Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT•

Journal of Functional Analysis 209 (2004) 332–366

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Invariant subspaces of the quasinilpotent DT-operator

Ken Dykema[a,*,1] and Uffe Haagerup[b,2]

[a] Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA
[b] Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark

Received 15 January 2003; revised 2 April 2003; accepted 2 April 2003

Communicated by D. Voiculescu

Abstract

In [4] we introduced the class of DT-operators, which are modeled by certain upper triangular random matrices, and showed that if the spectrum of a DT-operator is not reduced to a single point, then it has a nontrivial, closed, hyperinvariant subspace. In this paper, we prove that also every DT-operator whose spectrum is concentrated on a single point has a nontrivial, closed, hyperinvariant subspace. In fact, each such operator has a one-parameter family of them. It follows that every DT-operator generates the von Neumann algebra $L(\mathbf{F}_2)$ of the free group on two generators.

MSC: 47A15; 46L54

1. Introduction

Let $\mathscr{H}$ be a separable, infinite dimensional Hilbert space and let $\mathscr{B}(\mathscr{H})$ be the algebra of bounded operators on $\mathscr{H}$. Let $A \in \mathscr{B}(\mathscr{H})$. An *invariant subspace* of A is a subspace $\mathscr{H}_0 \subseteq \mathscr{H}$ such that $A(\mathscr{H}_0) \subseteq \mathscr{H}_0$, and a *hyperinvariant subspace* of A is a subspace $\mathscr{H}_0$ of $\mathscr{H}$ that is invariant for every operator $B \in \mathscr{B}(\mathscr{H})$ that commutes with A. A subspace of $\mathscr{H}$ is said to be *nontrivial* if it is neither $\{0\}$ nor $\mathscr{H}$ itself. The

*Corresponding author.
E-mail addresses: kdykema@math.tamu.edu (K. Dykema), haagerup@imada.sdu.dk (U. Haagerup).
[1] Supported in part by NSF Grant DMS-0070558.
[2] Affiliated with MaPhySto, Centre for Mathematical Physics and Stochastics, which is funded by a grant from The Danish National Research Foundation.

0022-1236/03/$ - see front matter
doi:10.1016/S0022-1236(03)00167-8

famous *invariant subspace problem* for Hilbert space asks whether every operator in $\mathcal{B}(\mathcal{H})$ has a closed, nontrivial, invariant subspace, and the *hyperinvariant subspace problem* asks whether every operator in $\mathcal{B}(\mathcal{H})$ that is not a scalar multiple of the identity operator has a closed, nontrivial, hyperinvariant subspace.

On the other hand, if $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ is *affiliated to* $\mathcal{M}$ if the projection p from $\mathcal{H}$ onto $\mathcal{H}_0$ belongs to $\mathcal{M}$. It is not difficult to show that every closed, hyperinvariant subspace of A is affiliated to the von Neumann algebra, $W^*(A)$, generated by A. The question of whether every element of a von Neumann algebra $\mathcal{M}$ has a nontrivial invariant subspace affiliated to $\mathcal{M}$ is called the invariant subspace problem *relative to* the von Neumann algebra $\mathcal{M}$.

In [3], we began using upper triangular random matrices to study invariant subspaces for certain operators arising in free probability theory, including Voiculescu's circular operator. In the sequel [4], we introduced the DT-operators; these form a class of operators including all those studied in [3]. (We note that the DT-operators were defined in terms of approximation by upper triangular random matrices, and have been shown in [5] to solve a maximization problem for free entropy.) We showed that DT-operators are decomposable in the sense of Foiaş, which entails that those DT-operators whose spectra contain more than one point have nontrivial, closed, hyperinvariant subspaces. In this paper, we show that also DT-operators whose spectra are singletons have (a continuum of) closed, nontrivial, hyperinvariant subspaces. These operators are all scalar translates of scalar multiples of a single operator, the DT$(\delta_0, 1)$-operator, which we will denote by T.

The free group factor $L(\mathbf{F}_2) \subseteq \mathcal{B}(\mathcal{H})$ is generated by a semicircular element X and a free copy of $L^\infty[0,1]$, embedded via a normal $*$-homomorphism $\lambda : L^\infty[0,1] \to L(\mathbf{F}_2)$ such that $\tau \circ \lambda(f) = \int_0^1 f(t)\,dt$, where τ is the tracial state on $L(\mathbf{F}_2)$. Thus X and the image of λ are free with respect to τ and together they generate $L(\mathbf{F}_2)$. As proved in [4, Section 4], the DT$(\delta_0, 1)$–operator T can be obtained by using projections from $\lambda(L^\infty[0,1])$ to cut out the "upper triangular part" of X; in the notation of [4, Section 4], $T = \mathcal{UT}(X, \lambda)$. It is clear from this construction that each of the subspaces $\mathcal{H}_t = \lambda(1_{[0,t]})\mathcal{H}$ is an invariant subspace of T. We will show that each of these subspaces is affiliated to $W^*(T)$ by proving $D_0 \in W^*(T)$, where $D_0 = \lambda(\mathrm{id}_{[0,1]})$ and $\mathrm{id}_{[0,1]}$ is the identity function from $[0,1]$ to itself. Since $X = T + T^*$, this will also imply $W^*(T) = L(\mathbf{F}_2)$. We will then show that each $\mathcal{H}_t$ is actually a hyperinvariant subspace of T, by characterizing $\mathcal{H}_t$ as the set of vectors $\xi \in \mathcal{H}$ such that $\|T^k\xi\|$ has a certain asymptotic property as $k \to \infty$.

2. Preliminaries and statement of results

In [4, Section 8], we showed that the distribution of T^*T is the probability measure μ on $[0, e]$ given by

$$d\mu(x) = \varphi(x)\,dx,$$

where $\varphi:(0,e)\to\mathbf{R}^+$ is the function given uniquely by

$$\varphi\left(\frac{\sin v}{v}\exp(v\cot v)\right)=\frac{1}{\pi}\sin v\exp(-v\cot v),\quad 0<v<\pi. \tag{2.1}$$

Proposition 2.1. *Let* $F(x)=\int_0^x\varphi(t)\,dt, x\in[0,e]$. *Then*

$$F\left(\frac{\sin v}{v}\exp(v\cot v)\right)=1-\frac{v}{\pi}+\frac{1}{\pi}\frac{\sin^2 v}{v},\quad 0<v<\pi. \tag{2.2}$$

Proof. From the proof of [4, Theorem 8.9] we have that

$$\sigma: v\mapsto\frac{\sin v}{v}\exp(v\cot v) \tag{2.3}$$

is a decreasing bijection from $(0,\pi)$ onto $(0,e)$. Hence

$$\begin{aligned}F(\sigma(v))&=\int_0^{\sigma(v)}\varphi(t)\,dt=-\int_v^{\pi}\varphi(\sigma(u))\sigma'(u)\,du\\&=-\left[\varphi(\sigma(u))\sigma(u)\right]_v^{\pi}+\int_v^{\pi}\left(\frac{d}{du}\varphi(\sigma(u))\right)\sigma(u)\,du\\&=-\frac{1}{\pi}\left[\frac{\sin^2 u}{u}\right]_v^{\pi}+\frac{1}{\pi}\int_v^{\pi}\frac{u}{\sin u}\cdot\frac{\sin u}{u}\,du=\frac{1}{\pi}\frac{\sin^2 v}{v}+1-\frac{v}{\pi}.\end{aligned}$$ □

The following is the central result of this paper.

Theorem 2.2. *Let* $S_k=k((T^k)^*T^k)^{\frac{1}{k}}$, $k=1,2,\ldots$. *Then* $\sigma(S_k)=[0,e]$ *for all* $k\in\mathbf{N}$ and

$$\lim_{k\to\infty}\|F(S_k)-D_0\|_2=0 \quad \textit{for} \quad k\to\infty.$$

In particular $D_0\in W^*(T)$. *Therefore* $\mathscr{H}_t=1_{[0,t]}(D_0)\mathscr{H}=\lambda(1_{[0,t]})\mathscr{H}$, $0<t<1$ *is a one-parameter family of nontrivial, closed, T-invariant subspaces affiliated with* $W^*(T)$.

Corollary 2.3. $W^*(T)\cong L(\mathbf{F}_2)$. *Moreover, if Z is any DT-operator, then* $W^*(Z)\cong L(\mathbf{F}_2)$.

Proof. As described in the introduction, with $T=\mathscr{UT}(X,\lambda)\in W^*(X\cup\lambda(L^\infty[0,1]))=L(\mathbf{F}_2)$, from Theorem 2.2 we have $D_0\in W^*(T)$. Since clearly $X\in W^*(T)$, we have $W^*(T)=L(\mathbf{F}_2)$. By [4, Theorem 4.4], Z can be realized as $Z=D+cT$ for some $D\in\lambda(L^\infty[0,1])$ and $c>0$. By [4, Lemma 6.2], $T\in W^*(Z)$, so $W^*(Z)=L(\mathbf{F}_2)$. □

We now outline the proof of Theorem 2.2. Let M be a factor of type II_1 with tracial state tr, and let $A, B \in M_{sa}$. By [1, Section 1], there is a unique probability measure $\mu_{A,B}$ on $\sigma(A) \times \sigma(B)$, such that for all bounded Borel functions f, g on $\sigma(A)$ and $\sigma(B)$, respectively, one has

$$\mathrm{tr}(f(A)g(B)) = \int\int_{\sigma(A)\times\sigma(B)} f(x)g(y)\, d\mu_{A,B}(x). \tag{2.4}$$

The following lemma is a simple consequence of (2.4) (cf. [1, Proposition 1.1]).

Lemma 2.4. *Let A, B and $\mu_{A,B}$ be as above, then for all bounded Borel functions f and g on $\sigma(A)$ and $\sigma(B)$, respectively,*

$$||f(A) - g(B)||_2^2 = \int\int_{\sigma(A)\times\sigma(B)} |f(x) - g(y)|^2\, d\mu_{A,B}(x, y). \tag{2.5}$$

We shall need the following key result of Śniady [6]. Strictly speaking, the results of [6] concern an operator that can be described as a generalized circular operator with a given variance matrix. It is not entirely obvious that the operator T studied in [4] and in the present article is actually of this form. A proof is supplied in Appendix A.

Theorem 2.5 (Śniady [6, Theorem 5]). *Let $E_{\mathscr{D}}$ be the trace preserving conditional expectation of $W^*(D_0, T)$ onto $\mathscr{D} = W^*(D_0)$, which we identify with $L^\infty[0,1]$ as in [6]. Let $k \in \mathbf{N}$ and let $(P_{k,n})_{n=0}^{\infty}$ be the sequence of polynomials in a real variable x determined by*

$$P_{k,0}(x) = 1, \tag{2.6}$$

$$P_{k,n}^{(k)}(x) = P_{k,n-1}(x+1), \quad n = 1, 2, \ldots, \tag{2.7}$$

$$P_{k,n}(0) = P_{k,n}{}'(0) = \cdots = P_{k,n}^{(k-1)}(0) = 0, \quad n = 1, 2, \ldots, \tag{2.8}$$

where $P_{k,n}^{(\ell)}$ denotes the ℓth derivative of $P_{k,n}$. Then for all $k, n \in \mathbf{N}$,

$$E_{\mathscr{D}}(((T^k)^* T^k)^n)(x) = P_{k,n}(x), \quad x \in [0,1].$$

Remark 2.6. The above Theorem is equivalent to [6, Theorem 5] because

$$E_{\mathscr{D}}(((T^k)^* T^k)^n)(x) = E_{\mathscr{D}}((T^k(T^k)^*)^n)(1-x), \quad x \in [0,1].$$

Śniady used Theorem 2.5 to prove the following formula, which was conjectured in [4, Section 9].

Theorem 2.7 (Śniady [6, Theorem 7]). *For all* $n, k \in \mathbf{N}$:

$$\mathrm{tr}(((T^k)^* T^k)^n) = \frac{n^{nk}}{(nk+1)!}. \tag{2.9}$$

Śniady proved that Theorem 2.5 implies Theorem 2.7 by a tricky and clever combinatorial argument. In the course of proving Theorem 2.2, we also obtained a purely analytic proof of Theorem 2.5 ⇒ Theorem 2.7 (see (3.2) and Remark 4.3). Note that it follows from Theorem 2.7 that $S_k^k = k^k (T^k)^* T^k$ has the same moments as $(T^*T)^k$. Hence the distribution measures μ_{S_k} and μ_{T^*T} in Prob($\mathbf{R}$) are equal. In particular their supports are equal. Hence, by [4, Theorem 8.9],

$$\sigma(S_k) = \sigma(T^*T) = [0, e]. \tag{2.10}$$

We will use Theorem 2.5 to derive in Theorem 2.8 an explicit formula for the measure μ_{D_0, S_k} defined in (2.4). The formula involves Lambert's W function, which is defined as the multivalued inverse function of the function $\mathbf{C} \ni z \mapsto ze^z$. We define a function ρ by

$$\rho(z) = -W_0(-z), \quad z \in \mathbf{C} \backslash \left[\frac{1}{e}, \infty\right), \tag{2.11}$$

where W_0 is the principal branch of Lambert's W-function. By [2, Section 4], ρ is an analytic bijection of $\mathbf{C} \backslash [\frac{1}{e}, \infty)$ onto

$$\Omega = \{x + iy \mid -\pi < y < \pi, \ x < y \cot y\},$$

where we have used the convention $0 \cot 0 = 1$. Moreover, ρ is the inverse function of the function f defined by

$$f(w) = we^{-w}, \quad w \in \Omega.$$

Note that f maps the boundary of Ω onto $[\frac{1}{e}, \infty)$, because

$$f(\theta \cot \theta \pm i\theta) = f\left(\frac{\theta}{\sin \theta} e^{\pm i\theta}\right) = \frac{\theta}{\sin \theta} e^{-\theta \cot \theta} \tag{2.12}$$

and $\theta \mapsto \frac{\sin \theta}{\theta} e^{\theta \cot \theta}$ is a bijection of $(0, \pi)$ onto $(0, e)$ (see [4, Section 8]). By (2.12), it also follows that if we define functions $\rho^+, \rho^- : [\frac{1}{e}, \infty) \to \mathbf{C}$ by

$$\rho^{\pm}\left(\frac{\theta}{\sin \theta} e^{-\theta \cot \theta}\right) = \theta \cot \theta \pm i\theta, \quad 0 \leqslant \theta < \pi, \tag{2.13}$$

then

$$\rho^{\pm}(x) = \lim_{y \downarrow 0} \rho(x \pm iy), \quad x \in \left[\frac{1}{e}, \infty\right).$$

In particular $\rho^{+}(\frac{1}{e}) = \rho^{-}(\frac{1}{e}) = 1$.

Theorem 2.8. *Let $k \in \mathbf{N}$ be fixed. Define for $t > \frac{1}{e}$ and $j = 0, \ldots, k$ the functions $a_j(t)$, $c_j(t)$ by*

$$\begin{aligned} a_0(t) &= \rho^{+}(t), \\ a_j(t) &= \rho\left(t \exp\left(i\frac{2\pi j}{k}\right)\right), \quad 1 \leqslant j \leqslant k-1, \\ a_k(t) &= \rho^{-}(t), \end{aligned} \tag{2.14}$$

and

$$c_j(t) = -k a_j(t) \prod_{\ell \neq j} \frac{a_\ell(t)}{a_\ell(t) - a_j(t)}. \tag{2.15}$$

Then the probability measure μ_{D_0, S_k} on $\sigma(D_0) \times \sigma(S_k) = [0,1] \times [0,e]$ is absolutely continuous with respect to the two-dimensional Lebesgue measure and, *with φ as in* (2.1), *has density*

$$\frac{d\mu_{D_0, S_k}(x,y)}{dx\, dy} = \varphi(y)\left(\sum_{j=0}^{k} c_j(y^{-1}) e^{k a_j(y^{-1}) x}\right) \tag{2.16}$$

for $x \in (0,1)$ and $y \in (0,e)$.

We will prove Theorem 2.2 by combining Lemma 2.4 and Theorem 2.8 (see Section 6).

Finally, we will prove the following characterization of the subspaces $\mathscr{H}_t$ (see Section 7).

Theorem 2.9. *For every $t \in [0,1]$,*

$$\mathscr{H}_t = \left\{\xi \in \mathscr{H} \mid \limsup_{n \to \infty}\left(\frac{k}{e} ||T^k \xi||^{2/k}\right) \leqslant t\right\}. \tag{2.17}$$

In particular, *$\mathscr{H}_t$ is a closed*, *hyperinvariant subspace of T.*

3. Proof of Theorem 2.8 for $k = 1$

This section is devoted to the proof of Theorem 2.8 in the special case $k = 1$, which is somewhat easier than in the general case. For $k = 1$ it is easy to solve Eqs. (2.6)–(2.8) explicitly to obtain

$$P_{1,n}(x) = \frac{1}{n!} x(x+n)^{n-1}, \quad (n \geqslant 1). \tag{3.1}$$

From (3.1) one immediately gets (2.9) for $k = 1$, because

$$\mathrm{tr}((T^*T)^n) = \int_0^1 P_{1,n}(x)\,dx = \left[\frac{1}{(n+1)!}(x-1)(x+n)^n\right]_0^1 = \frac{n^n}{(n+1)!}. \tag{3.2}$$

Lemma 3.1. *For $x \in \mathbf{R}$ and $z \in \mathbf{C}$, $|z| < \frac{1}{e}$, one has*

$$\sum_{n=0}^{\infty} P_{1,n}(x) z^n = e^{\rho(z)x}$$

where ρ: $\mathbf{C} \backslash [\frac{1}{e}, \infty) \to \mathbf{C}$ is the analytic function defined in Section 2.

Proof. Note that $\rho(0) = 0, \rho'(0) = 1$. Let $\rho(z) = \sum_{n=1}^{\infty} \gamma_n z^n$ be the power-series expansion of ρ in $B(0, \frac{1}{e})$. The convergence radius is $\frac{1}{e}$, because ρ is analytic in $B(0, \frac{1}{e})$ and $\frac{1}{e}$ is a singular point for ρ. Hence for $|z| < \frac{1}{e}$ and $x \in \mathbf{C}$, the function $(z, x) \mapsto e^{\rho(z)x}$ has a power-series expansion

$$e^{\rho(z)x} = \sum_{\ell,m=0}^{\infty} c_{\ell m} z^{\ell} x^m.$$

Since

$$e^{\rho(z)x} = \sum_{m=0}^{\infty} \frac{1}{m!} \rho(z)^m x^m$$

and since the first non-zero term in the power series for $\rho(z)^m$ is z^m, we have $c_{\ell m} = 0$ for $\ell < m$. Hence

$$e^{\rho(z)x} = \sum_{\ell=0}^{\infty} Q_{\ell}(x) z^{\ell} \tag{3.3}$$

where $Q_{\ell}(x)$ is the polynomial $\sum_{m=0}^{\ell} c_{\ell m} x^m$. Putting $z = 0$ in (3.3) we get $Q_0(x) = 1$ and putting $x = 0$ in (3.3) we get $Q_n(0) = 0$ for $n \geqslant 1$. Moreover since $\rho(z)e^{-\rho(z)} = z$

for $\mathbf{C}\backslash[\frac{1}{e}, \infty)$, we get

$$\frac{d}{dx}(e^{\rho(z)x}) = \rho(z)e^{\rho(z)x} = \rho(z)e^{-\rho(z)}e^{\rho(z)(x+1)} = ze^{\rho(z)(x+1)}.$$

Hence differentiating (3.3), we get

$$\sum_{\ell=0}^{\infty} Q_\ell{}'(x)z^\ell = \sum_{\ell=0}^{\infty} Q_\ell(x+1)z^{\ell+1} = \sum_{\ell=1}^{\infty} Q_{\ell-1}(x+1)z^\ell, \quad |z| < \frac{1}{e}.$$

Therefore $Q_\ell{}'(x) = Q_{\ell-1}(x+1)$ for $\ell \geqslant 1$. Together with $Q_0(x) = 1$, $Q_\ell(x) = 0$, $(\ell \geqslant 1)$, this proves that $Q_\ell(x) = P_{1,\ell}(x)$ for $\ell \geqslant 0$. $\square$

Remark 3.2. From Lemma 3.1 and (3.1) we can find the power-series expansion of $\rho(z)$, namely

$$\rho(z) = ze^{\rho(z)} = \sum_{n=0}^{\infty} P_{1,n}(1)z^{n+1} = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} z^{n+1} = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)!} z^n. \tag{3.4}$$

Similarly one gets

$$\begin{aligned} \frac{1}{\rho(z)} &= \frac{1}{z} e^{-\rho(z)} = \sum_{n=0}^{\infty} P_{1,n}(-1)z^{n-1} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{(n-1)^{n-1}}{n!} z^{n-1} \\ &= \frac{1}{z} - \sum_{n=0}^{\infty} \frac{n^n}{(n+1)!} z^n. \end{aligned} \tag{3.5}$$

The latter formula was also found in [4, Section 8] by different means. Actually, both formulae can be obtained from the Lagrange Inversion Formula, (cf. [8, Example 5.44]).

Lemma 3.3. *For every $x \in [0, 1]$ there is a unique probability measure v_x on $[0, e]$ such that*

$$\int_0^e y^n \, dv_x(y) = P_{1,n}(x), \quad n \in \mathbf{N}_0. \tag{3.6}$$

Proof. The uniqueness is clear by Weierstrass' approximation theorem. For existence, recall that $\sigma(D_0) = [0, 1]$ and, by [4, Section 8], $\sigma(T^*T) = [0, e]$. Let now $\mu = \mu_{D_0, T^*T}$ denote the joint distribution of D_0 and T^*T in the sense of (2.4). For $x = 0$, $v_x = \delta_0$ (the Dirac measure at 0) is a solution of (3.6). Assume now that $x > 0$

and let $\epsilon \in (0, x)$. Then for $n \in \mathbf{N}_0$,

$$\int_{x-\epsilon}^{x} P_{1,n}(x')\,dx' = \int_0^1 1_{[x-\epsilon,x]}(x')P_{1,n}(x')\,dx' = \operatorname{tr}(1_{[x-\epsilon,x]}(D_0)E_{\mathscr{D}}((T^*T)^n))$$
$$= \operatorname{tr}(1_{[x-\epsilon,x]}(D_0)(T^*T)^n) = \int\int_{[0,1]\times[0,e]} 1_{[x-\epsilon,x]}(x')y^n\,d\mu(x',y).$$

Let $v_{\epsilon,x}$ denote the Borel measure on $[0,e]$ given by $v_{\epsilon,x}(B) = \frac{1}{\epsilon}\mu([x-\epsilon,x]\times B)$ for any Borel set B in $[0,e]$. Then by the above calculation,

$$\int_0^e y^n\,dv_{\epsilon,x}(y) = \frac{1}{\epsilon}\int_{x-\epsilon}^{x} P_{1,n}(x')\,dx', \quad n \in \mathbf{N}_0. \tag{3.7}$$

Since $P_{1,0}(x') = 1$, $v_{\epsilon,x}$ is a probability measure. By (3.7), $v_{\epsilon,x}$ converges as $\epsilon \to 0$ in the w^*-topology on $\operatorname{Prob}([0,e])$ to a measure v_x satisfying (3.6). $\square$

Lemma 3.4. *Let $x \in [0,1]$.*

(a) *For $\lambda \in \mathbf{C}\backslash[0,e]$, the Stieltjes transform (or Cauchy transform) of v_x is given by*

$$G_x(\lambda) = \frac{1}{\lambda}\exp\left(\rho\left(\frac{1}{\lambda}\right)x\right). \tag{3.8}$$

(b) *If $x \in (0,1], dv_x(y) = h_x(y)dy$, where*

$$h_x(y) = \frac{1}{\pi y}\operatorname{Im}\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right)\right), \quad y \in (0,e]. \tag{3.9}$$

Proof. (a) Since $G_x(\lambda) = \int_0^e \frac{1}{\lambda-y}\,dv_x(y)$ is analytic in $\mathbf{C}\backslash[0,e]$, it is sufficient to check (3.8) for $|\lambda| > e$. In this case, we get from Lemmas 3.3 and 3.1 that

$$G_x(\lambda) = \sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}\int_0^e y^n\,dv_x(y) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\lambda^{-n}P_n(x) = \frac{1}{\lambda}\exp\left(\rho\left(\frac{1}{\lambda}\right)x\right).$$

(b) For $y \in (0,e]$, put

$$h_x(y) = -\frac{1}{\pi}\lim_{z\to 0^+}\operatorname{Im}(G_x(y+iz)) = -\frac{1}{\pi y}\operatorname{Im}\left(\exp\left(\rho^-\left(\frac{1}{y}\right)x\right)\right)$$
$$= \frac{1}{\pi y}\operatorname{Im}\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right)\right).$$

It is easy to see that the above convergence is uniform for y in compact subsets of $(0,e]$, so by the inverse Stieltjes transform, the restriction of v_x to $(0,e]$ is absolutely continuous with respect to the Lebesgue measure and has density $h_x(y)$. It remains to be proved that $\nu_x(\{0\}) = 0$. But

$$\lim_{\lambda\to 0^-} \lambda G_x(\lambda) = \nu_x(\{0\}) + \lim_{\lambda\to 0^-}\left(\int_{(0,e]} \frac{|\lambda|}{|\lambda|+y}\,d\nu_x(y)\right) = \nu_x(\{0\}).$$

However, $\lambda G_x(\lambda) = \exp(\rho(\frac{1}{\lambda})x)\to 0$ as $\lambda\to 0^-$, because $x>0$ and $\lim_{y\to-\infty}\rho(y) = -\infty$. Hence $\nu_x(\{0\}) = 0$, which completes the proof of (b). □

Proof of Theorem 2.8 for $k = 1$. Put $\mu = \mu_{D_0,T^*T}$ as defined in (2.4). For $m, n\in\mathbf{N}_0$ we get from Lemmas 3.3 and 3.4,

$$\begin{aligned}
&\int\int_{[0,1]\times[0,e]} x^m y^n\,d\mu(x,y)\\
&\quad = \mathrm{tr}(D_0^m(T^*T)^n) = \mathrm{tr}(D_0^m E_{\mathscr{D}}((T^*T)^n)) = \int_0^1 x^m P_{1,n}(x)\,dx\\
&\quad = \int_0^1 x^m \int_0^e y^n\,d\nu_x(y)\,dx = \int_0^1\left(\int_0^e x^m y^n h_x(y)\,dy\right)dx.
\end{aligned}$$

Hence by the Stone–Weierstrass Theorem, μ is absolutely continuous with respect to the two-dimensional Lebesgue measure on $[0,1]\times[0,e]$, and for $x\in(0,1)$, $y\in(0,e)$, we have

$$\frac{d\mu(x,y)}{dx\,dy} = h_x(y) = \frac{1}{\pi y}\mathrm{Im}\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right)\right). \tag{3.10}$$

We now have to compare (3.10) with (2.16) in Theorem 2.8. Putting $k = 1$ in (2.14) and (2.15) one gets for $t>\frac{1}{e}$,

$$a_0(t) = \rho^+(t), \quad a_1(t) = \overline{\rho^+(t)}$$

and

$$c_0(t) = \frac{|\rho^+(t)|^2}{2i\,\mathrm{Im}(\rho^+(t))}, \quad c_1(t) = -\frac{|\rho^+(t)|^2}{2i\,\mathrm{Im}(\rho^+(t))}.$$

Hence the RHS of (2.16) becomes

$$\begin{aligned}
&\varphi(y)c_0\left(\frac{1}{y}\right)\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right) - \exp\left(\overline{\rho^+\left(\frac{1}{y}\right)}x\right)\right)\\
&\quad = \frac{\varphi(y)|\rho^+(\frac{1}{y})|^2}{\mathrm{Im}\,\rho^+(\frac{1}{y})}\mathrm{Im}\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right)\right).
\end{aligned}$$

Substituting now $y = \frac{\sin v}{v} e^{v \cot v}$ with $0 < v < \pi$ as in (2.3), by (2.13) and (2.1) we get

$$\frac{\varphi(y)|\rho^+(\frac{1}{y})|^2}{\operatorname{Im} \rho^+\left(\frac{1}{y}\right)} = \frac{1}{\pi v}\left(\sin v e^{-v \cot v} \cdot \frac{v^2}{\sin^2 v}\right) = \frac{1}{\pi y}. \tag{3.11}$$

Hence (3.10) coincides with (2.16) for $k = 1$. □

4. A generating function for Śniady's polynomials for $k \geqslant 2$

Throughout this section and Section 5, k is a fixed integer, $k \geqslant 2$.

Lemma 4.1. *Let* $\alpha_1, \ldots, \alpha_k$ *be distinct complex numbers and put*

$$\gamma_j = \prod_{\ell \neq j} \frac{\alpha_\ell}{\alpha_\ell - \alpha_j}, \quad j = 1, \ldots, n. \tag{4.1}$$

Then

$$\begin{cases} \sum_{j=1}^{k} \gamma_j = 1, \\ \sum_{j=1}^{k} \gamma_j \alpha_j^p = 0 \quad \text{for } p = 1, 2, \ldots, k-1. \end{cases} \tag{4.2}$$

Proof. We can express (4.2) as

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & & \alpha_k \\ \vdots & & & \vdots \\ \alpha_1^{k-1} & \ldots & \ldots & \alpha_k^{k-1} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4.3}$$

where the determinant of the coefficient matrix is non-zero (Vandermonde's determinant), so we just have to check that (4.1) is the unique solution to (4.3). Let A denote the coefficient matrix in (4.3). Then the solution to (4.3) is given by

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_k \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence $\gamma_j = (-1)^{j+1}\frac{\det(A_{1j})}{\det(A)}$, where A_{1j} is the $(1,j)$th minor of A. By Vandermonde's formula,

$$\det A = \prod_{\ell < m} (a_m - a_\ell)$$

and

$$\det(A_{1j}) = (\alpha_1 \cdots \alpha_{j-1})(\alpha_{j+1} \cdots \alpha_k) \prod_{\substack{\ell < m \\ \ell, m \neq j}} (a_m - a_\ell).$$

Hence

$$\gamma_j = \frac{(-1)^{j+1} \prod_{\ell \neq j} \alpha_\ell}{\prod_{\ell < j}(\alpha_j - \alpha_\ell) \prod_{\ell > j}(\alpha_\ell - \alpha_j)} = \prod_{\ell \neq j} \frac{\alpha_\ell}{\alpha_\ell - \alpha_j}. \qquad \square$$

We prove next a generalization of Lemma 3.1 to $k \geqslant 2$.

Proposition 4.2. *Let $(P_{k,n})_{n=0}^{\infty}$ be the sequence of polynomials defined Theorem* 2.5. *For $z \in \mathbf{C}$, $|z| < \frac{1}{e}$ and $j = 1, \ldots, k$, put*

$$\alpha_j(z) = \rho(z e^{i\frac{2\pi j}{k}}), \tag{4.4}$$

$$\gamma_j(z) = \begin{cases} \displaystyle\prod_{\ell \neq j} \frac{\alpha_j(z)}{\alpha_\ell(z) - \alpha_j(z)}, & z \neq 0, \\ 1/k, & z = 0. \end{cases} \tag{4.5}$$

Then

$$\sum_{n=0}^{\infty} (kz)^{nk} P_{k,n}(x) = \sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x} \tag{4.6}$$

for all $z \in B(0, \frac{1}{e})$ and all $x \in \mathbf{R}$.

Proof. Since ρ is analytic and one-to-one on $\mathbf{C}\backslash[\frac{1}{e}, \infty)$, it is clear that $\alpha_j(z)$ is analytic in $B(0, \frac{1}{e})$ and $\gamma_j(z)$ is analytic in $B(0, \frac{1}{e})\backslash\{0\}$. Using $\rho(0) = 0$ and $\rho'(0) = 1$, one gets

$$\lim_{z \to 0} \gamma_j(z) = \prod_{\ell \neq j} \frac{1}{1 - \exp\left(i\frac{2\pi(j-\ell)}{k}\right)} = \prod_{m=1}^{k-1} \left(1 - \exp\left(i\frac{2\pi m}{k}\right)\right)^{-1}.$$

But the numbers $\exp(i\frac{2\pi m}{k})$, $m = 1, \ldots, k-1$ are precisely the $k-1$ roots of the polynomial

$$S(z) = \frac{z^k - 1}{z - 1} = z^{k-1} + z^{k-2} + \cdots + 1.$$

Hence

$$\lim_{z \to 0} \gamma_j(z) = \frac{1}{S(1)} = \frac{1}{k} = \gamma_j(0).$$

Thus γ_j is analytic in $B(0, \frac{1}{e})$. The RHS of (4.6) is equal to

$$\sum_{\ell=0}^{\infty} \beta_\ell(z) x^\ell,$$

where

$$\beta_\ell(z) = \sum_{j=1}^{k} \gamma_j(z) k^\ell \alpha_j(z)^\ell.$$

Since $\alpha_j(0) = 0$, the coefficients to $1, z, \ldots, z^{\ell-1}$ in the power-series expansion of $\beta_\ell(z)$ are equal to 0. Hence

$$\sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x} = \sum_{\ell,m=0}^{\infty} \beta_{\ell,m} x^\ell z^m, \tag{4.7}$$

where $\beta_{\ell,m} = 0$ when $m < \ell$. But, by the definition of $\alpha_j(z)$ and $\gamma_j(z)$ the LHS of (4.7) is invariant under the transformation $z \to e^{i\frac{2\pi}{k}} z$. Hence $\beta_{\ell,m} = 0$ unless m is a multiple of k. Therefore,

$$\sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x} = \sum_{n=0}^{\infty} R_n(x) z^{nk}, \tag{4.8}$$

where

$$R_n(x) = \sum_{\ell=0}^{nk} \beta_{\ell,nk} x^\ell \tag{4.9}$$

is a polynomial of degree at most nk. To complete the proof of Proposition 4.2, we now have to prove, that the sequence of polynomials

$$Q_n(x) = k^{-nk} R_n(x), \quad n = 0, 1, 2, \ldots \tag{4.10}$$

satisfies the same three conditions (2.6)–(2.8) as $P_{k,n}$. Putting $z = 0$ in (4.8) we get

$$Q_0(x) = R_0(x) = \sum_{j=1}^{k} \gamma_j(0) = 1.$$

Moreover by (4.8)

$$\frac{d^k}{dx^k}\left(\sum_{n=0}^{\infty} R_n(x)z^{nk}\right) = \sum_{j=1}^{k} \gamma_j(z)k^k\alpha_j(z)^k e^{k\alpha_j(z)x}.$$

By definition of ρ, $\rho(z)e^{-\rho(z)} = z$ for all $z \in \mathbf{C}\setminus(\frac{1}{e}, \infty)$. Hence

$$(\alpha_j(z)e^{-\alpha_j(z)})^k = (ze^{i\frac{2\pi}{k}j})^k = z^k, \quad j = 1, \dots, k. \tag{4.11}$$

Thus

$$\begin{aligned}\frac{d^k}{dx^k}\left(\sum_{n=0}^{\infty} R_n(z)z^{nk}\right) &= (kz)^k \sum_{j=1}^{k} \gamma_j(z)e^{k\alpha_j(z)(x+1)} = (kz)^k \sum_{n=0}^{\infty} R_n(x+1)z^{nk} \\ &= k^k \sum_{n=1}^{\infty} R_{n-1}(x+1)z^{nk}\end{aligned}$$

so differentiating termwise, we get

$$R_n^{(k)}(x) = k^k R_{n-1}(x+1), \quad n \geqslant 1$$

and thus $Q_n^{(k)}(x) = Q_{n-1}(x+1)$ for all $n \geqslant 1$. We next check the last condition (2.8) for the Q_n, i.e.

$$Q_n(0) = Q_n{}'(0) = \cdots = Q_n^{(k-1)}(0) = 0, \quad n \geqslant 1.$$

If we put $x = 0$ in (4.5), we get

$$\sum_{n=0}^{\infty} R_n(x)z^{nk} = \sum_{j=1}^{k} \gamma_j(z) = 1,$$

where the last equality follows from (4.2) in Lemma 4.1. Hence $Q_n(0) = R_n(0) = 0$ for $n \geqslant 1$. For $p = 1, \dots, k-1$ we have

$$\sum_{n=0}^{\infty} R_n^{(p)}(0)z^{nk} = \frac{d^p}{dx^p}\left(\sum_{j=1}^{k} \gamma_j(z)e^{k\alpha_j(z)x}\right)\Bigg|_{x=0} = k^p \sum_{j=1}^{k} \gamma_j(z)\alpha_j(z)^p = 0,$$

where we again use (4.2) from Lemma 4.1. Hence $Q_n^{(p)}(0) = k^{-nk}R_n^{(p)}(0) = 0$ for all $n = 0, 1, 2, \dots$ and $p = 1, \dots, k-1$.

Altogether we have shown that $(Q_n(x))_{n=0}^{\infty}$ satisfies the defining relations (2.6)–(2.8) for $P_{k,n}(x)$, and hence $Q_n(x) = P_{k,n}(x)$ for all n and. This proves (4.6). □

Remark 4.3. Based on Proposition 4.2, we give a new proof of the implication Theorem 2.5 ⇒ Theorem 2.7. Put

$$s_{k,n} = \mathrm{tr}(((T^k)^*T^k)^n) = \int_0^1 P_{k,n}(x)\,dx.$$

Then by (4.6)

$$\sum_{n=0}^{\infty} s_{k,n}(kz)^{nk} = \sum_{j=1}^{k} \gamma_j(k) \int_0^1 e^{k\alpha_j(z)x}\, dx \tag{4.12}$$

for all $z \in B(0, \frac{1}{e})$. Using (4.11), for every $z \in B(0, \frac{1}{e}) \setminus \{0\}$ we get

$$\int_0^1 e^{k\alpha_j(z)x}\, dx = \frac{1}{k\alpha_j(z)}(e^{k\alpha_j(z)} - 1) = \frac{1}{kz^k}\, \alpha_j(z)^{k-1} - \frac{1}{k\alpha_j(z)}.$$

By Lemma 4.1, we have $\sum_{j=0}^{k} \gamma_j(z)\alpha_j(z)^{k-1} = 0$. Hence by (4.12),

$$\sum_{n=0}^{\infty} s_{k,n}(kz)^{nk} = -\frac{1}{k} \sum_{j=1}^{k} \frac{\gamma_j(z)}{\alpha_j(z)}. \tag{4.13}$$

To compute the RHS of (4.13), we apply the residue theorem to the rational function $f(s) = \frac{1}{s^2} \prod_{\ell=1}^{k} \frac{\alpha_\ell}{\alpha_\ell - s}$, $s \in \mathbf{C} \setminus \{0, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. In the following computation z is fixed, so let us put $\alpha_j = \alpha_j(z)$, $\gamma_j = \gamma_j(z)$. Note that f has simple poles at $\alpha_1, \ldots, \alpha_k$ and

$$\mathrm{Res}(f; \alpha_j) = -\frac{1}{\alpha_j} \prod_{\ell \neq j} \frac{\alpha_\ell}{\alpha_\ell - \alpha_j} = -\frac{\gamma_j}{\alpha_j}.$$

Moreover f has a second-order pole at 0 and $\mathrm{Res}(f; 0)$ is the coefficient of s in the power-series expansion of $s^2 f(s) = \prod_{\ell=1}^{k} (1 - \frac{s}{\alpha_\ell})^{-1}$, i.e.

$$\mathrm{Res}(f; 0) = \sum_{j=1}^{\ell} \frac{1}{\alpha_j}.$$

Since $f(s) = O(|s|^{-(k+2)})$ as $|s| \to \infty$, we have

$$\lim_{R \to \infty} \int_{\partial B(0,R)} f(s)\, ds = 0.$$

Hence, by the residue Theorem, $\mathrm{Res}(f; 0) + \sum_{j=1}^{k} \mathrm{Res}(f; \alpha_j) = 0$, giving

$$\sum_{j=1}^{k} \frac{\gamma_j}{\alpha_j} = \sum_{j=1}^{k} \alpha_j^{-1}. \tag{4.14}$$

Thus, by (4.13), we get

$$\sum_{n=0}^{\infty} s_{k,n}(kz)^{nk} = -\frac{1}{k} \sum_{j=1}^{k} \alpha_j(z)^{-1} = -\frac{1}{k} \sum_{j=1}^{k} \rho(ze^{i\frac{2\pi j}{k}})^{-1}. \tag{4.15}$$

By (3.5), $\rho(z)^{-1} = \frac{1}{z} - \sum_{m=0}^{\infty} \frac{m^m}{(m+1)!} z^m$ whenever $0 < |z| < \frac{1}{e}$. Hence

$$\sum_{j=1}^{k} \rho(ze^{i\frac{2\pi j}{k}})^{-1} = -k \sum_{k \mid m} \frac{m^m}{(m+1)!} z^m = -k \sum_{n=0}^{\infty} \frac{(nk)^{nk}}{(nk+1)!} z^{nk}. \tag{4.16}$$

So by comparing the terms in (4.15) and (4.16), we get $s_{kn} = \frac{n^{nk}}{(nk+1)!}$ as desired.

5. Proof of Theorem 2.8 for $k \geqslant 2$

Lemma 5.1. *Put* $\Omega_k = \{z \in \mathbf{C} \mid z^k \notin [e^{-k}, \infty)\}$ *and define* $\alpha_j(z)$, $\gamma_j(z)$, $j = 1, \dots, k$ *by* (4.4) *and* (4.5) *for all* $z \in \Omega_k$. *Then for every* $x \in \mathbf{R}$, *the function*

$$M_x(z) = \sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x} \tag{5.1}$$

is analytic in Ω_k *and for every* $t \in [\frac{1}{e}, \infty)$, *the following two limits exist*:

$$M_x^+(t) = \lim_{\substack{z \to t \\ \operatorname{Im} z > 0}} M_x(z), \quad M_x^-(t) = \lim_{\substack{z \to t \\ \operatorname{Im} z < 0}} M_x(z).$$

Let $a_j(t)$ *and* $c_j(t)$ *for* $t > \frac{1}{e}$ *and* $j = 0, \dots, k$ *be as in Theorem* 2.8. *Then for* $t > \frac{1}{e}$,

$$\operatorname{Im} M_x^+(t) = \frac{\operatorname{Im} \rho^+(t)}{k|\rho^+(t)|^2} \sum_{j=0}^{k} c_j(t) e^{ka_j(t)x}. \tag{5.2}$$

Proof. Since $\rho : \mathbf{C} \backslash [\frac{1}{e}, \infty) \to \mathbf{C}$ is one-to-one and analytic, it is clear, that M_x is defined and analytic on Ω_k. Moreover for $t \geqslant \frac{1}{e}$,

$$\lim_{\substack{z \to t \\ \operatorname{Im} z > 0}} \alpha_j(z) = \begin{cases} \rho(te^{i\frac{2\pi j}{k}}), & j = 1, \dots, k-1, \\ \rho^+(t), & j = k, \end{cases}$$
$$= \begin{cases} a_j(t), & j = 1, \dots, k-1, \\ a_0(t), & j = k, \end{cases}$$

and similarly

$$\lim_{\substack{z \to t \\ \operatorname{Im} z < 0}} \alpha_j(z) = a_j(t), \quad j = 1, \dots, k.$$

Moreover

$$\lim_{\substack{z\to t\\ \mathrm{Im}\, z>0}} \gamma_j(z) = \begin{cases} \displaystyle\prod_{\substack{0\leqslant \ell\leqslant k-1\\ \ell\neq j}} \frac{a_\ell(t)}{a_\ell(t)-a_j(t)}, & j=1,\dots,k-1,\\[2ex] \displaystyle\prod_{\substack{0\leqslant \ell\leqslant k-1\\ \ell\neq 0}} \frac{a_\ell(t)}{a_\ell(t)-a_j(t)}, & j=k, \end{cases}$$

$$\lim_{\substack{z\to t\\ \mathrm{Im}\, z<0}} \gamma_j(z) = \prod_{\substack{1\leqslant \ell\leqslant k\\ \ell\neq j}} \frac{a_\ell(t)}{a_\ell(t)-a_j(t)}, \quad j=1,\dots,k.$$

Hence the two limits $M_x^+(t)$ and $M_x^-(t)$ are well defined and by relabeling the kth term to be the 0th term in case of $M_x^+(t)$ one gets:

$$M_\lambda^+(t) = \sum_{j=0}^{k-1}\left(\prod_{\substack{0\leqslant \ell\leqslant k-1\\ \ell\neq j}} \frac{a_\ell(t)}{a_\ell(t)-a_j(t)}\right) e^{ka_j(t)x}, \tag{5.3}$$

$$M_\lambda^-(t) = \sum_{j=1}^{k}\left(\prod_{\substack{1\leqslant \ell\leqslant k\\ \ell\neq j}} \frac{a_\ell(t)}{a_\ell(t)-a_j(t)}\right) e^{ka_j(t)x}. \tag{5.4}$$

It is clear, that $M_x(\bar{z}) = \overline{M_x(z)}$, $z\in\Omega_k$. Therefore, $M_\lambda^-(t) = \overline{M_\lambda^+(t)}$ and

$$\mathrm{Im}\, M_\lambda^+(t) = \frac{1}{2i}(M_\lambda^+(t) - M_\lambda^-(t)).$$

Hence for $t>\frac{1}{e}$,

$$\mathrm{Im}\, M_\lambda^+(t) = \sum_{j=0}^{k} b_j(t) e^{ka_j(t)x},$$

where

$$\begin{aligned} b_0(t) &= \frac{1}{2i}\prod_{1\leqslant \ell\leqslant k-1} \frac{a_\ell(t)}{a_\ell(t)-a_0(t)},\\ b_j(t) &= \frac{1}{2i}\left(\frac{a_0(t)}{a_0(t)-a_j(t)} - \frac{a_k(t)}{a_k(t)-a_j(t)}\right)\prod_{\substack{1\leqslant \ell\leqslant k-1\\ \ell\neq j}} \frac{a_\ell(t)}{a_\ell(t)-a_0(t)},\\ b_k(t) &= -\frac{1}{2i}\prod_{1\leqslant \ell\leqslant k-1} \frac{a_\ell(t)}{a_\ell(t)-a_k(t)}. \end{aligned}$$

Using (2.15) and the identity

$$\frac{a_0(t)}{a_0(t)-a_j(t)}-\frac{a_k(t)}{a_k(t)-a_j(t)}=\frac{a_j(t)(a_k(t)-a_0(t))}{(a_0(t)-a_j(t))(a_k(t)-a_j(t))},$$

one observes that for all $j\in\{0,1,\dots,k\}$

$$b_j(t)=\frac{1}{2i}\frac{a_0(t)-a_k(t)}{ka_0(t)a_k(t)}c_j(t)=\frac{\operatorname{Im}\rho^+(t)}{k|\rho^+(t)|^2}c_j(t).$$

This proves (5.2). □

We next prove results analogous to Lemmas 3.3 and 3.4 for $k\geqslant 2$.

Lemma 5.2. *For every $x\in[0,1]$, there is a unique probability measure v_x on $[0,e^k]$, such that*

$$\int_0^{e^k} u^n\,dv_x(u)=k^{nk}P_{k,n}(x),\quad n\in\mathbf{N}_0. \tag{5.5}$$

For $\lambda\in\mathbf{C}\backslash[0,e^k]$, the Cauchy transform of v_x is given by

$$G_x(\lambda)=\frac{1}{\lambda}\sum_{j=1}^{k}\gamma_j(\lambda^{-\frac{1}{k}})e^{k\alpha_j(\lambda^{-\frac{1}{k}})x} \tag{5.6}$$

where α_j,γ_j are given by (4.4) *and* (4.5) *and $\lambda^{-1/k}$ is the principal value of $(\sqrt[k]{\lambda})^{-1}$. Moreover, the restriction of v_x to $(0,e^k]$ is absolutely continuous with respect to Lebesgue measure, and its density is given by*

$$\frac{dv_x(u)}{du}=\frac{u^{\frac{1}{k}-1}\varphi(u^{1/k})}{k}\sum_{j=0}^{k}c_j(u^{-1/k})e^{ka_j(u^{-1/k})x} \tag{5.7}$$

for $u\in(0,e^k)$.

Proof. By Theorem 2.5

$$k^{nk}P_{k,n}(x)=E_{\mathscr{D}}(k^{nk}((T^k)^*T^k)^n)(x)=E_{\mathscr{D}}(S_k^{nk})(x),\quad x\in[0,1].$$

Moreover $\sigma(S_k^k)=\sigma(S_k)^k=[0,e^k]$ by (2.10). Hence the existence and uniqueness of v_x can be proved exactly as in Lemma 3.3. From Proposition 4.2, we get that for $|\lambda|>e^k$, the Stieltjes transform $G_x(\lambda)$ of v_x is given by

$$G_x(\lambda)=\frac{1}{\lambda}\sum_{n=1}^{\infty}\lambda^{-n}k^{nk}P_{k,n}(x)=\frac{1}{\lambda}\sum_{j=1}^{k}\gamma_j(\lambda^{-\frac{1}{k}})e^{k\alpha_j(\lambda^{-\frac{1}{k}})x}.$$

Let $M_x(z), z\in\Omega_k$ and $M_x^+(t), M_x^-(t), t\geqslant 1/e$ be as in Lemma 5.1. Then it is easy to see that the function

$$\widetilde{M}_x(z) = \begin{cases} M_x(z), & z\in\Omega_K, \\ M_x^-(z), & z\in[1/e,\infty) \end{cases}$$

is a continuous function on the set

$$\left\{x+iy \mid x\geqslant 0, \frac{-1}{ke}\leqslant y\leqslant 0\right\}.$$

Hence, by applying the inverse Stieltjes transform, we get that the restriction of ν_x to $(0,e^k]$ is absolutely continuous with respect to the Lebesgue measure with density

$$\begin{aligned} h_x(u) &= -\frac{1}{\pi}\lim_{v\to 0^+} \operatorname{Im}(G_x(u+iv)) = -\frac{1}{\pi u}\lim_{\substack{z\to u^{-1/k}\\ \operatorname{Im} z<0}}\left(\operatorname{Im}\sum_{j=1}^{k}\gamma_j(z)e^{k\alpha_j(z)x}\right) \\ &= -\frac{1}{\pi u}\operatorname{Im} M_x^-(u^{-1/k}) = \frac{1}{\pi u}\operatorname{Im} M_x^+(u^{-1/k}). \end{aligned}$$

Hence, by Lemma 5.1 we get that for $u\in(0,e^k)$,

$$h_x(u) = \frac{1}{\pi u}\frac{\operatorname{Im}(\rho^+(u^{-1/k}))}{k|\rho^+(u^{-1/k})|^2}\sum_{j=0}^{k} c_j(u^{-1/k})e^{ka_j(u^{-1/k})x}.$$

By (3.11),

$$\varphi(y) = \frac{1}{\pi y}\frac{\operatorname{Im}(\rho^+(1/y))}{|\rho^+(1/y)|^2}, \quad 0<y<e.$$

Hence

$$h_x(u) = \frac{u^{\frac{1}{k}-1}\varphi(u^{1/k})}{k}\sum_{j=0}^{k} c_j(u^{-1/k})e^{\,ka_j(u^{-1/k})x}. \qquad \square \tag{5.8}$$

Remark 5.3. In order to derive Theorem 2.8 from Lemma 5.2, we will have to prove $\nu_x(\{0\})=0$ for almost all $x\in[0,1]$ with respect to Lebesgue measure. This is done in the proof of Lemma 5.4 below. Actually it can be proved that $\nu_x(\{0\})=0$ for *all* $x>0$. This can be obtained from the formula

$$\nu_x(\{0\}) = \lim_{\lambda\to 0^-}\lambda G_x(\lambda)$$

(cf. proof of Lemma 3.4) together with the following asymptotic formula for $\rho(z)$ for large values of $|z|$:

$$\rho(z) = -\log(-z) + \log(\log(-z)) + O\left(\frac{\log(\log|z|))}{\log |z|}\right),$$

where $\log(-z)$ is the principal value of the logarithm. The latter formula can be obtained from [2, pp. 347–350] using (2.11).

Lemma 5.4. *Let* $\nu = \mu_{D_0, S_k^k}$ *be the measure on* $[0,1] \times [0, e^k]$ *defined in* (2.4). *Then* ν *is absolutely continuous with respect to the Lebesgue measure, and its density is given by*

$$\frac{d\nu(x,u)}{dx\,du} = h_x(u), \quad x \in (0,1), \quad u \in (0, e^k),$$

where $h_x(u)$ *is given by* (5.8).

Proof. For $m, n \in \mathbf{N}_0$ we have from Lemma 5.2 and Theorem 2.5 that

$$\begin{aligned} \int\int_{[0,1]\times[0,e^k]} x^m u^n \, d\nu(x,u) &= \mathrm{tr}(D_0^m S_k^{kn}) = \mathrm{tr}(D_0^m E_{\mathscr{D}}(S_k^{kn})) \\ &= \int_0^t x^m (k^{nk} P_{k,n}(x))\, dx \\ &= \int_0^1 x^m \left(\int_e^{e^k} u^n \, d\nu_x(u) \right) dx. \end{aligned} \tag{5.9}$$

Put $g(x) = \nu_x(\{0\})$, $x \in [0,1]$. From the definition of ν_x it is clear that $x \to \nu_x$ is a w^*-continuous function from $[0,1]$ to $\mathrm{Prob}([0, e^k])$, i.e.

$$x \to \int_0^{e^k} f(u)\, d\nu_x(u), \quad x \in [0,1]$$

is continuous for all $f \in C([0, e^k])$. Put for $j \in \mathbf{N}$,

$$f_j(u) = \begin{cases} j, & 0 \leqslant u \leqslant 1/j, \\ 0, & u > 1/j. \end{cases}$$

Then $g(x) = \lim_{j\to\infty} (\int_0^{e^k} f_j(u)\, d\nu_x(u))$, and hence g is a Borel function on $[0,1]$. Putting now $m = 0$ in (5.9) we get

$$\mathrm{tr}(S_k^{kn}) = \int_0^1 \left(\int_0^{e^k} u^n h_x(u)\, du \right) dx, \quad n = 1, 2, \ldots \tag{5.10}$$

and for $n = 0$ we get

$$1 = \int_0^1 g(x)\,dx + \int_0^1 \left(\int_0^{e^k} h_x(u)\,du \right) dx. \tag{5.11}$$

Let $\lambda \in \mathrm{Prob}([0, e^k])$ be the distribution of S_k^k. Then

$$\int_0^{e^k} u^n \, d\lambda(u) = \mathrm{tr}(S_k^{kn})$$

so by (5.10) and (5.11), $\lambda(\{0\}) = \int_0^1 g(x)\,dx$ and λ is absolutely continuous on $(0, e^k]$ with respect to Lebesgue measure, with density $u \to \int_0^1 h_x(u)\,dx$, $u \in (0, e^k)$. However by (2.9) S_k^k and $(T^*T)^k$ have the same moments. Thus S_k^k and $(T^*T)^k$ have the same distribution measure. By ([4, Section 8]), $\ker(T^*T) = \ker(T) = \{0\}$. Hence $\lambda(\{0\}) = 0$, which implies that $g(x) = 0$ for almost all $x \in [0, 1]$. Thus, using (5.9), we have for all $m, n \in \mathbf{N}_0$

$$\int_{[0,1]\times[0,e^k]} x^m u^n \, d\nu(x, u) = \int_0^1 x^m \left(\int_0^{e^k} u^n h_x(u)\,du \right) dx.$$

Hence by Stone–Weierstrass Theorem, ν is absolutely continuous with respect to two-dimensional Lebesgue measure, and

$$\frac{d\nu(x,u)}{dx\,du} = h_x(u), \quad x \in (0,1), \quad u \in (0, e^k). \qquad \square$$

Proof of Theorem 2.8 for $k \geqslant 2$. Let f, g be bounded Borel functions on $[0, 1]$ and $[0, e]$ respectively, and put

$$g_1(u) = g(u^{1/k}), \quad u \in [0, e^k].$$

By Lemma 5.4,

$$\begin{aligned} \mathrm{tr}(f(D_0)g(S_k)) = \mathrm{tr}(f(D_0)g_1(S_k^k)) &= \int\int_{[0,1]\times[0,e^k]} f(x)g_1(u)h_x(u)\,dx\,du \\ &= \int\int_{[0,1]\times[0,e]} f(x)g(y)h_x(y^k)ky^{k-1}\,dx\,dy, \end{aligned}$$

where the last equality is obtained by substituting $u = y^k$, $y \in [0, e]$. Hence the measure μ_{D_0,S_k} is absolutely continuous with respect to the two-dimensional

Lebesgue measure, and by (5.8) the density is given by

$$h_x(y^k)ky^{k-1} = \varphi(y) \sum_{j=0}^{\infty} c_j\left(\frac{1}{y}\right)e^{ka_j(\frac{1}{y})x}$$

for $x \in (0,1)$, $y \in (0,e)$. □

6. Proof of Theorem 2.8 ⇒ Theorem 2.2

Lemma 6.1. *Let $k \in \mathbf{N}$ and let $a_0, \dots, a_k$ be distinct numbers in $\mathbb{C}\backslash\{0\}$ and put*

$$b_j = \prod_{\substack{\ell=0\\ \ell\neq j}}^{k} \frac{a_\ell}{a_\ell - a_j}.$$

Then

$$\sum_{j=0}^{k} b_j a_j^p = 0 \quad p = 1,2,\dots,k, \tag{6.1}$$

$$\sum_{j=0}^{k} b_j = 1, \tag{6.2}$$

$$\sum_{j=0}^{k} b_j a_j^{-1} = \sum_{j=0}^{k} a_j^{-1}, \tag{6.3}$$

$$\sum_{j=0}^{k} b_j a_j^{-2} = \sum_{0\leqslant i\leqslant j\leqslant k} (a_i a_j)^{-1}. \tag{6.4}$$

Proof. By applying Lemma 4.1 to the $k+1$ numbers $a_0, \dots, a_k$, we get (6.1) and (6.2). Moreover, (6.3) follows from the residue calculus argument in Remark 4.3 (cf. (4.14)), and (6.4) follows by a similar argument. Indeed, letting g be the rational function

$$g(s) = \frac{1}{s^3}\prod_{\ell=0}^{k}\left(\frac{a_\ell}{a_\ell - s}\right), \quad s \in \mathbf{C}\backslash\{0, a_0, \dots, a_k\},$$

we have $\mathrm{Res}(g; a_j) = -\frac{1}{a_j^2}\prod_{\ell\neq j}\frac{a_\ell}{a_\ell - a_j} = -b_j a_j^{-2}$ and $\mathrm{Res}(g;0)$ is the coefficient of s^2 in the power-series expansion of

$$s^3 g(s) = \prod_{\ell=0}^{k}\left(1 - \frac{s}{a_\ell}\right)^{-1} = \prod_{\ell=0}^{k}\left(1 + \frac{s}{a_\ell} + \frac{s^2}{a_\ell^2} + \cdots\right).$$

Hence $\mathrm{Res}(g;0) = \sum_{0\leqslant i\leqslant j\leqslant k} (a_i a_j)^{-1}$. Since $g(s) = O(|s|^{-(k+4)})$ as $|s|\to\infty$, as in Remark 4.3 we get

$$\mathrm{Res}(g;0) + \sum_{j=0}^{k} \mathrm{Res}(g;a_j) = 0.$$

This proves (6.4). □

Lemma 6.2. *Let* $k\in\mathbf{N}$ *be fixed and let* $a_j(t)$, $c_j(t)$ *for* $t\in(\frac{1}{e},\infty)$ *and* $j = 0,\dots,k$ *be defined as in* (2.14) *and* (2.15). *Put*

$$H(x,t) = \sum_{j=0}^{k} c_j(t)e^{ka_j(t)x}, \quad x\in\mathbf{R}, \quad t>1/e, \tag{6.5}$$

$$m(t) = -\frac{1}{k}\sum_{j=0}^{k} a_j(t)^{-1}, \tag{6.6}$$

$$v(t) = \frac{1}{k^2}\sum_{j=0}^{k} a_j(t)^{-2}. \tag{6.7}$$

Then

$$\int_0^1 H(x,t)\,dx = 1. \tag{6.8}$$

Moreover, if $k\geqslant 2$, *then*

$$\int_0^1 xH(x,t)\,dx = m(t) \tag{6.9}$$

and if $k\geqslant 3$, *then*

$$\int_0^1 x^2H(x,t)\,dx = m(t)^2 + v(t). \tag{6.10}$$

Proof. For a fixed $t\in(\frac{1}{e},\infty)$, we will apply Lemma 6.1 to the numbers $a_j(t)$, $j = 0,\dots,k$ and

$$b_j(t) = \prod_{\ell\neq j}\frac{a_\ell(t)}{a_\ell(t) - a_j(t)}. \tag{6.11}$$

Note that by (2.15)

$$c_j(t) = -ka_j(t)b_j(t). \tag{6.12}$$

Since t is fixed, we will drop the t in $a_j(t)$, $b_j(t)$ and $c_j(t)$ in the rest of this proof. We have

$$\int_0^1 H(x,t)\,dx = \sum_{j=0}^{k} \frac{c_j}{ka_j}(e^{ka_j}-1) = \sum_{j=0}^{k} b_j(1-e^{ka_j}). \tag{6.13}$$

Recall that

$$\begin{cases} a_0 = \rho^+(t), \\ a_j = \rho(te^{i\frac{2\pi j}{k}}), \quad 1 \leqslant j \leqslant n, \\ a_k = \rho^-(t), \end{cases}$$

where $t \in (\frac{1}{e}, \infty)$. Since $\rho(z)e^{-\rho(z)} = z$ for $z \in \mathbf{C} \setminus [\frac{1}{e}, \infty)$ we get in the limit $z \to t$ with $\mathrm{Im}\, z > 0$, respectively $\mathrm{Im}\, z < 0$, that also

$$\rho^+(t)e^{-\rho^+(t)} = \rho^-(t)e^{-\rho^-(t)} = t.$$

Hence

$$(a_je^{-a_j})^k = \left(te^{i\frac{2\pi j}{k}}\right)^k = t^k, \quad j = 0, \dots, k,$$

which shows

$$e^{ka_j} = \left(\frac{a_j}{t}\right)^k, \quad j = 0, \dots, k. \tag{6.14}$$

Hence by (6.13), (6.1) and (6.2) we get

$$\int_0^1 H(x,t)\,dx = \sum_{j=0}^{k} b_j - \frac{1}{t^k}\sum_{j=0}^{k} b_j a_j^k = 1,$$

which proves (6.8). Moreover,

$$\int_0^1 xH(x,t)\,dx = \sum_{j=0}^{k} (-ka_jb_j)\left[x\frac{e^{ka_jx}}{ka_j} - \frac{e^{ka_jx}}{(ka_j)^2}\right]_0^1.$$

Using (6.14), (6.1) and (6.3) we get

$$\int_0^1 xH(x,t)\,dx = -\frac{1}{t^k}\sum_{j=0}^{k} b_ja_j^k + \frac{1}{kt^k}\sum_{j=0}^{k} b_ja_j^{k-1} - \frac{1}{k}\sum_{j=0}^{k}\frac{b_j}{a_j} = -\frac{1}{k}\sum_{j=0}^{k}\frac{1}{a_j} = m(t)$$

provided $k \geqslant 2$. This proves (6.9). Similarly

$$\int_0^1 x^2 H(x,t)\,dx = \sum_{j=0}^{k} (-ka_j b_j)\left[x^2 \frac{e^{ka_j x}}{ka_j} 2x \frac{e^{ka_j x}}{(ka_j)^2} + 2\frac{e^{ka_j x}}{(ka_j)^3}\right]_0^1$$
$$= -\frac{1}{t^k}\sum_{j=0}^{k} b_j a_j^k + \frac{2}{kt^k}\sum_{j=0}^{k} b_j a_k^{k-1} - \frac{2}{k^2 t^k}\sum_{j=0}^{k} b_j a_j^{k-2} + \frac{2}{k^2}\sum_{j=0}^{k}\frac{b_j}{a_j^2}.$$

Hence by (6.1) and (6.4), we get for $k \geqslant 3$

$$\int_0^1 x^2 H(x,t)\,dx = \frac{2}{k^2}\sum_{0\leqslant i\leqslant j\leqslant k} (a_i a_j)^{-1} = \frac{1}{k^2}\left(\left(\sum_{j=0}^{k} a_j^{-1}\right)^2 + \sum_{j=0}^{k} a_j^{-2}\right)$$
$$= m(t)^2 + v(t). \qquad \square$$

The functions H, m, v, a_j, c_j in Lemma 5.2 depend on $k \in \mathbf{N}$. Therefore we will in the rest of this section rename them $H_k, m_k, v_k, a_{kj}, c_{kj}$. Let $F(y) = \int_0^y \varphi(u)\,du$, $y \in [0, e]$ as in Proposition 2.1. Since φ is the density of a probability measure on $[0, e]$, we have

$$0 \leqslant F(y) \leqslant 1, \quad y \in [0, e]. \tag{6.15}$$

Lemma 6.3. *For* $t \in (\frac{1}{e}, \infty)$,

$$\lim_{k\to\infty} m_k(t) = F\left(\frac{1}{t}\right), \tag{6.16}$$

$$\lim_{k\to\infty} v_k(t) = 0. \tag{6.17}$$

Proof.

$$m_k(t) = -\frac{1}{k}\sum_{j=0}^{k} a_{kj}(t)^{-1} = -\frac{1}{k}\left(\sum_{j=0}^{k} f\left(\frac{j}{k}\right)\right),$$

where $f : [0,1] \to \mathbf{C}$ is the continuous function

$$f(u) = \begin{cases} \rho^+(t)^{-1}, & u = 0, \\ \rho(te^{i2\pi u})^{-1}, & 0 < u < 1, \\ \rho^-(t)^{-1}, & u = 1. \end{cases}$$

Hence

$$\lim_{k\to\infty} m_k(t) = -\int_0^1 f(u)\,du = -\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\rho(te^{i\theta})}\,d\theta = -\frac{1}{2\pi i}\int_{\partial B(0,t)}\frac{1}{z\rho(z)}\,dz. \tag{6.18}$$

To evaluate the RHS of (6.18) we apply the residue theorem to compute the integral of $(z\rho(z))^{-1}$ along the closed path C_ϵ, $0<\epsilon<\frac{1}{e}$, which is drawn in Fig. 1.

Since $\rho(z)\neq 0$ when $z\neq 0$ we have

$$\frac{1}{2\pi i}\int_{C_\epsilon}\frac{dz}{z\rho(z)} = \operatorname{Res}\left(\frac{1}{z\rho(z)};0\right)$$

and by (3.5), $\operatorname{Res}(\frac{1}{z\rho(z)},0) = -1$. Thus, taking the limit $\epsilon\to 0^+$, we get

$$\frac{1}{2\pi i}\left(\int_{1/e}^{t}\frac{dt}{t\rho^+(t)} + \int_{\partial B(0,t)}\frac{dz}{z\rho(z)} + \int_{t}^{1/e}\frac{dt}{t\rho^-(t)}\right) = -1.$$

Since $\rho^-(t) = \overline{\rho^+(t)}$, we get by (3.11)

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial B(0,t)}\frac{dz}{z\rho(z)}\\
&\quad = -1 - \frac{1}{\pi}\int_{1/e}^{t}\frac{1}{s}\operatorname{Im}\left(\frac{1}{\rho^+(s)}\right)ds = -1 + \frac{1}{\pi}\int_{1/e}^{t}\frac{\operatorname{Im}\rho^+(s)}{s|\rho^+(s)|^2}\,ds\\
&\quad = -1 + \int_{1/e}^{t}\frac{1}{s^2}\varphi\left(\frac{1}{s}\right)ds = -1 + \int_{1/t}^{e}\varphi(u)\,du\\
&\quad = -1 + F(1) - F(1/t) = -F(1/t).
\end{aligned}$$

Hence (6.16) follows from (6.18). In the same way we get

$$v_k(t) = \frac{1}{k^2}\sum_{j=0}^{k} f\left(\frac{j}{k}\right)^2.$$

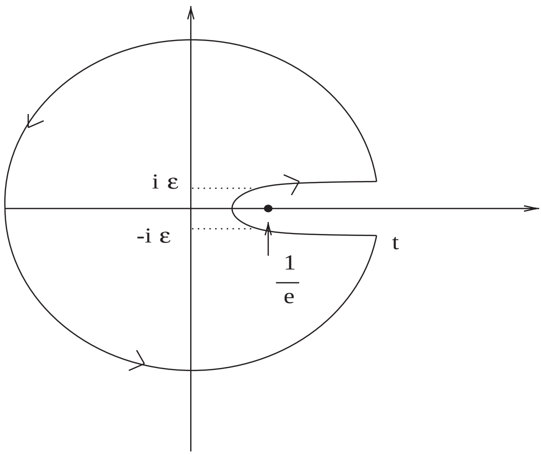

Fig. 1. The contour C_ε.

Hence

$$\lim_{k\to\infty} kv_k(t) = \int_0^1 f(u)^2\,du,$$

so in particular

$$\lim_{k\to\infty} v_k(t) = 0. \qquad \square$$

Proof of Theorem 2.2. By Lemma 2.4, Theorem 2.8 and (6.5),

$$\|D_0 - F(S_k)\|_2^2 = \int\int_{[0,1]\times[0,e]} |x - F(y)|^2 \varphi(y) H_k(x, \frac{1}{y})\,dx\,dy.$$

Moreover by (6.8)–(6.10) we have for $y \in (0,e)$ and $k \geqslant 3$,

$$\begin{aligned}\int_0^1 (x - F(y))^2 H_k\left(x, \frac{1}{y}\right) dx &= \left(v_k\left(\frac{1}{y}\right) + m_k\left(\frac{1}{y}\right)^2\right) - 2m_k\left(\frac{1}{y}\right)F(y) + F(y)^2 \\ &= \left(m_k\left(\frac{1}{y}\right) - F(y)\right)^2 + v_k\left(\frac{1}{y}\right).\end{aligned}$$

Hence for $k \geqslant 3$

$$\|D_0 - F(S_k)\|_2^2 = \int_0^e \left(\left(m_k\left(\frac{1}{y}\right) - F(y)\right)^2 + v_k\left(\frac{1}{y}\right)\right)\varphi(y)\,dy.$$

Since $\varphi(y)H_k(x, \frac{1}{y})$ is a continuous density function for the probability measure $\mu_{D_0 S_k}$ on $(0,1) \times (0,e)$, and since $\varphi(y) > 0$, $0 < y < e$, we have $H_k(x,t) \geqslant 0$ for all $x \in (0,1)$ and $t \in (\frac{1}{e}, \infty)$. Thus by (6.8)–(6.10), $m_k(t)$ and $v_k(t)$ are the mean and variance of a probability measure on $(0,1)$. In particular $0 \leqslant m_k(t) \leqslant 1$ and $0 \leqslant v_k(t) \leqslant 1$ for all $t > 1/e$. Hence by (6.16), (6.17) and Lebesgue's dominated convergence theorem

$$\lim_{k\to\infty} \|D_0 - F(S_k)\|_2^2 = 0.$$

Hence $D_0 \in W^*(T)$. For $0 < t < 1$, the subspace $\mathscr{H}_t = 1_{[0,t]}(D_0)\mathscr{H}$ is clearly T-invariant, and since $D_0 \in W^*(T)$, $\mathscr{H}_t$ is affiliated with $W^*(T)$. $\square$

7. Hyperinvariant subspaces for T

In this section, we prove Theorem 2.9. The proof relies on the following four results. Lemma 7.2 is probably well known, but we include a proof for convenience.

Lemma 7.1. *For every* $k \in \mathbf{N}$, $\|T^k\| = (\frac{e}{k})^{k/2}$.

Proof. By (2.10), $||T^k||^2 = ||(T^*)^k T^k|| = k^{-k}||S^k|| = (\frac{e}{k})^k$. □

Lemma 7.2. *Let* $(S_\lambda)_{\lambda\in\Lambda}$ *be a bounded net of selfadjoint operators on a Hilbert space* $\mathscr{H}$ *which converges in strong operator topology to the selfadjoint operator* $S\in\mathscr{B}(\mathscr{H})$, *and let* $\sigma_p(S)$ *denote the set of eigenvalues of* S. *Then for all* $t\in\mathbf{R}\backslash\sigma_p(S)$, *we have*

$$\lim_{\lambda\in\Lambda} 1_{(-\infty,t]}(S_\lambda) = 1_{(-\infty,t]}(S), \tag{7.1}$$

where the limit is in strong operator topology.

Proof. There is a compact interval $[a,b]$ such that $\sigma(S_\lambda)\subseteq[a,b]$ for all λ and $\sigma(S)\subseteq[a,b]$. Therefore, given a continuous function $\phi:\mathbf{R}\to\mathbf{R}$, approximating by polynomials we get

$$\lim_{\lambda\in\Lambda} \phi(S_\lambda) = \phi(S),$$

in strong operator topology. Let $t\in\mathbf{R}$, let $\varepsilon>0$ and choose a continuous function $\phi:\mathbf{R}\to\mathbf{R}$ such that $0\leqslant\phi\leqslant1$, $\phi(x)=1$ for $x\leqslant t-\varepsilon$ and $\phi(x)=0$ for $x\geqslant t$. Then for every $\xi\in\mathscr{H}$

$$\langle 1_{(-\infty,t-\varepsilon]}(S)\xi,\xi\rangle \leqslant \langle\phi(S)\xi,\xi\rangle = \lim_{\lambda\in\Lambda}\langle\phi(S_\lambda)\xi,\xi\rangle \leqslant \liminf_{\lambda\in\Lambda}\langle 1_{(-\infty,t]}(S_\lambda)\xi,\xi\rangle.$$

Hence taking the limit as ε^+, we get

$$\langle 1_{(-\infty,t)}(S)\xi,\xi\rangle \leqslant \liminf_{\lambda\in\Lambda}\langle 1_{(-\infty,t]}(S_\lambda)\xi,\xi\rangle. \tag{7.2}$$

Similarly, by using a continuous function $\psi:\mathbf{R}\to\mathbf{R}$ satisfying $\psi(x)=1$ for $x\leqslant t$ and $\psi(x)=0$ for $x\geqslant t+\varepsilon$, we get

$$\langle 1_{(-\infty,t]}(S)\xi,\xi\rangle \geqslant \limsup_{\lambda\in\Lambda}\langle 1_{(-\infty,t]}(S_\lambda)\xi,\xi\rangle. \tag{7.3}$$

If $t\notin\sigma_p(S)$, then $1_{(-\infty,t)}(S)=1_{(-\infty,t]}(S)$, and thus by (7.2) and (7.3), we have

$$\lim_{\lambda\in\Lambda} 1_{(-\infty,t]}(S_\lambda) = 1_{(-\infty,t]}(S), \tag{7.4}$$

with convergence in weak operator topology. However, the weak and strong operator topologies coincide on the set of projections in $\mathscr{B}(\mathscr{H})$. Hence we have convergence (7.1) in strong operator topology, as desired. □

Proposition 7.3. *Let* $F:[0,e]\to[0,1]$ *be the increasing function defined in Proposition* 2.1 *and fix* $t\in[0,1]$. *Let*

$$\mathscr{L}_t = \left\{\xi\in\mathscr{H} \mid \exists\xi_k\in\mathscr{H}, \lim_{k\to\infty}||\xi_k-\xi||=0, \quad \limsup_{k\to\infty}\left(\frac{k}{e}||T^k\xi_k||^{2/k}\right)\leqslant t\right\}.$$

Then $\mathscr{L}_t = \mathscr{H}_{F(et)}$.

Proof. For $t=1$, we have by Lemma 7.1 that $\mathscr{L}_1=\mathscr{H}=\mathscr{H}_1=\mathscr{H}_{F(e)}$. Assume now $0\leqslant t<1$, and let $\xi\in\mathscr{H}_{F(et)}=1_{[0,F(et)]}(D_0)\mathscr{H}=1_{[0,et]}(F(D_0))\mathscr{H}$. Since $\sigma_p(D_0)=\emptyset$ and since F is one-to-one, we also have $\sigma_p(F(D_0))=\emptyset$. Hence, by Theorem 2.8 and Lemma 7.2,

$$\lim_{k\to\infty} 1_{[0,et]}(S_k)\xi = 1_{[0,et]}(F(D_0))\xi=\xi.$$

Let $\xi_k=1_{[0,et]}(S_k)\xi$. Then as we just showed, $\lim_{k\to\infty}||\xi-\xi_k||=0$. Moreover, since $(T^*)^kT^k=k^{-k}S_k^k$, we have

$$||T^k\xi_k||^2=k^{-k}\langle S_k^k\xi_k,\xi_k\rangle\leqslant k^{-k}(et)^k||\xi_k||^2\leqslant\left(\frac{et}{k}\right)^k||\xi||^2.$$

Hence $\limsup_{k\to\infty}(\frac{k}{e}||T^k\xi_k||^{2/k})\leqslant t$, which proves $\mathscr{H}_{F(et)}\subseteq\mathscr{L}_t$. To prove the reverse inclusion, let $\xi\in\mathscr{L}_t$ and choose $\xi_k\in\mathscr{H}$ such that

$$\lim_{k\to\infty}||\xi_k-\xi||=0,\qquad \limsup_{k\to\infty}\left(\frac{k}{e}||T^k\xi_k||^{2/k}\right)\leqslant t. \tag{7.5}$$

By (2.10), $\sigma(S_k)=[0,e]$. Let E_k be the spectral measure of S_k and let

$$\gamma_k(B)=\langle E_k(B)\xi_k,\xi_k\rangle$$

for every Borel set $B\subseteq[0,e]$. Then γ_k is a finite Borel measure on $[0,e]$ of total mass $\gamma_k([0,e])=||\xi_k||^2$ and for all bounded Borel functions $f:[0,e]\to\mathbf{C}$, we have

$$\langle f(S_k)\xi_k,\xi_k\rangle=\int_0^e f d\gamma_k. \tag{7.6}$$

In particular,

$$\langle S_k^k\xi_k,\xi_k\rangle=\int_0^e x^k d\gamma_k(x).$$

Let $0<\varepsilon<1-t$. By (7.5), there exists $k_0\in\mathbf{N}$ such that $\frac{k}{e}||T^k\xi_k||^{2/k}\leqslant t+\frac{\varepsilon}{2}$ for all $k\geqslant k_0$. Thus,

$$\int_0^e x^k d\gamma_k(x)=\langle S_k^k\xi_k,\xi_k\rangle=k^k||T^k\xi_k||^2\leqslant\left(e\left(t+\frac{\varepsilon}{2}\right)\right)^k\quad(k\geqslant k_0).$$

Since $(\frac{x}{e(t+\varepsilon)})^k\geqslant 1$ for $x\in[e(t+\varepsilon),e]$, we have

$$\gamma_k([e(t+\varepsilon),e])\leqslant\int_0^e\left(\frac{x}{e(t+\varepsilon)}\right)^k d\gamma_k(x)\leqslant\left(\frac{t+\frac{\varepsilon}{2}}{t+\varepsilon}\right)^k||\xi_k||^2.$$

Hence, by (7.6),

$$||1_{(e(t+\varepsilon),\infty)}(S_k)\xi_k||^2=\langle 1_{(e(t+\varepsilon),\infty)}(S_k)\xi_k,\xi_k\rangle\leqslant\left(\frac{t+\frac{\varepsilon}{2}}{t+\varepsilon}\right)^k||\xi_k||^2,$$

which tends to zero as $k \to \infty$. Since $||\xi_k - \xi|| \to 0$ as $k \to \infty$, we get

$$\lim_{k \to \infty} ||1_{(e(t+\varepsilon),\infty)}(S_k)\xi|| = 0,$$

which is equivalent to

$$\lim_{k \to \infty} 1_{[0,e(t+\varepsilon)]}(S_k)\xi = \xi.$$

Hence, by Theorem 2.8 and Lemma 7.2,

$$1_{[0,F(e(t+\varepsilon))]}(D_0)\xi = 1_{[0,e(t+\varepsilon)]}(F(D_0))\xi = \xi,$$

i.e. $\xi \in \mathscr{H}_{F(e(t+\varepsilon))}$ for all $\varepsilon \in (0, 1-t)$. Since

$$\mathscr{H}_{F(et)} = \bigcap_{s \in (F(et),1)} \mathscr{H}_s,$$

it follows that $\mathscr{L}_t \subseteq \mathscr{H}_{F(et)}$, which completes the proof of the proposition. $\square$

Lemma 7.4. *Let $t \in (0,1)$ and define $(a_n)_{n=1}^{\infty}$ recursively by*

$$a_1 = F(et), \tag{7.7}$$

$$a_{n+1} = a_n F\left(\frac{et}{a_n}\right). \tag{7.8}$$

Then $(a_n)_{n=1}^{\infty}$ is a strictly decreasing sequence in $[0,1]$ and $\lim_{n\to\infty} a_n = t$.

Proof. The function $x \mapsto F(ex)$ is a strictly increasing, continuous bijection of $[0,1]$ onto itself. By definition, the restriction of F to $(0,e)$ is differentiable with continuous derivative

$$F'(x) = \phi(x), \quad x \in (0,e),$$

where ϕ is uniquely determined by

$$\phi\left(\frac{\sin v}{v}\exp(v\cot v)\right) = \frac{1}{\pi}\sin v \exp(-v\cot v).$$

As observed in the proof of [4, Theorem 8.9], the map $v \mapsto \frac{\sin v}{v}\exp(v\cot v)$ is a strictly decreasing bijection from $(0,\pi)$ onto $(0,e)$. Moreover,

$$\frac{d}{dv}(\sin v \exp(-v\cot v)) = \frac{v}{\sin v}\exp(-v\cot v) > 0$$

for $v \in (0, \pi)$. Hence ϕ is a strictly decreasing function on $(0, e)$, which implies that F is strictly convex on $[0, e]$. Hence

$$F(ex) > (1 - x)F(0) + xF(e) = x, \quad x \in (0, 1). \tag{7.9}$$

With $t \in (0, 1)$ and with $(a_n)_{n=1}^{\infty}$ defined by (7.7) and (7.8), from (7.9) we have $a_1 = F(et) \in (t, 1)$. If $a \in (t, 1)$ and if $a' = aF(\frac{et}{a})$, then clearly $a' < a$. Moreover, by (7.9),

$$a' = aF\left(\frac{et}{a}\right) > a \cdot \frac{t}{a} = t.$$

Hence $(a_n)_{n=1}^{\infty}$ is a strictly decreasing sequence in $(t, 1)$ and therefore converges. Let $a_\infty = \lim_{n\to\infty} a_n$. Then by the continuity of F on $[0, e]$, we have

$$a_\infty = a_\infty F\left(\frac{et}{a_\infty}\right).$$

Hence $F(\frac{et}{a_\infty}) = 1$, which implies $a_\infty = t$. $\square$

Proof of Theorem 2.9. Let $T = \mathcal{UT}(X, \lambda)$ be constructed using [4, Section 4], as described in the introduction. For $t \in [0, 1]$, let

$$\mathcal{K}_t = \left\{ \xi \in \mathcal{H} \mid \limsup_{n\to\infty} \left(\frac{k}{e} \|T^k \xi\|^{2/k} \right) \leqslant t \right\}. \tag{7.10}$$

We will show

$$\mathcal{H}_t \subseteq \mathcal{K}_t \subseteq \mathcal{H}_{F(et)}, \quad t \in [0, 1]. \tag{7.11}$$

The second inclusion in (7.11) follows immediately from Proposition 7.3. The first inclusion is trivial for $t = 0$, so we can assume $t > 0$. Letting $P_t = 1_{[0,t]}(D_0)$ be the projection onto $\mathcal{H}_t$, from [4, Lemma 4.10] we have

$$T_t \overset{\text{def}}{=} \frac{1}{\sqrt{t}} T\restriction_{\mathcal{H}_t} = P_t T P_t = \mathcal{UT}\left(\frac{1}{\sqrt{t}} P_t X P_t, \lambda_t \right), \tag{7.12}$$

where $\lambda_t : L^\infty[0, 1] \to P_t L(\mathbf{F}_2) P_t$ is the injective, normal $*$-homomorphism given by $\lambda_t(f) = \lambda(f_t)$, where

$$f_t(s) = \begin{cases} f(s/t) & \text{if } s \in [0, t], \\ 0 & \text{if } s \in (t, 1]. \end{cases}$$

Therefore, T_t is itself a DT$(\delta_0, 1)$-operator in $(P_t \mathcal{M} P_t, t^{-1}\tau\restriction_{P_t \mathcal{M} P_t})$. Hence, by Lemma 7.1 applied to T_t, we have, for all $\xi \in \mathcal{H}_t$,

$$\|T^k \xi\| = t^{k/2} \|T_t^k \xi\| \leqslant \left(\frac{te}{k} \right)^{k/2} \|\xi\|.$$

Therefore, $\limsup_{k\to\infty}(\frac{k}{e}||T^k\xi||^{2/k})\leqslant t$ and $\xi\in\mathscr{K}_t$. This completes the proof of (7.11).

From (7.11), we have in particular $\mathscr{K}_0=\mathscr{H}_0=\{0\}$ and $\mathscr{K}_1=\mathscr{H}_1=\mathscr{H}$. Let $t\in(0,1)$ and let $(a_n)_{n=1}^{\infty}$ be the sequence defined by Lemma 7.4. We will prove by induction on n that $\mathscr{K}_t\subseteq\mathscr{H}_{a_n}$. By (7.11), $\mathscr{K}_t\subseteq\mathscr{H}_{a_1}$. Let $n\in\mathbf{N}$ and assume $\mathscr{K}_t\subseteq\mathscr{H}_{a_n}$. Then

$$\mathscr{K}_t=\left\{\xi\in\mathscr{H}_{a_n}\mid\limsup_{k\to\infty}\left(\frac{k}{e}||T^k\xi||^{2/k}\right)\leqslant t\right\} \tag{7.13}$$

$$=\left\{\xi\in\mathscr{H}_{a_n}\mid\limsup_{k\to\infty}\left(\frac{k}{e}||T_{a_n}^k\xi||^{2/k}\right)\leqslant\frac{t}{a_n}\right\}. \tag{7.14}$$

But the space (7.14) is the analogue of $\mathscr{K}_{t/a_n}$ for the operator T_{a_n}. By (7.11) applied to the operator T_{a_n}, we have that $\mathscr{K}_t$ is contained in the analogue of $\mathscr{H}_{F(et/a_n)}$ for T_{a_n}. Using (7.12) (with a_n instead of t), we see that this latter space is

$$\lambda_{a_n}(1_{[0,F(et/a_n)]})\mathscr{H}_{a_n}=\lambda(1_{[0,a_nF(et/a_n)]})\mathscr{H}_{a_n}=\lambda(1_{[0,a_{n+1}]})\mathscr{H}_{a_n}=\mathscr{H}_{a_{n+1}}.$$

Thus $\mathscr{K}_t\subseteq\mathscr{H}_{a_{n+1}}$ and the induction argument is complete.

Now applying Lemma 7.4, we get $\mathscr{K}_t\subseteq\bigcap_{n=1}^{\infty}\mathscr{H}_{a_n}=\mathscr{H}_t$, as desired. □

Appendix A. $\mathscr{D}$-Gaussianity of T, T^*

The operator T was defined in [4] as the limit in $*$-moments of upper triangular Gaussian random matrices, and it was shown in [4] that T can be constructed as $T=\mathscr{UT}(X,\lambda)$ in a von Neumann algebra $\mathscr{M}$ equipped with a normal, faithful, tracial state τ, from a semicircular element $X\in\mathscr{M}$ with $\tau(X)=0$ and $\tau(X^2)=1$ and an injective, unital, normal $*$-homomorphism $\lambda:L^\infty[0,1]\to\mathscr{M}$ such that $\{X\}$ and $\lambda(L^\infty[0,1])$ are free with respect to τ and $\tau\circ\lambda(f)=\int_0^1 f(t)\,dt$. (See the description in the introduction and [4, Section 4].) Let $\mathscr{D}=\lambda(L^\infty[0,1])$ and let $E_{\mathscr{D}}:\mathscr{M}\to\mathscr{D}$ be the τ-preserving conditional expectation onto $\mathscr{D}$.

In [6], it was asserted that T is a generalized circular element with respect to $E_{\mathscr{D}}$ and with a particular variance. It is the purpose of this appendix to provide a proof.

Lemma A.1. *Let $f\in L^\infty[0,1]$. Then*

$$E_{\mathscr{D}}(T\lambda(f)T^*)=\lambda(g), \tag{A.1}$$

$$E_{\mathscr{D}}(T^*\lambda(f)T)=\lambda(h), \tag{A.2}$$

$$E_{\mathscr{D}}(T\lambda(f)T) = 0, \tag{A.3}$$

$$E_{\mathscr{D}}(T^*\lambda(f)T^*) = 0, \tag{A.4}$$

where

$$g(x) = \int_x^1 f(t)\,dt, \quad h(x) = \int_0^x f(t)\,dt. \tag{A.5}$$

Moreover,

$$E_{\mathscr{D}}(T) = 0. \tag{A.6}$$

Proof. From [4, Section 4], $\lim_{n\to\infty} ||T - T_n|| = 0$, where

$$T_n = \sum_{j=1}^{2^n-1} p\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] Xp\left[\frac{j}{2^n}, 1\right]$$

and $p[a,b] = \lambda(1_{[a,b]})$. Therefore,

$$\lim_{n\to\infty} ||E_{\mathscr{D}}(T\lambda(f)T^*) - E_{\mathscr{D}}(T_n\lambda(f)T_n^*)|| = 0.$$

We have

$$E_{\mathscr{D}}(T_n\lambda(f)T_n^*) = \sum_{j=1}^{2^n-1} p\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] E_{\mathscr{D}}\left(Xp\left[\frac{j}{2^n}, 1\right]\lambda(f)X\right).$$

Fixing n and letting $a = \int_{j/2^n}^1 f(t)\,dt$, we have

$$Xp\left[\frac{j}{2^n}, 1\right]\lambda(f)X = X\left(p\left[\frac{j}{2^n}, 1\right]\lambda(f) - a\right)X + a(X^2 - 1) + a,$$

and from this we see that $E_{\mathscr{D}}(Xp[\frac{j}{2^n}, 1]\lambda(f)X)$ is the constant $\int_{j/2^n}^1 f(t)\,dt$. Therefore, we get $E_{\mathscr{D}}(T_n\lambda(f)T_n^*) = \lambda(g_n)$, where

$$g_n(x) = \begin{cases} \int_{j/2^n}^1 f(t)\,dt & \text{if } \frac{j-1}{2^n} \leqslant x \leqslant \frac{j}{2^n},\ j \in \{1, \dots, 2^n - 1\}, \\ 0 & \text{if } \frac{2^n-1}{2^n} \leqslant x \leqslant 1. \end{cases}$$

Letting $n \to \infty$, we obtain (A.1) with g as in (A.5).

Eqs. (A.2)–(A.4) and (A.6) are obtained similarly. □

Comparing Śniady's definition of a generalized circular element (with respect to $\mathscr{D}$) in [6] with Speicher's algorithm for passing from $\mathscr{D}$-cummulants to $\mathscr{D}$-moments in [7, Sections 2.1 and 3.2], we see that an operator $S \in L(\mathbf{F}_2)$ is generalized circular if and only if all $\mathscr{D}$-cummulants of order $k \neq 2$ for the pair (S, S^*) vanish. Hence S is generalized circular if and only if the pair (S, S^*) is $\mathscr{D}$-Gaussian in the sense of [7, Definition 4.2.3]. Thus, in order to prove that T has the properties used in [6], it suffices to prove the following.

Proposition A.2. *The distribution of the pair T, T^* with respect to $E_{\mathscr{D}}$ is a $\mathscr{D}$-Gaussian distribution with covariance matrix determined by* (A.1)–(A.6).

Proof. Take $X_1, X_2, \ldots \in \mathscr{M}$, each a $(0,1)$-semicircular element such that

$$\mathscr{D}, (\{X_j\})_{j=1}^{\infty}$$

is a free family of sets of random variables. Then the family

$$(W^*(\mathscr{D} \cup \{X_j\}))_{j=1}^{\infty}$$

of $*$-subalgebras of $\mathscr{M}$ is free (over $\mathscr{D}$) with respect to $E_{\mathscr{D}}$. Let $T_j = \mathscr{U}\mathscr{T}(X_j, \lambda)$. Then each T_j has $\mathscr{D}$-valued $*$-distribution (with respect to $E_{\mathscr{D}}$) the same as T. Therefore, by Speicher's $\mathscr{D}$-valued free central limit theorem [7, Theorem 4.2.4], the $\mathscr{D}$-valued $*$-distribution of $\frac{T_1 + \cdots + T_n}{\sqrt{n}}$ converges as $n \to \infty$ to a $\mathscr{D}$-Gaussian $*$-distribution with the correct covariance. However, $\frac{X_1 + \cdots + X_n}{\sqrt{n}}$ is a $(0,1)$-semicircular element that is free from $\mathscr{D}$, and

$$\frac{T_1 + \cdots + T_n}{\sqrt{n}} = \mathscr{U}\mathscr{T}\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}, \lambda\right).$$

Thus $\frac{T_1 + \cdots + T_n}{\sqrt{n}}$ itself has the same $\mathscr{D}$-valued $*$-distribution as T. □

References

[1] A. Connes, Classification of injective factors, Ann. of Math. 104 (1976) 73–115.

[2] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth, On the Lambert W function, Adv. Comput. Math. 5 (1996) 329–359.

[3] K. Dykema, U. Haagerup, Invariant subspaces of Voiculescu's circular operator, Geom. Funct. Anal. 11 (2001) 693–741.

[4] K. Dykema, U. Haagerup, DT-operators and decomposability of Voiculescu's circular operator, Amer. J. Math. (to appear).

[5] P. Śniady, Inequality for Voiculescu's free entropy in terms of Brown measure, Internat. Math. Res. Notices, 2003, 51–64.
[6] P. Śniady, Multinomial identities arising from the free probability, J. Combin. Theory A 101 (2003) 1–19.
[7] R. Speicher, Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, Mem. Amer. Math. Soc. 132 (627) (1998) x + 88.
[8] R. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, Cambridge, 1999.