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Pointwise Inner Automorphisms of Injective Factors

UFFE HAAGERUP

Department of Mathematics and Computer Science, Odense University, Campusvej 55, DK-5230 Odense M, Denmark

AND

ERLING STØRMER

Department of Mathematics, University of Oslo, N-0316 Oslo 3, Norway

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It is shown that for the injective factor of type III_1 with separable predual an automorphism is pointwise inner if and only if it is the composition of an inner and a modular automorphism.

1. INTRODUCTION

If M is a von Neumann algebra an automorphism α of M is called *pointwise inner* if for each normal state φ of M there is a unitary operator $u = u(\varphi)$ in M such that $\varphi \circ \alpha = \varphi \circ \mathrm{Ad}(u)$, see [7]. It was shown in [7] that if M is semifinite with separable predual then each pointwise inner automorphism is inner. When M is a factor of type III_λ, $0 \leqslant \lambda < 1$, with separable predual we showed in [8] that pointwise inner automorphisms were all compositions of inner automorphisms and extended modular automorphisms. We conjectered that in the case of III_1-factors they were necessarily compositions of inner and modular automorphisms. In the present paper we prove this conjecture for the injective III_1-factor. Thus we obtain a complete classification of all pointwise inner automorphisms of injective factors with separable preduals. It turns out that this classification coincides with that of Connes [3] and Kawahigashi *et al.* [10] of centrally trivial automorphisms of injective factors, hence for such factors pointwise inner automorphisms are the same as centrally trivial automorphisms.

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2. Almost Periodic States

Let M be a von Neumann algebra, and suppose $\alpha \in \operatorname{Aut}(M)$—the automorphism group of M—is pointwise inner. Let φ be a normal state and let u be a unitary operator in M so that $\varphi \circ \alpha = u\varphi u^*$. Replacing α by $\alpha \circ \operatorname{Ad} u$ we may assume $\varphi \circ \alpha = \varphi$. In particular α leaves the centralizer M_φ of φ globally invariant. A crucial problem concerning α is whether $\alpha|_{M_\varphi}$ is inner. In [8] we showed this when M is a factor with separable predual and φ is a lacunary faithful normal semifinite weight with infinite multiplicity, and we could thus accomplish the classification for III_λ-factors, $0 \leqslant \lambda < 1$. In the present section we show $\alpha|_{M_\varphi}$ is inner when φ is an almost periodic state, where a faithful normal state is said to be almost periodic if its modular operator Δ_φ has a total set of eigenvectors [1, 3.7.1].

Theorem 2.1. *Let M be a von Neumann algebra with separable predual. Suppose φ is a faithful normal almost periodic state on M. Let α be a pointwise inner automorphism of M such that $\varphi \circ \alpha = \varphi$. Then the restriction $\alpha|_{M_\varphi}$ to the centralizer of φ is an inner automorphism.*

The proof of this proposition is divided into some lemmas. The first is of purely topological character. We say a topological space S is a *Cantor set* if S is homeomorphic with $\{0, 1\}^{\mathbb{N}}$.

Lemma 2.2. *Let $\Gamma \subset \mathbb{R}$ be a countable group. Then there exists a Cantor set $S \subset \mathbb{R}$ such that*

$$(\gamma_1 + S) \cap (\gamma_2 + S) = \varnothing \qquad \textit{for} \quad \gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2.$$

Proof. Put

$$A = \left\{ 3 \sum_{i=1}^{\infty} t_i 4^{-i} : t_i \in \{0, 1\} \text{ for all } i \right\}.$$

Then $A \subset [0, 1]$ and $0 \in A$, $1 \in A$. A is a Cantor set via the mapping $3 \sum t_i 4^{-i} \to (t_i) \in \{0, 1\}^{\mathbb{N}}$. We assert that $A - A$ has no interior points. Indeed,

$$A - A = \left\{ 3 \sum_{i=1}^{\infty} s_i 4^{-i} : s_i \in \{-1, 0, 1\} \right\}$$
$$= \left\{ 3 \sum_{i=1}^{\infty} s_i 4^{-i} : s_i \in \{0, 1, 2\} \right\} - 1,$$

using that $3\sum_{i=1}^{\infty} 4^{-i}=1$. Since $(s_i)_{i=1}^{\infty} \to \sum_{i=0}^{\infty} s_i 4^{-i}$ is a one-to-one function on $\{0, 1, 2\}^{\mathbb{N}}$, $A-A$ is homeomorphic to $\{0, 1, 2\}^{\mathbb{N}}$, hence it is totally disconnected, proving the assertion.

Let $\gamma_1, \gamma_2 \in \Gamma$. Then for a set S

$$(\gamma_1+S) \cap (\gamma_2+S)=\varnothing \Leftrightarrow \gamma_1-\gamma_2 \notin S-S.$$

It therefore suffices to find a Cantor set S such that $(S-S) \cap (\Gamma \backslash \{0\})=\varnothing$.

Let $\gamma \in \Gamma_+ = \Gamma \cap \mathbb{R}_+$. Put

$$O_\gamma=\{\lambda \in \mathbb{R}_+ : \gamma \notin \lambda^{-1}(A-A)\}=\mathbb{R}_+ \backslash \gamma^{-1}(A-A).$$

O_γ is open since $A-A$ is closed, and dense since $A-A$ has no interior points. By Baire's Category Theorem $\bigcap_{\gamma \in \Gamma_+} O_\gamma$ is dense in $\mathbb{R}$. In particular there is $\lambda_0 \in \bigcap_{\gamma \in \Gamma_+} O_\gamma$. Thus $\gamma \notin \lambda_0^{-1}(A-A)$ for all $\gamma \in \Gamma_+$. Since $A-A$ is symmetric we find $\Gamma \backslash \{0\}=\Gamma_- \cup \Gamma_+$ does not intersect $\bigcap_{\gamma \in \Gamma_+} O_\gamma$, hence $S=\lambda_0^{-1}(A-A)$ satisfies the requirements of the lemma. ∎

Lemma 2.3. *Let H be a Hilbert space and $h, k \in B(H)_+$ invertible operators. Suppose $b \in B(H)$ and $\lambda>0$ are such that $bh=\lambda kb$, and $Sp(h) \cap \lambda\, Sp(k)=\varnothing$. Then $b=0$.*

Proof. Taking adjoints we have $hb^*=\lambda b^*k$. Thus $b^*bh=\lambda b^*kb=hb^*b$, so $b^*b \in \{h\}'$, and similarly $bb^* \in \{k\}'$. Let $b=v\,|b|$ be polar decomposition, and put $v_\varepsilon=b(b^*b+\varepsilon 1)^{-1/2}$ for $\varepsilon>0$. Then $v=s.\lim_{\varepsilon \to 0} v_\varepsilon$ (strong limit). Since

$$v_\varepsilon h=bh(b^*b+\varepsilon 1)^{-1/2}=\lambda k v_\varepsilon$$

we find, letting $\varepsilon \to 0$,

$$vh=\lambda kv.$$

Taking adjoints we have $hv^*=\lambda v^*k$, and as above $v^*v \in \{h\}'$, $vv^* \in \{k\}'$. Furthermore we have

$$v(v^*vh)v^*=vhv^*=\lambda vv^*k,$$

hence v is an isometry of $v^*v(H)$ onto $vv^*(H)$ carrying $h|_{v^*v(H)}$ onto $\lambda k|_{vv^*(H)}$. In particular,

$$\mathrm{Sp}(h|_{v^*v(H)})=\lambda\, \mathrm{Sp}(k|_{vv^*(H)}).$$

However, since $v^*v \in \{h\}'$, $\mathrm{Sp}(h|_{v^*v(H)}) \subset \mathrm{Sp}(h)$ and similarly for k. Since by assumption $\mathrm{Sp}(h) \cap \lambda\, \mathrm{Sp}(k)=\varnothing$, $v=0$, and thus $b=0$. ∎

LEMMA 2.4. *Let M be a von Neumann algebra with separable predual. Suppose φ is a faithful normal almost periodic state. Let A be a maximal Abelian subalgebra of M_φ, and suppose α is a pointwise inner automorphism of M such that $\varphi \circ \alpha = \varphi$. Then there exists a unitary operator $u \in M_\varphi$ such that $\alpha|_A = \mathrm{Ad}(u)|_A$.*

Proof. We let $\mathrm{Sp}_d(\sigma^\varphi)$ denote the set of $\gamma \in \mathbb{R}$ for which the spectral subspace

$$M_\gamma = \{x \in M \mid \sigma_t^\varphi(x) = e^{+it\gamma}x,\ t \in \mathbb{R}\}$$

is non-zero. By [1, Section 3.7], one easily gets that $\mathrm{sp}_d(\sigma^\varphi)$ is just the logarithm of the discrete spectrum $\mathrm{sp}_d(\Delta_\varphi)$ of the modular operator Δ_φ, so in particular $\mathrm{sp}_d(\sigma^\varphi)$ is countable. Since φ is almost periodic, the function

$$t \to \varphi(\sigma_t^\varphi(a^*)b), \qquad t \in \mathbb{R},$$

is almost periodic for all $a, b \in M$ (cf. [1, Lemma 3.7.4]). Thus, if m denotes the invariant mean on the almost periodic functions, the map

$$E_\gamma(x) = \int_{-\infty}^{\infty} \sigma_t^\varphi(x)\, e^{-it\gamma}\, dm(t), \qquad x \in M,$$

defines a normal projection of M onto M_γ. In particular, $E_\gamma = 0$ when $\gamma \notin \mathrm{sp}_d(\sigma^\varphi)$.

Since an almost periodic function f on $\mathbb{R}$ is uniquely determined by its "Fourier coefficients"

$$f(\gamma) = \int_{-\infty}^{\infty} f(t)\, e^{-it\gamma}\, dm(t), \qquad \gamma \in \mathbb{R},$$

it follows that two elements $x, y \in M$ coincide if and only if $E_\gamma(x) = E_\gamma(y)$ for all $\gamma \in \mathrm{sp}_d(\sigma^\varphi)$.

Let by Lemma 2.2 $S \subset \mathbb{R}$ be a Cantor set such that $(S - S) \cap \Gamma = \{0\}$, where Γ is the countable group generated by $\mathrm{Sp}_d(\sigma^\varphi)$. Let $a \in A_+$ be a generator for A with $\mathrm{Sp}(a) = S$. This can be done since $A \cong L^\infty(S, d\mu)$ for some purely nonatomic measure μ with $\mathrm{supp}(\mu) = S$. Let $h = e^a$. Then h is a generator for A, and $\mathrm{Sp}(h) = e^S$.

Since α is pointwise inner there is a unitary operator $u \in M$ such that

$$u(h\varphi)u^* = (h\varphi) \circ \alpha^{-1} = \alpha(h)\varphi.$$

Thus

$$u(h\varphi) = (\alpha(h)\, \varphi)u.$$

If $x \in M$ we thus have

$$\begin{aligned}\varphi(x\sigma_t^\varphi(u)h) &= \varphi(\sigma_{-t}^\varphi(x)\, uh) \\ &= (h\varphi)(\sigma_{-t}^\varphi(x)u) \\ &= (\alpha(h)\varphi)(u\sigma_{-t}^\varphi(x)) \\ &= \varphi(\alpha(h)\, u\sigma_{-t}^\varphi(x)) \\ &= \varphi(\alpha(h)\, \sigma_t^\varphi(u)x).\end{aligned}$$

Integrating with respect to dm we thus obtain

$$\varphi(xE_\gamma(u)h) = \varphi(\alpha(h)\, E_\gamma(u)x), \qquad \gamma \in \mathrm{Sp}_d(\sigma^\varphi),$$

or, with $u_\gamma = E_\gamma(u)$,

$$u_\gamma(h\varphi) = (\alpha(h)\varphi)u_\gamma, \qquad \gamma \in \mathrm{Sp}_d(\sigma^\varphi).$$

Since $u_\gamma \in M_\gamma$, $\varphi u_\gamma = e^\gamma u_\gamma \varphi$, see [13] or [14, Lemma 1.6],

$$u_\gamma(h\varphi) = (\alpha(h)\varphi)u_\gamma = e^\gamma \alpha(h)\, u_\gamma \varphi.$$

Since φ is faithful

$$u_\gamma h = e^\gamma \alpha(h) u_\gamma.$$

By hypothesis $S \cap (\gamma + S) = \varnothing$ for $\gamma \in \mathrm{Sp}_d(\sigma^\varphi)\setminus\{0\}$, hence $e^S \cap e^\gamma e^S = \varnothing$. Since $e^S = \mathrm{Sp}(h) = \mathrm{Sp}(\alpha(h))$ it follows from Lemma 2.3 that $u_\gamma = 0$ for $\gamma \neq 0$. Therefore $E_\gamma(u) = E_\gamma(u_0)$ for all $\gamma \in \mathrm{Sp}_d(\sigma^\varphi)$. Hence $u = u_0 \in M_\varphi$. But then for $x \in M$,

$$\varphi(\alpha(h)x) = (h\varphi)(u^*xu) = \varphi(hu^*xu) = \varphi(uhu^*x),$$

so that $\alpha(h) = uhu^*$, proving the lemma. ∎

Proof of Theorem 2.1. By [12] there exists a maximal Abelian subalgebra A of M_φ such that if G is the group generated by $\alpha|_{M_\varphi}$ and the inner automorphisms of M_φ, then if $\beta \in G$ and $\beta|_A = \mathrm{id}$ then $\beta = Ad(v)$ with v unitary in A. Apply Lemma 2.4 to A and choose a unitary $u \in M_\varphi$ such that $\alpha|_A = \mathrm{Ad}(u)|_A$. Then $\beta = \alpha|_{M_\varphi} \circ \mathrm{Ad}(u^*) \in G$, and $\beta|_A = \mathrm{id}$. Thus $\beta = \mathrm{Ad}(v)$ with $v \in A$, hence $\alpha|_{M_\varphi} = \mathrm{Ad}(vu)$ is inner. ∎

3. The Injective Factor or Type III_1

Recall that by [6] all injective factors of type III_1 with separable preduals are isomorphic, so they can be identified.

THEOREM 3.1. *Let M be the injective factor of type* III_1 *with separable predual. Let $\alpha \in \mathrm{Aut}(M)$ and φ be a faithful normal state. Then α is pointwise inner if and only if there are $t \in \mathbb{R}$ and a unitary operator $v \in M$ such that*

$$\alpha = \sigma_t^\varphi \circ \mathrm{Ad}(v).$$

Before we prove the theorem we show a lemma which is probably well known.

LEMMA 3.2. *Let N and M be von Neumann algebras. Suppose τ is a faithful normal finite trace on N and φ a faithful normal state on M. Then the centralizer of $\tau \otimes \varphi$ is given by*

$$(N \otimes M)_{\tau \otimes \varphi} = N \otimes M_\varphi.$$

Proof. For any normal faithful state ω on a von Neumann algebra P we let $E_\omega : P \to M_\omega$ normal conditional expectation of P onto the centralizer M_ω for which $\omega \circ E_\omega = \omega$. Let $x \in P$. By [11, Section 2, Theorem 1.1], $E_w(x)$ is the unique element of the σ-weak closure of

$$\mathrm{conv}\{\sigma_t^\omega(x),\ t \in \mathbb{R}\}$$

which is containeld in M_ω.

With the notation of the lemma, we have

$$\sigma_t^{\tau \otimes \varphi} = i_N \otimes \sigma_t^\varphi.$$

Hence it follows from the above that

$$E_{\tau \otimes \varphi}(x \otimes y) = x \otimes E_\varphi(y)$$

for all $x \in N$ and $y \in M$. Thus $(N \otimes M)_{\tau \otimes \varphi}$ is the σ-weakly closed linear span of

$$\{x \otimes E_\varphi(y) \mid x \in N,\ y \in M\}.$$

This proves the lemma. ∎

In order to prove Theorem 3.1 we use the classification up to outer conjugacy of automorphisms of the injective factor M of type III_1 as given in [15]. An automorphism α is called *centrally trivial* if the *-strong limit $\lim_n(\alpha(x_n) - x_n) = 0$ for all bounded central sequences (x_n) in M. For $\alpha \in \mathrm{Aut}(M)$ the *asymptotic period* $p_a(\alpha)$ of α is the smallest positive integer p such that α^p is centrally trivial. If no such p exists we put $p_a(\alpha) = 0$.

Let ω be a dominant weight on M (cf. [5, § II.1]). Since the flow of weights of a III_1-factor is trivial, it follows from [10, Theorem 1] that if

$p = p_a(\alpha) > 0$, then $\alpha^p = \mathrm{Ad}(u) \circ \sigma_t^\omega$ for a unitary operator $u \in M$, and a real number $t = t(\alpha)$. The numbers $p_a(\alpha)$ and $t(\alpha)$ are easily seen to be outer conjugacy invariants of α. Moreover, by composing α by an inner automorphism, we may assume that $\omega \circ \alpha = \omega$. In this case, the identity $\alpha^p = \mathrm{Ad}(u) \circ \sigma_t^\omega$ implies that $\alpha(u) = \gamma u$ for a complex number $\gamma = \gamma(\alpha)$ satisfying $\gamma^p = 1$ (cf. [15, § 2]). By [15, Theorem 2.1] the triple $(p_a(\alpha), t(\alpha), \gamma(\alpha))$ is a complete invariant for outer conjugacy of α.

Let R denote the hyperfinite factor of type II_1, and let $\beta \in \mathrm{Aut}(R)$. From the work of Connes [4] the centrally trivial automorphisms of R are inner, hence $p = p_a(\alpha) = p_0(\alpha)$ is the outer period of α. Thus $\alpha^p = \mathrm{Ad}(u)$ for a unitary operator $u \in R$, and $\alpha(u) = \gamma u$ with $\gamma^p = 1$. By [2] and [4] $p_0(\alpha)$ and γ are complete invariants for outer conjugacy of α.

Proof of Theorem 3.1. Since $R \otimes M \simeq M$ by the isomorphism of all hyperfinite III_1-factors with separable preduals, we can write

$$M = R \otimes M_1,$$

where $M_1 \cong M$. Let τ be the trace state on R and let ω be a dominant weight on M_1. Then $\bar{\omega} = \tau \otimes \omega$ is a dominant weight on M. By the above classification of outer conjugacy classes of automorphisms of R and M, it follows that automorphisms of M of the form

$$\beta \otimes \sigma_t^\omega, \qquad \beta \in \mathrm{Aut}(R), \qquad t \in \mathbb{R},$$

run through all outer conjugacy classes in $\mathrm{Aut}(M)$. The algebra M_1 admits a normal, faithful, almost periodic sate ψ because it is isomorphic to a tensor product of two Powers factors R_γ and R_μ for which $\log \gamma / \log \mu \notin \mathbb{Q}$. By Connes' cocycle theorem for modular automorphisms $\sigma_t^\psi = \mathrm{Ad}(u_t) \circ \sigma_t^w$ for a one-parameter family $(u_t)_{t \in \mathbb{R}}$ of unitary operators in M_1. Hence also

$$\beta \otimes \sigma_t^\psi, \qquad \beta \in \mathrm{Aut}(R), \qquad t \in \mathbb{R},$$

run through all outer conjugacy classes of $\mathrm{Aut}(M)$. Since the property "pointwise inner" is invariant under outer conjugacy, it suffices to prove Theorem 3.1 for pointwise inner automorphisms of the form $\alpha = \beta \otimes \sigma_t^\psi$, $\beta \in \mathrm{Aut}(R)$, $t \in \mathbb{R}$. Clearly $\bar{\psi} = \tau \otimes \psi$ is an almost periodic state on M and $\bar{\psi} \circ \alpha = \bar{\psi}$. Thus by Theorem 2.1 and Lemma 3.2, the restriction of α to $R \otimes M_\psi$ is inner; i.e., $\beta \otimes i$ is inner on $R \otimes M_\psi$. Hence, by [9, Cor. 1.14], $\beta = \mathrm{Ad}(u)$ for a unitary $u \in R$. Therefore

$$\alpha = \sigma_t^{\bar{\psi}} \circ \mathrm{Ad}(u \otimes 1).$$

Thus by Connes' cocycle theorem, α is of the form

$$\alpha = \sigma_t^\varphi \circ \mathrm{Ad}(v),$$

$v \in U(M)$, for any fixed normal faithful state φ on M. This completes the proof. ∎

From the classification of pointwise inner automorphisms in [7, 8] and of centrally trivial automorphisms in [10, 15], it follows that if M is an injective factor with separable predual and not of type III_1, then pointwise inner and centrally trivial automorphisms are the same, see [10, Remark 19]. In the III_1-case this is also true by Theorem 3.1 and [10]. Thus we have

COROLLARY 3.3 *Let M be an injective factor with separable predual, and let $\alpha \in \mathrm{Aut}(M)$. Then α is pointwise inner if and only if α is centrally trivial.*

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