PACIFIC JOURNAL OF MATHEMATICS
Vol. 253, No. 2, 2011

FUSION RULES ON A PARAMETRIZED SERIES OF GRAPHS

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A series of pairs of graphs (Γ_k, Γ'_k), $k = 0, 1, 2, \ldots$, has been considered as candidates for dual pairs of principal graphs of subfactors of small Jones index above 4 and it has recently been proved that the pair (Γ_k, Γ'_k) comes from a subfactor if and only if $k = 0$ or $k = 1$. We show that nevertheless there exists a unique fusion system compatible with this pair of graphs for all nonnegative integers k.

1. Introduction

A subfactor $N \subset M$ with finite index and finite depth generates finitely many isomorphism classes of bimodules with four different combinations of left and right coefficients. They form a bigraded fusion category. Its Grothendieck ring forms a *fusion ring* or a *fusion hypergroup*, namely a bigraded $\mathbb{Z}$-algebra $\mathscr{A}$ satisfying:

- $\mathscr{A}$ has a basis given by finitely many irreducible bimodules of four different kinds: $\mathscr{X} = {}_N\mathscr{X}_N \sqcup {}_N\mathscr{X}_M \sqcup {}_M\mathscr{X}_N \sqcup {}_M\mathscr{X}_M$ (we call the labels N and M right or left coefficients, depending on the position).
- An involution $X \in {}_P\mathscr{X}_Q \to \overline{X} \in {}_Q\mathscr{X}_P$ is defined, where $P, Q \in \{N, M\}$.
- A product is defined for a pair of bimodules with "matching" coefficients, namely, for a pair $(X, Y) \in \mathscr{X} \times \mathscr{X}$ such that the right coefficient of X and the left coefficient of Y match, XY is defined. It decomposes as
$$XY = \sum N_{X,Y}^{Z} Z,$$
where the sum is taken over those $Z \in \mathscr{X}$ that have the same left (respectively, right) coefficient as X (respectively, Y), and $N_{X,Y}^{Z} \in \mathbb{N}_0$. Moreover, Frobenius reciprocity holds:
$$N_{X,Y}^{Z} = N_{Z,\overline{Y}}^{X} = N_{\overline{X},Z}^{Y} = N_{\overline{Y},\overline{X}}^{\overline{Z}} = N_{\overline{Z},X}^{\overline{Y}} = N_{Y,\overline{Z}}^{\overline{X}}.$$
- There are identity objects $\mathbf{1}_N \in {}_N\mathscr{X}_N$, $\mathbf{1}_M \in {}_M\mathscr{X}_M$ that act as identity with respect to the product, whenever it is defined.

Asaeda was sponsored in part by NSF grant number DMS-0504199.
MSC2010: 46L37.
Keywords: fusion algebra, operator algebra, subfactor.

The involution extends linearly to define an involution on $\mathcal{A}$. For a fusion ring $\mathcal{A}$, there is a unique weight function $\mu : \mathcal{A} \to \mathbb{R}_{\geq}$ satisfying

$$\mu(\mathbf{1}_N) = \mu(\mathbf{1}_M) = 1,$$
$$\mu(XY) = \mu(X)\mu(Y),$$
$$\mu(X + Z) = \mu(X) + \mu(Z),$$

where $X, Y, Z \in \mathscr{X}$ are with suitable coefficients for each equality, so that XY and $X + Z$ are defined. The (*dual*) *principal graph* of the subfactor encodes partial information of the fusion algebra: namely, the (dual) principal graph has the vertices corresponding to ${}_N\mathscr{X}_N \sqcup {}_N\mathscr{X}_M$ (respectively, ${}_M\mathscr{X}_N \sqcup {}_M\mathscr{X}_M$), with the number of the edges between vertices ${}_NX_N$ and ${}_NY_M$ (respectively, ${}_MX_M$ and ${}_MY_N$) given by $N^Y_{X,{}_NM_M}$ (respectively, $N^Y_{X,{}_MM_N}$.)

On the other hand, one may start with a pair of graphs and may consider if there is a fusion algebra compatible with the fusion constraints determined by the graphs. Such investigation may be used to exclude graphs as (dual) principal graphs of subfactors. For example, type E_7 and D_{2n+1} Dynkin diagrams are proved *not* to be (dual) principal graphs of subfactors, by showing that the fusion constraints given by the graphs give rise to inconsistency in fusion rules [Izumi 1991; Sunder and Vijayarajan 1993]. Note that the existence of a fusion algebra compatible with a given pair of graphs does not imply the existence of a subfactor with given graphs as (dual) principal graphs.

In this paper, we deal with the series of pairs of graphs shown in Figure 1.

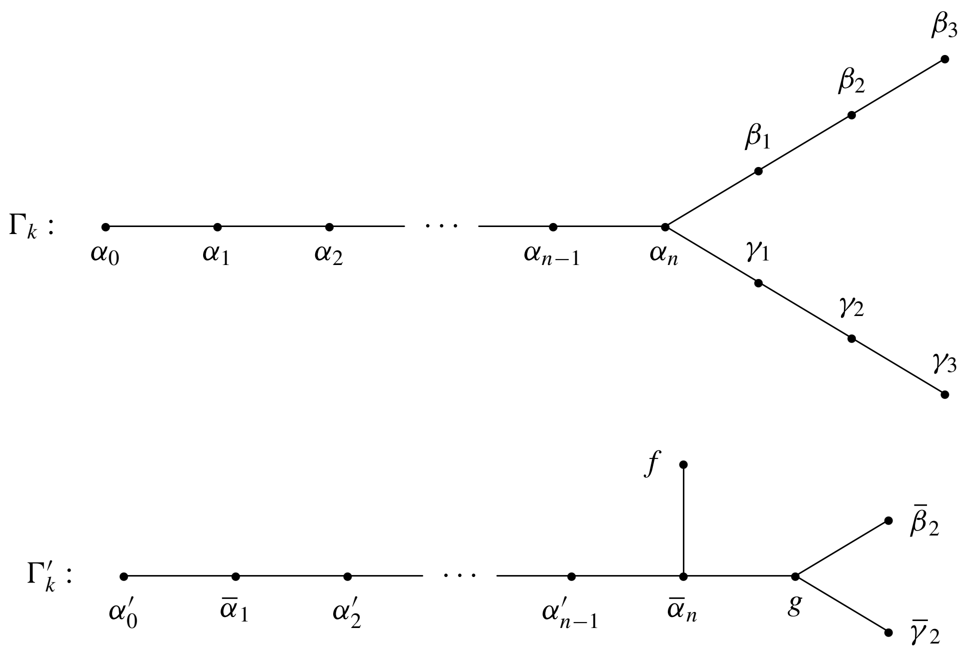

Figure 1. $n = 4k + 3$, $k = 0, 1, \ldots$

These graphs are a part of the list of the graphs that were candidates for (dual) principal graphs of a subfactor with indices between 4 and $3+\sqrt{3}$ given by [Haagerup 1994]. The notation used here is somewhat different from the one used in [Haagerup 1994]. It has been already proved that, for $k = 0, 1$, the graphs Γ_k (respectively, Γ'_k) are (dual) principal graphs of a subfactors [Asaeda and Haagerup 1999; Bigelow et al. 2009], and for $k > 1$, they are not realized as (dual) principal graphs [Asaeda and Yasuda 2009]. In this paper, we prove that, despite that the Γ_k (respectively, Γ'_k) are not principal graphs for $k > 1$, there are still fusion algebras consistent with the graphs, and moreover such fusion algebras are unique for each k.

Theorem 1.1. *Let* $V_{11} := \{\textit{even vertices of } \Gamma_k\}$, $V_{12} := \{\textit{odd vertices of } \Gamma_k\}$, $V_{21} := \{\textit{odd vertices of } \Gamma'_k\}$, $V_{22} := \{\textit{even vertices of } \Gamma'_k\}$, *and* $V := V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22}$. *For each* k, *there is a unique fusion algebra* $\mathcal{A} = \mathbb{Z}\mathcal{X}$, *where*

$$\mathcal{X} = {}_N\mathcal{X}_N \sqcup {}_N\mathcal{X}_M \sqcup {}_M\mathcal{X}_N \sqcup {}_M\mathcal{X}_M$$

is compatible with the graphs Γ_k, Γ'_k. *Namely,*

$$\begin{aligned} {}_N\mathcal{X}_N &= V_{11}, \\ {}_N\mathcal{X}_M &= V_{12}, \\ {}_M\mathcal{X}_N &= V_{21}, \\ {}_M\mathcal{X}_M &= V_{22} \end{aligned}$$

as sets, and

$$N^Y_{X,\alpha_1} (\textit{respectively, } N^Y_{X,\bar{\alpha}_1}) = \begin{cases} 1 & \textit{if } X \textit{ and } Y \textit{ are connected by an edge}, \\ 0 & \textit{else,} \end{cases}$$

$$N^Y_{X,1} = \delta_{X,Y},$$

where $X, Y \in \mathcal{X}$, *and* 1 *denotes identity objects* $1_N = \alpha_0 \in {}_N\mathcal{X}_N$ *or* $1_M = \alpha'_0 \in {}_M\mathcal{X}_M$.

In Section 2 we show that if there is a fusion system compatible with the graphs Γ_k, Γ'_k, it must be unique. In Section 3 we show the existence of such a fusion system.

2. Uniqueness, positivity, and integrality of the fusion rules

In this section we prove that if there is a fusion algebra compatible with the graphs, it is unique. Positivity and integrality of fusion coefficients is derived: we do not impose them in showing uniqueness of the fusion rules.

2A. ***Fusion rules for the even vertices.*** In this subsection we show that there is a unique fusion algebra structure on $\mathcal{A}_1 = \mathbb{Z}_N\mathcal{X}_N$ compatible with the graph Γ_k.

The main issue is to determine the fusion rule among $\beta_1, \beta_3, \gamma_1, \gamma_3$. The rest will follow easily from this.

In the following we assume there is a fusion algebra compatible with (Γ_k, Γ_k'). The involution $\gamma \in V \to \bar{\gamma} \in V$ extends linear to a map on $\mathbb{R}V$. For simplicity, we refer to the objects in $\mathscr{X}$ by corresponding vertices in V. For $X := \sum N_X^Z Z \in \mathbb{R}V$ and $Y \in V$, denote

$$\langle X, Y\rangle = \langle Y, X\rangle := N_X^Y.$$

Observe that $\langle\,\cdot\,,\,\cdot\,\rangle$ expends linearly to define a bilinear form on $\mathbb{R}V$, and

$$\langle XY, Z\rangle = \langle X, Z\bar{Y}\rangle = \langle Y, \bar{X}Z\rangle$$

holds by Frobenius reciprocity. The graph Γ_k encodes the decomposition of $X\alpha_1$ for X in V_{11} as a direct sum of vertices from V_{12} and the decomposition of $Y\bar{\alpha}_1$ as a direct sum of vertices from V_{11}. Let G be the adjacency matrix for (V_{11}, V_{12}), that is,

$$G = (G_{X,Y})_{X\in V_{11}, Y\in V_{12}},$$

where $G_{X,Y}$ is the number of the edges connecting X and Y, namely

$$G_{X,Y} = \langle X\alpha_1, Y\rangle = \langle Y\bar{\alpha}_1, X\rangle.$$

G has dimensions $\left(\frac{n+1}{2}+4\right)\times\left(\frac{n+1}{2}+2\right)$ and can be written as

$$G = \begin{array}{c} \\ \beta_3 \\ \beta_1 \\ \gamma_3 \\ \gamma_1 \\ \alpha_{n-1} \\ \vdots \\ \alpha_2 \\ \alpha_0 \end{array} \begin{array}{c} \begin{array}{ccccccc} \beta_2 & \gamma_2 & \alpha_n & \alpha_{n-2} & \cdots & \cdots & \alpha_1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix} \end{array}. \tag{1}$$

Letting

$$\Delta := \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix},$$

we have

$$\Delta^2 = \begin{pmatrix} GG^t & 0 \\ 0 & G^tG \end{pmatrix}.$$

Put $\mathbb{D} := GG^t$, which acts on $\bar{\mathscr{A}}_1 := \mathbb{R}V_{11}$. We utilize certain eigenvectors of $\mathbb{D}$ to determine the fusion structure of $\mathscr{A}_1$.

Observe from the graph that

$$\begin{aligned} \Delta\beta_1 &= \alpha_n + \beta_2, & \Delta\gamma_1 &= \alpha_n + \gamma_2, \\ \Delta\beta_2 &= \beta_1 + \beta_3, & \Delta\gamma_2 &= \gamma_1 + \gamma_2, \\ \Delta\beta_3 &= \beta_2, & \Delta\gamma_3 &= \gamma_2. \end{aligned}$$

Put

$$\begin{aligned} \xi &= (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3), \\ \eta &= (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3). \end{aligned}$$

Then

$$\begin{aligned} \mathbb{D}\xi &= \Delta^2\xi = \Delta(2\beta_2 - 2\gamma_2) = 2\xi, \\ \mathbb{D}\eta &= \Delta^2\eta = 0. \end{aligned}$$

Let $E(\mathbb{D}, c)$, $c \in \mathbb{R}$, be the eigenspace of the eigenvalue c for $\mathbb{D}$ in $\mathbb{R}(V_{11})$.

Lemma 2.1. $\dim E(\mathbb{D}, 2) = E(\mathbb{D}, 0) = 2$.

Proof. The matrix $\mathbb{D}$ is

$$\mathbb{D} = \begin{array}{c} \\ \beta_3 \\ \beta_1 \\ \gamma_3 \\ \gamma_1 \\ \alpha_{n-1} \\ \alpha_{n-3} \\ \vdots \\ \vdots \\ \alpha_2 \\ \alpha_0 \end{array} \begin{array}{c} \begin{array}{cccccccccc} \beta_3 & \beta_1 & \gamma_3 & \gamma_1 & \alpha_{n-1} & \cdots & \cdots & \cdots & \alpha_2 & \alpha_0 \end{array} \\ \left(\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 2 & 0 & 1 & 1 & 0 & & & & \vdots \\ 0 & 0 & 1 & 1 & 0 & 0 & & & & \vdots \\ 0 & 1 & 1 & 2 & 1 & 0 & & & & \vdots \\ 0 & 1 & 0 & 1 & 2 & 1 & 0 & & & \vdots \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & 0 & 1 & 2 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 2 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{array}\right) \end{array}.$$

Recall that $n = 4k + 3$. Let $\rho_k(x) := \det(tI - \mathbb{D})$ be the characteristic polynomial of $\mathbb{D} = GG^t$. It was proved in [Asaeda 2007] that the characteristic polynomial of G^tG is equal to $(t-2)^2 q_k(t)$, where the polynomials $q_k(t)$, $k \geq 0$, can be defined recursively by

$$\begin{aligned} q_0(t) &= t^2 - 5t + 3, \\ q_1(t) &= (t-1)(t^3 - 8t^2 + 17t - 5), \\ q_k(t) &= (t^2 - 4t + 2)q_{k-1}(t) - q_{k-2}(t), \quad k \geq 2. \end{aligned}$$

Since the matrix G has $2k+6$ rows and $2k+4$ columns, GG^t is a unitary conjugate of $G^tG \oplus 0_2$, where 0_2 is the zero 2×2 matrix. Hence

$$\begin{aligned}\rho_k(t) &= t^2 \det(tI - G^tG)\\ &= t^2(t-2)^2 q_k(t).\end{aligned}$$

The recursion formula for $q_k(t)$ gives $q_k(0) = 2k+3$ and $q_k(2) = (-1)^{(k+1)}(2k+3)$ In particular neither 0 nor 2 is a root of q_k. Hence 0 and 2 are roots of multiplicity 2 in ρ_k. Since $\mathbb{D} = GG^t$ is a symmetric matrix, the dimensions of the eigenspaces for $\mathbb{D}$ for the eigenvalues 0 and 2 are both equal to 2.

Bases of $E(\mathbb{D}, 2)$, $E(\mathbb{D}, 0)$ may be taken as

$$\begin{aligned}E(\mathbb{D}, 2) &:= \operatorname{span}\{x_1, x_2\},\\ E(\mathbb{D}, 0) &:= \operatorname{span}\{y_1, y_2\},\end{aligned}$$

where

$$\begin{aligned}x_1 &:= 2(\alpha_0+\alpha_2) - 2(\alpha_4+\alpha_6) + \cdots + (-1)^k 2(\alpha_{4k}+\alpha_{4k+2})\\ &\qquad\qquad + (-1)^{k+1}(\beta_1+\gamma_1+\beta_3+\gamma_3),\\ x_2 &:= \xi = (\beta_1-\gamma_1) + (\beta_3-\gamma_3),\\ y_1 &:= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1+\gamma_1) - (\beta_3+\gamma_3),\\ y_2 &:= \eta = (\beta_1-\gamma_1) - (\beta_3-\gamma_3).\end{aligned}$$

Assume that we have a fusion algebra compatible with the pair of the graphs (Γ_k, Γ'_k), and let π and π' be the conjugate maps $\gamma \mapsto \bar{\gamma}$ on V_{11} and V_{22}. By the argument used in [Haagerup 1994, pp 28–31], the map π' fixes every element of V_{22}. For π, there are only two possibilities:

Case 1 [Haagerup 1994, Case (b), p 31].

$$\bar{\beta}_1 = \beta_1, \quad \bar{\gamma}_1 = \gamma_1, \quad \bar{\beta}_3 = \gamma_3 \ (\Leftrightarrow \bar{\gamma}_3 = \beta_3).$$

Case 2 [Haagerup 1994, Case (a), p 31]. (This case will be eliminated.)

$$\bar{\beta}_1 = \gamma_1 \ (\Leftrightarrow \bar{\gamma}_1 = \beta_1), \quad \bar{\beta}_3 = \beta_3, \quad \bar{\gamma}_3 = \gamma_3.$$

In both cases, $\bar{\alpha}_{2j} = \alpha_{2j}$ for $j = 0, 1, \ldots, 2k+1$. Note that π extends linearly to $\mathscr{A}_1$ and $\bar{\mathscr{A}}_1 = \mathbb{R}V_{11}$. Let $E(\mathbb{D}, c)_{\mathrm{sc}} := E(\mathbb{D}, c)^\pi$. Observe that

$$c_1\bar{x}_1 + c_2\bar{x}_2 = c_1x_1 + c_2x_2, \quad c_1, c_2 \in \mathbb{R},$$

holds if and only if $c_2 = 0$ in both Cases 1 and 2, and similarly

$$c_1c_1\bar{y}_1 + c_2\bar{y}_2 = c_1y_1 + c_2y_2, \quad c_1, c_2 \in \mathbb{R},$$

if and only if $c_2 = 0$ in both cases. Therefore

$$E(\mathbb{D}, 2)_{sc} = \mathbb{R}x_1,$$
$$E(\mathbb{D}, 0)_{sc} = \mathbb{R}y_1.$$

By the definition of principal graphs, the matrix $\mathbb{D} : \mathbb{R}V_{11} \to \mathbb{R}V_{11}$ corresponds to the fusion rule of the right tensor product by $\alpha\bar{\alpha}$, where $\alpha = \alpha_1$. Therefore

$$\mathbb{D}(\bar{\xi}\xi) = \bar{\xi}\mathbb{D}(\xi) = 2\bar{\xi}\xi,$$
$$\mathbb{D}(\bar{\eta}\eta) = \bar{\eta}\mathbb{D}(\eta) = 0.$$

Hence

$$\bar{\xi}\xi \in E(\mathbb{D}, 2)_{sc} = \mathbb{R}x_1,$$
$$\bar{\eta}\eta \in E(\mathbb{D}, 0)_{sc} = \mathbb{R}y_1.$$

Thus

$$\langle \bar{\xi}\xi, \alpha_0 \rangle = \langle \xi, \xi\alpha_0 \rangle = \langle \xi, \xi \rangle = 4.$$

Hence the coefficient of $\bar{\xi}\xi$ at α_0 is 4. Since $\bar{\xi}\xi \in \mathbb{R}x_1$, we have $\bar{\xi}\xi = 2x_1$. Likewise we obtain $\bar{\eta}\eta = 2y_1$. Noting that

$$\bar{\xi} = \begin{cases} \eta & \text{in Case 1,} \\ -\eta & \text{in Case 2,} \end{cases}$$

we have

$$\begin{cases} \xi\eta = 2y_1, & \eta\xi = 2x_1 \quad \text{in Case 1,} \\ \xi\eta = -2y_1, & \eta\xi = -2x_1 \quad \text{in Case 2,} \end{cases}$$

which completes the proof. □

Lemma 2.2. $\xi^2 = 0$ *and* $\eta^2 = 0$.

Proof. The equality $\mathbb{D}(\xi^2) = \xi\mathbb{D}(\xi) = 2\xi^2$ implies $\xi^2 = c_1x_1 + c_2x_2$ for some $c_1, c_2 \in \mathbb{R}$. Moreover, since $\langle \xi, \eta \rangle = 0$, we have

$$\langle \xi^2, \alpha_0 \rangle = \langle \xi, \bar{\xi}\alpha_0 \rangle = \pm\langle \xi, \eta \rangle = 0.$$

Together with $\langle c_1x_1 + c_2x_2, \alpha_0 \rangle = 2c_1$, $c_1, c_2 \in \mathbb{R}$, we obtain

$$\xi^2 = c_2x_2 = c_2\xi.$$

We show that $c_2 = 0$, using that $\bar{\xi}\xi = 2x_1$ and $\xi\bar{\xi} = 2y_1$ in Cases 1 and 2:

$$\begin{aligned} 4c_2 = \langle c_2\xi, c_2\xi \rangle = \langle \xi^2, \xi^2 \rangle = \langle \bar{\xi}\xi, \xi\bar{\xi} \rangle = 4\langle x_1, y_1 \rangle \\ = (2-2) - (2-2) + \cdots + (-1)^k(2-2) + (1+1-1-1) = 0. \end{aligned}$$

Thus $\xi^2 = 0$. Then $\bar{\xi}^2 = \eta^2 = 0$ for both cases. □

Since $\beta_3-\gamma_3=\frac{1}{2}(\xi-\eta)$, we get

$$\begin{aligned}(\beta_3-\gamma_3)^2&=\tfrac{1}{4}(\xi-\eta)^2\\&=\tfrac{1}{4}(\xi^2+\eta^2-\xi\eta-\eta\xi)\\&=-\tfrac{1}{4}(\xi\eta+\eta\xi)\\&=\begin{cases}-\frac{1}{2}(x_1+y_1) & \text{in Case 1,}\\ \frac{1}{2}(x_1+y_1) & \text{in Case 2.}\end{cases}\end{aligned}$$

Remark 2.3. For k even, that is, $n=3 \pmod 8$ and $k=2l$,

$$\tfrac{1}{2}(x_1+y_1)=2(\alpha_0-\alpha_6+\alpha_8-\alpha_{14}+\alpha_{16}-\cdots+\alpha_{8l})-(\beta_3+\gamma_3)$$

and for k odd, that is, $n=7 \pmod 8$ and $k=2l+1$,

$$\tfrac{1}{2}(x_1+y_1)=2(\alpha_0-\alpha_6+\alpha_8-\alpha_{14}+\alpha_{16}-\cdots+\alpha_{8l}-\alpha_{8l+6})+(\beta_1+\gamma_1).$$

Consider next the sequence of polynomials R_n given recursively by

$$R_0(t)=1,\ R_1(t)=t,\ R_m(t)=tR_{m-1}(t)-R_{m-2}(t),\quad n\ge 2,$$

as in [Haagerup 1994, pp 33–34]. Note that $R_m(t)=U_m(\frac{t}{2})$, where U_m is the m-th Chebyshev polynomial of second kind [Erdélyi et al. 1981, Section 10.11]. Moreover,

$$R_m(2\cos\theta)=\frac{\sin(m+1)\theta}{\sin\theta},\quad 0<\theta<\pi.$$

By the recursion formula for R_n,

$$\begin{aligned}R_j(\Delta)\alpha_0&=\alpha_j,\quad 0\le j\le n,\\R_{n+1}(\Delta)\alpha_0&=\beta_1+\gamma_1,\\R_{n+2}(\Delta)\alpha_0&=\alpha_n+\beta_2+\gamma_2,\\R_{n+3}(\Delta)\alpha_0&=\alpha_{n-1}+\beta_1+\gamma_1+\beta_3+\gamma_3.\end{aligned}$$

Hence

$$\begin{aligned}\beta_3+\gamma_3&=(R_{n+3}(\Delta)-R_{n+1}(\Delta)-R_{n-1}(\Delta))\alpha_0\\&=(R_{4k+6}(\Delta)-R_{4k+4}(\Delta)-R_{4k+2}(\Delta))\alpha_0.\end{aligned}$$

For m even, $R_m(t)$ is an even polynomial in t, thus there is are unique polynomials $(Q_j)_{j=0,1,2,\dots}$ with $\deg(Q_l)=l$, such that

$$Q_j(t^2)=R_{2j}(t),\quad t\in\mathbb{R},\ j=0,1,2,\dots.$$

With this notation, we have

$$\begin{aligned}\beta_3+\gamma_3&=(Q_{2k+3}(\mathbb{D})-Q_{2k+2}(\mathbb{D})-Q_{2k+1}(\mathbb{D}))\alpha_0\\&=(Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\alpha\overline{\alpha}).\end{aligned}$$

Therefore

$$\begin{aligned}(\beta_3-\gamma_3)(\beta_3+\gamma_3) &= (Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\mathbb{D})(\beta_3-\gamma_3)\\ &= \tfrac{1}{2}(Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\mathbb{D})(\xi-\eta).\end{aligned}$$

Since $\mathbb{D}\xi = 2\xi$ and

$$\begin{aligned}Q_m(2) = R_{2j}(\sqrt{2}) &= \frac{\sin(2j+1)\pi/4}{\sin\pi/4}\\ &= \begin{cases} 1 & j=0,1 \pmod 4,\\ -1 & j=2,3 \pmod 4,\end{cases}\end{aligned}$$

we have

$$Q_j(\mathbb{D})\xi = \begin{cases} \xi & j=0,1 \pmod 4,\\ -\xi & j=2,3 \pmod 4.\end{cases}$$

Similarly, since $\mathbb{D}\eta = 0$ and

$$Q_j(0) = R_{2j}(0) = \frac{\sin(2j+1)\pi/2}{\sin\pi/2} = (-1)^j,$$

we have

$$Q_j(\mathbb{D})\eta = (-1)^j\eta, \quad j=0,1,2\dots.$$

Therefore,

$$\begin{aligned}&(Q_{2k+3}(\mathbb{D})-Q_{2k+2}(\mathbb{D})-Q_{2k+1}(\mathbb{D}))\xi\\ &\quad= \begin{cases}(Q_{4l+3}(\mathbb{D})-Q_{4l+2}(\mathbb{D})-Q_{4l+1}(\mathbb{D}))\xi = -\xi & \text{for } k=2l,\ l\in\mathbb{N}_0,\\ (Q_{4l+5}(\mathbb{D})-Q_{4l+4}(\mathbb{D})-Q_{4l+3}(\mathbb{D}))\xi = \xi & \text{for } k=2l+1,\ l\in\mathbb{N}_0,\end{cases}\end{aligned}$$

and in both cases

$$(Q_{2k+3}(\mathbb{D})-Q_{2k+2}(\mathbb{D})-Q_{2k+1}(\mathbb{D}))\eta = -\eta.$$

Hence

$$\begin{aligned}(\beta_3-\gamma_3)(\beta_3+\gamma_3) &= \tfrac{1}{2}(Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\mathbb{D})(\xi-\eta)\\ &= \begin{cases}\tfrac{1}{2}(-\xi+\eta) = \gamma_3-\beta_3 & k \text{ even},\\ \tfrac{1}{2}(\xi+\eta) = \beta_1-\gamma_1 & k \text{ odd}.\end{cases}\end{aligned}$$

Using the contragredient map we get in Case 1 that

$$\begin{aligned}(\beta_3+\gamma_3)(\beta_3-\gamma_3) &= \overline{(\bar\beta_3-\bar\gamma_3)(\bar\beta_3+\bar\gamma_3)}\\ &= \overline{(\gamma_3-\beta_3)(\gamma_3+\beta_3)}\\ &= -\overline{(\beta_3-\gamma_3)(\beta_3+\gamma_3)}\\ &= \begin{cases}-(\bar\gamma_3-\bar\beta_3) = -(\beta_3-\gamma_3) & k \text{ even},\\ -(\bar\beta_1-\bar\gamma_1) = -(\beta_1-\gamma_1) & k \text{ odd},\end{cases}\end{aligned}$$

and in Case 2 (to be eliminated) that

$$\begin{aligned}(\beta_3+\gamma_3)(\beta_3-\gamma_3) &= \overline{(\bar\beta_3-\bar\gamma_3)(\bar\beta_3+\bar\gamma_3)}\\ &= \overline{(\beta_3-\gamma_3)(\beta_3+\gamma_3)}\\ &= \begin{cases}\bar\gamma_3-\bar\beta_3=\gamma_3-\beta_3 & k \text{ even},\\ \bar\beta_1-\bar\gamma_1=\gamma_1-\beta_1 & k \text{ odd}.\end{cases}\end{aligned}$$

Thus in both cases,

$$(\beta_3+\gamma_3)(\beta_3-\gamma_3)=\begin{cases}\gamma_3-\beta_3 & k \text{ even},\\ \gamma_1-\beta_1 & k \text{ odd}.\end{cases}$$

So far, we have obtained the three formulae

$$\text{(A)}\qquad (\beta_3-\gamma_3)^2=\begin{cases}-\frac12(x_1-y_1) & \text{in Case 1},\\ \ \ \frac12(x_1-y_1) & \text{in Case 2},\end{cases}$$

$$\text{(B)}\qquad (\beta_3-\gamma_3)(\beta_3+\gamma_3)=\begin{cases}\frac12(-\xi+\eta)=\gamma_3-\beta_3 & k \text{ even},\\ \frac12(\xi+\eta)=\beta_1-\gamma_1 & k \text{ odd},\end{cases}$$

$$\text{(C)}\qquad (\beta_3+\gamma_3)(\beta_3-\gamma_3)=\begin{cases}\gamma_3-\beta_3 & k \text{ even},\\ \gamma_1-\beta_1 & k \text{ odd}.\end{cases}$$

Next we compute $(\beta_3+\gamma_3)^2$, in order to find β_3^2, γ_3^2, $\beta_3\gamma_3$ and $\gamma_3\beta_3$.

Claim 2.4. *We have*

$$\text{(D)}\quad (\beta_3+\gamma_3)^2=2(c_0\alpha_0+c_1\alpha_2+\cdots+c_{2k+1}\alpha_{4k+2})+c_{2k+2}(\beta_1+\gamma_1)+c_{2k}(\beta_3+\gamma_3),$$

where the c_j are defined by

$$\begin{aligned}&c_0=1,\\ &c_1=c_2=0,\\ &c_j=c_{j-1}+c_{j-2}+c_{j-3}\quad \textit{for } j\ge 3.\end{aligned}$$

Proof. Recall that

$$\begin{aligned}(\beta_3+\gamma_3) &= (Q_{2k+3}-Q_{2k+2}-Q_{2k+1})(\mathbb{D})\alpha_0\\ &= (R_{4k+6}(\Delta)-R_{4k+4}(\Delta)-R_{4k+2}(\Delta))\alpha_0;\end{aligned}$$

thus

$$(\sharp)\qquad (\beta_3+\gamma_3)^2=(R_{4k+6}(\Delta)-R_{4k+4}(\Delta)-R_{4k+2}(\Delta))(\beta_3+\gamma_3).$$

Our strategy of the proof is as follows: First we find a sequence of polynomials (S_j) such that $S_j(\Delta)(\beta_3+\gamma_3)$ is given by a simple formula. Next we rewrite the right-hand side of $(\sharp)$ using the S_j.

From the graph, we obtain

$$\begin{aligned}
R_0(\Delta)(\beta_3+\gamma_3) &= (\beta_3+\gamma_3),\\
R_1(\Delta)(\beta_3+\gamma_3) &= (\beta_2+\gamma_2),\\
R_2(\Delta)(\beta_3+\gamma_3) &= \Delta(\beta_2+\gamma_2)-(\beta_3+\gamma_3) = \beta_1+\gamma_1,\\
R_3(\Delta)(\beta_3+\gamma_3) &= \Delta(\beta_1+\gamma_1)-(\beta_2+\gamma_2) = 2\alpha_n,\\
R_4(\Delta)(\beta_3+\gamma_3) &= 2\Delta\alpha_n-(\beta_1+\gamma_1) = 2\alpha_{n-1}+\beta_1+\gamma_1.
\end{aligned}$$

Define the polynomials $(S_j(t))_{j\geq 3}$ by the recursive formula

$$\begin{aligned}
S_3(t) &= R_3(t),\\
S_4(t) &= R_4(t)-R_2(t),\\
S_j(t) &= tS_{j-1}(t)-S_{j-2}(t), \quad j\geq 5.
\end{aligned}$$

By definition $S_3(\Delta)(\beta_3+\gamma_3)=2\alpha_n$ and $S_4(\Delta)(\beta_3+\gamma_3)=2\alpha_{n-1}$. Since $\alpha_{l-1}=\Delta\alpha_l-\alpha_{l+1}$ for $l=1,2,\dots,n-1$, we easily obtain

$$S_j(\Delta)(\beta_3+\gamma_3)=2\alpha_{n-j+3}$$

for $j=3,4,\dots,n+3$. Next we express the R_j in terms of the S_j.

Lemma 2.5. *For $j\geq 2$,*

$$\begin{aligned}
R_{2j-1} &= d_0S_{2j-1}+d_1S_{2j-3}+\cdots+d_{j-2}S_3+(d_{j-1}-d_{j-2})R_1,\\
R_{2j} &= d_0S_{2j}+d_1S_{2j-2}+\cdots+d_{j-2}S_4+d_{j-1}R_2+d_{j-3}R_0,
\end{aligned}$$

where the d_j satisfy

$$d_{-1}=0, \quad d_0=d_1=1, \quad d_j=d_{j-1}+d_{j-2}+d_{j-3}.$$

Proof. For $j=2$ this is obvious by the definition of the S_j. We proceed with induction. Assume the statement is true for $j\geq 2$. Using the recursion formulae for the R_j and S_j, we have

$$\begin{aligned}
R_{2j+1}(t) &= tR_{2j}(t)-R_{2j-1}(t)\\
&= t(d_0S_{2j}+d_1S_{2j-2}+\cdots+d_{j-2}S_4+d_{j-1}R_2+d_{j-3})\\
&\qquad -(d_0S_{2j-1}+d_1S_{2j-3}+\cdots+d_{j-2}S_3+(d_{j-1}-d_{j-2})R_1)\\
&= d_0S_{2j+1}+d_1S_{2j-1}+\cdots+d_{j-2}S_5+t(d_{j-1}R_2+d_{j-3})-(d_{j-1}-d_{j-2})R_1\\
&= d_0S_{2j+1}+d_1S_{2j-1}+\cdots+d_{j-2}S_5+d_{j-1}(tR_2-R_1)+td_{j-3}-d_{j-2}R_1\\
&= d_0S_{2j+1}+d_1S_{2j-1}+\cdots+d_{j-2}S_5+d_{j-1}S_3+(d_{j-3}-d_{j-2})R_1.
\end{aligned}$$

The last equality was obtained using $S_3 = R_3$, $R_1 = t$, and $d_{j-2}+d_{j-3} = d_j - d_{j-1}$. Likewise we have

$$\begin{aligned} R_{2j+2}(t) &= tR_{2j+1}(t) - R_{2j}(t) \\ &= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 \\ &\qquad + t(d_{j-1}S_3 + (d_j - d_{j-1})R_1) - (d_{j-1}R_2 + d_{j-3}R_0) \\ &= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 + d_{j-1}R_4 \\ &\qquad + (d_j - d_{j-1})(R_2 + R_0) - d_{j-3}R_0 \\ &= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 + d_{j-1}S_4 \\ &\qquad + d_jR_2 + (d_j - d_{j-1} - d_{j-3})R_0 \\ &= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 + d_{j-1}S_4 + d_jR_2 + d_{j-2}R_0, \end{aligned}$$

which completes the proof of Lemma 2.5. □

We return to (♯). Using Lemma 2.5,

$$\begin{aligned} &R_{4k+6} - R_{4k+4} - R_{4k+2} \\ &= d_0S_{4k+6} + (d_1 - d_0)S_{4k+4} + d_{-1}S_{4k+2} + d_0S_{4k} + d_1S_{4k-2} \\ &\qquad + \cdots + d_{2k-2}S_4 + d_{2k-1}R_2 + d_{2k-3}R_0 \\ &= S_{4k+6} + d_0S_{4k} + d_1S_{4k-2} + \cdots + d_{2k-2}S_4 + d_{2k-1}R_2 + d_{2k-3}R_0. \end{aligned}$$

Recall

$$S_j(\Delta)(\beta_3+\gamma_3) = 2\alpha_{n-j+3},$$

$$R_2(\beta_3+\gamma_3) = \beta_1+\gamma_1.$$

Letting $c_0 := 1$, $c_1 = c_2 = 0$ and $c_j := d_{j-3}$ for $j \geq 3$, we obtain Equation (D), which concludes the proof of Claim 2.4. □

Thus far we have obtained the formulae for $(\beta_3-\gamma_3)^2$, $(\beta_3-\gamma_3)(\beta_3+\gamma_3)$, $(\beta_3+\gamma_3)(\beta_3-\gamma_3)$ and $(\beta_3+\gamma_3)^2$ in Equations (A), (B), (C) and (D). This enables us to understand the fusion rules among β_3, γ_3 and their conjugates.

Proposition 2.6. *Case 2 does not occur. Namely, β_1 and γ_1 are self conjugate and $\bar{\beta}_3 = \gamma_3$ if there is a fusion algebra compatible with the graphs Γ_k and Γ'_k.*

Proof. First observe that, by the definition of c_j, $j \geq 0$, in Claim 2.4, it follows that c_j (mod 4) is periodic in j with period 8. The values are:

j (mod 8)	0	1	2	3	4	5	6	7
c_j (mod 4)	1	0	0	1	1	2	0	0

In particular,

$$(*)\qquad \begin{cases} c_{2j} = 1 \pmod 4 & \text{for j even,} \\ c_{2j} = 0 \pmod 4 & \text{for j odd.} \end{cases}$$

In the following we assume Case 2 and derive a contradiction.

First consider the case when k is even. By (B) and (C), we have

$$(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = (\beta_3 + \gamma_3)(\beta_3 - \gamma_3),$$

hence

$$\begin{aligned} \beta_3\gamma_3 = \gamma_3\beta_3 &= \tfrac{1}{2}(\beta_3\gamma_3 + \gamma_3\beta_3) \\ &= \tfrac{1}{4}((\beta_3 + \gamma_3)^2 - (\beta_3 - \gamma_3)^2). \end{aligned}$$

From (A) for Case 2, (D) and Remark 2.3, the coefficient of β_3 in the expansion of $\beta_3\gamma_3$ in irreducible objects is equal to

$$\frac{c_{2k} + 1}{4}.$$

Since k is even, $c_{2k} = 1 \bmod 4$ by ($\star$), so $(c_{2k} + 1)/4$ is not an integer. This implies that Case 2 does not occur if k is even.

Next consider the case when k is odd. From (B) and (C), we get

$$(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = -(\beta_3 + \gamma_3)(\beta_3 - \gamma_3).$$

Hence

$$\begin{aligned} \beta_3^2 = \gamma_3^2 &= \tfrac{1}{2}(\beta_3^2 + \gamma_3^2) \\ &= \tfrac{1}{4}((\beta_3 + \gamma_3)^2 + (\beta_3 - \gamma_3)^2). \end{aligned}$$

From (A) for Case 2, (D) and Remark 2.3, it follows that the coefficient of β_1 in the expansion of β_3^2 in irreducible objects is equal to

$$\frac{c_{2k+2} + 1}{4}.$$

Since k is odd, $c_{2k+2} = 1$ mod 4 by ($\star$), so $(c_{2k} + 1)/4$ is not an integer. This excludes Case 2 for k odd as well. □

In the following we determine all the irreducible decompositions for the products of any two objects in V and show that the coefficients are nonnegative integers. Since we excluded Case 2, we rewrite (A) as

$$(\mathrm{A}')\quad (\beta_3 - \gamma_3)^2 = \begin{cases} -2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l}) - (\beta_3 + \gamma_3) \\ \qquad\qquad k = 2l,\ l = 0, 1, 2, \ldots, \\ -2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l} - \alpha_{8l+6}) + (\beta_1 + \gamma_1) \\ \qquad\qquad k = 2l + 1,\ l = 0, 1, 2, \ldots. \end{cases}$$

Put

$$A := (\beta_3-\gamma_3)^2, \qquad B := (\beta_3-\gamma_3)(\beta_3+\gamma_3),$$
$$C := (\beta_3+\gamma_3)(\beta_3-\gamma_3), \quad D := (\beta_3+\gamma_3)^2.$$

Then

$$\beta_3\gamma_3 = \frac{(D-A)+(B-C)}{4}, \quad \beta_3^2 = \frac{(D+A)+(B+C)}{4},$$
$$\gamma_3\beta_3 = \frac{(D-A)-(B-C)}{4}, \quad \gamma_3^2 = \frac{(D+A)-(B+C)}{4}.$$

We introduce new constants $(f_j)_{j\geq 0}$, $(g_j)_{j\geq 0}$ by

$$\begin{cases} f_j = \frac{1}{2}(c_j+1),\ g_j = \frac{1}{2}(c_j-1) & \text{for } j = 0 \pmod 4, \\ f_j = \frac{1}{2}(c_j-1),\ g_j = \frac{1}{2}(c_j+1) & \text{for } j = 3 \pmod 4, \\ f_j = g_j = \frac{1}{2}c_j & \text{for } j = 1, 2 \pmod 4. \end{cases}$$

Note that $f_j + g_j = c_j$ for all j. Further, from the table on page 268, observe that f_j, g_j is an nonnegative integer for all $j \geq 0$. Here are some values of f_j and g_j:

j	0	1	2	3	4	5	6	7	8	9	10	11	12
f_j	1	0	0	0	1	1	2	3	7	12	22	40	75
g_j	0	0	0	1	0	1	2	4	6	12	22	41	74

For k even, using (A′), (B), (C), (D), we have

$$\frac{D-A}{4} = f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} + \tfrac{1}{4}c_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{4}(c_{2k}-1)(\beta_3+\gamma_3),$$
$$\frac{D+A}{4} = g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{4k+2} + \tfrac{1}{4}c_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{4}(c_{2k}+1)(\beta_3+\gamma_3),$$
$$\frac{B-C}{4} = 0,$$
$$\frac{B+C}{4} = \tfrac{1}{2}(\gamma_3-\beta_3).$$

Since k is even, $c_{2k+2} = 2f_{2k+2} = 2g_{2k+2}$, $c_{2k}+1 = 2f_{2k}$ and $c_{2k}-1 = 2g_{2k}$. Hence we obtain the following theorem:

Theorem 2.7. *For k even,*

$$\beta_3\gamma_3 = \gamma_3\beta_3 = f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} + \tfrac{1}{2}f_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{2}(f_{2k}-1)(\beta_3+\gamma_3),$$
$$\beta_3^2 = g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{4k+2} + \tfrac{1}{2}g_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{2}g_{2k}\beta_3 + \tfrac{1}{2}(g_{2k}+2)\gamma_3,$$
$$\gamma_3^2 = g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{2k+2} + \tfrac{1}{2}g_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{2}(g_{2k}+2)\beta_3 + \tfrac{1}{2}g_{2k}\gamma_3.$$

All the coefficients of irreducible elements are nonnegative integers.

Proof. The only remaining thing to prove is that f_{2k+2} is even, f_{2k} is odd and g_{2j} is even for any j. Since k is even, $c_{2k+2} = 0 \pmod 4$. Thus $f_{2k+2} = \frac{1}{2}c_{2k+2}$ is even. Likewise $c_{2k} = 1 \pmod 4$, thus $f_{2k} = \frac{1}{2}(c_{2k}+1)$ is odd. Now,

$$g_{2j} = \begin{cases} \frac{1}{2}(c_{2j}-1) & \text{for } j \text{ even,} \\ \frac{1}{2}c_{2j} & \text{for } j \text{ odd.} \end{cases}$$

Since $c_{2j} - 1 = 0 \pmod 4$ for j even and $c_{2j} = 0 \pmod 4$ for j odd, we have that g_{2j} is even for any j. □

In the same way, we get for k odd,

$$\begin{aligned}
\frac{D-A}{4} &= f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} + \tfrac{1}{4}(c_{2k+2}+1)(\beta_1+\gamma_1) + \tfrac{1}{4}c_{2k}(\beta_3+\gamma_3), \\
\frac{D+A}{4} &= g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{2k+2} + \tfrac{1}{4}(c_{2k+2}-1)(\beta_1+\gamma_1) + \tfrac{1}{4}c_{2k}(\beta_3+\gamma_3), \\
\frac{B-C}{4} &= \tfrac{1}{2}(\beta_1 - \gamma_1), \\
\frac{B+C}{4} &= 0.
\end{aligned}$$

Since k is odd, $c_{2k+2} + 1 = 2f_{2k+2}$, $c_{2k+2} - 1 = 2g_{2k+2}$ and $c_{2k} = 2f_{2k} = 2g_{2k}$. Hence we get:

Theorem 2.8. *For k odd,*

$$\begin{aligned}
\beta_3\gamma_3 &= f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} \\
&\qquad + \tfrac{1}{2}(f_{2k+2}+1)\beta_1 + \tfrac{1}{2}(f_{2k+2}-1)\gamma_1 + \tfrac{1}{2}f_{2k}(\beta_3+\gamma_3), \\
\gamma_3\beta_3 &= f_0\alpha_0 + f_1\alpha_2 + \cdots + f_{2k+1}\alpha_{4k+2} \\
&\qquad + \tfrac{1}{2}(f_{2k+2}-1)\beta_1 + \tfrac{1}{2}(f_{2k+2}+1)\gamma_1 + \tfrac{1}{2}f_{2k}(\beta_3+\gamma_3), \\
\beta_3^2 = \gamma_3^2 &= g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{4k+2} + \tfrac{1}{2}g_{2k+2}(\beta_1+\gamma_1) + \tfrac{1}{2}g_{2k}(\beta_3+\gamma_3).
\end{aligned}$$

All the coefficients of irreducible elements are nonnegative integers.

Proof. It remains to show that f_{2k+2} is odd and f_{2k} is even. In the proof of Theorem 2.7, it has been already proved that g_{2j} is even for any j.

Since k is odd, $c_{2k+2} = 1 \pmod 4$. Thus $f_{2k+2} - 1 = \frac{1}{2}(c_{2k+2} - 1)$ is even, that is, f_{2k+2} is odd. Likewise $c_{2k} = 0 \pmod 4$, thus $f_{2k} = \frac{1}{2}c_{2k}$ is even. □

Thus far we determined that β_1 and γ_1 are self-conjugate and computed the full irreducible decompositions of β_3 and γ_3, in particular, $\overline{\beta_3} = \gamma_3$. This determines the rest of the fusion rule. Note that the conjugate map π on $\mathbb{Z}V_{11}$ is now determined.

First, for α_{2j}, $j = 0, 1, \ldots, 2k+1$, the right and left multiplication of α_{2j} on any other object from V_{11} is represented by the matrices $Q_j(\mathbb{D})$ and $Q_j(\pi\mathbb{D}\pi)$ respectively.

Claim 2.9. *The entries of the matrices $R_i(\Delta)$ for $i = 0, 1, \dots, 4k+3$ are nonnegative integers. In particular, the entries of the matrices $Q_j(\mathbb{D})$ for $j = 0, 1, \dots, 2k+1$ are nonnegative integers.*

Proof. This immediate from the result in [de la Harpe and Wenzl 1987], which states that when Δ is an adjacency matrix of a graph with norm greater than 2, the matrix $R_i(\Delta)$ has nonnegative integer entries for any i. □

It remains to determine the decomposition of tensor product of β_1 and γ_1 with themselves and β_3 and γ_3.

Since by the graph $\beta_1 = \beta_3\alpha_2$ and $\gamma_1 = \gamma_3\alpha_2$, the fusion among β_3 and γ_3 together with the fusion of α_2 with all the objects determine $\beta_3\beta_1$, $\gamma_3\gamma_1$, $\beta_3\gamma_1$, $\gamma_3\beta_1$ by imposing associativity. Taking the conjugate, we obtain $\beta_1\beta_3$, $\gamma_1\gamma_3$, $\beta_1\gamma_3$, $\gamma_1\beta_3$ as well. Thus $\beta_1^2 = \beta_1\gamma_3\alpha_2$, $\gamma_1^2 = \gamma_1\gamma_3\alpha_2$, $\beta_1\gamma_1 = \beta_1\gamma_3\alpha_2$, $\gamma_1\beta_1 = \gamma_1\beta_3\alpha_2$ are all determined. Since there is no division, subtraction of objects are involved in the process of determining each desired fusion rule, the coefficients are all nonnegative integers.

2B. ***Fusion rules on $_N\mathscr{X}_N \times {}_N\mathscr{X}_M$.*** We identify $_N\mathscr{X}_N$ with V_{11} and $_N\mathscr{X}_M$ with V_{12}. Claim 2.9 implies that $\alpha_i Y$ for i even and any $Y \in V_{12}$ are determined, and so are $X\alpha_i$ for $X \in V_{11}$ and i odd. Thus it remains to obtain $\beta_i Y$ and $\gamma_i Y$, where $i = 1, 3$, $Y = \beta_2$ or γ_2. They are easily determined, since $\beta_2 = \beta_3\alpha_1$, $\gamma_2 = \gamma_3\alpha_1$, and the fusion among β_i, γ_j, $i, j = 1, 3$ are already determined. (Here we used associativity again.) Since the fusion coefficients among the β_i and the γ_j are nonnegative integers and the product of α_1 from the right gives fusion with nonnegative integers, the fusion coefficients of $\beta_i Y$ and $\gamma_i Y$ are nonnegative integers as well.

2C. ***Fusion rules on $_N\mathscr{X}_M \times {}_M\mathscr{X}_N$.*** Let $X \in {}_N\mathscr{X}_M$. Then for j odd,

$$X\bar{\alpha}_j = R_j(\Delta)X.$$

Claim 2.9 implies that $R_j(\Delta)X$ is a linear combination of the objects in $_N\mathscr{X}_N$ with nonnegative integer coefficients. It remains to show that $\beta_2\bar{\beta}_2$, $\beta_2\bar{\gamma}_2$, $\gamma_2\bar{\beta}_2$ and $\gamma_2\bar{\gamma}_2$ also have this property. It is immediate, since $\bar{\beta}_2 = \bar{\alpha}_1\bar{\beta}_3$, $\bar{\gamma}_2 = \bar{\alpha}_1\bar{\gamma}_3$, $\beta_2\bar{\alpha} = \beta_1 + \beta_3$, $\gamma_2\bar{\alpha} = \gamma_1 + \gamma_3$, and all the fusion rules involved have decompositions into simple objects with $\mathbb{Z}_{\geq 0}$-coefficients.

2D. ***Fusion rules on $_M\mathscr{X}_M \times {}_M\mathscr{X}_M$ and $_M\mathscr{X}_M \times {}_M\mathscr{X}_N$.*** Recall that we have identification $_M\mathscr{X}_M = V_{22}$ and $_M\mathscr{X}_N = V_{21}$. Let Δ' be the adjacency matrix for Γ'. Then the fusion rules of the tensor products of the α'_j for $j = 0, 2, \dots, n-1$, as well as the $\bar{\alpha}_k$ for $k = 1, 3, \dots, n-1$ with any objects in $V_{21} \sqcup V_{22}$ are given by the matrices $R_l(\Delta')$, where $l = 0, 1, \dots, n$. Similarly to Claim 2.9, the entries of $R_l(\Delta')$ are all nonnegative integers. Furthermore, using Frobenius reciprocity, this also takes care of the coefficients of the α'_j and $\overline{\alpha}_k$ in the tensor product of two bimodules.

2E. *Fusion rules on $_M\mathscr{X}_M \times {}_M\mathscr{X}_M$.* The remaining issue is to determine the fusion rule among f and g. Observing the Perron–Frobenius weights shows that $\overline{f} = f$, $\overline{g} = g$. Since for j even, each α'_j is self-conjugate as well, $fg = gf$.

Theorem 2.10. *We have*

$$\langle f^2, f\rangle = d_{2k-1}, \quad \langle fg, f\rangle = d_{2k},$$
$$\langle fg, g\rangle = d_{2k+1}, \quad \langle g^2, g\rangle = d_{2k+2},$$

where the d_k are defined as in the proof of Claim 2.4 by

$$d_j = d_{j-1} + d_{j-2} + d_{j-3}, \quad d_{-1} = 0, \quad d_0 = d_1 = 1.$$

Lemma 2.11. *We have*

$$\langle f^2, f\rangle - \langle fg, g\rangle = d_{2k-1} - d_{2k+1},$$
$$\langle fg, f\rangle - \langle g^2, g\rangle = d_{2k} - d_{2k+2},$$
$$\langle fg, g\rangle - \langle g^2, g\rangle = d_{2k+1} - d_{2k+2}.$$

Proof of Lemma 2.11. We use a similar strategy to the proof of Claim 2.4. Let G' be the adjacency matrix for (V_{22}, V_{21}) corresponding to the graph Γ'_k (see Figure 1), and let

$$\Delta' := \begin{pmatrix} 0 & G' \\ G'^t & 0 \end{pmatrix}.$$

Observe that

$$R_0(\Delta')(g - f) = (g - f),$$
$$R_1(\Delta')(g - f) = \overline{\gamma}_2 + \overline{\beta}_2,$$
$$R_2(\Delta')(g - f) = g + f,$$
$$R_3(\Delta')(g - f) = 2\alpha'_n,$$
$$R_4(\Delta')(g - f) = 2\alpha'_{n-1} + f + g,$$

where $\alpha'_j = \overline{\alpha}_j$ for j odd. Then we have

$$S_j(\Delta')(g - f) = 2\alpha'_{n-j+3}$$

for $j = 3, 4, \ldots, n+3$, where the polynomial S_j is defined in the proof of Claim 2.4. On the other hand,

$$g + f = R_{n+1}(\mathbb{D}')\alpha'_0 = R_{4k+4}(\mathbb{D}')\alpha'_0 = Q_{2k+2}(\overline{\alpha}_1\alpha_1).$$

Using Lemma 2.5,

$$\begin{aligned}
&(g+f)(g-f)\\
&\quad=(d_0S_{2(2k+2)}+d_1S_{2(2k+1)}+\cdots+d_{2k}S_4+d_{2k+1}R_2+d_{2k-1}R_0)(\Delta')(g-f)\\
&\quad=(\text{linear combination of the }\alpha'_*)+d_{2k+1}(g+f)+d_{2k-1}(g-f)\\
&\quad=(\text{linear combination of the }\alpha'_*)+(d_{2k+1}+d_{2k-1})g+(d_{2k+1}-d_{2k-1})f.
\end{aligned}$$

Therefore we have

$$(\flat 1)\qquad \begin{aligned}
&\langle (g-f)(g+f),g\rangle=\langle g^2,g\rangle-\langle f^2,g\rangle=d_{2k+1}+d_{2k-1}=d_{2k+2}-d_{2k},\\
&\langle (g-f)(g+f),f\rangle=\langle g^2,f\rangle-\langle f^2,f\rangle=d_{2k+1}-d_{2k-1}.
\end{aligned}$$

We obtain further information by investigating $R_2(\Delta')(g+f)(g-f)$. Note that $R_2(\Delta')(g+f)=2\alpha'_{n-1}+f+3g$. Therefore

$$\begin{aligned}
(\sharp 1)\quad &R_2(\Delta')(g+f)(g-f)\\
&\quad=(2\alpha'_{n-1}+f+3g)(g-f)\\
&\quad=2\alpha'_{n-1}(g-f)+3g^2-f^2-2fg\\
&\quad=(\alpha'_*\text{'s})+2(d_{2k}(g+f)+d_{2k-2}(g-f))+3g^2-f^2-2fg\\
&\quad=(\alpha'_*\text{'s})+2(d_{2k}+d_{2k-2})g+2(d_{2k}-d_{2k-2})f+3g^2-f^2-2fg.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\sharp 2)\quad &R_2(\Delta')(g+f)(g-f)\\
&\quad=R_2(\Delta')(2(d_0\alpha'_2+d_1\alpha'_4+\cdots+d_{2k}\alpha'_{4k+2}))+(d_{2k+1}+d_{2k-1})R_2(\Delta')g\\
&\qquad\qquad+(d_{2k+1}-d_{2k-1})R_2(\Delta')f\\
&\quad=(\alpha'_*\text{'s})+2d_{2k}(f+g)+(d_{2k+1}+d_{2k-1})(\alpha'_{n-1}+f+2g)\\
&\qquad\qquad+(d_{2k+1}-d_{2k-1})(\alpha'_{n-1}+g)\\
&\quad=(\alpha'_*\text{'s})+(2d_{2k}+d_{2k+1}+d_{2k-1})f+(2d_{2k}+3d_{2k+1}+d_{2k-1})g.
\end{aligned}$$

Comparing ($\sharp$1) and ($\sharp$2) we obtain

$$(\flat 2)\qquad \begin{aligned}
3\langle g^2,g\rangle-\langle f^2,g\rangle-2\langle fg,g\rangle&=3d_{2k+1}+d_{2k-1}-2d_{2k-2},\\
3\langle g^2,f\rangle-\langle f^2,f\rangle-2\langle fg,f\rangle&=d_{2k+1}+d_{2k-1}+2d_{2k-2}.
\end{aligned}$$

Combining Equations ($\flat$1) and ($\flat$2), we obtain the statement of the lemma. Note that we use Frobenius reciprocity such as $\langle fg,f\rangle=\langle f^2,g\rangle$, etc. □

The next lemma, together with Lemma 2.11, implies Theorem 2.10.

Lemma 2.12. $\langle g^2,g\rangle=d_{2k+2}$.

Proof. Since $g = \bar{\beta}_2\alpha_1 = \bar{\gamma}_2\alpha_1$,

$$2g = (\bar{\beta}_2 + \bar{\gamma}_2)\alpha_1 = \overline{(\beta_3 + \gamma_3)\alpha_1}\alpha_1 = \bar{\alpha}_1(\beta_3 + \gamma_3)\alpha_1.$$

Also, $\bar{\gamma}_2 = \bar{\gamma}_3\alpha_1 = \bar{\alpha}_1\beta_3$. Therefore

$$\begin{aligned} 4\langle g^2, g\rangle &= \langle \bar{\alpha}_1(\beta_3+\gamma_3)\alpha_1\bar{\alpha}_1(\beta_3+\gamma_3)\alpha_1, \bar{\alpha}_1\beta_3\alpha_1\rangle \\ &= \langle \alpha_1\bar{\alpha}_1(\beta_3+\gamma_3)\alpha_1\bar{\alpha}_1(\beta_3+\gamma_3)\alpha_1\bar{\alpha}_1, \beta_3\rangle \\ &= \langle (\beta_3+\gamma_3)^2(\alpha_1\bar{\alpha}_1)^3, \beta_3\rangle = \langle (\beta_3+\gamma_3)^2, \beta_3(\alpha_1\bar{\alpha}_1)^3\rangle, \end{aligned}$$

where we used

$$\begin{aligned} \alpha_1\bar{\alpha}_1(\beta_3+\gamma_3) &= \beta_1+\beta_3+\gamma_1+\gamma_3 \\ &= \overline{\beta_1+\beta_3+\gamma_1+\gamma_3} = \overline{(\beta_3+\gamma_3)}\alpha_1\bar{\alpha}_1 = (\beta_3+\gamma_3)\alpha_1\bar{\alpha}_1. \end{aligned}$$

A computation using the graph Γ_k gives

$$\beta_3(\alpha_1\bar{\alpha}_1)^3 = 5\beta_3 + 10\beta_1 + 6\alpha_{n-1} + 6\gamma_1 + \alpha_{n-3} + \gamma_3.$$

Using the formula for $(\beta_3+\gamma_3)^2$ given in Claim 2.4, we obtain

$$\begin{aligned} \langle (\beta_3+\gamma_3)^2, \beta_3(\alpha_1\bar{\alpha}_1)^3\rangle &= 8c_{2k} + 12c_{2k+1} + 16c_{2k+2} = 4c_{2k+1} + 8c_{2k+2} + 8c_{2k+3} \\ &= 4c_{2k+2} + 4c_{2k+3} + 4c_{2k+4} = 4c_{2k+5} = 4d_{2k+2}. \end{aligned}$$

Therefore $\langle g^2, g\rangle = d_{2k+2}$. □

2F. *Fusion rules on $_M\mathscr{X}_M \times {}_M\mathscr{X}_N$.* The remaining problem is to determine the fusion rule on $\{f, g\} \times \{\bar{\beta}_2, \bar{\gamma}_2\}$:

$$\begin{aligned} \langle f\bar{\beta}_2, \bar{\beta}_2\rangle = \langle f, \bar{\beta}_2\beta_2\rangle = \langle f, \bar{\alpha}_1\beta_3^2\alpha_1\rangle &= \langle \alpha_1 f\bar{\alpha}_1, \beta_3^2\rangle = \langle \alpha_n\bar{\alpha}_1, \beta_3^2\rangle \\ &= \langle \beta_3^2, \beta_1\rangle + \langle \beta_3^2, \gamma_1\rangle + \langle \beta_3^2, \alpha_{n-1}\rangle. \end{aligned}$$

Theorems 2.7 and 2.8 imply that

$$\langle f\bar{\beta}_2, \bar{\beta}_2\rangle = g_{2k+2} + g_{2k+1}.$$

Both values are nonnegative integers. Similarly we obtain

$$\begin{aligned} &\langle f\bar{\beta}_2, \bar{\gamma}_2\rangle = \langle f\bar{\gamma}_2, \bar{\beta}_2\rangle = f_{2k+2} + f_{2k+1}, \\ &\langle f\bar{\gamma}_2, \bar{\gamma}_2\rangle = g_{2k+2} + g_{2k+1}, \\ &\langle g\bar{\beta}_2, \bar{\beta}_2\rangle = \langle \bar{\beta}_2\alpha_1\bar{\beta}_2, \bar{\beta}_2\rangle = \langle \bar{\alpha}_1\bar{\beta}_3\alpha_1\bar{\alpha}_1\bar{\beta}_3, \bar{\alpha}_1\bar{\beta}_3\rangle = \langle \alpha_1\bar{\alpha}_1\gamma_3\alpha_1\bar{\alpha}_1, \gamma_3\beta_3\rangle \\ &\qquad = \langle \overline{\overline{(\gamma_1+\gamma_3)}\alpha_1\bar{\alpha}_1}, \gamma_3\beta_3\rangle, \end{aligned}$$

$$\begin{aligned} \overline{(\gamma_1+\gamma_3)}\alpha_1\bar{\alpha}_1 &= (\gamma_1+\beta_3)\alpha_1\bar{\alpha}_1 = (\alpha_{n-1}+\beta_1+2\gamma_1+\gamma_3)+\beta_1+\beta_3 \\ &= \alpha_{n-1}+2(\beta_1+\gamma_1)+\gamma_3+\beta_3 = \overline{\alpha_{n-1}+2(\beta_1+\gamma_1)+\gamma_3+\beta_3}. \end{aligned}$$

Thus, using Theorems 2.7 and 2.8 we obtain

$$\langle g\bar{\beta}_2, \bar{\beta}_2\rangle = \begin{cases} f_{2k+1} + 2f_{2k+2} + f_{2k} - 1 & \text{for } k \text{ even,} \\ f_{2k+1} + 2f_{2k+2} + f_{2k} & \text{for } k \text{ odd.} \end{cases}$$

Similarly,

$$\begin{aligned} \langle g\bar{\beta}_2, \bar{\gamma}_2\rangle &= \langle g\bar{\gamma}_2, \bar{\beta}_2\rangle \\ &= \begin{cases} g_{2k+1} + 2g_{2k+2} + g_{2k} + 2 & \text{for } k \text{ even,} \\ g_{2k+1} + 2g_{2k+2} + g_{2k} & \text{for } k \text{ odd,} \end{cases} \quad \langle g\bar{\gamma}_2, \bar{\gamma}_2\rangle = \langle g\bar{\beta}_2, \bar{\beta}_2\rangle. \end{aligned}$$

3. Existence of the fusion algebra

Let $k \in \mathbb{N}_0$, and put $n = 4k+3$ as before. In this section we will reserve the symbols

$$(\alpha_j)_{0\le k\le n}, \quad (\beta_j)_{1\le j\le 3}, \quad (\gamma_j)_{1\le j\le 3}$$

for elements in a certain bigraded $\mathbb{Z}$-algebra $\mathscr{A}$ which we define later. Therefore we relabel the vertices of the graph Γ_k as in Figure 2.

As in Section 2A, let G be the adjacency matrix for $(\Gamma_k^{\text{even}}, \Gamma_k^{\text{odd}})$, where

$$\begin{aligned} \Gamma_k^{\text{even}} &= \{a_0, a_2, \dots, a_{n-1}, b_1, c_1, b_3, c_3\}, \\ \Gamma_k^{\text{odd}} &= \{a_1, a_3, \dots, a_n, b_2, c_2\}. \end{aligned}$$

Set $\mathbb{D} = GG^t$ and

$$\Delta := \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}.$$

Let $(q_k)_{k=0}^{\infty}$ be the sequence of polynomials defined by

$$\begin{aligned} q_0(t) &= t^2 - 5t + 3, \\ q_1(t) &= (t-1)(t^3 - 8t^2 + 17t - 5), \\ q_k(t) &= (t^2 - 4t + 2)q_{k-1}(t) - q_{k-2}(t), \quad k \ge 2, \end{aligned}$$

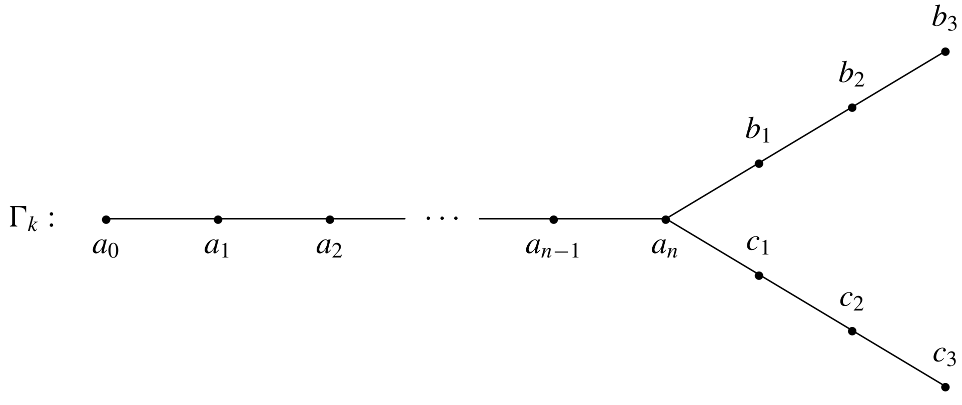

Figure 2

as in Section 2A. Then the characteristic polynomial for $\mathbb{D}$ is

$$\chi_k(t) = t^2(t-2)^2 q_k(t)$$

(see Section 2A). Moreover $q_k(t)$ is a polynomial of degree $2k+2$ with $2k+2$ distinct roots, because by [Asaeda and Yasuda 2009], either $q_k(t)$ or $q_k(t)/(t-1)$ is an irreducible polynomial. The recursion formula for the q_k-polynomials implies

$$q_k(0) = 2k+3,$$
$$q_k(2) = (-1)^{k+1}(2k+3).$$

In particular, 0 and 2 are not roots of q_k. Let $k \in \mathbb{N}_0$ be fixed. Then $\chi_k(t)$ has exactly $2k+4$ distinct roots $(t_j)_{k=1}^{2k+4}$, where $t_1 = 0$, $t_2 = 2$ and $t_3, \dots, t_{2k+4}$ are the roots of $q_k(t)$. Since $\mathbb{D} = GG^t$ is a positive operator, $t_j \geq= 0$ for $1 \leq j \leq 2k+4$.

Lemma 3.1. *Let E_j be the orthogonal projection on the eigenspace of $\mathbb{D}$ corresponding to the eigenvalue t_j, $1 \leq j \leq 2k+4$, and put*

$$\mu_j = \langle E_j a_0, a_0 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $l^2(\Gamma_k^{\text{even}})$. Then

(a) $\sum_{j=1}^{2k+4} \mu_j = 1$,

(b) $\mu_j > 0$ *for* $1 \leq j \leq 2k+4$,

(c) $\mu_1 = \mu_2 = 1/(2k+3)$.

Proof. (a) Since $\mathbb{D}$ is a symmetric matrix, $\sum_{j=1}^{2k+4} E_j = I$, thus $\sum_{j=1}^{2k+4} \mu_j = 1$.

(b) From Section 2A, we have

$$Q_j(\mathbb{D})a_0 = R_{2j}(\Delta)a_0 = a_{2j}, \quad 0 \leq j \leq 2k+1,$$
$$Q_{2k+2}(\mathbb{D})a_0 = R_{4k+4}(\Delta)a_0 = b_1 + c_1,$$
$$Q_{2k+3}(\mathbb{D})a_0 = R_{4k+6}(\Delta)a_0 = b_1 + c_1 + b_3 + c_3.$$

Since $\{a_0, a_2, \dots, a_{4k+2}, b_1+c_1, b_1+c_1+b_3+c_3\}$ is a set of $2k+4$ linearly independent vectors in $l^2(\Gamma_k^{\text{even}})$, and since $(Q_j)_{0 \leq j \leq 2k+3}$ spans the set of polynomials of degree less or equal to $2k+3$, we have

$$P(\mathbb{D})a_0 \neq 0$$

for every nonzero polynomial $P \in \mathbb{R}[x]$ with $\deg(P) \leq 2k+3$. On the other hand, $\mathbb{D}$ is diagonalizable with eigenvalues $(t_j)_{j=1}^{2k+4}$, so

$$E_j = P_j(\mathbb{D}),$$

where

$$P_j(t) = \prod_{i \neq j} \frac{t - t_i}{t_j - t_i}, \quad t \in \mathbb{R},$$

is a polynomial of degree $2k+3$. Hence

$$\mu_j = \langle E_k a_0, a_0\rangle = \|E_j a_0\|^2 > 0, \quad 1 \le j \le 2k+4.$$

(c) From Section 2A, we have

$$\begin{aligned}\operatorname{rg}(E_1) &= E(\mathbb{D}, 0) = \operatorname{span}\{y_1, y_2\},\\ \operatorname{rg}(E_2) &= E(\mathbb{D}, 2) = \operatorname{span}\{x_1, x_2\},\end{aligned}$$

where

$$\begin{aligned}x_1 &:= 2(a_0+a_2) - 2(a_4+a_6) + \cdots + (-1)^k 2(a_{4k}+a_{4k+2}),\\ &\qquad\qquad\qquad\qquad +(-1)^{k+1}(b_1+c_1+b_3+c_3),\\ x_2 &:= (b_1-c_1)+(b_3-c_3),\\ y_1 &:= 2a_0 - 2a_2 + \cdots + 2a_{4k} - 2a_{4k+2} + (b_1+c_1) - (b_3+c_3),\\ y_2 &:= (b_1-c_1)-(b_3-c_3).\end{aligned}$$

Since $y_1 \perp y_2$ and $y_2 \perp a_0$, we get

$$\mu_1 = \langle E_1 a_0, a_0\rangle = \frac{|\langle y_1, a_0\rangle|^2}{\|y_1\|^2} = \frac{1}{2k+3},$$

and similarly,

$$\mu_2 = \langle E_2 a_0, a_0\rangle = \frac{|\langle x_1, a_0\rangle|^2}{\|x_1\|^2} = \frac{1}{2k+3}.$$ □

Corollary 3.2. *Let $(e_{ij})_{i,j=1}^{2k+4}$ be the matrix units of $M_{2k+4}(\mathbb{R})$. Put*

$$\begin{aligned}\mathcal{B} &= \operatorname{span}_{\mathbb{R}}\{e_{11}, e_{12}, e_{21}, e_{22}, e_{33}, e_{44}, \dots, e_{2k+4,2k+4}\}\\ &\cong M_2(\mathbb{R}) \oplus l^\infty(\{3,4,\dots,2k+4\}, \mathbb{R}).\end{aligned}$$

Then $\mathcal{B}$ is a finite dimensional real C^*-algebra and the map $\mu : \mathcal{B} \to \mathbb{R}$ given by

$$\mu(b) := \sum_{j=1}^{2k+4} \mu_j b_{jj}, \quad b = (b_{ij})_{i,j=1}^{2k+4} \in \mathcal{B},$$

is a faithful trace state on $\mathcal{B}$.

Proof. It is clear from Lemma 3.1(a), (b) that μ is a faithful state on $\mathcal{B}$. The trace property

$$\mu(bc) = \mu(cb), \quad b, c \in \mathcal{B},$$

follows from Lemma 3.1(c). □

Lemma 3.3. *Fix $k \in \mathbb{N}_0$, let $\mu : \mathcal{B} \to \mathbb{R}$ be the trace in Corollary 3.2, and put*

$$A := \operatorname{diag}(0, \sqrt{2}, \sqrt{t_3}, \dots \sqrt{t_{2k+4}})),$$

where $t_3, \dots, t_{2k+4}$ *are the roots of* q_k.

(a) *For every even polynomial* $P \in \mathbb{R}[x]$,

$$\mu(P(A)) = \langle P(\Delta)a_0, a_0 \rangle.$$

(b) *Let* $P, Q \in \mathbb{R}[x]$ *be two polynomials, which are either both even or both odd. Then*

$$\mu(P(A)Q(A)) = \langle P(\Delta)a_0, Q(\Delta)a_0 \rangle.$$

(c) *Let* $n = 4k + 3$ *(as usual). Then*

$$R_{n+4}(A) - R_{n+2}(A) - R_n(A) - R_{n-2}(A) = 0.$$

Proof. (a) Choose $Q \in \mathbb{R}[x]$ so that $P(t) = Q(t^2)$. Then

$$\langle P(\Delta)a_0, a_0 \rangle = \langle Q(\mathbb{D})a_0, a_0 \rangle.$$

Let E_j denote the spectral projection of $\mathbb{D}$ corresponding to the eigenvalue t_j, $1 \le j \le 2k + 4$, as before, where $t_1 = 0$ and $t_2 = 2$. Then

$$Q(\mathbb{D}) = \sum_{j=1}^{2k+4} Q(t_j)E_j.$$

Hence

$$\langle Q(\mathbb{D})a_0, a_0 \rangle = \sum_{j=1}^{2k+4} Q(t_j)\langle E_j a_0, a_0 \rangle = \sum_{j=1}^{2k+4} \mu_j Q(t_j) = \mu(Q(A^2)) = \mu(P(A)).$$

(b) Under the assumption on P and Q, the product PQ is an even polynomial. Hence by (a) we have

$$\begin{aligned} \mu(P(A)Q(A)) &= \langle P(\Delta)Q(\Delta)a_0, a_0 \rangle \\ &= \langle P(\Delta)a_0, Q(\Delta)a_0 \rangle. \end{aligned}$$

(c) Put $P = Q = R_{n+4} - R_{n+2} - R_n - R_{n-2}$, which is an odd polynomial. By (b),

$$\mu(P(A)^2) = \|P(\Delta)a_0\|_2^2.$$

From the recursive formula for the polynomials R_j,

$$\begin{aligned} R_{n-2}(\Delta)a_0 &= a_{n-2}, \\ R_n(\Delta)a_0 &= a_n, \\ R_{n+2}(\Delta)a_0 &= a_n + b_2 + c_2, \\ R_{n+4}(\Delta)a_0 &= a_{n-2} + 2a_n + b_2 + c_2 \\ &= (R_{n+2}(A) + R_n(A) + R_{n-2}(A))a_0. \end{aligned}$$

Hence $\mu(P(A)^2) = \|P(\Delta)a_0\|_2^2 = 0$, and since μ is a faithful trace on $\mathcal{B}$, we have $P(A) = 0$. □

Remark 3.4. Since $P = R_{n+4} - R_{n+2} - R_n - R_{n-2}$ is an odd polynomial and $P(A) = 0$, we know that $P(t)$ has at least $n + 4 = 4k + 7$ roots

$$0, \pm\sqrt{2}, \pm\sqrt{t_3}, \dots, \sqrt{t_{2k+4}},$$

which are exactly the distinct roots of $t(t^2 - 2)q_k(t^2)$. Since P and $t(t^2 - 2)q_k(t^2)$ are both monic polynomial of degree $4k + 7$, it follows that

$$(R_{n+4} - R_{n+2} - R_n - R_{n-2})(t) = t(t^2 - 2)q_k(t^2).$$

It is not hard to prove this identity directly by using the recursion formulas for the polynomials $\{q_k\}$ and $\{R_j\}$.

Definition 3.5. Let $k \in \mathbb{N}_0$, $n = 4k + 3$, and let $\mathcal{B}$ and μ be as in Corollary 3.2 and $A = \operatorname{diag}(\sqrt{t_1}, \sqrt{t_2}, \dots, \sqrt{t_{2k+4}}) \in \mathcal{B}$ be as in Lemma 3.3. Let $(f_{ij})_{i,j=1}^2$ be the matrix units in $M_2(\mathbb{R})$, and put

$$V := V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22},$$

where $V_{ij} \subset \mathcal{B} \otimes f_{ij}$, $i, j = 1, 2$, are as follows:

(a) $V_{11} = \{\alpha_0, \alpha_2, \alpha_4, \dots, \alpha_{4k+2}, \beta_1, \gamma_1, \beta_3, \gamma_3\}$, where

$$\begin{aligned}
\alpha_{2j} &= R_{2j}(A) \otimes f_{11}, \quad 0 \le j \le 2k+1,\\
\beta_1 &= \tfrac{1}{2}(R_{n+1}(A) + \sqrt{2k+3}(e_{12} + e_{21})) \otimes f_{11},\\
\gamma_1 &= \tfrac{1}{2}(R_{n+1}(A) - \sqrt{2k+3}(e_{12} + e_{21})) \otimes f_{11},\\
\beta_3 &= \tfrac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) + \sqrt{2k+3}(e_{12} - e_{21})) \otimes f_{11},\\
\gamma_3 &= \tfrac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) - \sqrt{2k+3}(e_{12} - e_{21})) \otimes f_{11}.
\end{aligned}$$

(b) $V_{12} = \{\alpha_1, \alpha_3, \alpha_5, \dots, \alpha_{4k+3}, \beta_2, \gamma_2\}$, where

$$\begin{aligned}
\alpha_{2j+1} &= R_{2j+1}(A) \otimes f_{12}, \quad 0 \le j \le 2k+1,\\
\beta_2 &= \tfrac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{12}) \otimes f_{12},\\
\gamma_2 &= \tfrac{1}{2}((R_{n+2} - R_n)(A) - \sqrt{2(2k+3)}e_{12}) \otimes f_{12}.
\end{aligned}$$

(c) $V_{21} = \{\bar{\alpha}_1, \bar{\alpha}_3, \bar{\alpha}_5, \dots, \bar{\alpha}_{4k+3}, \bar{\beta}_2, \bar{\gamma}_2\}$, where

$$\begin{aligned}
\bar{\alpha}_{2j+1} &= R_{2j+1}(A) \otimes f_{21}, \quad 0 \le j \le 2k+1,\\
\bar{\beta}_2 &= \tfrac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{21}) \otimes f_{21},\\
\bar{\gamma}_2 &= \tfrac{1}{2}((R_{n+2} - R_n)(A) - \sqrt{2(2k+3)}e_{21}) \otimes f_{21}.
\end{aligned}$$

(d) $V_{22} = \{\alpha'_0, \alpha'_2, \ldots, \alpha'_{4k+2}, f, g\}$, where

$$\begin{aligned}\alpha'_j &= R_{2j}(A) \otimes f_{22}, \quad 0 \le j \le 2k+1,\\ f &= \tfrac{1}{2}(R_{n-1} + 2R_{n+1} - R_{n+3})(A) \otimes f_{22},\\ g &= \tfrac{1}{2}(R_{n+3} - R_{n-1})(A) \otimes f_{22}.\end{aligned}$$

(e) The conjugation map $V_{12} \to V_{21}$ and $V_{21} \to V_{12}$ is already defined earlier. For V_{11} and V_{22}, all the elements are defined to be self-conjugate except β_3 and γ_3 which are defined to be conjugate of each other. Note that for every $X \in V_{ij}$, the conjugate $\overline{X}$ is equal to X^* (or X^t, since all the matrices here are real).

(f) Equip $\mathbb{R}V_{ij} \subset \mathcal{B} \otimes f_{ij}$ with inner products given by

$$\langle b \otimes f_{ij}, c \otimes f_{ij}\rangle_\mu := \mu(c^t b) = \mu(bc^t)$$

for every $b, c \in \mathbb{R}V_{ij}$, $i, j = 1, 2$.

Lemma 3.6. *Let $i, j \in \{1, 2\}$. For $X, Y \in V_{ij}$,*

$$\langle X, Y\rangle_\mu = \begin{cases} 1 & \text{if } X = Y,\\ 0 & \text{if } X \ne Y.\end{cases}$$

Proof. Let $(b, c)_\mu := \mu(c^t b) = \mu(bc^t)$, $b, c \in \mathcal{B}$, be the inner product on $\mathcal{B}$ given by μ, and put $\|b\|_\mu (b, b)_\mu^{1/2}$, $b \in \mathcal{B}$.

(a) Case $(i, j) = (1, 1)$. It suffices to show that

$$S_1 := \{R_0(A), R_2(A), \ldots, R_{n+1}(A), (R_{n+3} - R_{n+1} - R_{n-1})(A), e_{12} + e_{21}, e_{12} - e_{21}\}$$

is an orthogonal set in $\mathcal{B}$ and that

$$\begin{aligned}\|R_{2j}(A)\|_\mu^2 &= 1, \quad 0 \le j \le \frac{n-1}{2},\\ \|R_{n+1}(A)\|_\mu^2 &= 2,\\ \|(R_{n+3} - R_{n+1} - R_{n-1})(A)\|_\mu^2 &= 2,\\ \|e_{12} + e_{21}\|_\mu^2 = \|e_{12} - e_{21}\|_\mu^2 &= \frac{2}{2k+3}.\end{aligned}$$

By the definition of μ in Corollary 3.2, it is clear that $e_{12} + e_{21}$ and $e_{12} - e_{21}$ are μ-orthogonal to the remaining matrices in S_1, because $R_j(A)$ is a diagonal matrix for all $j \in \mathbb{N}_0$. Moreover, by Lemma 3.1,

$$\begin{aligned}\langle e_{12} + e_{21}, e_{12} - e_{21}\rangle_\mu &= \mu(e_{11} - e_{22}) = \mu_1 - \mu_2 = 0,\\ \|e_{12} + e_{21}\|_\mu^2 = \|e_{12} - e_{21}\|_\mu^2 &= \mu(e_{11} + e_{22}) = \mu_1 + \mu_2 = \frac{2}{2k+3}.\end{aligned}$$

By Lemma 3.3(b), the remaining part of the proof in the V_{11} case reduces to showing that

$$T_1 := \{R_0(\Delta)a_0, R_2(\Delta)a_0, \ldots, R_{n+1}(\Delta)a_0, (R_{n+3}(\Delta) - R_{n+1}(\Delta) - R_{n-1}(\Delta))a_0\}$$

is an orthogonal set in $l^2(\Gamma_k)$ with

$$\|R_{2j}(\Delta)a_0\|^2 = 1, \quad 0 \le j \le n-1,$$
$$\|R_{n+1}(\Delta)a_0\|^2 = 2,$$
$$\|(R_{n+3} - R_{n+1} - R_{n-1})(\Delta)a_0\|^2 = 2.$$

This follows from the fact that

$$T_1 = \{a_0, a_2, \ldots, a_{n-1}, b_1 + c_1, b_3 + c_3\}.$$

(b) Cases $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$. It suffices to show that

$$S_2 := \{R_1(A), R_3(A), \ldots R_n(A), (R_{n+2} - R_n)(A), e_{12}\}$$

is an orthonormal set in $\mathcal{B}$ and that

$$\|R_{2j+1}(A)\|_\mu^2 = 1, \quad 0 \le j \le \frac{n-1}{2},$$
$$\|(R_{n+2} - R_n)(A)\|_\mu^2 = 2,$$
$$\|e_{12}\|_\mu^2 = \frac{1}{2k+3}.$$

It is easy to check that e_{12} is orthogonal to the remaining elements of S_2 and that $\|e_{12}\|_\mu^2 = (2k+3)^{-1}$ by Lemma 3.3(b). The remaining statement about the set S_2 follow from the fact that

$$T_2 = \{R_1(\Delta)a_0, R_3(\Delta)a_0, \ldots, R_n(\Delta)a_0, (R_{n+2} - R_n)(\Delta)a_0\}$$
$$= \{a_1, a_3, \ldots, a_n, b_2 + c_2\}$$

is an orthonormal set in $l^2(\Gamma_k)$, and from the equalities

$$\|b_2 + c_2\|^2 = 2, \qquad \|a_{2j+1}\|^2 = 1 \quad \text{for } 0 \le j \le \frac{n-1}{2}.$$

(c) Case $(i, j) = (2, 2)$. The statement follows in this case if we can show that

$$S_3 := \Big\{R_0(A), R_2(A), \ldots, R_{n-1}(A), \tfrac{1}{2}(R_{n-1} + 2R_{n+1} - R_{n+3})(A), \tfrac{1}{2}(R_{n+3} - R_{n-1})(A)\Big\}$$

is a μ-orthogonal set in $\mathcal{B}$. By Lemma 3.3(b) this reduces to showing that

$$T_3 := \Big\{ a_0, a_2, \ldots, a_{n-1}, \tfrac{1}{2}(b_1 + c_1 + b_3 + c_3), \tfrac{1}{2}(b_1 + c_1 - b_3 - c_3)\Big\}$$

is an orthogonal set in $l^2(\Gamma_k)$, which is obvious. □

Theorem 3.7. *Let $V = V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22}$ as in Definition 3.5. Then $\mathbb{Z}V \subset M_2(\mathcal{B})$ forms a fusion ring, with coefficients given by*

$$N_{X,Y}^{Z} = \langle XY, Z\rangle_{\mu},$$

where $X \in V_{ij}$, $Y \in V_{jk}$, $Z \in V_{ik}$, $(i, j, k) \in \{1, 2\}^3$, and with units $\alpha_0 \in V_{11}$ and $\alpha_0' \in V_{22}$. Moreover the graph with vertices $V_{11} \sqcup V_{12}$ obtained by right multiplication by $\alpha = \alpha_1$ is Γ_k and the graph with vertices $V_{21} \sqcup V_{22}$ obtained by right multiplication by $\overline{\alpha}$ is Γ_k'.

Proof. By Lemma 3.6, for all $i, j \in \{1, 2\}$, the set V_{ij} is linearly independent in $\mathcal{B} \otimes f_{ij}$. Hence

$$\begin{aligned}
&\dim(\mathbb{R}V_{11}) = |V_{11}| = 2k + 6,\\
&\dim(\mathbb{R}V_{12}) = \dim(\mathbb{R}V_{21}) = \dim(\mathbb{R}V_{22}) = 2k + 4.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{R}V_{11} = \mathcal{B} \otimes f_{11},\\
&\mathbb{R}V_{12} = \operatorname{span}\{e_{12}, e_{22}, e_{33}, \dots, e_{2k+4,2k+4}\} \otimes f_{12},\\
&\mathbb{R}V_{21} = \operatorname{span}\{e_{21}, e_{22}, e_{33}, \dots, e_{2k+4,2k+4}\} \otimes f_{21},\\
&\mathbb{R}V_{22} = \operatorname{span}\{e_{11}, e_{22}, e_{33}, \dots, e_{2k+4,2k+4}\} \otimes f_{22},
\end{aligned}$$

because the four inclusions $\subset$ are obvious, and the right-hand sides have dimensions $2k + 6$ (respectively, $2k + 4$, $2k + 4$, $2k + 4$). Therefore

$$\mathbb{R}V = \mathbb{R}V_{11} \oplus \mathbb{R}V_{12} \oplus \mathbb{R}V_{21} \oplus \mathbb{R}V_{22}$$

forms a bigraded $\mathbb{R}$-algebra, and the conjugation $X \to \bar{X}$ extends by linearity to all of $\mathbb{R}V$ and it is given by transposition of matrices. Moreover, for $X \in V_{ij}$, $Y \in V_{jk}$, $i, j, k \in \{1, 2\}$, we have a unique decomposition

$$XY = \sum_{Z \in V_{ik}} N_{X,Y}^{Z} Z,$$

where by Lemma 3.6,

$$N_{X,Y}^{Z} = \langle XY, Z\rangle_{\mu} \in \mathbb{R}.$$

The identities

$$N_{X,Y}^{Z} = N_{Z,\bar{Y}}^{X} = N_{\bar{X},Z}^{Y} = N_{\bar{Z},X}^{\bar{Y}} = N_{Y,\bar{Z}}^{\bar{X}}$$

are now a simple consequence of the fact that μ is a trace state on the real C^*-algebra $\mathcal{B}$, so in particular

$$\begin{aligned}
\mu(b) &= \mu(b^t), \quad b \in \mathcal{B},\\
\mu(bc) &= \mu(cb), \quad b, c \in \mathcal{B}.
\end{aligned}$$

It remains to prove that $N_{X,Y}^Z \in \mathbb{N}_0$ and that multiplication from the right by $\alpha = \alpha_1$ (respectively, $\overline{\alpha}$) on V_{11} (respectively, V_{22}) generates the graph Γ_k (respectively, Γ'_k).

Lemma 3.8. *Let* $\alpha = \alpha_1$.

(a) *For* $X \in V_{11}$, $Y \in V_{12}$,

$$\langle X\alpha, Y\rangle_\mu = \langle X, Y\overline{\alpha}\rangle_\mu \in \mathbb{N}_0,$$

and $(\langle X\alpha, Y\rangle_\mu)_{X\in V_{11}, Y\in V_{12}}$ *is the adjacency matrix* G_k *for* Γ_k.

(b) *For* $X \in V_{22}$, $Y \in V_{21}$,

$$\langle X\overline{\alpha}, Y\rangle_\mu = \langle X, Y\alpha\rangle_\mu \in \mathbb{N}_0,$$

and $(\langle X\overline{\alpha}, Y\rangle_\mu)_{X\in V_{22}, Y\in V_{21}}$ *is the adjacency matrix* G'_k *for* Γ'_k.

Proof. This follows from simple computations using Definition 3.5, Lemma 3.6, the recursion formula

$$(\star) \qquad tR_n(t) = R_{n+1}(t) + R_{n-1}(t), \quad n \geq 1,$$

and the identity from Lemma 3.3(c)

$$(\star\star) \qquad R_{n+4}(A) - R_{n+2}(A) - R_n(A) - R_{n-2}(A) = 0.$$

(a) It follows immediately from $(\star)$ that for $1 \leq j \leq 2k+1$,

$$\alpha_{2j}\alpha = \alpha_{2j+1} + \alpha_{2j-1},$$

which shows that $\alpha_{2j} \in V_{11}$ is connected to α_{2j+1} and α_{2j-1} in V_{12} (with simple edges) and not connected to any other $Y \in V_{12}$. To prove that we recover the graph Γ_k this way we just have to check that $\alpha_0\alpha = \alpha_1$, which is obvious, and that $\beta_1\alpha = \alpha_n + \beta_2$ and $\beta_3\alpha = \beta_2$. The last equality follows from

$$\begin{aligned}
\beta_3\alpha &= \tfrac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) + \sqrt{2k+3}(e_{12}+e_{21}))A) \otimes f_{12} \\
&= \tfrac{1}{2}(R_{n+4} - 2R_n - R_{n-2})(A) + \sqrt{2(2k+3)}e_{12}) \otimes f_{12} \\
&= \tfrac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{12}) \otimes f_{12} \\
&= \beta_2,
\end{aligned}$$

where we used $(\star)$ and $(\star\star)$ and the fact that $e_{12}A = \sqrt{2}e_{12}$, $e_{21}A = 0$. The proof of $\beta_1\alpha = \alpha_n + \beta_2$ is similar.

(b) To recover the graph Γ_k from $V_{22} \sqcup V_{21}$, it suffices to prove that

$$\begin{aligned}
\alpha_0'\overline{\alpha} &= \overline{\alpha}_1,\\
\alpha_{2j}'\overline{\alpha} &= \overline{\alpha}_{2j+1} + \overline{\alpha}_{2j-1}, \quad 1 \le j \le 2k+1,\\
f\overline{\alpha} &= \overline{\alpha}_n,\\
g\overline{\alpha} &= \overline{\alpha}_n + \overline{\beta}_2 + \overline{\gamma}_2.
\end{aligned}$$

The first two are obvious. A computation proves $f\overline{\alpha} = \overline{\alpha}_n$:

$$\begin{aligned}
f\overline{\alpha} &= \tfrac{1}{2}((R_{n-1}(A) + 2R_{n+1}(A) - R_{n+3}(A))A \otimes f_{21}\\
&= \tfrac{1}{2}(R_{n-2} + 3R_n + R_{n+2} - R_{n+4})(A) \otimes f_{21}\\
&= \tfrac{1}{2} \cdot 2R_n(A) \otimes f_{21}\\
&= \overline{\alpha}_n,
\end{aligned}$$

where we again used ($\star$) and ($\star\star$). The formula for $g\overline{\alpha}$ is obtained similarly. □

Lemma 3.9. *Put*

$$\xi := (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3).$$

Then

$$\overline{\xi} := (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3),$$

and

$$\begin{aligned}
\tfrac{1}{2}\xi\overline{\xi} &= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3),\\
\tfrac{1}{2}\overline{\xi}\xi &= 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k 2(\alpha_{4k} + \alpha_{4k+2})\\
&\qquad + (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3).
\end{aligned}$$

Proof. Clearly $\overline{\xi} = (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3)$. By Lemma 3.8, the linear maps

$$\begin{aligned}
R_\alpha &: \mathbb{R}V_{11} \to \mathbb{R}V_{12},\\
R\overline{\alpha} &: \mathbb{R}V_{12} \to \mathbb{R}V_{11}
\end{aligned}$$

obtained by right multiplication by α (respectively, by $\overline{\alpha}$) have the matrices G^t (respectively, G) expressed with respect to bases V_{11} for $\mathbb{R}V_{11}$ and V_{11} for $\mathbb{R}V_{12}$. Hence

$$R_{\alpha\overline{\alpha}} := R\overline{\alpha}R_\alpha : \mathbb{R}V_{11} \to \mathbb{R}V_{12}$$

has the matrix $\mathbb{D} = GG^t$ with respect to the basis V_{11} for $\mathbb{R}V_{11}$. We can now argue exactly as in Case 1 of Section 2A to get

$$\begin{aligned}
\xi\overline{\xi} &\in E(\mathbb{D}, 0)_{sc} = \mathbb{R}y_1,\\
\overline{\xi}\xi &\in E(\mathbb{D}, 2)_{sc} = \mathbb{R}x_1,
\end{aligned}$$

where

$$\begin{aligned} y_1 &= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3), \\ x_1 &= 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k 2(\alpha_{4k} + \alpha_{4k+2}), \\ &\qquad + (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3). \end{aligned}$$

Since $\langle \xi\bar{\xi}, \alpha_0\rangle_\mu = \langle \bar{\xi}\xi, \alpha_0\rangle_\mu = \langle \xi, \xi\rangle_\mu = 4$ and $\langle y_1, \alpha_0\rangle_\mu = \langle x_1, \alpha_0\rangle_\mu = 2$, it follows that $\xi\bar{\xi} = 2y_1$ and $\bar{\xi}\xi = 2x_1$. □

End of proof of Theorem 3.7. It remains to prove that $N_{X,Y}^Z \in \mathbb{N}_0$ for all $X \in V_{ij}$, $Y \in V_{jk}$ and $Z \in V_{ik}$, $(i, j \in \{1, 2, 3\})$.Having established the formulas for $\xi\bar{\xi}$ and $\bar{\xi}\xi$ in Lemma 3.8, the proof that $N_{X,Y}^Z \in \mathbb{N}_0$ can be obtained from Section 2: Using

$$N_{X,Y}^Z = N_{Z,\bar{Y}}^X = N_{\bar{X},Z}^Y,$$

if X, Y or Z is one of the elements $(\alpha_j)_{0\le j\le n}$, $(\alpha'_j)_{0\le j\le n}$ (where $\alpha'_{2k+1} = \bar{\alpha}_{2k+1}$), then $N_{X,Y}^Z$ is an entry of the matrix $R_j(\Delta)$ or $R_j(\Delta')$, which is a nonnegative integer by [de la Harpe and Wenzl 1987]. In the remaining cases, X, Y and Z are compatible and come from the list

$$\beta_1,\ \gamma_1,\ \beta_3,\ \gamma_3,\ \beta_2,\ \gamma_2,\ \bar{\beta}_2,\ \bar{\gamma}_2,\ f,\ g.$$

For $X, Y, Z \in \{\beta_1, \gamma_1, \beta_3, \gamma_3\}$, we have $N_{X,Y}^Z \in \mathbb{N}_0$ by Theorems 2.7 and 2.8, and the remark at the end of Section 2A. The case $X, Y, Z \in \{f, g\}$ is treated in Theorem 2.10 and the remaining cases can easily be reduced to these two cases by using $\beta_2 = \beta_3\alpha$ and $\gamma_2 = \gamma_3\alpha$ (see Sections 2B and 2F). □

Remark 3.10. From Definition 3.5, we have

$$\begin{aligned} \xi &= (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3) = 2\sqrt{2k+3}e_{12} \otimes f_{11}, \\ \bar{\xi} &= (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3) = 2\sqrt{2k+3}e_{21} \otimes f_{11}. \end{aligned}$$

Thus

$$\begin{aligned} \xi\bar{\xi} &= 4(2k+3)e_{11} \otimes f_{11}, \\ \bar{\xi}\xi &= 4(2k+3)e_{22} \otimes f_{11}. \end{aligned}$$

Since $A = \mathrm{diag}(0, \sqrt{2}, \sqrt{t_3}, \ldots, \sqrt{t_{2k+4}})$, where $t_3, \ldots, t_{2k+4}$ are the distinct roots of $q_k(t)$, and since $0, 2 \notin \{t_3, \ldots, t_{2k+4}\}$, the maps e_{11} and e_{22} are the projections on the eigenspaces for A with eigenvalues 0 and 2, respectively. Using $q_k(0) = 2k+3$ and $q_k(2) = (-1)^{k+1}(2k+3)$ gives

$$\begin{aligned} (2 - A^2)q_k(A^2) &= 2(2k+3)e_{11}, \\ A^2 q_k(A^2) &= (-1)^{k+1}(2k+3)e_{22}, \end{aligned}$$

because the polynomial $(2-t)q_k(t)$ vanishes at $t = 2$ and $t = t_j$, $3 \le j \le 2k+4$, and has the value $2(2k+3)$ at $t = 0$. Similarly $tq_k(t)$ vanishes at $t = 0$ and

$t = t_j$, $3 \le j \le 2k+4$, and has the value $(-1)^{k+1}2(2k+3)$ at $t = 2$. Hence the two identities

$$\xi\bar{\xi} = 2(2 - A^2)q_k(A^2) \otimes f_{11} = 2(1_N - \alpha\bar{\alpha})q_k(\alpha\bar{\alpha}),$$
$$\bar{\xi}\xi = (-1)^{k+2}2A^2 q_k(A^2) \otimes f_{11} = (-1)^{k+2}2\alpha\bar{\alpha}q_k(\alpha\bar{\alpha})$$

hold, where $1_N = \alpha_0$ and $\alpha = \alpha_1$. Let Q_j denote as usual the polynomial for which $R_{2j}(t) = Q_j(t^2)$, $t \in \mathbb{R}$. Then by Definition 3.5,

$$\begin{aligned} \alpha_{2j} &= Q_j(\alpha\bar{\alpha}), \\ \beta_1 + \gamma_1 &= Q_{2k+2}(\alpha\bar{\alpha}), \\ \beta_3 + \gamma_3 &= (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\alpha\bar{\alpha}). \end{aligned}$$

Hence a more direct proof of Lemma 3.8 can be obtained if the two polynomial identities hold:

$$\begin{aligned} r_k &= (2Q_0 - 2Q_1 + \cdots + 2Q_{2k} - 2Q_{2k+1}) + (Q_{2k+1} + 2Q_{2k+2} - Q_{2k+3}), \\ s_k &= 2(Q_0 + Q_2) - 2(Q_2 + Q_4) + \cdots + (-1)^k 2(Q_{2k} + Q_{2k+1}) \\ &\qquad + (-1)^{k+1}(Q_{2k+3} - Q_{2k+1}), \end{aligned}$$

where

$$r_k(t) = (2 - t)q_k(t), \quad s_k(t) = (-1)^{k+1} t q_k(t).$$

These two polynomials identities are actually true, and they can be proved using the recursion formulas for $(q_k)_{k=0}^{\infty}$ and $(R_j)_{j=0}^{\infty}$. □

Acknowledgements

Part of the work on this paper was conducted while both authors were visiting the Department of Mathematics at Universität Münster in April 2009. The authors wish to thank the Department and especially Joachim Cuntz for their hospitality.

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Received September 5, 2010. Revised September 8, 2010.

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