h_n''(y)\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{5.20}$$

and it suffices then to establish asymptotic expansions of the type set out in the theorem for each of the integrals appearing on the right-hand side.

By Fubini's theorem and integration by parts, it follows that

$$\int_{\mathbb{R}^2} f(x,y)h_n'(x)h_n'(y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} \frac{\partial^2}{\partial x \partial y} f(x,y)h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y, \tag{5.21}$$

and since $\frac{\partial^2}{\partial x \partial y} f(x,y) \in C_b^\infty(\mathbb{R}^2)$, Proposition 5.7 yields an asymptotic expansion of the desired kind for this integral. Similarly

$$\int_{\mathbb{R}^2} f(x,y)h_n''(x)h_n''(y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} \frac{\partial^4}{\partial x^2 \partial y^2} f(x,y)h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y, \tag{5.22}$$

where $\frac{\partial^4}{\partial x^2 \partial y^2} f(x,y) \in C_b^\infty(\mathbb{R}^2)$, and another application of Proposition 5.7 yields the desired asymptotic expansion. Finally, using again Fubini's theorem and integration by parts,

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(x,y)\tilde{h}_n(x)\tilde{h}_n(y)\mathrm{d}x\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^2} f(x,y)(h_n(x) - xh_n'(x))(h_n(y) - yh_n'(y))\mathrm{d}x\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^2} f(x,y)[h_n(x)h_n(y) - xh_n'(x)h_n(y) - yh_n'(y)h_n(x) \\
&\qquad + xyh_n'(x)h_n'(y)]\mathrm{d}x\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^2} \Big[f(x,y) + \frac{\partial}{\partial x}(xf(x,y)) \\
&\qquad + \frac{\partial}{\partial y}(yf(x,y)) + \frac{\partial^2}{\partial x \partial y}(xyf(x,y)) \Big] h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^2} \Big[4f(x,y) + 2x\frac{\partial}{\partial x} f(x,y) \\
&\qquad + 2y\frac{\partial}{\partial y} f(x,y) + xy\frac{\partial^2}{\partial x \partial y} f(x,y) \Big] h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{5.23}$$

In the latter integral, the function inside the brackets is clearly a polynomially bounded C^∞-function on $\mathbb{R}^2$, and hence Proposition 5.8 provides an asymptotic expansion of the desired kind. This completes the proof. □

Corollary 5.10. *For any functions f, g in $C_b^\infty(\mathbb{R})$, there exists a sequence $(\beta_j(f,g))_{j \in \mathbb{N}}$ of complex numbers, such that for any k in $\mathbb{N}_0$*

$$\begin{aligned}
\mathrm{Cov}\{\mathrm{Tr}_n[f(X_n)], \mathrm{Tr}_n[g(X_n)]\} &= \int_{\mathbb{R}^2} \left(\frac{f(x) - f(y)}{x - y} \right) \left(\frac{g(x) - g(y)}{x - y} \right) \rho_n(x,y)\mathrm{d}x\mathrm{d}y \\
&= \sum_{j=0}^{k} \frac{\beta_j(f,g)}{n^{2j}} + O(n^{-2k-2}).
\end{aligned} \tag{5.24}$$

Proof. The first equality in (5.24) was established in Proposition 5.2(ii). Appealing then to Theorem 5.9, the existence of a sequence $(\beta_j(f,g))_{j\in\mathbb{N}_0}$ satisfying the second equality will follow, if we establish that the function

$$\Delta f(x,y) = \begin{cases} \dfrac{f(x)-f(y)}{x-y}, & \text{if } x \neq y, \\ f'(x), & \text{if } x = y, \end{cases}$$

belongs to $C_b^\infty(\mathbb{R}^2)$ for any function f from $C_b^\infty(\mathbb{R})$. But this follows from the formula

$$\Delta f(x,y) = \int_0^1 f'(sx+(1-s)y)\mathrm{d}s, \quad ((x,y)\in\mathbb{R}^2),$$

which together with the usual theorem on differentiation under the integral sign shows that Δf is a C^∞-function on $\mathbb{R}^2$ with derivatives given by

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l}\Delta f(x,y) = \int_0^1 f^{(k+l+1)}(sx+(1-s)y)s^k(1-s)^l\mathrm{d}s, \quad ((x,y)\in\mathbb{R}^2),$$

for any k,l in $\mathbb{N}_0$. $\square$

We close this section by giving a short proof of the previously mentioned fact that the measures $\rho_n(x,y)\mathrm{d}x\mathrm{d}y$ converge weakly to the measure $\rho(x,y)\mathrm{d}x\mathrm{d}y$ given by (5.4). As indicated at the end of the Introduction, this fact is well-known in the physics literature (see Ref. 11 and references therein).

Proposition 5.11. *For each n in $\mathbb{N}$, let μ_n denote the measure on $\mathbb{R}^2$ with density ρ_n with respect to Lebesgue measure on $\mathbb{R}^2$. Then μ_n is a probability measure on $\mathbb{R}^2$, and μ_n converges weakly, as $n\to\infty$, to the probability measure μ on $\mathbb{R}^2$ with density*

$$\rho(x,y) = \frac{1}{4\pi^2}\frac{4-xy}{\sqrt{4-x^2}\sqrt{4-y^2}}1_{(-2,2)}(x)1_{(-2,2)}(y),$$

with respect to Lebesgue measure on $\mathbb{R}^2$.

Proof. We prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}\rho_n(x,y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}\rho(x,y)\mathrm{d}x\mathrm{d}y, \tag{5.25}$$

for all z,w in $\mathbb{R}$. Given such z and w, we apply formulas (5.20)–(5.23) to the case where $f(x,y) = \mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}$, and it follows that

$$\begin{aligned}&\int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}\rho_n(x,y)\mathrm{d}x\mathrm{d}y \\ &\quad = \frac{1}{4}\int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}[\tilde{h}_n(x)\tilde{h}_n(y) - 4h_n'(x)h_n'(y) - n^{-2}h_n''(x)h_n''(y)]\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$= \frac{1}{4}\int_{\mathbb{R}^2}[4 + 2\mathrm{i}zx + 2\mathrm{i}wy - zwxy + 4zw - n^{-2}z^2w^2]\mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{4}\int_{\mathbb{R}^2}[(4 + 4zw - n^{-2}z^2w^2) + 2\mathrm{i}zx + 2\mathrm{i}wy - zwxy]\mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}h_n(x)h_n(y)\mathrm{d}x\mathrm{d}y. \tag{5.26}$$

In the case $z = w = 0$, it follows in particular that μ_n is indeed a probability measure, and hence, once (5.25) has been established, so is μ. By linearity the resulting expression in (5.26) may be written as a linear combination of four integrals of tensor products (a function of x times a function of y). Therefore, by Fubini's theorem and Wigner's semi-circle law, it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}\rho_n(x,y)\mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{4}\int_{\mathbb{R}^2}[4 + 4zw + 2\mathrm{i}zx + 2\mathrm{i}wy - zwxy]\mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}h_\infty(x)h_\infty(y)\mathrm{d}x\mathrm{d}y,$$

where $h_\infty(x) = \frac{1}{2\pi}\sqrt{4-x^2}1_{[-2,2]}(x)$. For x in $(-2,2)$ it is easily seen that

$$h'_\infty(x) = \frac{-x}{2\pi\sqrt{4-x^2}}, \quad \text{and} \quad \tilde{h}_\infty(x) := h_\infty(x) - xh'_\infty(x) = \frac{2}{\pi\sqrt{4-x^2}}, \tag{5.27}$$

so in particular h'_∞ and $\tilde{h}_\infty$ are both $\mathcal{L}^1$-functions (with respect to Lebesgue measure). This enables us to perform the calculations in (5.26) in the reversed order and with h_n replaced by h_∞. We may thus deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}\rho_n(x,y)\mathrm{d}x\mathrm{d}y$$
$$= \int_{(-2,2)\times(-2,2)}\mathrm{e}^{\mathrm{i}zx+\mathrm{i}wy}\left[\frac{1}{4}\tilde{h}_\infty(x)\tilde{h}_\infty(y) - h'_\infty(x)h'_\infty(y)\right]\mathrm{d}x\mathrm{d}y. \tag{5.28}$$

Finally it follows from (5.27) and a straightforward calculation that

$$\frac{1}{4}\tilde{h}_\infty(x)\tilde{h}_\infty(y) - h'_\infty(x)h'_\infty(y) = \frac{4-xy}{4\pi^2\sqrt{4-x^2}\sqrt{4-y^2}}, \tag{5.29}$$

for all x, y in $(-2,2)$. Combining (5.28) with (5.29), we have established (5.25). □

6. Asymptotic Expansion for the Two-Dimensional Cauchy Transform

In this section we study in greater detail the asymptotic expansion from Corollary 5.10 in the case where $f(x) = \frac{1}{\lambda - x}$ and $g(x) = \frac{1}{\mu - x}$ for λ, μ in $\mathbb{C}\backslash\mathbb{R}$.

In this setup we put

$$G_n(\lambda,\mu) = \mathrm{Cov}\{\mathrm{Tr}_n[(\lambda\mathbf{1} - X_n)^{-1}], \mathrm{Tr}_n[(\mu\mathbf{1} - X_n)^{-1}]\},$$

where as before X_n is a $\mathrm{GUE}(n, \frac{1}{n})$ random matrix.

Recall from Corollary 5.2, Proposition 5.11 that $\lim_{n\to\infty} G_n(\lambda,\mu) = G(\lambda,\mu)$ for any λ,μ in $\mathbb{C}\backslash\mathbb{R}$, where

$$G(\lambda,\mu) = \int_{\mathbb{R}^2} \left(\frac{(\lambda-x)^{-1} - (\lambda-y)^{-1}}{x-y}\right)\left(\frac{(\mu-x)^{-1} - (\mu-y)^{-1}}{x-y}\right)\rho(x,y)\mathrm{d}x\mathrm{d}y. \tag{6.1}$$

Lemma 6.1. *Let G_n be the Cauchy transform of a* $\mathrm{GUE}(n, \frac{1}{n})$ *random matrix. Then for any λ in $\mathbb{C}\backslash\mathbb{R}$ we have that*

$$\tilde{G}_n(\lambda)^2 - 4G_n'(\lambda)^2 + 4G_n'(\lambda) - \frac{1}{n^2}G_n''(\lambda)^2 = 0,$$

where $\tilde{G}_n(\lambda) = G_n(\lambda) - \lambda G_n'(\lambda)$.

Proof. For λ in $\mathbb{C}\backslash\mathbb{R}$ we put

$$K_n(\lambda) = \tilde{G}_n(\lambda)^2 - 4G_n'(\lambda)^2 + 4G_n'(\lambda) - \frac{1}{n^2}G_n''(\lambda)^2.$$

Observing that $\tilde{G}_n'(\lambda) = -\lambda G_n''(\lambda)$, it follows that for any λ in $\mathbb{C}\backslash\mathbb{R}$

$$\begin{aligned} K_n'(\lambda) &= 2\tilde{G}_n(\lambda)\tilde{G}_n'(\lambda) - 8G_n'(\lambda)G_n''(\lambda) + 4G_n''(\lambda) - \frac{2}{n^2}G_n''(\lambda)G_n'''(\lambda) \\ &= 2G_n''(\lambda)\left[-\lambda G_n(\lambda) + \lambda^2 G_n'(\lambda) - 4G_n'(\lambda) + 2 - \frac{1}{n^2}G_n'''(\lambda)\right] \\ &= 2G_n''(\lambda)\left[-\frac{1}{n^2}G_n'''(\lambda) + (\lambda^2-4)G_n'(\lambda) - \lambda G_n(\lambda) + 2\right] \\ &= 0, \end{aligned}$$

where the last equality follows from Lemma 4.1. We may thus conclude that K_n is constant on each of the two connected components of $\mathbb{C}\backslash\mathbb{R}$. However, for y in $\mathbb{R}$ we have by dominated convergence that

$$|\mathrm{i}yG_n'(\mathrm{i}y)| = \left|y\int_{\mathbb{R}} \frac{1}{(\mathrm{i}y-x)^2}h_n(x)\mathrm{d}x\right| \le \int_{\mathbb{R}} \frac{|y|}{y^2+x^2}h_n(x)\mathrm{d}x \to 0,$$

as $|y| \to \infty$, and similarly $G_n(\mathrm{i}y) \to 0$ and $G_n''(\mathrm{i}y) \to 0$ as $|y| \to \infty$. It thus follows that $K_n(\mathrm{i}y) \to 0$, as $|y| \to \infty$, $y \in \mathbb{R}$, and hence we must have $K_n \equiv 0$, as desired. ∎

Theorem 6.2. *Let X_n be a* GUE$(n, \frac{1}{n})$ *random matrix, and consider for λ, μ in $\mathbb{C}\backslash\mathbb{R}$ the two-dimensional Cauchy transform*:

$$G_n(\lambda, \mu) = \mathrm{Cov}\{\mathrm{Tr}_n[(\lambda\mathbf{1} - X_n)^{-1}], \mathrm{Tr}_n[(\mu\mathbf{1} - X_n)^{-1}]\}.$$

(i) *If $\lambda \neq \mu$, we have that*

$$G_n(\lambda, \mu) = \frac{-1}{2(\lambda-\mu)^2}(\tilde{G}_n(\lambda)\tilde{G}_n(\mu) - (2G_n'(\lambda) - 1)(2G_n'(\mu) - 1) + 1 - \frac{1}{n^2}G_n''(\lambda)G_n''(\mu)),$$

where $G_n(\lambda)$ is the Cauchy transform of X_n at λ, and where $\tilde{G}_n(\lambda) = G_n(\lambda) - \lambda G_n'(\lambda)$.

(ii) *If $\lambda = \mu \in \mathbb{C}\backslash\mathbb{R}$ we have that*

$$\mathbb{V}\{\mathrm{Tr}_n[(\lambda\mathbf{1} - X_n)^{-1}]\} = G_n(\lambda, \lambda) = \frac{1}{4}(\lambda^2 - 4)G_n''(\lambda)^2 - \frac{1}{4n^2}G_n'''(\lambda)^2,$$

with $G_n(\lambda)$ as in (i).

Proof. (i) Assume that $\lambda, \mu \in \mathbb{C}\backslash\mathbb{R}$, and that $\lambda \neq \mu$. Using Corollary 5.2(ii) we find that

$$\begin{aligned} G_n(\lambda, \mu) &= \int_{\mathbb{R}^2} \left(\frac{(\lambda - x)^{-1} - (\lambda - y)^{-1}}{x - y}\right)\left(\frac{(\mu - x)^{-1} - (\mu - y)^{-1}}{x - y}\right)\rho_n(x, y)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^2} \frac{1}{(\lambda - x)(\mu - x)(\lambda - y)(\mu - y)}\rho_n(x, y)\mathrm{d}x\mathrm{d}x \\ &= \frac{1}{(\mu - \lambda)^2}\int_{\mathbb{R}^2}\left(\frac{1}{\lambda - x} - \frac{1}{\mu - x}\right)\left(\frac{1}{\lambda - y} - \frac{1}{\mu - y}\right)\rho_n(x, y)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Using now Proposition 5.4 and Fubini's theorem, it follows that

$$G_n(\lambda, \mu) = \frac{1}{4(\mu - \lambda)^2}((H_n(\lambda) - H_n(\mu))^2 - 4(I_n(\lambda) - I_n(\mu))^2 - \frac{1}{n^2}(J_n(\lambda) - J_n(\mu))^2), \tag{6.2}$$

where e.g.,

$$H_n(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - x}\tilde{h}_n(x)\mathrm{d}x, \quad I_n(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - x}h_n'(x)\mathrm{d}x, \quad \text{and}$$

$$J_n(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - x}h_n''(x)\mathrm{d}x.$$

Note here that by partial integration and (4.1)

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{\lambda - x}\tilde{h}_n(x)\mathrm{d}x &= \int_{\mathbb{R}} \frac{1}{\lambda - x}(h_n(x) - xh_n'(x))\mathrm{d}x \\ &= G_n(\lambda) - \int_{\mathbb{R}} \left(\frac{\lambda}{\lambda - x} - 1\right) h_n'(x)\mathrm{d}x \\ &= G_n(\lambda) + \lambda \int_{\mathbb{R}} \frac{1}{(\lambda - x)^2} h_n(x)\mathrm{d}x = G_n(\lambda) - \lambda G_n'(\lambda) \\ &= \tilde{G}_n(\lambda).\end{aligned}$$

We find similarly that

$$\int_{\mathbb{R}} \frac{1}{\lambda - x} h_n'(x)\mathrm{d}x = G_n'(\lambda), \quad \text{and} \quad \int_{\mathbb{R}} \frac{1}{\lambda - x} h_n''(x)\mathrm{d}x = G_n''(\lambda).$$

Inserting these expressions into (6.2), it follows that

$$\begin{aligned}4(\lambda - \mu)^2 G(\lambda, \mu) &= (\tilde{G}_n(\lambda) - \tilde{G}_n(\mu))^2 - 4(G_n'(\lambda) - G_n'(\mu))^2 \\ &\quad - \frac{1}{n^2}(G_n''(\lambda) - G_n''(\mu))^2 \\ &= \left[\tilde{G}_n(\lambda)^2 - 4G_n'(\lambda)^2 - \frac{1}{n^2}G_n''(\lambda)^2\right] \\ &\quad + \left[\tilde{G}_n(\mu)^2 - 4G_n'(\mu)^2 - \frac{1}{n^2}G_n''(\mu)^2\right] \\ &\quad - 2\tilde{G}_n(\lambda)\tilde{G}_n(\mu) + 8G_n'(\lambda)G_n'(\mu) + \frac{2}{n^2}G_n''(\lambda)G_n''(\mu) \\ &= -4G_n'(\lambda) - 4G_n'(\mu) - 2\tilde{G}_n(\lambda)\tilde{G}_n(\mu) \\ &\quad + 8G_n'(\lambda)G_n'(\mu) + \frac{2}{n^2}G_n''(\lambda)G_n''(\mu),\end{aligned}$$

where the last equality uses Lemma 6.1. We may thus conclude that

$$\begin{aligned}G_n(\lambda, \mu) &= \frac{-1}{2(\lambda - \mu)^2}\Big(\tilde{G}_n(\lambda)\tilde{G}_n(\mu) + 2G_n'(\lambda) + 2G_n'(\mu) \\ &\quad - 4G_n'(\lambda)G_n'(\mu) - \frac{1}{n^2}G_n''(\lambda)G_n''(\mu)\Big) \\ &= \frac{-1}{2(\lambda - \mu)^2}\Big(\tilde{G}_n(\lambda)\tilde{G}_n(\mu) - (2G_n'(\lambda) - 1) \\ &\quad \cdot (2G_n'(\mu) - 1) + 1 - \frac{1}{n^2}G_n''(\lambda)G_n''(\mu)\Big),\end{aligned}$$

which completes the proof of (i).

(ii) Proceeding as in the proof of (i) we find by application of Proposition 5.4 that

$$\begin{aligned}4G_n(\lambda,\lambda) &= 4\int_{\mathbb{R}} \frac{1}{(\lambda-x)^2(\lambda-y)^2}\rho_n(x,y)\mathrm{d}x\mathrm{d}y \\ &= \left(\int_{\mathbb{R}} \frac{\tilde{h}_n(x)}{(\lambda-x)^2}\mathrm{d}x\right)^2 - 4\left(\int_{\mathbb{R}} \frac{h_n'(x)}{(\lambda-x)^2}\mathrm{d}x\right)^2 - \frac{1}{n^2}\left(\int_{\mathbb{R}} \frac{h_n''(x)}{(\lambda-x)^2}\mathrm{d}x\right)^2.\end{aligned} \tag{6.3}$$

By calculations similar to those in the proof of (i), we have here that

$$\int_{\mathbb{R}} \frac{\tilde{h}_n(x)}{(\lambda-x)^2}\mathrm{d}x = \lambda G_n''(\lambda), \quad \int_{\mathbb{R}} \frac{h_n'(x)}{(\lambda-x)^2}\mathrm{d}x = -G_n''(\lambda),$$

$$\int_{\mathbb{R}} \frac{h_n''(x)}{(\lambda-x)^2}\mathrm{d}x = -G_n'''(\lambda),$$

which inserted into (6.3) yields the formula in (ii). ☐

Corollary 6.3. *Consider the coefficients η_j, $j \in \mathbb{N}_0$, in the asymptotic expansion of $G_n(\lambda)$ (cf. Proposition 4.5), and adopt as before the notation $\tilde{\eta}_j(\lambda) = \eta_j(\lambda) - \lambda\eta_j'(\lambda)$.*

(i) *For any distinct λ, μ from $\mathbb{C}\backslash\{0\}$ and k in $\mathbb{N}_0$ we have the asymptotic expansion:*

$$\begin{aligned}G_n(\lambda,\mu) = \frac{1}{2(\lambda-\mu)^2}\bigg[\Gamma_0(\lambda,\mu) + \frac{\Gamma_1(\lambda,\mu)}{n^2} + \frac{\Gamma_2(\lambda,\mu)}{n^4} + \cdots \\ + \frac{\Gamma_k(\lambda,\mu)}{n^{2k}} + O(n^{-2k-2})\bigg],\end{aligned} \tag{6.4}$$

where

$$\Gamma_0(\lambda,\mu) = (2\eta_0'(\lambda)-1)(2\eta_0'(\mu)-1) - \tilde{\eta}_0(\lambda)\tilde{\eta}_0(\mu) - 1,$$

and for l in $\{1,2,\ldots,k\}$

$$\begin{aligned}\Gamma_l(\lambda,\mu) &= 2\eta_l'(\lambda)(2\eta_0'(\mu)-1) + 2\eta_l'(\mu)(2\eta_0'(\lambda)-1) + 4\sum_{j=1}^{l-1}\eta_j'(\lambda)\eta_{l-j}'(\mu) \\ &\quad + \sum_{j=0}^{l-1}\eta_j''(\lambda)\eta_{l-1-j}''(\mu) - \sum_{j=0}^{l}\tilde{\eta}_j(\lambda)\tilde{\eta}_{l-j}(\mu)\end{aligned} \tag{6.5}$$

(*the third term on the right-hand side should be neglected, when $l = 1$*).

(ii) *For any λ in $\mathbb{C}\backslash\mathbb{R}$ and any k in $\mathbb{N}_0$ we have that*

$$G_n(\lambda,\lambda) = \frac{1}{4}\left[\Upsilon_0(\lambda) + \frac{\Upsilon_1(\lambda)}{n^2} + \frac{\Upsilon_2(\lambda)}{n^4} + \cdots + \frac{\Upsilon_k(\lambda)}{n^{2k}} + O(n^{-2k-2})\right],$$

where

$$\Upsilon_0(\lambda) = (\lambda^2-4)\eta_0''(\lambda)^2,$$

and for l in $\{1, 2, \ldots, k\}$

$$\Upsilon_l(\lambda) = (\lambda^2 - 4)\sum_{j=0}^{l} \eta_j''(\lambda)\eta_{l-j}''(\lambda) - \sum_{j=0}^{l-1} \eta_j'''(\lambda)\eta_{l-1-j}'''(\lambda). \tag{6.6}$$

Proof. From the asymptotic expansion of $G_n(\lambda)$ (cf. Proposition 4.5) it follows that

$$2G_n'(\lambda) - 1 = (2\eta_0'(\lambda) - 1) + \frac{2\eta_1'(\lambda)}{n^2} + \cdots + \frac{2\eta_k'(\lambda)}{n^{2k}} + O(n^{-2k-2}),$$

$$G_n''(\lambda) = \eta_0''(\lambda) + \frac{\eta_1''(\lambda)}{n^2} + \cdots + \frac{\eta_k''(\lambda)}{n^{2k}} + O(n^{-2k-2}),$$

$$\tilde{G}_n(\lambda) = \tilde{\eta}_0(\lambda) + \frac{\tilde{\eta}_1(\lambda)}{n^2} + \cdots + \frac{\tilde{\eta}_k(\lambda)}{n^{2k}} + O(n^{-2k-2}),$$

where we also use that the derivatives of the remainder terms are controlled via Lemma 4.2.

Inserting the above expressions (and the corresponding expressions for $2G_n'(\mu) - 1, G_n''(\mu)$ and $\tilde{G}_n(\mu)$) into the formula in Theorem 6.2(i), it is straightforward to establish (i) by collecting the n^{-2l}-terms for each l in $\{0, 1, \ldots, k\}$. The proof of (ii) follows similarly from Theorem 6.2(ii). $\square$

Remark 6.4. Using that $\eta_0(\lambda) = \frac{\lambda}{2} - \frac{1}{2}(\lambda^2 - 4)^{\frac{1}{2}}$ (cf. Proposition 4.5) it follows from Corollary 6.3(i) and a straightforward calculation that for distinct λ and μ from $\mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned} G(\lambda, \mu) = \lim_{n\to\infty} G_n(\lambda, \mu) &= \frac{\Gamma_0(\lambda, \mu)}{2(\lambda - \mu)^2} \\ &= \frac{1}{2(\lambda - \mu)^2}\left(\frac{\lambda\mu - 4}{(\lambda^2 - 4)^{\frac{1}{2}}(\mu^2 - 4)^{\frac{1}{2}}} - 1\right), \end{aligned} \tag{6.7}$$

where $G(\lambda, \mu)$ was initially given by (6.1). If $\lambda = \mu$, it follows similarly from Corollary 6.3(ii) that

$$G(\lambda, \lambda) = \lim_{n\to\infty} G_n(\lambda, \lambda) = \frac{1}{4}(\lambda^2 - 4)\eta_0''(\lambda)^2 = \frac{1}{(\lambda^2 - 4)^2},$$

which may also be obtained by letting λ tend to μ in (6.7).

Using also that $\eta_1(\lambda) = (\lambda^2 - 4)^{-5/2}$, it follows from (6.5) and a rather tedious calculation that

$$\begin{aligned} \Gamma_1(\lambda, \mu) = \frac{(\lambda - \mu)^2}{(\lambda^2 - 4)^{\frac{7}{2}}(\mu^2 - 4)^{\frac{7}{2}}}&(5\lambda\mu^5 + 4\lambda^2\mu^4 + 4\mu^4 - 52\lambda\mu^3 + 3\lambda^3\mu^3 \\ &- 16\mu^2 + 4\lambda^4\mu^2 - 52\lambda^2\mu^2 + 208\lambda\mu + 5\lambda^5\mu \\ &- 52\lambda^3\mu - 16\lambda^2 + 320 + 4\lambda^4). \end{aligned}$$

Inserting this into (6.4) and letting λ tend to μ we obtain that

$$\Upsilon_1(\lambda) = 4(21\lambda^2 + 20)(\lambda^2 - 4)^{-5},$$

which is in accordance with (6.6).

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