Random Matrices and Non-Exact C^*-Algebras

U. Haagerup* and S. Thorbjørnsen*

Department of Mathematics and Computer Science
SDU, Odense University
Campusvej 55, 5230 Odense M
Denmark
E-mail: haagerup@imada.sdu.dk
E-mail: steenth@imada.sdu.dk

1 Introduction

In the paper [HT2], we gave new proofs based on random matrix methods of the following two results:

(1) Any unital exact stably finite C^*-algebra has a tracial state.
(2) If $\mathcal{A}$ is a unital exact C^*-algebra, then any state on $K_0(\mathcal{A})$ comes from a tracial state on $\mathcal{A}$.

For each of the results (1) and (2), one may ask whether or not it holds without the assumption that the C^*-algebra be exact. These two problems are still open, and both problems are equivalent to Kaplansky's famous problem, whether all AW^*-factors of type II_1 are von Neumann algebras (cf. [Ha] and [BR]).

In the present note, we provide examples which show that the method used in [HT2] cannot be employed to show that (1) and (2) hold for all C^*-algebras.

As in [HT2], we let $\mathrm{GRM}(m, n, \sigma^2)$ denote the class of complex Gaussian $m \times n$ random matrices of the form

$$B = (b(i,j))_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

for which the $2mn$ real random variables $\mathrm{Re}(b(i,j))$, $\mathrm{Im}(b(i,j))$ are independent and Gaussian distributed random variables with mean 0 and variance $\sigma^2/2$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Moreover, for any bounded operator A on a Hilbert space, we denote by $\mathrm{sp}(A)$ the spectrum of A.

The proofs of (1) and (2) above given in [HT2] were both based on the following theorem:

* MaPhySto - Centre for Mathematical Physics and Stochastics, funded by a grant from The Danish National Research Foundation.

J. Cuntz et al. (eds.), *C*-Algebras*

Theorem 1.1 (cf. [HT2]). *Let $a_1, a_2, \dots, a_r$ be elements of a unital exact C^*-algebra $\mathcal{A}$. Let further $(\Omega, \mathcal{F}, P)$ be a fixed probability space, and let, for each n in $\mathbb{N}$, $Y_1^{(n)}, \dots, Y_r^{(n)}$ be independent Gaussian random matrices defined on Ω and lying in the class* $\mathrm{GRM}(n, n, \frac{1}{n})$ *defined below. Put*

$$S_n = \sum_{i=1}^{r} a_i \otimes Y_i^{(n)}, \qquad (n \in \mathbb{N}),$$

and let c be a positive real number. We then have

(i) *If $\|\sum_{i=1}^r a_i^* a_i\| \le c$ and $\|\sum_{i=1}^r a_i a_i^*\| \le 1$, then for almost all ω in Ω,*

$$\limsup_{n\to\infty} \max\big\{\mathrm{sp}\big(S_n(\omega)^* S_n(\omega)\big)\big\} \le \big(\sqrt{c}+1\big)^2.$$

(ii) *If $\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, $\|\sum_{i=1}^r a_i a_i^*\| \le 1$, and $c \ge 1$, then for almost all ω in Ω,*

$$\liminf_{n\to\infty} \min\big\{\mathrm{sp}\big(S_n(\omega)^* S_n(\omega)\big)\big\} \ge \big(\sqrt{c}-1\big)^2. \qquad \square$$

The upper and lower bounds $(\sqrt{c}+1)^2$ and $(\sqrt{c}-1)^2$ in Theorem 1.1 are best possible. This follows from

Theorem 1.2 (cf. [Th]). *Let $\mathcal{B}$ be a unital exact C^*-algebra and let $b_1, b_2, \dots, b_s$ be elements of $\mathcal{B}$ satisfying that*

$$\sum_{i=1}^{s} b_i^* b_i = c\mathbf{1}_{\mathcal{B}} \quad \text{and} \quad \sum_{i=1}^{s} b_i b_i^* = \mathbf{1}_{\mathcal{B}},$$

for some real number c in $[1, \infty[$. Consider further, for each n in $\mathbb{N}$, independent random matrices $Y_1^{(n)}, Y_2^{(n)}, \dots, Y_s^{(n)}$ in $\mathrm{GRM}(n, n, \frac{1}{n})$, *and put $T_n = \sum_{i=1}^s b_i \otimes Y_i^{(n)}$. Then for almost all ω in Ω,*

$$\max\big\{\mathrm{sp}(T_n(\omega)^* T_n(\omega))\big\} \to (\sqrt{c}+1)^2, \quad \text{as } n \to \infty,$$

and

$$\min\big\{\mathrm{sp}(T_n(\omega)^* T_n(\omega))\big\} \to (\sqrt{c}-1)^2, \quad \text{as } n \to \infty. \qquad \square$$

Let $C^*(\mathbb{F}_r)$ denote the full C^*-algebra associated with the free group $\mathbb{F}_r$ on r generators, and let $u_1, \dots, u_r$ denote the unitary generators of $C^*(\mathbb{F}_r)$. In [HT2, Proposition 4.9] it was proved, that with $a_i = r^{-1/2} u_i$, $i = 1, \dots, r$, and $S_n = \sum_{i=1}^r a_i \otimes Y_i^{(n)}$ as in Theorem 1.1, one has:

$$\liminf_{n\to\infty} \max\{\mathrm{sp}(S_n(\omega)^* S_n(\omega))\} \ge \big(\tfrac{8}{3\pi}\big)^2 r.$$

In particular, for $c \ge 1$ and $r \ge 6c$, the upper bound in Theorem 1.1 is violated because $6c > (\frac{3\pi}{8})^2 4c > (\frac{3\pi}{8})^2(\sqrt{c}+1)^2$. The upper bound in Theorem 1.2 is also violated in the general non-exact case provided that $c \ge 1$ and $r \ge 8c$ (see Remark 4.5 at the end of this paper). The main result in this note concerns the lower bound in Theorem 1.1 and Theorem 1.2: