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BOUNDED LINEAR OPERATORS BETWEEN C^*-ALGEBRAS

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CONTENTS

Introduction. Let $u\colon A \to B$ be a bounded linear operator between two C^*-algebras A, B. The following result was proved in [P1].

THEOREM 0.1. *There is a numerical constant K_1 such that for all finite sequences $x_1, \ldots, x_n$ in A we have*

$$(0.1)_1 \qquad \max\{\|(\sum u(x_i)^* u(x_i))^{1/2}\|_B, \|(\sum u(x_i)u(x_i)^*)^{1/2}\|_B\} \leqslant K_1 \|u\| \max\{\|(\sum x_i^* x_i)^{1/2}\|_A, \|(\sum x_i x_i^*)^{1/2}\|_A\}.$$

A simpler proof was given in [H1]. More recently, another alternate proof appeared in [LPP]. In this paper we give a sequence of generalizations of this inequality.

The above inequality $(0.1)_1$ appears as the case of "degree one" in this sequence. The next case of degree 2 seems particularly interesting, and so we now formulate it explicitly.

Let us assume that $A \subset B(H)$ (embedded as a C^*-subalgebra) for some Hilbert space H, and similarly that $B \subset B(K)$. Let (a_{ij}) be an $n \times n$ matrix of elements of A. We define

$$[(a_{ij})]_{(2)} = \max\{\|(a_{ij})\|_{M_n(A)}, \|(a_{ij}^*)\|_{M_n(A)}, \|(\textstyle\sum_{ij} a_{ij}^* a_{ij})^{1/2}\|_A, \|(\sum_{ij} a_{ij} a_{ij}^*)^{1/2}\|_A\}.$$

Then we have the following result:

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THEOREM 0.2. *There is a numerical constant K_2 such that for all n and for all (a_{ij}) in $M_n(A)$ we have*

$$[(u(a_{ij}))]_{(2)} \leqslant K_2 \|u\| [(a_{ij})]_{(2)}. \tag{0.1$_2$}$$

We recall in passing the following identities for $a_{ij} \in A$ and $a_i \in A$:

$$\|(a_{ij})\|_{M_n(A)} = \sup\left\{\left|\sum_{ij} \langle y_i, a_{ij}x_j\rangle\right|, x_j, y_i \in H, \sum \|x_j\|^2 \leqslant 1, \sum \|y_i\|^2 \leqslant 1\right\},$$

and

$$\|(\sum a_i^* a_i)^{1/2}\|_A = \sup\{|\sum \langle y_i, a_i x_0\rangle|, x_0 \in H, y_i \in H, \|x_0\| \leqslant 1, \sum \|y_i\|^2 \leqslant 1\}.$$

We will denote

$$[(a_i)]_{(1)} = \max\{\|(\sum a_i^* a_i)^{1/2}\|_A, \|(\sum a_i a_i^*)^{1/2}\|_A\}. \tag{0.2}$$

More generally, let us explain the general case of "degree k" of our main result. Let $k \geqslant 1$. Let n be a fixed integer. We will denote $[n] = \{1, 2, \ldots, n\}$. Let $\{a_J | J \in [n]^k\}$ be a family of elements of A indexed by $[n]^k$. Let us denote by P_k the set of all the 2^k subsets (including the void set) of $\{1, 2, \ldots, k\}$.

For any $\alpha \subset \{1, \ldots, k\}$ we denote by α^c the complement of α and by

$$\pi_\alpha \colon [n]^k \to [n]^\alpha$$

the canonical projection, i.e.

$$\forall\, J = (j_1, \ldots, j_k) \in [n]^k, \qquad \pi(J) = (j_i)_{i \in \alpha}.$$

For any α with $\alpha \neq \emptyset$ and $\alpha^c \neq \emptyset$ we define

$$\|(a_J)\|_\alpha = \sup\left\{\left|\sum_{J \in [n]^k} \langle a_J x_{\pi_\alpha(J)}, y_{\pi_{\alpha^c}(J)}\rangle\right|\right\} \tag{0.3}$$

where the supremum runs over all families

$$\{x_l | l \in [n]^\alpha\} \quad \text{and} \quad \{y_m | m \in [n]^{\alpha^c}\}$$

of elements of H such that $\sum \|x_l\|^2 \leqslant 1$ and $\sum \|y_m\|^2 \leqslant 1$. There is an alternate description: we can identify $[n]^k$ with $[n]^{\alpha^c} \times [n]^\alpha$ so that $J \in [n]^k$ is identified with (i, j) with $i = \pi_{\alpha^c}(J)$, $j = \pi_\alpha(J)$. Then $\|(a_J)\|_\alpha$ is nothing but the norm of the matrix (a_{ij}) acting from $l_2([n]^\alpha, H)$ into $l_2([n]^c, H)$. For $\alpha = \emptyset$, this definition extends naturally to

$$\|(a_J)\|_\emptyset = \sup\left\{\left|\sum_{J\in[n]^k}\langle a_J x_0, y_J\rangle\right|\right\} = \left\|\left(\sum_{J\in[n]^k} a_J^* a_J\right)^{1/2}\right\|_A$$

where the supremum runs over all $x_0 \in H$, $y_J \in H$ such that $\|x_0\| \leqslant 1$ and $\sum \|y_J\|^2 \leqslant 1$. Similarly, for $\alpha = \{1, \ldots, k\}$ we set

$$\|(a_J)\|_\alpha = \|(\sum a_J a_J^*)^{1/2}\|_A.$$

We then define

$$[(a_J)]_{(k)} = \max_{\alpha\in P_k}\{\|(a_J)\|_\alpha\}. \tag{0.4}$$

We can now state one of our main results.

THEOREM 0.k. *For each $k \geqslant 1$, there is a constant K_k such that, for any bounded linear operator $u: A \to B$, for any $n \geqslant 1$, and for any family $\{a_J | J \in [n]^k\}$ in A, we have*

$$[(u(a_J))]_{(k)} \leqslant K_k\|u\|\,[(a_J)]_{(k)}. \tag{0.1$_k$}$$

Moreover, we have $K_k \leqslant 2^{(3k/2)-1}$.

The proof is essentially in Section 1. (It is completed in Section 2.)

We now reformulate this result in a fashion which emphasizes the connection with the notion of complete boundedness for which we refer to [Pa].

Let $A \subset B(H)$ be a C^*-algebra embedded as a C^*-subalgebra. (H is a Hilbert space.) We denote as usual by M_n the set of all $n \times n$ complex matrices (equipped with the norm of the space $B(l_2^n)$) and by $M_n(A)$ the space $M_n \otimes A$ equipped with its natural C^*-norm, induced by $B(l_2^n(H))$. More generally, let $S \subset B(\mathscr{H})$ be any closed linear subspace of $B(\mathscr{H})$. ($\mathscr{H}$ is a Hilbert space.) We call S an "operator space".

We denote by $S \otimes A$ the completion of the linear space $S \otimes A$ equipped with the norm induced by $B(\mathscr{H} \otimes_2 H)$. (Here $\mathscr{H} \otimes_2 H$ denotes the Hilbert space tensor product of $\mathscr{H}$ and H.) We will repeatedly use the following fact. (For a proof see Lemma 1.5 in [DCH].) Let K be an arbitrary Hilbert space. Whenever $u: S \to B(K)$ is completely bounded, the map $I_A \otimes u: A \otimes S \to A \otimes B(K)$ is bounded and we have

$$\|I_A \otimes u\|_{A\otimes S\to A\otimes B(K)} \leqslant \|u\|_{cb}. \tag{0.5}$$

Clearly, $S \otimes A$ is again an operator space embedded into $B(\mathscr{H} \otimes_2 H)$.

For example, we will need to consider a particular embedding of the Euclidean space l_2^n into $M_n \oplus M_n$ as follows. (We equip $M_n \oplus M_n$ with the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$, for which it clearly is an operator space embedded—say, into M_{2n} in a block diagonal way.) We denote by E_n the subspace of $M_n \oplus M_n$ formed by all the elements of the form

$$\begin{bmatrix} x_1 & \\ \vdots & \bigcirc \\ x_n & \end{bmatrix} \oplus \begin{bmatrix} x_1 \cdots x_n \\ \\ \bigcirc \end{bmatrix}$$

with $x_1, \ldots, x_n \in \mathbf{C}$. Let (e_{ij}) be the usual basis of M_n. We denote by

$$\delta_i = e_{i1} \oplus e_{1i}$$

the natural basis of E_n (so that the above element can be written as $\sum x_i \delta_i$). As a Banach space, E_n is clearly isometric to l_2^n. More precisely, for any C^*-algebra A and for any $a_1, \ldots, a_n$ in A, we have (this known fact is easy to check)

$$\|\sum \delta_i \otimes a_i\|_{E_n \otimes A} = \max\{\|(\sum a_i^* a_i)^{1/2}\|, \|(\sum a_i a_i^*)^{1/2}\|\} \tag{0.6}$$

or equivalently,

$$= [(a_i)]_{(1)},$$

in the preceding notation.

Let us denote by E_n^k the tensor product

$$E_n \otimes \cdots \otimes E_n \qquad (k \text{ times}).$$

Then Theorem 0.k implies (and is actually equivalent to) the following.

PROPOSITION 0.k. *For any* $u\colon A \to B$

$$\|I_{E_n^k} \otimes u\|_{E_n^k \otimes A \to E_n^k \otimes B} \leqslant 2^{(3k/2)-1} \|u\|.$$

This proposition is proved in Section 1. In Section 2 we extend (0.6) and compute the norm of an element of $E_n^k \otimes A$ for $k > 1$ to deduce Theorem 0.k from Proposition 0.k.

In Section 3, we develop the viewpoint of [LPP] which dualizes inequalities such as $(0.1)_1$ or $(0.1)_k$ to compute (an equivalent of) the norm of certain random series with coefficients in a noncommutative L_1-space. Let $(\varepsilon_j)_{j \in \mathbf{N}}$ be an i.i.d. sequence of random variables each distributed uniformly over the unimodular complex numbers. (Such variables are sometimes called Steinhaus variables.) Let A_* be a noncommutative L_1-space. Roughly, while [LPP] treats the case of A_*-valued random variables which depend *linearly* on the sequence (ε_j), we can treat variables which depend bilinearly or multilinearly in the variables (ε_j). For a precise statement see Theorem 3.6 below.

It might be useful for some readers to emphasize that the variables (ε_j) can be replaced by independent choices of signs, or more importantly, by i.i.d. *Gaussian* variables. All our results remain true in this setting, but with different numerical constants; this follows from the fact (due to N. Tomczak-Jaegermann) that A_* is of cotype 2, see e.g. [P3, p. 36] for more details. We also would like to draw the reader's attention to Kwapień's paper [K] which contains "decoupling inequalities" quite relevant to the situation considered in Theorem 3.6 below. Using [K], one can deduce from (3.1) below some "nondecoupled" inequalities. For instance, we can find an equivalent of integrals of the form $\int \|\sum_{1 \leqslant i<j \leqslant n} \varepsilon_i \varepsilon_j x_{ij}\|_{A_*}\, dP$ where $x_{ij} \in A_*$ and $(\varepsilon_j)_{j \geqslant 1}$ is an i.i.d. sequence of symmetric ± 1-valued random variables on a probability space (Ω, P), and similarly in the multilinear case. We will not spell out the details.

The results of the first three sections of this paper rely heavily on the following factorization result proved in Section 1: The identity map I_{E_n} on the operator space E_n has a completely bounded factorization through the von Neumann algebra $VN(F_n)$ associated with the left regular representation of the free group with n generators; i.e. there are $w_n\colon E_n \to VN(F_n)$ and $v_n\colon VN(F_n) \to E_n$ such that

$$I_{E_n} = v_n w_n \quad \text{and} \quad \|v_n\|_{cb} \|w_n\|_{cb} \leqslant 2.$$

In Section 4, we show that for any sequence of factorizations $I_{E_n} = v_n w_n\ (n = 1, 2, \ldots)$ of the identity maps I_{E_n} through *injective* von Neumann algebras we have

$$\lim_{n \to \infty} \|v_n\|_{cb} \|w_n\|_{cb} = +\infty.$$

Combining these two facts about the factorization of I_{E_n} with Voiculescu's recent result ([V1]) that the algebra of all $n \times n$ matrices over $VN(F_\infty)$ is isomorphic (as a von Neumann algebra) to $VN(F_\infty)$, we show at the end of Section 4 that the von Neumann algebra $VN(F_n)$ is not a complemented subspace of $B(H)$ for any $n \geqslant 2$. (For very recent results on similar questions, see [P4, CS].) We also include several general remarks about the relation between the existence of a *completely bounded* linear projection from $B(H)$ onto a subspace S and that of a *bounded* linear projection from $B(l_2) \otimes B(H)$ onto $B(l_2) \otimes S$. For instance, if S is weak-$*$ closed and if $B(l_2) \overline{\otimes} S$ denotes the weak-$*$ closure of $B(l_2) \otimes S$ in $B(l_2 \otimes H)$, we show that there is a bounded linear projection from $B(l_2 \otimes H)$ onto $B(l_2) \overline{\otimes} S$ if and only if there is a completely bounded one from $B(H)$ onto S.

Finally, we compare the space E_n with the linear span S_n of a free system of random variables $\{x_1, \ldots, x_n\}$ in a C^*-probability space (A, φ) in the sense of Voiculescu [V1, 2]. In particular, in the case of a semicircular (or circular) system in Voiculescu's sense, we show that there is an isomorphism u from E_n onto the operator space S_n such that

$$\|u\|_{cb} \|u^{-1}\|_{cb} \leqslant 2.$$

1. Operators between C^*-algebras. We will use repeatedly the following fact which has been known to the first author for some time. The main point ((1.2) below) is a refinement of one of the inequalities of [H2]. (We remind the reader that we denote simply by $C^*_\lambda(F_n) \otimes A$ the minimal or spatial tensor product which is often denoted by $C^*_\lambda(F_n) \otimes_{\min} A$.)

PROPOSITION 1.1. *Let F_n denote the free group on n generators $g_1, \ldots, g_n$, and let $C^*_\lambda(F_n)$ be the reduced C^*-algebra of F_n, i.e. the C^*-algebra generated by the left regular representation λ: $F_n \to B(l^2(F_n))$. Then*

(1) *for any C^*-algebra A and for any set $(a_g)_{g \in S}$ of elements of A indexed by a finite subset S of F_n,*

$$\left\| \sum_{g \in S} \lambda(g) \otimes a_g \right\|_{C^*_\lambda(F_n) \otimes A} \geqslant \max\left\{ \left\| \sum_{g \in S} a_g^* a_g \right\|^{1/2}, \left\| \sum_{g \in S} a_g a_g^* \right\|^{1/2} \right\}; \tag{1.1}$$

(2) *for any C^*-algebra A and for any set $(a_g)_{g \in G}$ of elements of A indexed by a subset S of $\{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$,*

$$\left\| \sum_{g \in S} \lambda(g) \otimes a_g \right\|_{C^*_\lambda(F_n) \otimes A} \leqslant 2 \max\left\{ \left\| \sum_{g \in S} a_g^* a_g \right\|^{1/2}, \left\| \sum_{g \in S} a_g a_g^* \right\|^{1/2} \right\}. \tag{1.2}$$

Proof. (1) Let $(\delta_g)_{g \in G}$ be the standard basis of $l^2(F_n)$. We may assume that $A \subset B(K)$ for some Hilbert space K. Since the min-tensor product coincides with the spatial tensor product, we have for all unit vectors $\xi \in K$

$$\begin{aligned} \left\|\sum \lambda(g) \otimes a_g\right\|_{C^*_\lambda(F_n) \otimes A} &\geqslant \left\| \sum_{g \in S} (\lambda(g) \otimes a_g)(\delta_e \otimes \xi) \right\| \\ &= \left\| \sum_{g \in S} \delta_g \otimes a_g \xi \right\| \\ &= \left(\sum_{g \in G} \|a_g \xi\|^2 \right)^{1/2} \\ &= \left(\left(\sum_{g \in G} a_g^* a_g \right) \xi, \xi \right)^{1/2}. \end{aligned}$$

Taking the supremum over all unit vectors $\xi \in K$, we get

$$\left\| \sum_{g \in S} \lambda(g) \otimes a_g \right\|_{C^*_\lambda(F_n) \otimes A} \geqslant \left\| \sum_{g \in S} a_g^* a_g \right\|^{1/2}.$$

The same argument applied to the norm of $(\lambda(g) \otimes a_g)^* = \lambda(g^{-1}) \otimes a_g^*$ gives

$$\left\| \sum_{g \in G} \lambda(g) \otimes a_g \right\|_{C_\lambda^*(F_n) \otimes A} \geqslant \left\| \sum_{g \in S} a_g a_g^* \right\|^{1/2}.$$

This proves (1). Note that the statement (1) actually holds in $C_\lambda^*(\Gamma) \otimes A$ for any discrete group Γ.

(2) Consider first the case $S = \{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$. We can write F_n as a disjoint union:

$$F_n = \{e\} \cup \left\{ \bigcup_{i=1}^{n} \Gamma_i^+ \right\} \cup \left\{ \bigcup_{i=1}^{n} \Gamma_i^- \right\}$$

where

Γ_i^+ = set of reduced words starting with a positive power of g_i,

Γ_i^- = set of reduced words starting with a negative power of g_i.

Let e_0, e_i^+, and e_i^- denote the orthogonal projection of $l^2(F_n)$ onto the subspaces $\mathbf{C}\delta_e$, $l^2(\Gamma_i^+)$, and $l^2(\Gamma_i^-)$ respectively. Then these projections are pairwise orthogonal and

$$e_0 + \sum_{i=1}^{n} e_i^+ + \sum_{i=1}^{n} e_i^- = I_{l^2(F_n)}.$$

For any $g \in G$ and for any generator g_i, the length of the reduced word for $g_i g$ is either

$$|g_i g| = |g| + 1 \quad \text{or} \quad |g_i g| = |g| - 1.$$

The first case exactly occurs when $g_i g$ starts with an element of Γ_i^+, and the second case when g starts with an element of Γ_i^-. Hence for all $g \in G$,

$$\lambda(g_i)\delta_g = \begin{cases} e_i^+ \lambda(g_i)\delta_g & \text{if } |g_i g| = |g| + 1 \\ \lambda(g_i) e_i^- \delta_g & \text{if } |g_i g| = |g| - 1 \end{cases}$$

$$= e_i^+ \lambda(g_i)\delta_g + \lambda(g_i) e_i^- \delta_g \qquad \text{(all cases)}.$$

Therefore

$$\lambda(g_i) = e_i^+ \lambda(g_i) + \lambda(g_i) e_i^-,$$

and by taking adjoints

$$\lambda(g_i^{-1}) = e_i^- \lambda(g_i^{-1}) + \lambda(g_i^{-1}) e_i^+ .$$

Set

$$\left.\begin{array}{ll} u_i = e_i^+ \lambda(g_i), & u_{n+i} = e_i^- \lambda(g_i^{-1}) \\ v_i = \lambda(g_i^{-1}) e_i^- , & v_{n+i} = \lambda(g_i^{-1}) e_i^+ \end{array}\right\} i = 1, \ldots, n$$

and, for simplicity of notation, set also $g_{n+i} = g_i^{-1}$, $i = 1, \ldots, n$. Then

$$\lambda(g_i) = u_i + v_i , \qquad i = 1, \ldots, 2n.$$

Since $\sum_{i=1}^n (e_i^+ + e_i^-) = 1 - e_0$, we have

$$\sum_{i=1}^{2n} u_i u_i^* = \sum_{i=1}^{2n} v_i^* v_i = 1 - e_0 \leqslant 1 .$$

So

$$\left\| \sum_{i=1}^{2n} u_i u_i^* \right\| \leqslant 1 \quad \text{and} \quad \left\| \sum_{i=1}^{2n} v_i^* v_i \right\| \leqslant 1 .$$

For elements $c_1, \ldots, c_m, d_1, \ldots, d_m$ of a C^*-algebra B, one easily has that

$$\left\| \sum_{i=1}^{m} c_i d_i \right\| \leqslant \left\| \sum_{i=1}^{m} c_i c_i^* \right\|^{1/2} \left\| \sum_{i=1}^{m} d_i^* d_i \right\|^{1/2} .$$

Hence, with $u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n}$ as above, and $a_1, \ldots, a_{2n} \in A$,

$$\begin{aligned} \left\| \sum_{i=1}^{2n} u_i \otimes a_i \right\|_{C_r^*(F_n) \otimes A} &= \left\| \sum_{i=1}^{n} (u_i \otimes 1)(1 \otimes a_i) \right\|_{C_r^*(F_n) \otimes A} \\ &\leqslant \left\| \sum u_i u_i^* \right\|^{1/2} \left\| \sum_{i=1}^{2n} a_i^* a_i \right\|^{1/2} \\ &\leqslant \left\| \sum a_i^* a_i \right\|^{1/2}, \end{aligned}$$

and similarly

$$\begin{aligned} \left\| \sum v_i \otimes a_i \right\|_{C_r^*(F_n) \otimes A} &\leqslant \left\| \sum_{i=1}^{n} (1 \otimes a_i)(v_i \otimes 1) \right\|_{C_r^*(F_n) \otimes A} \\ &\leqslant \left\| \sum a_i a_i^* \right\|^{1/2} \left\| \sum_{i=1}^{n} v_i^* v_i \right\|^{1/2} \\ &\leqslant \left\| \sum a_i a_i^* \right\|^{1/2}; \end{aligned}$$

so altogether

$$\left\| \sum_{i=1}^{2n} \lambda(g_i) \otimes a_i \right\| = \left\| \sum_{i=1}^{2n} u_i \otimes a_i + \sum_{i=1}^{2n} v_i \otimes a_i \right\|$$

$$\leqslant \left\| \sum_{i=1}^{2n} a_i^* a_i \right\|^{1/2} + \left\| \sum_{i=1}^{2n} a_i a_i^* \right\|^{1/2}$$

$$\leqslant 2 \max \left\{ \left\| \sum_{i=1}^{2n} a_i^* a_i \right\|^{1/2}, \left\| \sum_{i=1}^{2n} a_i a_i^* \right\|^{1/2} \right\}.$$

This proves (2) in the case $S = \{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$, and the remaining cases follow from this by setting some of the a_g's equal to 0. ■

Remark. The preceding statement remains true (with the obvious modifications) for the free group on infinitely many generators. See also Proposition 4.9 below for a generalization of (1.1) and (1.2).

Remark 1.2. The proof of (2) is an illustration of the following general principle. Let $T_1, \ldots, T_n$ be operators on a Hilbert space H and let c be a constant. The following properties are essentially equivalent:

$(\mathrm{i})_c$ For any C^*-algebra A and any set $(a_i)_{i \leqslant n}$ in A, we have

$$\|\sum T_i \otimes a_i\| \leqslant c \max\{\|(\sum a_i^* a_i)^{1/2}\|, \|(\sum a_i a_i^*)^{1/2}\|\}.$$

$(\mathrm{ii})_c$ There are operators u_i, v_i in $B(H)$ such that $T_i = u_i + v_i$ and

$$\|(\sum u_i^* u_i)^{1/2}\| + \|(\sum v_i v_i^*)^{1/2}\| \leqslant c.$$

More precisely, we have $(\mathrm{ii})_c \Rightarrow (\mathrm{i})_c$ and $(\mathrm{i})_c \Rightarrow (\mathrm{ii})_{2c}$ The implication $(\mathrm{ii})_c \Rightarrow (\mathrm{i})_c$ follows as above from the triangle inequality. To prove the converse, note that $(\mathrm{i})_c$ equivalently means that the operator $u\colon E_n \to B(H)$ which maps δ_i to T_i satisfies $\|u\|_{cb} \leqslant 1$. By the extension property of *cb* maps (cf. [Pa, p. 100]) there is an extension $\tilde{u}\colon M_n \oplus M_n \to B(H)$ such that $\tilde{u}(\delta_i) = T_i$ and $\|\tilde{u}\|_{cb} \leqslant 1$. Letting $u_i = \tilde{u}(e_{i1} \oplus 0)$ and $v_i = \tilde{u}(0 \oplus e_{1i})$, we obtain a decomposition satisfying $(\mathrm{ii})_{2c}$. This shows that $(\mathrm{i})_c$ implies $(\mathrm{ii})_{2c}$.

PROPOSITION 1.3. *Let $E_n \subset M_n \oplus M_n$ be the operator space*

$$E_n = \left\{ \begin{bmatrix} c_1 & \\ \vdots & \bigcirc \\ c_n & \end{bmatrix} \oplus \begin{bmatrix} c_1 \cdots c_n \\ \\ \bigcirc \end{bmatrix} \middle| c_1, \ldots, c_n \in \mathbf{C} \right\}.$$

Then there are linear mappings

$$w\colon E_n \to C^*_\lambda(F_n) \quad \textit{and} \quad v\colon C^*_\lambda(F_n) \to E_n$$

such that

$$vw = I_{E_n} \quad \textit{and} \quad \|v\|_{cb}\|w\|_{cb} \leqslant 2.$$

Similarly, for the von Neumann algebra $VN(F_n)$ generated by λ, there are linear mappings

$$w_1\colon E_n \to VN(F_n) \quad \textit{and} \quad v_1\colon VN(F_n) \to E_n$$

such that

$$v_1 w_1 = I_{E_n} \quad \textit{and} \quad \|v_1\|_{cb}\|w_1\|_{cb} \leqslant 2.$$

In particular, E_n is cb-isomorphic to a cb-complemented subspace of $C^*_\lambda(F_n)$ (resp. of $VN(F_n)$).

Proof. Let $(\delta_1, \ldots, \delta_n)$ be the basis of E_n determined by

$$\sum_{i=1}^n c_i\delta_i = \begin{bmatrix} c_1 & \\ \vdots & \bigcirc \\ c_n & \end{bmatrix} \oplus \begin{bmatrix} c_1 \cdots c_n \\ \\ \bigcirc \end{bmatrix}$$

for $c_1, \ldots, c_n \in \mathbf{C}$. Define $w\colon E_n \to C^*_\lambda(F_n)$ by

$$w\left(\sum_{i=1}^n c_i\delta_i\right) = \sum_{i=1}^n c_i\lambda(g_i)$$

and $v\colon C^*_\lambda(F_n) \to E_n$ by

$$v(x) = \sum_{i=1}^n \tau(\lambda(g_i)^* x)\delta_i$$

where τ is the trace on $C^*_\lambda(F_n)$ given by

$$\tau(y) = (y\delta_e, \delta_e), \qquad y \in C^*_\lambda(F_n). \qquad \text{(Cf. [KR, p. 433].)}$$

For any set $a_1, \ldots, a_n$ of n elements in a C^*-algebra A,

$$(w \otimes I_A)\left(\sum_{i=1}^n \delta_i \otimes a_i\right) = \sum_{i=1}^n \lambda(g_i) \otimes a_i.$$

Since

$$\left\| \sum_{i=1}^{n} e_i \otimes a_i \right\| = \left\| \begin{bmatrix} a_1 & \\ \vdots & \bigcirc \\ a_n & \end{bmatrix} \oplus \begin{bmatrix} a_1 \cdots a_n \\ \\ \bigcirc \end{bmatrix} \right\|$$

$$= \max\left\{ \left\| \sum_{i=1}^{n} a_i^* a_i \right\|^{1/2}, \left\| \sum_{i=1}^{n} a_i a_i^* \right\|^{1/2} \right\},$$

it follows from Theorem 1.1(2) that $\|w \otimes I_A\| \leqslant 2$. Hence $\|w\|_{cb} \leqslant 2$. Since

$$\tau(\lambda(g)^* \lambda(h)) = \begin{cases} 1 & g = h \\ 0 & g \neq h, \end{cases}$$

we get for any finite subset $S \subset F_n$ and scalars $(c_g)_{g \in S}$

$$v\left(\sum_{g \in S} c_g \lambda(g) \right) = \sum_{\substack{i=1 \\ (g_i \in S)}}^{n} c_{g_i} \delta_i ,$$

and hence

$$(v \otimes I_A)\left(\sum_{g \in S} \lambda(g) \otimes a(g) \right) = \sum_{\substack{i=1 \\ (g_i \in S)}}^{n} \delta_i \otimes a(g_i).$$

Let $S^1 = S \cap \{g_1, \ldots, g_n\}$. Then

$$\left\| \sum_{\substack{i=1 \\ g_i \in S}}^{n} \delta_i \otimes a(g_i) \right\| = \max\left\{ \left\| \sum_{g \in S^1} a(g)^* a(g) \right\|^{1/2}, \left\| \sum_{g \in S^1} a(g) a(g)^* \right\|^{1/2} \right\}$$

$$\leqslant \max\left\{ \left\| \sum_{g \in S} a(g)^* a(g) \right\|^{1/2}, \left\| \sum_{g \in S} a(g) a(g)^* \right\|^{1/2} \right\},$$

which by Theorem 1.1(1) is smaller than or equal to

$$\left\| \sum_{g \in S} \lambda(g) \otimes a(g) \right\|_{C_r^*(F_n) \otimes A} .$$

Hence $\|v \otimes I_A\| \leqslant 1$ and thus $\|v\|_{cb} \leqslant 1$. Therefore

$$\|v\|_{cb} \|w\|_{cb} \leqslant 2$$

and by construction $vw = I_{E_n}$. This implies that w is a cb-isomorphism of E_n onto its range

$$w(E_n) = \operatorname{span}\{\lambda(g_i) | i = 1, \ldots, n\}$$

and

$$\|w\|_{cb} \|w^{-1}\|_{cb} \leqslant 2.$$

Moreover, $P = wv$ is a completely bounded projection of $C_\lambda^*(F_n)$ onto $w(E_n)$ and $\|P\|_{cb} \leqslant 2$. The proof with $VN(E_n)$ in the place of $C_\lambda^*(F_n)$ is easy since v admits an extension v_1: $VN(F_n) \to E_n$ with $\|v_1\|_{cb} \leqslant 1$. We leave the details to the reader. ■

LEMMA 1.4 ([P1, H1, LPP]). *Let u: $A \to B$ be a bounded linear operator between two C^*-algebras A and B. Then for every $n \in \mathbf{N}$*

$$\|I_{E_n} \otimes u\|_{E_n \otimes A \to E_n \otimes B} \leqslant \sqrt{2}\|u\|.$$

Proof. The statement of the lemma is equivalent to: For all $a_1, \ldots, a_n \in A$

$$\max\{\|\textstyle\sum u(a_i)^* u(a_i)\|, \|\sum u(a_i) u(a_i)^*\|\} \leqslant 2\|u\|^2 \max\{\|\sum a_i^* a_i\|, \|\sum a_i a_i^*\|\}. \tag{1.3}$$

This is essentially [P1] (see also [H1, LPP]). However in order to get the constant 2 in (1.3), one has to modify the proof of [H1, Cor. 3.4] slightly:

Let T: $A \to H$ be a bounded linear operator from the C^*-algebra A with values in a Hilbert space. By [H1, Thm. 3.2],

$$\sum \|T(a_k)\|^2 \leqslant \|T\|^2 (\|\textstyle\sum a_k^* a_k\| + \|\sum a_k a_k^*\|). \tag{1.4}$$

We can assume that $B \subseteq B(K)$ for some Hilbert space K. By the above inequality (1.4) we get for any $\xi \in K$ that

$$\sum \|u(a_k)\xi\|^2 \leqslant \|\xi\|^2 \|u\|^2 (\|\textstyle\sum a_k^* a_k\| + \|\sum a_k a_k^*\|).$$

Clearly, (1.4) also holds for conjugate linear maps, and so

$$\sum \|u(a_k)^* \xi\|^2 \leqslant \|\xi\|^2 \|u\|^2 (\|\textstyle\sum a_k^* a_k\| + \|\sum a_k a_k^*\|).$$

Thus

$$\max\left\{\|\textstyle\sum u(a_k)^* u(a_k)\|, \|\sum u(a_k) u(a_k)^*\|\right\} \leqslant \|u\|^2 (\|\sum a_k^* a_k\| + \|\sum a_k a_k^*\|)$$

which implies (1.3). ■

THEOREM 1.5. *Let $u\colon A \to B$ be a bounded linear operator between two C^*-algebras A and B. Then for every $k, n \in \mathbf{N}$,*

$$\|I_{E_n^k} \otimes u\|_{E_n^k \otimes A \to E_n^k \otimes B} \leqslant 2^{(3/2)k-1}\|u\|.$$

Proof. The theorem is proved by induction on k. By Lemma 1.4 the theorem holds for $k = 1$. Assume next that the theorem is true for a particular $k \in \mathbf{N}$. Let

$$w\colon E_n \to C_\lambda^*(F_n) \quad \text{and} \quad v\colon C_\lambda^*(F_n) \to E_n$$

be as in Proposition 1.2 and let $u\colon A \to B$ be a linear map between two C^*-algebras A and B. Clearly

$$I_{E_n} \otimes u = (v \otimes u)(w \otimes I_A) \tag{1.5}$$

where

$$\begin{aligned}\|v \otimes u\| &= \|(v \otimes I_B)(I_{E_n} \otimes u)\|\\ &\leqslant \|v\|_{cb}\|I_{E_n} \otimes u\|\\ &\leqslant \sqrt{2}\|u\|\,\|v\|_{cb}\end{aligned}$$

by Lemma 1.4. Moreover, $v \otimes u$ maps the C^*-algebra $C_\lambda^*(F_n) \otimes A$ into the C^*-algebra $M_n(B) \oplus M_n(B)$, and so by the induction hypothesis

$$\|I_{E_n^k} \otimes v \otimes u\| \leqslant 2^{(3/2)k-1}\|v \otimes u\| \leqslant 2^{(3/2)k-1/2}\|u\|\,\|v\|_{cb}.$$

On the other hand, by (0.5)

$$\|I_{E_n^k} \otimes w \otimes I_A\| = \|I_{E_n^k \otimes A} \otimes w\| \leqslant \|w\|_{cb}.$$

Now by (1.5)

$$I_{E_n^{k+1}} \otimes u = (I_{E_n^k} \otimes v \otimes u)(I_{E_n^k} \otimes w \otimes I_A).$$

Thus, by Proposition 1.3

$$\begin{aligned}\|I_{E_n^{k+1}} \otimes u\| &\leqslant 2^{(3/2)k-1/2}\|u\|\,\|v\|_{cb}\|w\|_{cb}\\ &\leqslant 2^{(3/2)k+1/2}\|u\|\\ &= 2^{(3/2)(k+1)-1}\|u\|.\end{aligned}$$

Hence Theorem 1.5 follows by induction on k. ∎

2. Description of E_n^k. In this section, we will identify the norm in the space $E_n^k \otimes A$ with the norm previously introduced in (0.3) and (0.4) as $[\ ]_{(k)}$.

PROPOSITION 2.1. *Let A be any C^*-algebra. Let $n \geqslant 1, k \geqslant 1$, and let $\{a_J | J \in [n]^k\}$ be elements of A. Then*

$$[(a_J)]_{(k)} = \left\| \sum_{J \in [n]^k} \delta_J \otimes a_J \right\|_{E_n^k \otimes A} \tag{2.1}$$

where, if $J = (j_1, \ldots, j_k)$, we denote

$$\delta_J = \delta_{j_1} \otimes \cdots \otimes \delta_{j_k}.$$

The proof below is easy, but the notation is a bit painful. Using Proposition 2.1, we can complete the proof of the results announced in the introduction.

Proof of Theorem 0.k. Consider an operator $u: A \to B$ between C^*-algebras. By Theorem 1.5 we have for all (a_J) in A

$$\|\sum \delta_J \otimes u(a_J)\|_{E_n^k \otimes B} \leqslant 2^{(3k/2)-1} \|u\| \, \|\sum \delta_J \otimes a_J\|_{E_n^k \otimes A}.$$

Taking (2.1) into account, this immediately implies $(0.1)_k$ and completes the proof of Theorem 0.k.

We now check (2.1). We will need the following elementary fact.

LEMMA 2.2. *Let H, H_1, H_2, H_3, H_4 be Hilbert spaces. Let $e \in H_1, f \in H_4$ be norm one vectors. Let $(\varphi_j)_{j \in J}$ and $(\psi_i)_{i \in I}$ be orthonormal finite sequences in H_2 and H_3 respectively. Let a_{ij} be elements of a C^*-algebra A embedded into $B(H)$. Then we have*

$$\left\| \sum_{\substack{i \in I \\ j \in J}} (e \otimes \varphi_j) \otimes (\psi_i \otimes f) \otimes a_{ij} \right\| \tag{2.2}$$

$$= \sup_{\substack{y_i \in H \\ x_j \in H}} \left\{ \left| \sum_{i,j} \langle y_i, a_{ij} x_j \rangle \right| \,\middle|\, \sum \|x_j\|^2 \leqslant 1, \sum \|y_i\|^2 \leqslant 1 \right\}.$$

Here the norm on the left-hand side means the norm in the space of all bounded operators from $H_1 \otimes_2 H_2 \otimes_2 H$ into $H_3 \otimes_2 H_4 \otimes_2 H$.

Proof. We may clearly assume without loss of generality that $H_1 = \mathbb{C}e, H_4 = \mathbb{C}f$ and that (φ_j) (resp. (ψ_i)) is a basis of H_2 (resp. H_3). Then the norm we want to compute is clearly equal to the norm of the operator

$$\tilde{T} = \sum_{ij} \varphi_j \otimes \psi_i \otimes a_{ij}$$

as an operator from $H_2 \otimes_2 H$ to $H_3 \otimes_2 H$. But then the general form of an element in the unit ball of $H_2 \otimes_2 H$ (resp. $H_3 \otimes_2 H$) is given by $\sum \varphi_j \otimes x_j$ (resp. $\sum \psi_i \otimes y_i$) with $x_j \in H_2$ (resp. $y_i \in H_3$) such that $\sum \|x_j\|^2 \leqslant 1$ (resp. $\sum \|y_i\|^2 \leqslant 1$). Hence the norm of $\tilde{T}$ (or of T) is equal to the right-hand side of (2.2). ∎

We need to introduce more notation. Recall that $E_n \subset M_n \oplus M_n$ and $\delta_i = e_{i1} \oplus e_{1i}$. We consider, of course, $M_n \oplus M_n$ as a subset of the set of all operators on $l_2^n \oplus l_2^n$. It will be convenient to denote $e_{ij}^0 = e_{ij} \oplus 0$ and $e_{ij}^1 = 0 \oplus e_{ij}$ in $M_n \oplus M_n$. Also $e_i^0 = e_i \oplus 0$ and $e_i^1 = 0 \oplus e_i$ in $l_2^n \oplus l_2^n$. As usual, for e and f in H, we will identify the tensor $e \otimes f$ with the operator $x \to \langle e, x\rangle f$ (defined on H). Hence in tensor product notation we have (with the usual matricial conventions) $e_{ij} = e_j \otimes e_i$ and $\delta_i = e_1^0 \otimes e_i^0 + e_i^1 \otimes e_1^1$. Let us denote by H_0 the span of $\{e_{i1} | i = 1, \ldots, n\}$ in M_n and by H_1 the span of $\{e_{1i} | i = 1, \ldots, n\}$ in M_n, so that $E_n \subset H_0 \oplus H_1$. Let $P_0\colon H_0 \oplus H_1 \to H_0$ (resp. $P_1\colon H_0 \oplus H_1 \to H_1$) denote the canonical projection. We have $E_n^{\otimes k} \subset (H_0 \oplus H_1)^{\otimes k}$. For $\alpha \in \{0, 1\}^k$ we denote

$$P_\alpha\colon (H_0 \oplus H_1)^k \to (H_0 \oplus H_1)^k$$

the projection defined by

$$P_\alpha = P_{\alpha(1)} \otimes P_{\alpha(2)} \otimes \cdots \otimes P_{\alpha(k)}.$$

Let us denote by I_X the identity on X. Then we have

$$\begin{aligned} I_{(H_0 \oplus H_1)^{\otimes k}} &= (I_{H_0 \oplus H_1})^{\otimes k} \\ &= (P_0 + P_1)^{\otimes k} \\ &= \sum_{\alpha \in \{0,1\}^k} P_{\alpha(0)} \otimes \cdots \otimes P_{\alpha(k)} \\ &= \sum_\alpha P_\alpha. \end{aligned} \tag{2.3}$$

Proof of Proposition 2.1. Let $T = \sum_{J \in [n]^k} \delta_J \otimes a_J$. By (2.3) we have

$$T = \sum_\alpha T_\alpha$$

where

$$T_\alpha = \sum_J P_\alpha(\delta_J) \otimes a_J.$$

We now claim that

$$\|T_\alpha\| = \|(a_J)\|_\alpha. \tag{2.4}$$

To check this, we can assume for simplicity (up to a permutation of the factors in the tensor product) that α is the indicator function of the set $\{1, 2, \dots, p\}$ for some p with $1 \leqslant p \leqslant k$. Then if $J = (j_1, \dots, j_k)$, we have

$$P_\alpha(\delta_J) = e^1_{j_1 1} \otimes \cdots \otimes e^1_{j_p 1} \otimes e^0_{1 j_{p+1}} \otimes \cdots \otimes e^0_{1 j_k}. \tag{2.5}$$

(Recall the convention that the tensor $e \otimes f$ represents the operator $x \to \langle e, x \rangle f$.) Let $e^1(\alpha) = e^1_1 \otimes \cdots \otimes e^1_1$ (p times) and $f^0(\alpha) = e^0_0 \otimes \cdots \otimes e^0_1$ ($k - p$ times). Then (2.5) yields

$$P_\alpha(\delta_J) = (e^1(\alpha) \otimes e^0_{j_{p+1}} \otimes \cdots \otimes e^0_{j_k}) \otimes (e^1_{j_1} \otimes \cdots \otimes e^1_{j_p} \otimes f^0(\alpha)).$$

If we now write $e^\varepsilon_{\{j_1, \dots, j_p\}}$ instead of $e^\varepsilon_{j_1} \otimes \cdots \otimes e^\varepsilon_{j_p}$ for $\varepsilon = 0$ or 1, we can rewrite the last identity as

$$P_\alpha(\delta_J) = (e^1(\alpha) \otimes e^0_{\pi_\alpha(J)}) \otimes (e^1_{\pi_{\alpha^c}(J)} \otimes f^0(\alpha)), \tag{2.6}$$

where we recall that π_α: $[n]^k \to [n]^\alpha$ denotes the canonical projection. Then in the present particular case, Lemma 2.2 above gives

$$\|T_\alpha\| = \left\| \sum_J (e^1(\alpha) \otimes e^0_{\pi_\alpha(J)}) \otimes (e^1_{\pi_{\alpha^c}(J)} \otimes f^0(\alpha)) \otimes a_J \right\| = \|(a_J)\|_\alpha.$$

This proves our claim (2.4).

Now we can finish. Let us denote $h^0 = l^n_2 \oplus 0$ and $h^1 = 0 \oplus l^n_2$ in $l^n_2 \oplus l^n_2$. Let K_α be the support of T_α (i.e., the orthogonal of its kernel) and let R_α be the range of T_α. Then the preceding formula (2.6) shows that K_α is equal to the tensor product $F_1 \otimes F_2 \otimes \cdots \otimes F_k$ where

$$F_j = \mathbb{C} e^1_1 \qquad \text{if } j \in \alpha$$

and

$$F_j = h^0 \qquad \text{if } j \notin \alpha.$$

It follows that the subspaces (K_α) are mutually orthogonal. Similarly, the family (R_α) is mutually orthogonal. By a well-known estimate it follows that

$$\|\textstyle\sum T_\alpha\| = \max_\alpha \|T_\alpha\|.$$

This completes the proof. ∎

3. Random series in noncommutative L_1-spaces. Let A be a von Neumann algebra with a predual denoted by A_*. Let $\xi_1, \ldots, \xi_n \in A_*$ and let (recall (0.2))

$$[(\xi_i)]^*_{(1)} = \sup\{|\sum \langle \xi_i, a_i\rangle| \, | a_i \in A \; [(a_i)]_{(1)} \leqslant 1\}.$$

For instance, if $A = B(H)$, $A_* = C_1(H)$ (the space of trace class operators on H), and we have clearly

$$[(\xi_i)]^*_{(1)} = \inf\{tr(\sum x_i^* x_i)^{1/2} + tr(\sum y_i y_i^*)^{1/2}\}$$

where the infimum runs over all decompositions $\xi_i = x_i + y_i$ in $C_1(H)$.

Let $\mathbf{T}^{\mathbf{N}}$ be the infinite-dimensional torus equipped with its normalized Haar measure μ. The following result is proved in [LPP].

For all $\xi_1, \ldots, \xi_n$ in A_*

$$\frac{1}{2}[(\xi_i)]^*_{(1)} \leqslant \int \left\| \sum_{j=1}^{n} e^{it_j} \xi_j \right\|_{A^*} d\mu(t) \leqslant [(\xi_i)]^*_{(1)}. \tag{3.1}$$

(See Theorem 3.3 below and its proof.)

It is easy to deduce from (3.1) a necessary and sufficient condition for a series of the form

$$S(t) = \sum_{j=1}^{\infty} e^{it_j} \xi_j, \qquad t = (t_j)_{j \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$$

to converge in $L_2(\mathbf{T}^{\mathbf{N}}, \mu; A_*)$. The aim of this section is to prove a natural extension of (3.1) to double series of the form

$$S(t', t'') = \sum_{j,k=1}^{\infty} e^{it'_j} e^{it''_k} \xi_{jk}$$

with $\xi_{jk} \in A_*$, $t', t'' \in \mathbf{T}^{\mathbf{N}}$. More generally, we will consider for any $k \geqslant 1$, elements $\xi_{j_1 j_2 \cdots j_k}$ in A_* and will find an equivalent for the expression

$$\int \left\| \sum_{j_1 \leqslant n, \ldots, j_k \leqslant n} e^{it^1_{j_1}} \cdots e^{it^k_{j_k}} \xi_{j_1 j_2 \cdots j_k} \right\|_{A^*} d\mu(t^1) \cdots d\mu(t^k).$$

See Theorem 3.6 below for an explicit statement.

Let A be a C^*-algebra throughout this section. We will denote simply

$$C_n = C^*_\lambda(F_n)$$

and

$$C_n^k = C_n \otimes \cdots \otimes C_n \qquad (k \text{ times}).$$

We always equip the tensor products such as $E_n \otimes A$, $C_n \otimes A$, $C_n^k \otimes A$ with the spatial (or minimal) tensor product. More precisely, whenever $S \subset B(K)$ is an operator space and $A \subset B(H)$ is a C^*-algebra, we will denote by $S \otimes A$ the linear tensor product equipped with the norm induced by $B(K \otimes_2 H)$.

Let G be a discrete group. For $t \in G$, let $\lambda_*(t)$ denote the element of $C_\lambda^*(G)^*$ given by

$$\forall a \in C_\lambda^*(G) \qquad \langle \lambda_*(t), a \rangle = \langle a\delta_e, \delta_t \rangle.$$

Clearly

$$\langle \lambda_*(s), \lambda(t) \rangle = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $C_\lambda^*(G)^*$ is identified with $B_\lambda(G)$ in the usual way (see for instance [E]), then $\lambda_*(t)$ simply corresponds to the function δ_t.

For any $J = (j_1, \ldots, j_k) \in [n]^k$ we denote by g_J the element of $(F_n)^k$ defined by

$$g_J = (g_{j_1}, \ldots, g_{j_k}).$$

Then with the obvious identification

$$C_\lambda^*((F_n)^k) = C_n^k$$

we have $\lambda(g_J) = \lambda(g_{j_1}) \otimes \cdots \otimes \lambda(g_{j_k})$. We will also consider the dual E_n^* of the space E_n considered in Section 1 and will denote by $\{\delta_j^*\}$ the basis of E_n^* which is biorthogonal to $\{\delta_j\}$. We will also consider $E_n^k = E_n \otimes \cdots \otimes E_n$ (k times) and its dual $(E_n^k)^*$. We will denote for any $J = (j_1, \ldots, j_k)$ in $[n]^k$

$$\delta_J^* = \delta_{j_1}^* \otimes \cdots \otimes \delta_{j_k}^* \in (E_n^k)^*$$

and

$$\lambda_*(g_J) = \lambda_*(g_{j_1}) \otimes \cdots \otimes \lambda_*(g_{j_k}) \in (C_n^k)^*.$$

We will denote by Ω the infinite-dimensional torus; i.e., we set

$$\Omega = \mathbf{T}^{\mathbf{N}},$$

and we equip Ω with the normalized Haar measure μ. (In most of what follows, it would be more appropriate to replace Ω by $\Omega_n = \mathbf{T}^n$, but we try to simplify the notation.) We will denote by

$$\varepsilon_j \colon \Omega \to \mathbf{T}$$

the sequence of the coordinate functions on Ω. Moreover, we will consider the product Ω^k equipped with the product measure μ^k. For any $J = (j_1, \dots, j_k) \in [n]^k$, let $\varepsilon_J \colon \Omega^k \to \mathbf{T}$ be the function defined by

$$\forall (t_1, \dots, t_k) \in \Omega^k \qquad \varepsilon_J(t_1, \dots, t_k) = \varepsilon_{j_1}(t_1) \cdots \varepsilon_{j_k}(t_k).$$

Equivalently, $\varepsilon_J = \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k}$. We first record a simple consequence of Proposition 1.3.

LEMMA 3.1. *For any $\{\xi_j | j \leqslant n\}$ in A^* we have*

$$\tfrac{1}{2}\|\textstyle\sum \delta_j^* \otimes \xi_j\|_{(E_n \otimes A)^*} \leqslant \|\sum \lambda_*(g_j) \otimes \xi_j\|_{(C_n \otimes A)^*} \leqslant \|\sum \delta_j^* \otimes \xi_j\|_{(E_n \otimes A)^*}.$$

Proof. Let v, w be as in Proposition 1.3. Since $w\delta_j = \lambda(g_j)$ and $v(\lambda(g_j)) = \delta_j$, we have $(w \otimes I_A)^*(\lambda_*(g_j) \otimes \xi_j) = \delta_j^* \otimes \xi_j$ and $(v \otimes I_A)^*(\delta_j^* \otimes \xi_j) = \lambda_*(g_j) \otimes \xi_j$. Hence, recalling (0.5), Lemma 3.1 follows from $\|w\|_{cb} \leqslant 2$ and $\|v\|_{cb} \leqslant 1$. ■

The next lemma is rather elementary.

LEMMA 3.2. (i) *Consider $\{\xi_{ij} | i, j = 1, \dots, n\}$ in A^*. For any orthonormal systems $\varphi_1, \dots, \varphi_n$ and $\psi_1, \dots, \psi_n$ in $L_2(\mu)$ (where μ is a probability as above), we have*

$$\int \|\textstyle\sum \varphi_i(t)\psi_j(s)\xi_{ij}\|_{A^*}\, d\mu(t)\, d\mu(s) \leqslant \|(\xi_{ij})\|_{M_n(A)^*}. \tag{3.2}$$

(ii) *For any $k \geqslant 1$ and any (ξ_J) in A^* we have*

$$\left\| \sum_{J \in [n]^k} \varepsilon_J \xi_J \right\|_{L_1(\mu; A^*)} \leqslant \left\| \sum_{J \in [n]^k} \delta_J^* \otimes \xi_J \right\|_{(E_n^k \otimes A)^*}. \tag{3.3}$$

Proof. (i) To prove this, it clearly suffices to assume that A is a von Neumann algebra and that $\xi_{ij} \in A_*$. Since $M_n(A)$ is a subspace of $M_n(B(H))$ for some Hilbert space H, by duality its predual $M_n(A)_*$ is a quotient of $M_n(B(H))_*$. This shows that it suffices to prove (i) for $A = B(H)$ and $\xi_{ij} \in B(H)_*$. Then we can identify $M_n(B(H))_*$ with the projective tensor product $l_2^n(H)^* \hat{\otimes} l_2^n(H)$. Consider an element x (resp. y) in the unit ball of $l_2^n(H)$ (resp. $l_2^n(H)^*$). Let ξ be the element of $M_n(B(H))_*$ defined by $\xi = y \otimes x$ or equivalently, $\xi = (\xi_{ij})$ with $\xi_{ij} = y_j \otimes x_i$. For such a ξ we have

$$\left(\int \|\textstyle\sum \varphi_i(t)\psi_j(s)\xi_{ij}\|_{A^*}^2\, d\mu(t)\, d\mu(s)\right)^{1/2} = \left(\int \|\sum \varphi_i(t) x_i\|^2\, d\mu(t) \int \|\sum \psi_j(s) y_j\|^2\, d\mu(s)\right)^{1/2}$$

$$= \|x\|\, \|y\| \leqslant 1.$$

Since the unit ball of $M_n(B(H))_*$ is the closed convex hull of elements of this form, we obtain (3.2).

(ii) Consider a subset $\alpha \subset \{1, \dots, k\}$. We denote by α^c its complement. Recall that for elements $(a_J)_{J \in [n]^k}$ in A the norm $\|(a_J)\|_\alpha$ defined in (0.3) can be viewed as the norm of a matrix acting from $l_2([n]^\alpha, H)$ into $l_2([n]^{\alpha^c}, H)$. Therefore we deduce from (3.2) that for any $(\xi_J)_{J \in [n]^k}$ in A^* we have

$$\int \left\| \sum_{J \in [n]^k} \varepsilon_J \xi_J \right\|_{A^*} d\mu^k \leqslant \|(\xi_J)\|_\alpha^*. \tag{3.4}$$

Observe that by duality (2.1) has the following consequence. If $\|\sum_{J \in [n]^k} \delta_J^* \otimes \xi_J\|_{(E_n^k \otimes A)^*} \leqslant 1$, then there is a decomposition

$$\xi_J = \sum_{\alpha \subset \{1,\dots,k\}} \xi_J^\alpha \qquad \text{with} \sum_\alpha \|(\xi_J^\alpha)\|_\alpha^* \leqslant 1.$$

Therefore (3.3) follows from (3.4) and the triangle inequality. ■

We now reformulate the main result of [LPP] in our framework.

THEOREM 3.3. *For any $\{\xi_j | j \leqslant n\}$ in A^* we have*

$$\|\sum \varepsilon_j \xi_j\|_{L_1(\mu;A^*)} \leqslant \|\sum \delta_j^* \otimes \xi_j\|_{(E_n \otimes A)^*} \leqslant 2\|\sum \varepsilon_j \xi_j\|_{L_1(\mu;A^*)}. \tag{3.5}$$

Proof. The left side is (3.3) above for $k = 1$. By our earlier analysis of $E_n \otimes A$, the right side is clearly equivalent to the following fact.

Assume $\|\sum \varepsilon_j \xi_j\|_{L_1(\mu;A^*)} < 1$. Then there is a decomposition $\xi_j = x_j + y_j$ in A^* such that

$$\forall (a_j) \in A \qquad |\sum \langle x_j, a_j \rangle| \leqslant \|(\sum a_j^* a_j)^{1/2}\|$$

$$\text{and} \qquad |\sum \langle y_j, a_j \rangle| \leqslant \|(\sum a_j a_j^*)^{1/2}\|.$$

This is precisely what is proved in section II of [LPP], except that the sequence (ε_j) on Ω is replaced by the sequence $(e^{i3^j t})$ on the one-dimensional torus. By a routine averaging argument, one can then obtain the preceding fact as stated above with (ε_j). (Note that the approach of [LPP] can actually be developed directly for the functions (ε_j); this is explicitly done in [P2].)

We now relate certain series on $\mathbf{Z}^n$ (formed by iterating the expressions appearing in Theorem 3.3) with the corresponding series on the free group $F_n = \mathbf{Z} * \cdots * \mathbf{Z}$. In other words, our aim is to compare for these series the free group F_n with n generators with its commutative counterpart $\mathbf{Z}^n$.

LEMMA 3.4. *For any $\{\xi_J | J \in [n]^k\}$ in A^* we have (the summation being over all J in $[n]^k$)*

$$2^{-k}\|\sum \varepsilon_J \xi_J\|_{L_1(\mu^k;A^*)} \leqslant \|\sum \lambda_*(g_J) \otimes \xi_J\|_{(C_n^k \otimes A)^*} \leqslant 2^k \|\sum \varepsilon_J \xi_J\|_{L_1(\mu^k;A^*)}.$$

Proof. By the preceding three statements, we know that this holds for $k = 1$. We now argue by induction. Assume that Lemma 3.4 is proved for an integer $k \geqslant 1$, and let us prove it for $k + 1$. Consider elements $\{\xi_{J_j} | J \in [n]^k, j \in [n]\}$ in A^*. We have

$$\sum_{J' \in [n]^{k+1}} \lambda_*(g_{J'}) \otimes \xi_{J'} = \sum_{J \in [n]^k} \lambda_*(g_J) \otimes \left(\sum_{j \leqslant n} \lambda_*(g_j) \otimes \xi_{J_j} \right).$$

By the induction hypothesis, we have

$$\left\| \sum_{J'} \lambda_*(g_{J'}) \otimes \xi_{J'} \right\|_{(C_N^{k+1} \otimes A)^*} \leqslant 2^k \int_{\Omega^k} \|\sum \varepsilon_J(t) \eta_J\|_{(C \otimes A)^*} \, d\mu^k(t) \tag{3.6}$$

where $\eta_J = \sum_j \lambda_*(g_j) \otimes \xi_{J_j}$. Now for each fixed t in Ω^k, we have by (3.5) and Lemma 3.1

$$\|\sum \varepsilon_J(t) \eta_J\|_{(C \otimes A)^*} \leqslant 2 \int \|\sum \varepsilon_J(t) (\sum \varepsilon_j(s) \xi_{J_j})\|_{A^*} \, d\mu(s).$$

Integrating over $t \in \Omega^k$, this yields

$$\int \|\sum \varepsilon_J(t) \eta_J\|_{(C \otimes A)^*} \, d\mu^k(t) \leqslant 2 \int \left\| \sum_{J' \in [n]^{k+1}} \varepsilon_{J'} \xi_{J'} \right\|_{A^*} d\mu^{(k+1)}, \tag{3.7}$$

and hence (3.6) and (3.7) yield the induction step for $k + 1$. This concludes the proof for the right-side inequality in Lemma 3.4. The proof of the other inequality is entirely similar. ■

We now come to the main result of this section.

THEOREM 3.5. *For any $\{\xi_J | J \in [n]^k\}$ in A^* we have*

$$\|\sum \varepsilon_J \xi_J\|_{L_1(\mu^k, A^*)} \leqslant \|\sum \delta_J^* \otimes \xi_J\|_{(E_n^k \otimes A)^*} \leqslant 2^{2k} \|\sum \varepsilon_J \xi_J\|_{L_1(\mu^k, A^*)}.$$

Proof. With v and w as in Proposition 1.1, we have $\|w^{\otimes k}\|_{cb} \leqslant 2^k$, hence by (0.5)

$$\|w^{\otimes k} \otimes I_A\|_{E_n^k \otimes A \to C_n^k \otimes A} \leqslant 2^k.$$

Moreover, we have $w^{\otimes k}(\delta_J) = \lambda(g_J)$ hence $(w^{\otimes k} \otimes I_A)^*(\lambda_*(g_J) \otimes \xi_J) = \delta_J^* \otimes \xi_J$. This yields

$$\|\sum \delta_J^* \otimes \xi_J\|_{(E_n^k \otimes A)^*} \leqslant 2^k \|\sum \lambda_*(g_J) \otimes \xi_J\|_{(C_n^k \otimes A)^*}.$$

Combined with Lemma 3.4, this gives the right side in Theorem 3.5. The left side has already been proved in Lemma 3.2. ■

Remark. A slight modification of our proof yields Theorem 3.5 with the constant 2^{2k-1} instead of 2^{2k}.

Remark. Let k be a fixed integer. Consider the mapping

$$Q_k\colon C(\Omega^k) \to E_n^k$$

defined by

$$\forall f \in C(\Omega^k) \qquad Q_k(f) = \sum_{J \in [n]^k} \hat{f}(J)\delta_J,$$

where $\hat{f}$ is the Fourier transform of f, i.e., $\hat{f}(J) = \int f(t)\bar{\varepsilon}_J(t)\, d\mu^k(t)$. Let $N_k = Ker(Q_k)$. Dualizing (3.3), we find that $\|Q_k\|_{cb} \leqslant 1$. Hence, considering Q_k modulo its kernel and equipping $C(\Omega^k)/N_k$ with its quotient operator space structure (in the sense of [BP, ER]), we find a map

$$U_k\colon C(\Omega^k)/N_k \to E_n^k \qquad \text{with } \|U_k\|_{cb} \leqslant 1.$$

Then Theorem 3.5 admits the following dual reformulation: $U_k\colon C(\Omega^k)/N_k \to E_n^k$ is a complete isomorphism and $\|U_k^{-1}\|_{cb} \leqslant 2^{2k}$. In other words, the space $C(\Omega^k)/N_k$ is, for each k, completely isomorphic (uniformly with respect to n) to E_n^k.

Assume now that A is a von Neumann algebra and let A_* be its predual. We define for any family $(x_J)_{J \in [n]^k}$ in A_* the norm which is dual to the norm $\|\ \|_\alpha$ defined in (0.3). We set

$$\|(x_J)\|_\alpha^* = \sup\left\{\left|\sum_{J \in [n]^k} \langle a_J, x_J\rangle\right| \,\middle|\, a_J \in A,\ \|(a_J)\|_\alpha \leqslant 1\right\}. \tag{3.8}$$

Then we define

$$[(x_J)]_{(k)}^* = \inf \sum_{\alpha \in \{0,1\}^k} \|x_J^\alpha\|_\alpha^* \tag{3.9}$$

where the infimum runs over all x_J^α in A_* such that $x_J = \sum_{\alpha \in \{0,1\}^k} x_J^\alpha$.

Assume that $A = (A_*)^*$ is a von Neumann subalgebra of $B(H)$ and let q: $N(H) \to A_*$ be the quotient mapping which is the preadjoint of the embedding $A \hookrightarrow B(H)$. We can also write

$$\|(x_J)\|_\alpha^* = \inf\{\textstyle\sum |\lambda_m|\} \tag{3.10}$$

where the infimum runs over all the possibilities to write (x_J) as a series

$$x_J = \sum_m \lambda_m h_{\pi_\alpha(J)}^m \otimes k_{\pi_{\alpha^c}(J)}^m$$

where $(h_i^m)_{i\in[n]^\alpha}$ and $(k_j^m)_{j\in[n]^{\alpha^c}}$ are elements of H such that $\sum_i \|h_i^m\|^2 \leqslant 1$ and $\sum_j \|k_j^m\|^2 \leqslant 1$ for each m.

The identity of (3.8) and (3.10) is clear since the dual norms are the same by (0.3). Similarly, it is clear that the dual space to $(A_*)^{n^k}$ equipped with the norm $[\ ]_{(k)}^*$ can be identified with $(A)^{n^k}$ equipped with the norm $[\ ]_{(k)}$. By Proposition 2.1, this means that $(A_*)^{n^k}$ equipped with the norm $[\ ]_{(k)}^*$ can be viewed as a predual (isometrically) of $E_n^k \otimes A$. Hence, we can now rewrite Theorem 3.5 a bit more explicitly. For all (x_J) in A_*, we have (as announced in the beginning of this section)

$$(2^{2k})^{-1}[(x_J)]_{(k)}^* \leqslant \|\sum \varepsilon_J x_J\|_{L_1(\Omega^k, A_*)} \leqslant [(x_J)]_{(k)}^*. \tag{3.11}$$

In particular, we can make the following statement for emphasis.

THEOREM 3.6. *Let $A \subset B(H)$ be a von Neumann subalgebra with predual A_* and let $q\colon H \hat{\otimes} H \to A_*$ be the corresponding quotient mapping. Consider $\{x_J | J \in [n]^k\}$ in A_* such that*

$$\|\sum \varepsilon_J x_J\|_{L_1(\Omega^k, A_*)} < 1.$$

Then (x_J) admits a decomposition as

$$x_J = \sum_{\alpha \in \{0,1\}^k} x_J^\alpha$$

with

$$x_J^\alpha = q\left(\sum_n \lambda_m^\alpha h_{\pi_\alpha(J)}^m \otimes k_{\pi_{\alpha^c}(J)}^m\right)$$

where for each α, $\{h_i^m | i \in [n]^\alpha\}$ and $\{k_j^m | j \in [n]^{\alpha^c}\}$ are elements of H such that $\sum_i \|h_i^m\|^2 \leqslant 1$ and $\sum_j \|k_j^m\|^2 \leqslant 1$ and where λ_m^α are scalars such that

$$\sum_\alpha \sum_m |\lambda_m^\alpha| < 2^{2k}.$$

Conversely, if (x_J) admits such a decomposition, we must have $\|\sum \varepsilon_J x_J\|_{L_1(\Omega^k, A_*)} < 2^{2k}$.

Proof. The proof is nothing but (3.9), (3.10), and (3.11) spelled out explicitly.

Remark. The preceding theorem proves one of the conjectures formulated in [P2] in the case $A = B(H)$, $A_* = H \hat{\otimes} H$.

4. Complements. The following result shows that, in Proposition 1.3, the algebra $(C_\lambda^*(F_n))_{n=1}^\infty$ cannot be substituted by any sequence of nuclear algebras.

THEOREM 4.1. *Let A be either a nuclear C^*-algebra or an injective von Neumann algebra and let $I_{E_n} = vw$ be a factorization of I_{E_n} through A. Then*

$$\|v\|_{cb}\|w\|_{cb} \geqslant \frac{1}{2}(1+\sqrt{n}).$$

For the proof we need the following.

LEMMA 4.2. *Consider the subspace S_n of $M_n \oplus M_n$ given by*

$$S_n = \left\{ \begin{bmatrix} x_1 & \\ x_2 & \\ \vdots & \bigcirc \\ x_n & \end{bmatrix} \oplus \begin{bmatrix} y_1 \cdots y_n \\ \\ \bigcirc \end{bmatrix} \,\middle|\, x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{C} \right\}$$

and define $R\colon S_n \to S_n$ by

$$R(x \oplus y) = y^t \oplus x^t, \qquad x \oplus y \in S_n.$$

Then

(a) $(1/2)(I_{S_n} + R)$ *is a projection of S_n onto E_n and*

$$\left\|\frac{1}{2}(I_{S_n} + R)\right\|_{cb} = \frac{1}{2}(1+\sqrt{n});$$

(b) *for any projection Q of S_n onto E_n (resp. $M_n \oplus M_n$ onto E_n) one has*

$$\|Q\|_{cb} \geqslant \frac{1}{2}(1+\sqrt{n}).$$

Proof. (a) Obviously $R^2 = I_{S_n}$ and $E_n = \{a \in S_n \mid Ra = a\}$. Hence $(1/2)(I_{S_n} + R)$ is a projection of S_n onto E_n. Let A be a C^*-algebra. Then

$$S_n \otimes A = \left\{ \begin{bmatrix} a_1 & \\ \vdots & \bigcirc \\ a_n & \end{bmatrix} \oplus \begin{bmatrix} b_1 \cdots b_n \\ \\ \bigcirc \end{bmatrix} \,\middle|\, a_1, \ldots, a_n, b_1, \ldots, b_n \in A \right\}$$

and

$$(R \otimes I_A)\left[\begin{bmatrix} a_1 & \\ \vdots & \bigcirc \\ a_n & \end{bmatrix} \oplus \begin{bmatrix} b_1 \cdots b_n \\ \\ \bigcirc \end{bmatrix} \right] = \begin{bmatrix} b_1 & \\ \vdots & \bigcirc \\ b_n & \end{bmatrix} \oplus \begin{bmatrix} a_1 \cdots a_n \\ \\ \bigcirc \end{bmatrix}.$$

Since

$$\max(\|\sum b_i^* b_i\|^{1/2}, \|\sum a_i a_i^*\|^{1/2}) \leqslant \sqrt{n} \max\{\|a_1\|, \ldots, \|a_n\|, \|b_1\|, \ldots, \|b_n\|\}$$
$$\leqslant \sqrt{n} \max\{\|\sum a_i^* a_i\|^{1/2}, \|\sum b_i b_i^*\|^{1/2}\},$$

it follows that $\|R \otimes 1_A\| \leqslant \sqrt{n}$. Hence $\|R\|_{cb} \leqslant \sqrt{n}$, and thus

$$\left\| \frac{1}{2}(I_{S_n} + R) \right\|_{cb} \leqslant \frac{1}{2}(1 + \sqrt{n}).$$

To prove the converse inequality, it suffices to consider $n \geqslant 2$. Let A be the Cuntz algebra O_n (cf. [C]), which is generated by n isometries $s_1, \ldots, s_n \in B(H)$ satisfying

$$s_i^* s_j = \delta_{ij} I, \tag{4.1}$$

$$\sum_{i=1}^{n} s_i s_i^* = 1. \tag{4.2}$$

By (4.2) the element

$$z = \begin{bmatrix} s_1^* & \\ \vdots & \bigcirc \\ s_n^* & \end{bmatrix} \oplus \begin{bmatrix} s_1 \cdots s_n \\ \bigcirc \end{bmatrix}$$

in $S_n \otimes A$ has norm $\|z\| = 1$, while

$$\left(\frac{1}{2}(I_{S_n} + R) \otimes I_A\right)(z) = \frac{1}{2}\left[\begin{bmatrix} s_1 + s_1^* & \\ \vdots & \bigcirc \\ s_n + s_n^* & \end{bmatrix} \oplus \begin{bmatrix} s_1 + s_1^* \cdots s_n + s_n^* \\ \bigcirc \end{bmatrix} \right]$$

has norm

$$\frac{1}{2}\left\| \sum_{i=1}^{n} (s_i + s_i^*)^2 \right\|^{1/2}$$
$$= \frac{1}{2} \sup\left\{ \sum_{i=1}^{n} \|(s_i + s_i^*)\xi\|^2 \,|\, \xi \in H, \|\xi\| = 1 \right\}^{1/2}$$
$$= \frac{1}{2} \sup\left\{ \left(\left(\sum_{i=1}^{n} s_i^* s_i + s_i s_i^* + s_i^2 + (s_i^*)^2 \right) \xi, \xi \right) \middle| \xi \in H, \|\xi\| = 1 \right\}^{1/2}.$$

By (4.1) and (4.2), $\sum_{i=1}^n s_i^* s_i = nI$ and $\sum_{i=1}^n s_i s_i^* = I$. Set

$$v = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i^2 .$$

By (4.1), $v^* v = I$, and so v is an isometry. By (4.1) the range of v is orthogonal to the range of the isometry $s_1 s_2$: Indeed, for $\xi, \eta \in H$

$$\begin{aligned}(v\xi, s_1 s_2 \eta) &= \frac{1}{\sqrt{n}}\left(\sum_i (s_2^* s_1^* s_i^2 \xi, \eta)\right) \\ &= \frac{1}{\sqrt{n}}(s_2^* s_1 \xi, \eta) \\ &= 0 .\end{aligned}$$

Hence v is a nonunitary isometry. Therefore the point spectrum of v^* contains the open unit disk D (cf., e.g., [KP], p. 253). Hence also the "numerical range" of v

$$\{(v\xi, \xi) \mid \|\xi\| = 1\} = \{(\xi, v^*\xi) \mid \|\xi\| = 1\}$$

contains the open unit disk. In particular, the number 1 is in the closure of this set. Therefore

$$\begin{aligned}&\sup_{\|\xi\|=1}\left(\left(\sum_{i=1}^{n} s_i^* s_i + s_i s_i^* + s_i^2 + (s_i^*)^2\right)\xi, \xi\right) \\ &= n + 1 + 2\sqrt{n} \sup_{\|\xi\|=1} (\mathrm{Re}(v\xi, \xi)) \\ &\geqslant n + 1 + 2\sqrt{n} \\ &= (1 + \sqrt{n})^2 .\end{aligned}$$

Hence $\|((1/2)(I_{S_n} + R) \otimes I_A)(z)\| \geqslant (1/2)(1 + \sqrt{n})\|z\|$, which proves (a).

(b) Let Q be a projection from S_n onto E_n. Set $\hat{Q} = QR = RQR$. Then $\hat{Q}$ is also a projection from S_n to E_n. Let i_m denote the identity on M_m and t_m the transposition of M_m. Then

$$\hat{Q} \otimes i_m = (R \otimes t_m)(Q \otimes i_m)(R \otimes t_m).$$

Since $t_n \otimes t_m$ can be identified with transposition on M_{nm}, $\|t_n \otimes t_m\| = 1$. Hence by the definition of R,

$$\|R \otimes t_m\| \leqslant 1 .$$

Therefore

$$\|\hat{Q} \otimes i_m\| \leqslant \|Q \otimes i_m\|, \qquad m \in \mathbf{N},$$

and so $\|\hat{Q}\|_{cb} \leqslant \|Q\|_{cb}$; therefore also

$$\|\tfrac{1}{2}(Q + \hat{Q})\|_{cb} \leqslant \|Q\|_{cb}.$$

But

$$\tfrac{1}{2}(Q + \hat{Q}) = Q(\tfrac{1}{2}(I_{S_n} + R)),$$

and since Q is the identity on E_n, which is the range of $(1/2)(I_{S_n} + R)$, we have $(1/2)(Q + \hat{Q}) = (1/2)(I_{S_n} + R)$. Thus

$$\|Q\|_{cb} \geqslant \|\tfrac{1}{2}(I_{S_n} + R)\| = \tfrac{1}{2}(1 + \sqrt{n}).$$

If $\psi\colon M_n \oplus M_n \xrightarrow{\text{onto}} E_n$ is a projection of norm 1, then from the above

$$\|\psi\|_{cb} \geqslant \|\psi_{|S_n}\|_{cb} \geqslant \tfrac{1}{2}(1 + \sqrt{n}),$$

proving (b). ■

Proof of Theorem 4.1. Let $I_{E_n} = vw$ be a factorization of I_{E_n} through an injective von Neumann algebra A. By the injectivity of A, w can be extended to a linear map $\tilde{w}\colon M_n \oplus M_n \to A$ such that $\|\tilde{w}\|_{cb} \leqslant \|w\|_{cb}$ (cf. [Pa] Theorem 7.2). Clearly, $Q = v\tilde{w}$ is a projection of $M_n \oplus M_n$ onto E_n. Hence by (b) in the preceding lemma

$$\|v\|_{cb}\|w\|_{cb} \geqslant \|v\|_{cb}\|\tilde{w}\|_{cb} \geqslant \|Q\|_{cb} \geqslant \tfrac{1}{2}(1 + \sqrt{n}).$$

This proves the announced result when A is an injective von Neumann algebra. If A is a nuclear C^*-algebra, and $I_{E_n} = vw$ as above, we can extend v to a $\sigma(A^{**}, A^*)$-continuous linear map $\tilde{v}\colon A^{**} \to E_n$ such that $\|\tilde{v}\|_{cb} = \|v\|_{cb}$. Since A^{**} is an injective von Neumann algebra (cf., e.g., [CE]), we are now reduced to the preceding case. ■

Remark 4.3. The constant $(1/2)(1 + \sqrt{n})$ is the best possible in Theorem 4.1: Namely, let $A = M_n \oplus M_n$, let $w\colon E_n \to M_n \oplus M_n$ be the inclusion map, and define a projection $v\colon M_n \oplus M_n \to E_n$ by

$$v(x \oplus y) = \tfrac{1}{2}(I_{S_n} + R)(xp \oplus py), \qquad x \oplus y \in M_n \oplus M_n,$$

where $p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \bigcirc & \\ 0 & & & \end{bmatrix}$. Then clearly $vw = I_{E_n}$ and

$$\|v\|_{cb} = \|v\|_{cb}\|w\|_{cb} \leqslant \|\tfrac{1}{2}(I_{S_n} + S)\|_{cb} = \tfrac{1}{2}(1 + \sqrt{n}),$$

which indeed shows that Theorem 4.1 is sharp.

In connection with Lemma 4.2(b), note that there is obviously a projection $P\colon M_n \oplus M_n \to E_n$ with (ordinary) norm $\|P\| \leqslant 1$. (Simply take $P = v$ with v as in Remark 4.3.) However, we will show below that the projection constant of $E_n \otimes M_n$ in $(M_n \otimes M_n) \oplus M_n$ goes to infinity when $n \to \infty$. To see this, it is clearer to place the discussion in a broader context.

Let $S \subset B(H)$ be a closed subspace. We define $\lambda(S)$ (resp. $\lambda_{cb}(S)$, $\lambda_n(S)$) to be the infimum of the constants λ such that there is a projection $P\colon B(H) \to S$ satisfying $\|P\| \leqslant \lambda$ (resp. $\|P\|_{cb} \leqslant \lambda$, resp. $\|I_{M_n} \otimes P\|_{M_n(B(H)) \to M_n(S)} \leqslant \lambda$). Then by the extension theorem of *cb* maps (cf. [W, Pa]), these constants are invariants of the "operator space" structure of S. By this we mean that, if $S_1 \subset B(K)$ is another operator space which is completely isometric to S (resp. such that for some constant λ there is an isomorphism $u\colon S \to S_1$ with $\|u\|_{cb}\|u^{-1}\|_{cb} \leqslant \lambda$), then $\lambda(S_1) = \lambda(S)$, $\lambda_{cb}(S_1) = \lambda_{cb}(S)$, $\lambda_n(S_1) = \lambda_n(S)$ (resp. $(1/\lambda)\lambda(S) \leqslant \lambda(S_1) \leqslant \lambda\lambda(S)$ and similarly for the other constants).

By a simple averaging argument, we can prove the following statement.

PROPOSITION 4.4. *Let $S \subset B(H)$ be a closed subspace. Consider $M_n(S) = M_n \otimes S \subset B(l_2^n(H))$. Then*

(i) $\lambda_n(S) = \lambda(M_n(S))$;
(ii) *if S is $\sigma(B(H), B(H)_*)$-closed in $B(H)$, then*

$$\lambda_{cb}(S) = \sup_{n \geqslant 1} \lambda_n(S). \tag{4.3}$$

For any infinite-dimensional Hilbert space K we have

$$\lambda_{cb}(S) \leqslant \lambda(B(K) \otimes S). \tag{4.4}$$

Moreover, let $B(K) \overline{\otimes} S$ denote the weak- closure of $B(K) \otimes S$ in $B(K \otimes H)$. Then*

$$\lambda_{cb}(S) = \lambda(B(K) \overline{\otimes} S).$$

Proof. (i) The inequality $\lambda(M_n \otimes S) \leqslant \lambda_n(S)$ is obvious, and so we turn to the converse. Assume that there is a projection

$$P\colon M_n \otimes B(H) \to M_n \otimes S$$

with $\|P\| \leqslant \lambda$.

Let $\mathscr{U}_n$ be the group of all $n \times n$ unitary matrices. Consider then the group $G = \mathscr{U}_n \times \mathscr{U}_n$ equipped with its normalized Haar measure m. We will use the representation

$$\pi: G \to B(M_n, M_n)$$

defined by

$$\pi(u, v)x = uxv^*.$$

We can define an operator $\tilde{P}: M_n \otimes B(H) \to M_n \otimes B(H)$ by the formula

$$\tilde{P} = \int (\pi(u, v) \otimes I_{B(H)})P(\pi(u, v) \otimes I_{B(H)})^{-1}\, dm(u, v). \tag{4.5}$$

Note that $\pi(u, v)$ leaves $M_n \otimes S$ invariant so that the range of $\tilde{P}$ is included in $M_n \otimes S$ and $\tilde{P}$ restricted to $M_n \otimes S$ is the identity; hence $\tilde{P}$ is a projection from $M_n \otimes B(H)$ onto $M_n \otimes S$. Moreover, by Jensen's inequality (notice that $\pi(u, v) \otimes I_{B(H)}$ is an isometry on $M_n \otimes B(H)$) we have

$$\|\tilde{P}\| \leqslant \|P\| \leqslant \lambda.$$

Furthermore, using the translation invariance of m in (4.5) we find

$$\forall (u_0, v_0) \in G \qquad \tilde{P}(\pi(u_0, v_0) \otimes I_{B(H)}) = (\pi(u_0, v_0) \otimes I_{B(H)})\tilde{P}, \tag{4.6}$$

so that $\tilde{P}$ commutes with $\pi(u_0, v_0) \otimes I_{B(H)}$. By well-known facts, this implies that $\tilde{P}$ is of the form

$$\tilde{P} = I_{M_n} \otimes Q$$

for some operator Q which has to be a projection onto S. Indeed, since M_n is spanned by $\mathscr{U}_n$, the above formula (4.6) is equivalent to: For all a, b in M_n and for all x in $M_n \otimes B(H)$,

$$\tilde{P}((a \otimes 1)x(b \otimes 1)) = (a \otimes 1)\tilde{P}(x)(b \otimes 1). \tag{4.7}$$

Let $(e_{ij})_{i,j=1,\dots,n}$ denote the matrix units in M_n. Set $x = e_{ij} \otimes y$, where y is in $B(H)$ and i, j are in $\{1, \dots, n\}$. Applying (4.7) to $a = 1 - e_{ii}$ and $b = 1 - e_{jj}$, one gets $(1 - e_{ii})\tilde{P}(e_{ij} \otimes y)(1 - e_{jj}) = 0$; i.e., $\tilde{P}(e_{ij} \otimes y) = e_{ij} \otimes z$ for some z in $B(H)$ depending on y, i, and j. However applying (4.7) again, this time with $a = e_{ki}$ and $b = e_{jl}$, it follows that z is independent of i and j. Hence $\tilde{P} \cong I_{M_n} \otimes Q$, for some operator Q (which has to be a projection onto S). Finally, we conclude

$$\|I_{M_n} \otimes Q\| = \|\tilde{P}\| \leqslant \lambda,$$

and hence $\lambda_n(S) \leqslant \lambda(M_n \otimes S)$. This proves (i).

We now check (ii). Consider an arbitrary closed subspace $S \subset B(H)$ and let $\bar{S}$ be the $\sigma(B(H), B(H)_*)$-closure of S. We claim that there is an operator $Q: B(H) \to \bar{S}$ such that $Q_{|S} = I_S$ and $\|Q\|_{cb} \leqslant \sup_n \lambda_n(S)$.

Let $\varepsilon_n > 0$ be such that $\varepsilon_n \to 0$. For each n there is a projection $P_n: B(H) \to S$ such that

$$\|I_{M_n} \otimes P_n\|_{M_n(B(H)) \to M_n(S)} \leqslant (1 + \varepsilon_n)\lambda_n(S). \tag{4.8}$$

Let $\mathscr{U}$ be a nontrivial ultrafilter on $\mathbf{N}$. For any bounded sequence (α_n) of real numbers (or for any relatively compact sequence in a topological space), we will denote simply by $\lim_{\mathscr{U}} \alpha_n$ the limit of α_n when $n \to \infty$ along $\mathscr{U}$. For any x in $B(H)$ let

$$Q(x) = \lim_{\mathscr{U}} P_n(x)$$

where the limit is in the $\sigma(B(H), B(H)_*)$-sense. Observe that $\|Q\| \leqslant \lim_{\mathscr{U}} \|P_n\| \leqslant \sup_n \lambda_n(S)$. More generally, for any integer $m \geqslant 1$ we clearly have

$$\forall y \in M_m \otimes B(H) \qquad (I_{M_m} \otimes Q)(y) = \lim_{\mathscr{U}} (I_{M_m} \otimes P_n)(y),$$

and hence $\|I_{M_m} \otimes Q\| \leqslant \lim_{\mathscr{U}} \|I_{M_m} \otimes P_n\|$. But when $n \geqslant m$, we obviously have

$$\|I_{M_m} \otimes P_n\| \leqslant \|I_{M_n} \otimes P_n\|,$$

and hence by (4.8) we obtain

$$\|I_{M_m} \otimes Q\| \leqslant \lim_{\mathscr{U}} (1 + \varepsilon_n)\lambda_n(S) \leqslant \sup_n \lambda_n(S),$$

so that $\|Q\|_{cb} \leqslant \sup_n \lambda_n(S)$. Clearly, $Q(B(H)) \subset \bar{S}$ and $Q_{|S} = I_S$. This proves our claim, and in the case $\bar{S} = S$ we obtain (4.3). (Note that $\lambda_{cb}(S) \geqslant \sup_n \lambda_n(S)$ is trivial.) We now turn to (4.4). We may clearly assume $K = l_2$. Recall that there is obviously a completely contractive projection $\pi_n: B(l_2) \to M_n$ (here M_n is considered as a subspace of $B(l_2)$ in the usual way), and hence

$$\lambda_n(S) = \lambda(M_n \otimes S) \leqslant \|\pi_n\| \lambda(B(l_2) \otimes S) = \lambda(B(l_2) \otimes S)$$

which implies by (4.3)

$$\lambda_{cb}(S) \leqslant \lambda(B(l_2) \otimes S).$$

This concludes the proof of (4.4).

To prove the last assertion, note that $M_n(S)$ is clearly contractively complemented in $B(l_2) \overline{\otimes} S$, and hence we have

$$\lambda_{cb}(S) \leqslant \sup_{n \geqslant 1} \lambda_n(S) = \sup_{n \geqslant 1} \lambda(M_n(S)) \leqslant \lambda(B(l_2) \overline{\otimes} S).$$

To prove the converse inequality, note that $B(l_2) \overline{\otimes} B(H)$ can be identified with the space of matrices $a = (a_{ij})_{i,j \in \mathbb{N}}$ which are bounded on $l_2(H)$, and $B(l_2) \overline{\otimes} S$ can be identified with the subspace formed by all matrices with entries in S. Then if P is a completely bounded projection from $B(H)$ onto S, defining

$$\tilde{P}(a) = (P(a_{ij}))_{i,j \in \mathbb{N}}$$

we obtain a projection from $B(l_2) \overline{\otimes} B(H)$ to $B(l_2) \overline{\otimes} S$ with $\|\tilde{P}\| \leqslant \|P\|_{cb}$. To check this last estimate, observe that the norm of an element $a = (a_{ij})_{i,j \in \mathbb{N}}$ in $B(l_2) \overline{\otimes} B(H)$ is the supremum over n of the norms in $M_n(B(H))$ of the matrices $(a_{ij})_{i,j \leqslant n}$. This yields the last assertion. ■

COROLLARY 4.5. *Let H, K be Hilbert spaces. Consider a completely isometric embedding $E_n \to B(H)$. Then if* $\dim K = \infty$, *for any projection P from $B(K) \otimes B(H)$ to $B(K) \otimes E_n$ we have*

$$\|P\| \geqslant \tfrac{1}{2}(\sqrt{n} + 1).$$

A fortiori, the same holds for any projection P from $B(K \otimes H)$ onto $B(K) \otimes E_n$.

Proof. By the preceding statement, this follows from Theorem 4.1.

COROLLARY 4.6. *Let $M \subset B(H)$ be a von Neumann subalgebra such that M is isomorphic (as a von Neumann algebra) to $M_n(M)$ for some integer $n \geqslant 2$. Then if there is a bounded linear projection from $B(H)$ onto M, there is also a completely bounded one.*

Proof. Note that if M is isomorphic to $M_n(M)$, then obviously it is isomorphic to $M_n(M_n(M)) = M_{n^2}(M)$, and similarly to $M_{n^3}(M)$, and so on. Hence this follows clearly from the first two parts of Proposition 4.4 and the observation preceding Proposition 4.4. ■

In particular, using [V1], we have the following.

COROLLARY 4.7. *Let $M \subset B(H)$ be a von Neumann subalgebra. If M is isomorphic to the von Neumann algebra $VN(F_n)$ (resp. $VN(F_\infty)$) associated to the free group with $n > 1$ generators (resp. countably many generators), then there is no bounded linear projection from $B(H)$ onto M.*

Proof. First note that $VN(F_n)$ trivially embeds into $VN(F_\infty)$ as a subalgebra which is the range of a completely contractive projection. Therefore, by Proposition 1.3 and Theorem 4.1 there is no *completely* bounded projection from $B(H)$ onto M if M is isomorphic to $VN(F_\infty)$. By [V1] $M_n(VN(F_\infty))$ is isomorphic to $VN(F_\infty)$ for all n. Hence Corollary 4.7 for $VN(F_\infty)$ follows from the preceding corollary. To obtain the case of finitely many generators, recall the well-known fact that F_∞ can be embedded in F_n for all $n \geqslant 2$. (If a, b are two of the generators of F_n, then it is easy to check, that $b, aba^{-1}, \ldots, a^n b a^{-n}, \ldots$ are free generators of a subgroup isomorphic to F_∞.) Therefore if $M = VN(F_n)$ for $n > 1$, then $VN(F_\infty)$ is isomorphic to a von Neumann subalgebra $N \subset M$, and since M is a finite von Neumann algebra, N is the range of a conditional expectation; hence there is a bounded projection from M onto N. Since there is no bounded projection from $B(H)$ onto N by the first part of the proof, a fortiori there cannot exist a bounded projection from $B(H)$ onto M. ∎

For two operator spaces E and F of the same finite dimension n, one can define the complete version of the Banach-Mazur distance between E and F by

$$d_{cb}(E, F) = \inf\{\|u\|_{cb}, \|u^{-1}\|_{cb}\},$$

where the infimum is taken over all invertible linear maps u from E to F. By Proposition 1.3 it follows that

$$d_{cb}(E_n, \operatorname{span}\{\lambda(g_i) | i = 1, \ldots, n\}) \leqslant 2$$

for all $n \in \mathbf{N}$. The next proposition shows that the same inequality holds if the unitary operators $\lambda(g_1), \ldots, \lambda(g_n)$ are replaced by a semicircular or circular system of operators in the sense of Voiculescu [V1].

PROPOSITION 4.8. *Let $n \in \mathbf{N}$ and let $x_1, \ldots, x_n$ be a semicircular or circular system of operators on a Hilbert space; then the map $u: E_n \to \operatorname{span}\{x_1, \ldots, x_n\}$ given by*

$$u: \sum_{k=1}^{n} c_k \delta_k \to \sum_{k=1}^{n} c_k x_k, \qquad c_k \in \mathbf{C},$$

satisfies $\|u\|_{cb} \|u^{-1}\|_{cb} \leqslant 2$.

Proof. Assume first that $x_1, \ldots, x_n$ is a semicircular system of selfadjoint operators in the sense of [V1]. By [V2], we can exchange $x_1, \ldots, x_n$ with the operators

$$x_k = \tfrac{1}{2}(s_k + s_k^*), \qquad k = 1, \ldots, n,$$

where $s_1, \ldots, s_n$ are the "creation operators" $\xi \to e_i \otimes \xi$ on the full Fock space

$$\mathscr{H} = \mathbf{C} \otimes \left(\bigoplus_{n=1}^{\infty} H^{\otimes n} \right)$$

based on a Hilbert space H with orthonormal basis $(e_1, \ldots, e_n)$. In particular, $s_1, \ldots, s_n$ are n isometries with orthogonal ranges, and therefore

$$\sum_{k=1}^{n} s_k s_k^* \leqslant 1.$$

Hence, as in the proof of Proposition 1.1, we get that for any n-tuple $a_1, \ldots, a_n$ of elements in a C^*-algebra A,

$$\begin{aligned}\left\|\sum_k x_k \otimes a_k\right\| &\leqslant \frac{1}{2}\left(\left\|\sum_k s_k \otimes a_k\right\| + \left\|\sum_k s_k^* \otimes a_k\right\|\right)\\ &\leqslant \frac{1}{2}\left(\left\|\sum_k s_k s_k^*\right\|^{1/2}\left\|\sum_k a_k^* a_k\right\|^{1/2} + \left\|\sum_k s_k s_k^*\right\|^{1/2}\left\|\sum_k a_k a_k^*\right\|^{1/2}\right)\\ &\leqslant \max\left\{\left\|\sum_k a_k^* a_k\right\|^{1/2}, \left\|\sum_k a_k a_k^*\right\|^{1/2}\right\}.\end{aligned}$$

Hence $\|u\|_{cb} \leqslant 1$. To prove that $\|u^{-1}\|_{cb} \leqslant 2$, notice that by [V1], [V2], the C^*-algebra generated by $x_1, \ldots, x_n$ and 1 has a trace

$$\tau\colon C^*(x_1, \ldots, x_n, 1) \to \mathbf{C}$$

(namely the vector-state given by a unit vector in the $\mathbf{C}$-part of the Fock space $\mathscr{H}$), with the properties:

$$\tau(1) = 1, \qquad \tau(x_k^2) = \frac{1}{4} \text{ and } \tau(x_k x_l) = 0, \qquad k \neq l.$$

Let $a_1, \ldots, a_n$ be n operators in a C^*-algebra A and let $S(A)$ denote the state space of A. Then

$$\begin{aligned}\left\|\sum_k x_k \otimes a_k\right\|^2 &\geqslant \sup_{\omega \in S(A)} (\tau \otimes \omega)\left(\left(\sum_k x_k \otimes a_k\right)^*\left(\sum_l x_l \otimes a_l\right)\right)\\ &= \frac{1}{4} \sup_{\omega \in S(A)} \omega\left(\sum_k a_k^* a_k\right)\\ &= \frac{1}{4}\left\|\sum_k a_k^* a_k\right\|,\end{aligned}$$

and similarly $\|\sum_k x_k \otimes a_k\|^2 \geqslant \frac{1}{4}\|\sum_k a_k a_k^*\|$. Hence

$$\left\|\sum_k x_k \otimes a_k\right\| \geqslant \frac{1}{2} \max\left\{\left\|\sum_k a_k^* a_k\right\|^{1/2}, \left\|\sum_k a_k a_k^*\right\|^{1/2}\right\},$$

proving that $\|u^{-1}\|_{cb} \leqslant 2$.

Assume finally that $y_1, \ldots, y_n$ is a circular system. Then

$$y_k = \frac{1}{\sqrt{2}}(x_{2k-1} + i x_{2k}), \qquad k = 1, \ldots, n,$$

where $(x_1, \dots, x_{2n})$ is a semicircular system of selfadjoint operators. Therefore the statement about circular systems in Proposition 4.8 follows from the one on semicircular systems by observing that the map

$$\sum_{k=1}^{n} c_k \delta_k \to \frac{1}{\sqrt{2}} \sum_{k=1}^{n} c_k (e_{2n-1} + e_{2k})$$

defines a *cb*-isometry of E_n onto its range in E_{2n}. ■

To conclude this paper we give a generalization of Proposition 1.1 to free products of discrete groups, or more generally free products of C^*-probability spaces in the sense of [V1] and [V2]. We refer to [V1] and [V2] for the terminology.

PROPOSITION 4.9. *Let (A, φ) be a C^*-algebra equipped with a faithful state φ. Let $(A_i)_{i \in I}$ be a free family of unital C^*-subalgebras of A in the sense of [V1] or [V2]. Consider elements $x_i \in A_i$ such that for some $\delta > 0$*

$$\forall i \in I \qquad \|x_i\| \leqslant 1, \quad \varphi(x_i) = 0, \quad \textit{and} \min\{\varphi(x_i^* x_i), \varphi(x_i x_i^*)\} \geqslant \delta^2.$$

Then for all finitely supported families $(a_i)_{i \in I}$ in $B(H)$ (H Hilbert), we have

(4.9)
$$\delta \max\{\|\textstyle\sum a_i^* a_i\|^{1/2}, \|\sum a_i a_i^*\|^{1/2}\} \leqslant \|\sum x_i \otimes a_i\| \leqslant 2 \max\{\|\sum a_i^* a_i\|^{1/2}, \|\sum a_i a_i^*\|^{1/2}\}.$$

Proof. We may assume that I is finite. The lower bound in (4.9) is proved exactly as in the semicircular case. To prove the upper bound we will prove that A can be faithfully represented as a C^*-algebra of operators on a Hilbert space H, such that x_i admits a decomposition $x_i = u_i + v_i$ with u_i, v_i in $B(H)$ and

$$\|\textstyle\sum u_i^* u_i\| \leqslant 1 \quad \text{and} \quad \|\sum v_i v_i^*\| \leqslant 1. \tag{4.10}$$

The upper bound in (4.9) then follows as in the semicircular case.

Following the notation of [V2, pp. 558–559], we let (H_i, ξ_i) be the space of the GNS-representation $\pi_i = \pi_{\varphi|A_i}$. In particular, ξ_i is a unit vector in H_i and

$$\varphi(x) = (\pi_i(x)\xi_i, \xi_i) \qquad \text{when } x \in A_i.$$

Then A can be realized as the C^*-algebra of operators on the Hilbert space

$$(H, \xi) = *_{i \in I}(H_i, \xi_i)$$

generated by $\bigcup_{i \in I} \lambda_i \circ \pi_i(A_i)$, where $\lambda_i\colon B(H_i) \to B(H)$ is the $*$-representation defined in [V2, sect. 1.2]. For simplicity of notation we will identify A_i with its range in $B(H)$; i.e., we set

$$\lambda_i \circ \pi_i(x) = x \qquad \text{when } x \in A_i.$$

Let $x \in A_i$. Corresponding to the decomposition

$$H_i = H_i^0 \oplus \mathbb{C}\xi_i,$$

we can write $\pi_i(x)$ as a 2×2 matrix

$$\pi_i(x) = \begin{pmatrix} b & \eta \\ \zeta^* & t \end{pmatrix}$$

where $b \in B(H_i^0)$, $\eta, \zeta \in H_i^0$, and $t \in \mathbb{C}$. (Here we identify η, ζ with the corresponding linear maps from $\mathbb{C}$ to H_i^0, and we also identify $\mathbb{C}$ with $\mathbb{C}\xi_i$.) The action of $x = \lambda_i \circ \pi_i(x)$ on $*_{i \in I}(H_i, \xi_i)$ can now be explicitly computed from [V2, sect. 1.2]. One finds that

$$x\xi = \eta \otimes \xi + t\xi, \tag{4.11}$$

$$x(h_1 \otimes \cdots \otimes h_n) = bh_1 \otimes \cdots \otimes h_n + (h_1, \zeta)h_2 \otimes \cdots \otimes h_n \text{ when } n \geqslant 1, \quad h_k \in H_{i_k}^0, i = i_1 \neq i_2 \neq \cdots \neq i_n, \tag{4.12}$$

$$x(h_1 \otimes \cdots \otimes h_n) = \eta \otimes h_1 \otimes \cdots \otimes h_n + th_1 \otimes \cdots \otimes h_n, \text{ when } n \geqslant 1, \quad h_k \in H_{i_k}^0, i \neq i_1 \neq i_2 \neq \cdots \neq i_n \tag{4.13}$$

where $h_2 \otimes \cdots \otimes h_n = \xi$ for $n = 1$.

Let $e_i \in B(H)$ be the orthogonal projection of H onto the subspace

$$H_i = \bigoplus_{n=1}^{\infty} (\oplus(H_{i_1} \otimes \cdots \otimes H_{i_n}))$$

where the second direct sum contains all n-tuples $(i_1, \ldots, i_n)$ for which $i = i_1 \neq i_2 \neq \cdots \neq i_n$. From (4.11), (4.12), and (4.13), one gets for all x in A_i

$$(1 - e_i)x(1 - e_i) = \varphi(x)(1 - e_i) \tag{4.14}$$

where we have used that

$$t = (\pi_i(x)\xi_i, \xi_i) = \varphi(x).$$

Let now $x_i \in A_i$, $\|x_i\| \leqslant 1$, $\varphi(x_i) = 0$. Then by (4.14)

$$(1 - e_i)x_i(1 - e_i) = 0.$$

Thus $x_i = u_i + v_i$, where

$$u_i = x_i e_i \quad \text{and} \quad v_i = e_i x_i (1 - e_i).$$

Since $\|x_i\| \leqslant 1$ and since $(e_i)_{i \in I}$ is a set of pairwise orthogonal projections,

$$\sum_{i \in I} u_i^* u_i \leqslant \sum_{i \in I} e_i \leqslant 1$$

and

$$\sum_{i \in I} v_i v_i^* \leqslant \sum_{i \in I} e_i \leqslant 1.$$

This completes the proof of Proposition 4.9. ■

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