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Almost L^2 matrix coefficients

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The purpose of this note is to insert into the literature two long-but-not-well-known facts about matrix coefficients of unitary representations. Our first theorem concerns general locally compact groups G: a positive definite function belonging to $L^{2+\varepsilon}(G)$ for every ε in $\mathbb{R}^+$ is in fact a matrix coefficient of a unitary representation weakly contained in the regular representation. This generalises the familiar theory of square integrable representations (see e.g. [Dix]) to give information "about almost square integrable representations". Our second theorem is about real and p-adic semisimple algebraic groups: we obtain growth estimates for the K-finite matrix coefficients of unitary representations weakly contained in the regular representation which imply in particular that they belong to $L^{2+\varepsilon}(G)$ for every ε in $\mathbb{R}^+$. This is related to estimates of Harish-Chandra and other authors on the behaviour of matrix coefficients at infinity. Both our results are simple to state and to prove. Together, they characterise matrix coefficients of tempered representations of semisimple groups, and spotlight the dramatic difference between abelian and semisimple harmonic analysis.

Let G be a locally compact group, with a left-invariant Haar measure denoted m or dx. The left and right modular functions of G are denoted δ_G and Δ_G respectively: thus

$$\int_G f(xy)\,dx = \delta_G(y) \int_G f(x)\,dx = \Delta_G(y)^{-1} \int_G f(x)\,dx \qquad \forall f \in L^1(G),\ \forall y \in G.$$

(In abstract harmonic analysis, Δ is usually used (see [Dix], [HR], [L]), while semisimple harmonic analysts usually use δ (see [HC1-6]) (note however that [W] uses δ for our Δ and [Kir] uses Δ for our δ). We shall continue this somewhat schizophrenic tradition.) The group convolution algebra $L^1(G)$, equipped with the involution $f \to f^*$, where

$$f^*(x) = \Delta_G(x)^{-1} \bar{f}^\vee(x) \qquad \forall x \in G,$$

$^-$ denoting complex conjugation, and $^\vee$ reflection ($f^\vee(x) = f(x^{-1})$ for all x in G), is a Banach $*$-algebra.

Let ϱ be a unitary representation of G on a Hilbert space $\mathbf{H}_\varrho$, with inner product denoted $(\,,)$. A matrix coefficient of ϱ is a function on G of the form

$$\phi: x \longrightarrow (\varrho(x)\,\xi, \eta), \tag{1}$$

where $\xi, \eta \in \mathbf{H}_\varrho$. If $\xi = \eta$, then ϕ is called a diagonal matrix coefficient, and is a positive definite function, in the sense of Bochner. Vice versa, any positive definite function can be realised as a diagonal matrix coefficient by the Gelfand-Naimark-Segal construction (see [Dix], [HR], or [L]). The representation ϱ extends to a *-representation of $L^1(G)$: for f in $L^1(G)$, $\varrho(f)$ is the operator on $\mathbf{H}$ for which

$$(\varrho(f)\,\xi, \eta) = \int_G f(x)\,(\varrho(x)\,\xi, \eta)\,dx \qquad \forall \xi, \eta \in \mathbf{H}_\varrho.$$

In particular, for the left regular representation λ, $\lambda(f)$ is the operator of left convolution by f on $L^2(G): \lambda(f)\,g = f * g$ for any g in $L^2(G)$.

Let σ be another unitary representation of G. Following Fell [F], we say that σ is weakly contained in ϱ if any diagonal matrix coefficient of σ can be approximated, uniformly on compacta, by convex combinations of diagonal matrix coefficients of ϱ. Eymard [Ey] developed equivalent formulations of this notion using the theory of C^*-algebras. For instance, σ is weakly contained in ϱ if and only if

$$\|\sigma(f)\| \leqq \|\varrho(f)\| \qquad \forall f \in L^1(G), \tag{2}$$

or, by density, if the same inequality holds for all f in $C_c(G)$, the space of compactly supported continuous functions on G, or equivalently, by duality, if and only if each matrix coefficient $x \longrightarrow (\sigma(x)\,\theta, \zeta)$ of σ is the limit, uniformly on compacta, of sums of matrix coefficients of ϱ

$$x \longrightarrow \sum_{i=1}^{n} (\varrho(x)\,\xi_i, \eta_i),$$

subject to the condition that

$$\sum_{i=1}^{n} \|\xi_i\|\;\|\eta_i\| \leqq \|\theta\|\;\|\zeta\|. \tag{3}$$

Loosely speaking, an irreducible representation is weakly contained in the regular representation if it appears in the Plancherel formula.

Here is our first main result.

Theorem 1. *Let σ be a unitary representation of the locally compact group G on the Hilbert space $\mathbf{H}_\sigma$. Let ξ be a cyclic vector in $\mathbf{H}_\sigma$* (i.e., *the set of vectors* $\{\sigma(x)\,\xi : x \in G\}$ *spans a dense subspace of* $\mathbf{H}_\sigma$). *Suppose that the diagonal matrix coefficient $\phi: x \longrightarrow (\sigma(x)\,\xi, \xi)$ lies in $L^{2+\varepsilon}(G)$ for each ε in $\mathbb{R}^+$. Then σ is weakly contained in the regular representation.*

Remark. The assumption that ξ is cyclic could be avoided by restricting σ to the closed subspace spanned by $\{\sigma(x)\,\xi : x \in G\}$.

Our proof of Theorem 1 provides a formula for the norm of a convolution operator on $L^2(G)$ which may be of independent interest. (Special cases were certainly known [Lei]; see remark (d) following the proof.)

The statement of our second main result requires a few more preliminaries.

Let G be a semisimple algebraic group over a local field; denote by P and K a minimal parabolic and a maximal compact subgroup of G, such that the Iwasawa decomposition holds:

$$G = KP. \tag{4}$$

(For groups over $\mathbb{R}$, there is only one choice of K up to conjugacy: for p-adic groups, one must choose a "good" K [BT].) Let δ be the left modular function of P, so that if dp is left-invariant Haar measure on P, $\delta(p)\,dp$ is a right-invariant Haar measure on P. As G is unimodular, Haar measure on G is both left-K-invariant and right-P-invariant; it follows that the Haar measures dx, dk and dp of G, K, and P can be normalized so that

$$\int_G f(x)\,dx = \int_K \int_P f(kp)\,\delta(p)\,dp\,dk \qquad \forall f \in C_c(G), \tag{5}$$

and

$$\int_K dk = 1.$$

Extend δ to a left-K-invariant function on G, still denoted δ, by the formula

$$\delta(kp) = \delta(p) \qquad \forall k \in K,\ \forall p \in P. \tag{6}$$

Let Ξ be the function defined by Harish-Chandra [HC 1], [HC 5]:

$$\Xi(x) = \int_K \delta^{-\frac{1}{2}}(xk)\,dk \qquad \forall x \in G. \tag{7}$$

The K-bi-invariant function Ξ is the matrix coefficient $x \to (\varrho(x)\,\xi, \xi)$, where ξ is a unit K-invariant vector for the representation σ of G unitarily induced from the trivial representation of P. As P is amenable, the trivial representation of P is weakly contained in the regular representation of P; by continuity of induction [F], ϱ is weakly contained in the regular representation of G. Harish-Chandra has shown that Ξ is almost square integrable: $\Xi \in L^{2+\varepsilon}(G)$ for all ε in $\mathbb{R}^+$ ([HC 2], [HC 5]).

Theorem 2. *Let σ be a unitary representation of the semi-simple algebraic group G, weakly contained in the regular representation of G. Let ξ and η be vectors in the Hilbert space $\mathbf{H}_\sigma$ of σ. Suppose that the decomposition of the closed subspaces $\mathbf{H}_\xi$ and $\mathbf{H}_\eta$ of $\mathbf{H}_\sigma$ generated by $\sigma(K)\xi$ and $\sigma(K)\eta$ involves finitely many distinct irreducible representations of K of arbitrary multiplicities,* i.e. *that, as K-representation spaces,*

$$\mathbf{H}_\xi = \sum_{\tau \in S_\xi}^{\oplus} m_\tau \mathbf{H}_\tau$$

and

$$\mathbf{H}_\eta = \sum_{\tau \in S_\eta}^{\oplus} n_\tau \mathbf{H}_\tau,$$

where S_ξ *and* S_η *are finite subsets of* $\hat{K}$. *Denote by* D_ξ *and* D_η *the sums* $\sum_{\tau \in S_\xi} d_\tau^2$ *and* $\sum_{\tau \in S_\eta} d_\tau^2$ *of the squares of the dimensions* d_τ *of the distinct* K-*irreducible constituents of* $\sigma|_{\mathbf{H}_\xi}$ *and* $\sigma|_{\mathbf{H}_\eta}$, *and by* d_ξ *and* d_η *the dimensions* $\dim \mathbf{H}_\xi$ *and* $\dim \mathbf{H}_\eta$ *of* $\mathbf{H}_\xi$ *and* $\mathbf{H}_\eta$. *Then* $d_\xi \leqq D_\xi$, $d_\eta \leqq D_\eta$, *and*

$$|\langle \sigma(x)\,\xi, \eta\rangle| \leqq (d_\xi d_\eta)^{\frac{1}{2}} \|\xi\|\,\|\eta\|\,\Xi(x) \qquad \forall x \in G. \tag{8}$$

Related estimates have appeared in the work of Harish-Chandra and other authors [A], [Co], [CM], [HC1-6], [Hw], [K], [M], [TV], [V1-3]. Our proof of Theorem 2 utilises a simple analytic consequence of the Schur orthogonality relations (see the proposition in the proof of Theorem 2). This fact has doubtless been observed by others, but we would like to point it out to the reader as a group theoretic version of the Sobolev inequality.

We shall now prove these theorems, and then give a few applications.

Proofs of Theorems 1 and 2

Before we prove our first main result, we recall that, if X is a compact subset of the locally compact group G, with non-void interior, then the Haar measure $m(X^n)$ of X^n grows at most exponentially, i.e.

$$m(X^n) = \int_{X^n} 1\,dx \leqq C M^{n-1} \qquad \forall n \in \mathbf{N}, \tag{9}$$

for appropriate (X-dependent) constants C and M. Indeed, since X^2 is compact, we can find finitely many points y_i, $1 \leqq i \leqq M$, in G, so that

$$X^2 \subseteqq \bigcup_{i=1}^{M} y_i X.$$

Inductively, it can be seen that

$$X^n \subseteqq \bigcup y_{i_1} y_{i_2} \cdots y_{i_{n-1}} X,$$

from which (9) follows immediately.

Proof of Theorem 1. By (2), it suffices to show that

$$\|\sigma(f)\| \leqq \|\lambda(f)\| \qquad \forall f \in C_c(G). \tag{10}$$

For any unitary representation ϱ of G,

$$\|\varrho(f)\|^2 = \|\varrho(f)^* \varrho(f)\| = \|\varrho(f^* * f)\| \qquad \forall f \in C_c(G),$$

so it will suffice to calculate $\|\varrho(f)\|$ for self-adjoint ($f^* = f$) elements of $C_c(G)$ in order to calculate $\|\varrho(f)\|$ in general. Further, if f is selfadjoint in $C_c(G)$, $\varrho(f)$ is self-adjoint.

For self-adjoint f, let $\mu_{\theta,\theta}$ denote the measure on the spectrum of $\varrho(f)$ induced by θ in $\mathbf{H}_\varrho$ via spectral theory. By measure theory,

$$\int t^2 d\mu_{\theta,\theta} \leqq \left[\int t^{2n} d\mu_{\theta,\theta}\right]^{\frac{1}{n}} \left[\int d\mu_{\theta,\theta}\right]^{1-\frac{1}{n}}$$

and

$$\lim_{n\to\infty} \left[\int t^{2n} d\mu_{\theta,\theta}\right]^{\frac{1}{n}} = \sup\{t^2 : t \in \operatorname{supp}(\mu_{\theta,\theta})\}.$$

Consequently

$$\|\varrho(f)\theta\| = (\varrho(f^{(*2)})\theta, \theta)^{\frac{1}{2}} \leqq \lim_{n\to\infty} (\varrho(f^{(*2n)})\theta, \theta)^{\frac{1}{2n}} \|\theta\|,$$

whence

$$\|\varrho(f)\| = \sup_{\theta\in\mathbf{H}} \|\varrho(f)\theta\| \, \|\theta\|^{-1} \leqq \sup_{\theta\in\mathbf{H}} \lim_{n\to\infty} (\varrho(f^{(*2n)})\theta, \theta)^{\frac{1}{2n}},$$

where $\mathbf{H}$ is a dense subspace of $\mathbf{H}_\varrho$. The converse inequality being trivial, equality holds, and so after rewriting in terms of matrix coefficients, we have that, for any f in $C_c(G)$,

$$\|\varrho(f)\| = \sup_{\theta\in\mathbf{H}} \lim_{n\to\infty} \left(\int_G (f^* * f)^{(*2n)}(x)\,(\varrho(x)\theta, \theta)\,dx\right)^{\frac{1}{4n}}. \tag{11}$$

We shall use this expression to estimate $\sigma(f)$ and $\lambda(f)$ separately, and so prove (10).

We consider the represention σ, and take $\{\sigma(g)\xi : g \in C_c(G)\}$ as the dense subspace $\mathbf{H}$ of $\mathbf{H}_\sigma$ in (11). We fix temporarily g in $C_c(G)$, and define θ, ϕ, and ψ as follows:

$$\theta = \sigma(g)\,\xi,$$
$$\phi(x) = (\sigma(x)\xi, \xi) \qquad \forall x \in G,$$
$$\psi(x) = (\sigma(x)\theta, \theta) \qquad \forall x \in G.$$

Recalling that $g^\vee$ denotes the reflection of g, we see that

$$\begin{aligned} \psi(x) &= (\sigma(x)\,\sigma(g)\xi, \sigma(g)\xi) \\ &= \int_G \int_G g(z)\,\bar{g}(y)\,(\sigma(x)\,\sigma(z)\xi, \sigma(y)\xi)\,dy\,dz \\ &= \int_G \int_G g(z)\,\bar{g}(y)\,\phi(y^{-1}xz)\,dy\,dz \\ &= \bar{g} * \phi * g^\vee(x); \end{aligned} \tag{12}$$

consequently ψ also lies in $L^{2+\varepsilon}(G)$ for each ε in $\mathbb{R}^+$. Let X be the support of $h = f^* * f * f^* * f$ in G. Then, by Schwarz' inequality, Hölder's inequality and (9),

$$\begin{aligned} \left|\int_G (f^* * f)^{(*2n)}(x)\,\psi(x)\,dx\right|^2 &= \left|\int_{X^n} (f^* * f * f^* * f)^{(*n)}(x)\,\psi(x)\,dx\right|^2 \\ &\leqq \left(\int_{X^n} |h^{(*n)}(x)|^2\,dx\right)\left(\int_{X^n} |\psi(x)|^2\,dx\right) \\ &\leqq \|h^{(*n)}\|_2^2 \left(\int_{X^n} |\psi(x)|^{2+\varepsilon}\,dx\right)^{\frac{2}{2+\varepsilon}} \left(\int_{X^n} 1\,dx\right)^{\frac{\varepsilon}{2+\varepsilon}} \\ &\leqq \|h^{(*n)}\|_2^2\, \|\psi\|_{2+\varepsilon}^2\,(CM^{n-1})^{\frac{\varepsilon}{2+\varepsilon}}. \end{aligned} \tag{13}$$

We extract roots and let n tend to infinity. From (11), it follows that

$$\|\sigma(f)\| \leqq \liminf_{n\to\infty} \|(f^* * f)^{(*2n)}\|_2^{\frac{1}{4n}} M^{\frac{\varepsilon}{8(2+\varepsilon)}};$$

since ε is arbitrary,

$$\|\sigma(f)\| \leqq \liminf_{n\to\infty} \|(f^* * f)^{(*2n)}\|_2^{\frac{1}{4n}}. \tag{14}$$

For the representation λ, we take $C_c(G)$ to be the dense subspace of $\mathbf{H}_\lambda$ in formula (11). For g in $C_c(G)$, $x \to (\lambda(x)g, g)$ lies in $C_c(G)$, so that, by the Schwartz inequality,

$$\Big|\int_G (f^* * f)^{(*2n)}(x)\,(\lambda(x)g, g)\,dx\Big| \leqq \|(f^* * f)^{(*2n)}\|_2 \, \|(\lambda(\,.\,)g, g)\|_2,$$

whence, from (11),

$$\|\lambda(f)\| \leqq \liminf_{n\to\infty} \|(f^* * f)^{(*2n)}\|_2^{\frac{1}{4n}}. \tag{15}$$

However, by definition,

$$(f^* * f)^{(*2n)} = (f^* * f)^{(*2n-2)} * (f^* * f)^{(*2)}$$

so

$$\|(f^* * f)^{(*2n)}\|_2 \leqq \|\lambda(f^* * f)\|^{2n-2} \, \|(f^* * f)^{(*2)}\|_2,$$

and

$$\limsup_{n\to\infty} \|(f^* * f)^{(*2n)}\|_2^{\frac{1}{4n}} \leqq \|\lambda(f)\|.$$

Combining this inequality with (15), we conclude that

$$\lim_{n\to\infty} \|(f^* * f)^{(*2n)}\|_2^{\frac{1}{4n}} = \|\lambda(f)\|;$$

this inequality, together with (14), proves (10) and hence the theorem. ∎

Remarks. (a) In the proof of (13), we saw that

$$\Big|\int_G (f^* * f)^{(*2n)}(x)\,\psi(x)\,dx\Big|^2 \leqq \Big(\int_G |(f^* * f)^{(*2n)}(x)|^2\,dx\Big)\Big(\int_{X^n} |\psi(x)|^2\,dx\Big).$$

If the Haar measure of X^n grows slower than exponentially, i.e. if

$$\lim_{n\to\infty} m(X^n)^{\frac{1}{n}} = 1 \tag{16}$$

for any compact subset X of G, then, since

$$\int_{X^n} |\psi(x)|^2\,dx \leqq \|\psi\|_\infty^2 \, m(X^n),$$

we obtain (14) without any hypotheses on ϕ. This implies that if (16) holds, then every unitary representation is weakly contained in the regular representation, i.e. that G is amenable.

(b) It can be similarly shown that, if $w: G \longrightarrow \mathbb{R}^+$ is continuous and satisfies the conditions that

$$\lim_{n \to \infty} \left(\sup_{x \in X^n} w(x)^{\frac{1}{n}}\right) = 1$$

for all compact subsets X of G, and that w is submultiplicative, i.e.

$$w(xy) \leqq w(x)\, w(y) \qquad \forall x, y \in G,$$

then, if the function $x \longrightarrow w^{-1}(x)\phi(x)$ lies in $L^2(G)$, the associated representation σ is weakly contained in the regular representation. Indeed, in (13), it suffices to estimate as follows:

$$\int_{X^n} |\psi(x)|^2\, dx \leqq \left(\int_{X^n} |w^{-1}(x)\, \psi(x)|^2\, dx\right) \sup_{x \in X^n} w^2(x),$$

whence (14) follows as before. Examples of functions w with this property include the functions $[1+d]^k$, $k \in \mathbb{N}$, where $d(x)$ is the geodetic distance of x from the origin in a connected Lie group G with a left-invariant Riemannian metric, or $d(x)$ is the distance of $x \cdot o$ from o in a metric structure (a symmetric space or a building, for instance) on which G acts by isometries. This is somewhat more directly related to Harish-Chandra's concept of tempered representations.

(c) If ϕ is a diagonal matrix coefficient of a unitary representation weakly contained in the regular representation, then so is $\phi^{\vee}$. Consequently, if G is not unimodular, and σ is a unitary representation of G with a cyclic vector ξ such that $\phi: x \longrightarrow (\sigma(x)\xi, \xi)$ does not belong to $L^{2+\varepsilon}(G)$ for all ε in $\mathbb{R}^+$, but such that $\phi^{\vee}$ does have this property, then σ is still weakly contained in the regular representation.

(d) As part of the proof of Theorem 1, we showed that

$$\|\lambda(f)\| = \lim_{n \to \infty} \|(f^* * f)^{(*2n)}\|_2^{\frac{1}{4n}} \qquad \forall f \in C_c(G).$$

A little extra work shows that

$$\|\lambda(f)\| = \lim_{n \to \infty} \|(f^* * f)^{(*n)}\|_2^{\frac{1}{2n}} \qquad \forall f \in C_c(G),$$

and

$$\|\lambda(f)\| = \lim_{n \to \infty} \|f^{(*n)}\|_2^{\frac{1}{n}}$$

for self-adjoint f in $C_c(G)$. This was known for special cases [Lei].

Proof of Theorem 2. There are three steps in our proof: reduction to consideration of the regular representation, reduction to the consideration of K-fixed vectors, and the K-fixed vector case.

Given a function v on K, we identify it with the singular measure on G, also denoted v, by the formula

$$v(u) = \int_K u(k)\, v(k)\, dk \qquad \forall u \in C_c(G).$$

Let ξ be one of the vectors of Theorem 2, and let $\mathbf{H}_\xi$ be the closed subspace of $\mathbf{H}_\sigma$ generated by $\sigma(K)\xi$. It is a standard result in the representation theory of compact groups that cyclic representations are equivalent to subrepresentations of the regular representation λ. Consequently there is a unitary mapping which intertwines the action of K by σ on $\mathbf{H}_\xi$ with the left translation action on a left-invariant subspace (i.e. left-ideal) J_ξ of $L^2(K)$. The hypotheses of Theorem 2 tell us that, in the decomposition of J_ξ into irreducible components, only representations in the finite subset S_ξ of $\hat{K}$ appear. It follows immediately that

$$d_\xi = \dim(\mathbf{H}_\xi) = \dim(J_\xi) \leqq \sum_{\tau \in S_\xi} d_\tau^2.$$

However, the representation theory of compact groups gives us more information. Consideration of the left ideal J_ξ in $L^2(K)$ shows that there is a unique function e_ξ in $C(K)$ with the properties that

$$e_\xi = e_\xi * e_\xi = e_\xi^*$$

and

$$C(K) * e_\xi = J_\xi.$$

It also follows readily from the theory that

$$d_\xi = \dim(\mathbf{H}_\xi) = \dim(J_\xi) = \|e_\xi\|_2^2$$

(this is seen by breaking J_ξ into irreducible components and treating these individually) and (since $e_\xi^\vee = \bar{e}_\xi$)

$$\sigma(\bar{e}_\xi)\,\xi = \sigma(e_\xi^\vee)\,\xi = \xi.$$

We note a consequence of this discussion. If ϕ denotes the matrix coefficient $k \to \langle \sigma(k)\xi, \zeta \rangle$, then (cf. (12))

$$\begin{aligned} \phi(k) &= \langle \sigma(k)\, \sigma(\bar{e}_\xi)\, \xi, \zeta \rangle \\ &= e_\xi * \phi(k) \qquad \forall k \in K; \end{aligned}$$

from the Cauchy-Schwartz inequality, we conclude that

$$\|\phi\|_\infty \leqq \|e_\xi\|_2\, \|\phi\|_2 = d_\xi^{\frac{1}{2}}\, \|\phi\|_2.$$

This discussion can be, for convenience, summarised in the following proposition.

Proposition. *Let σ be a unitary representation of a compact group K, and let ξ be a K-finite vector in $\mathbf{H}_\sigma$* (i.e. *the span $\mathbf{H}_\xi$ of $\sigma(K)\xi$ is finite-dimensional*). *Then there exists a unique function e_ξ in $C(K)$ so that*

$$e_\xi = e_\xi * e_\xi = e_\xi^*,$$

$$\sigma(\bar{e}_\xi)\,\xi = \xi$$

and

$$d_\xi = \dim \operatorname{span} \lambda(K) e_\xi = \|e_\xi\|_2^2.$$

Consequently, for any ζ in $\mathbf{H}_\sigma$, if ϕ denotes the matrix coefficient $k \longrightarrow (\sigma(k)\xi, \zeta)$, then

$$\|\phi\|_\infty \leqq d_\xi^{\frac{1}{2}} \|\phi\|_2.$$

Remark. The last inequality is our group theoretic version of the Sobolev inequality. In particular, if we take σ to be the regular representation, it follows that, if the left translates of ϕ span a d-dimensional space, then $\|\phi\|_\infty \leqq d^{\frac{1}{2}} \|\phi\|_2$. It is easy to show that this inequality is sharp — just consider the characters of irreducible representations τ of dimension d, which are matrix coefficients of the restriction of the regular representation to the τ-isotypic subspace of $L^2(K)$.

Proof of Theorem 2 (*continued*). We now consider our matrix coefficient $\phi\colon x \longrightarrow (\sigma(x)\xi, \eta)$. Then from the proposition, there are self-adjoint projections e_ξ and e_η in $C(K)$ so that

$$e_\xi * \phi * e_\eta = \phi.$$

By (3), we can approximate ϕ, uniformly on compacta, by sums $\sum_{i=1}^{n} \psi_i$ of matrix coefficients $\psi_i\colon x \longrightarrow (\lambda(x) g_i, h_i)$, satisfying the condition that

$$\sum_{i=1}^{n} \|g_i\|_2 \, \|h_i\|_2 \leqq \|\xi\| \, \|\eta\|.$$

Since e_ξ and e_η have compact support, and further $\lambda(\bar{e}_\xi)$ and $\lambda(\bar{e}_\eta)$ are projections, we can approximate ϕ by sums of matrix coefficients $\sum_{i=1}^{n} e_\xi * \psi_i * e_\eta$, where

$$e_\xi * \psi_i * e_\eta(x) = (\lambda(x)\,\lambda(\bar{e}_\xi) g_i, \lambda(\bar{e}_\eta) h_i)$$

and

$$\sum_{i=1}^{n} \|\lambda(\bar{e}_\xi) g_i\|_2 \, \|\lambda(\bar{e}_\eta) h_i\|_2 \leqq \|\xi\| \, \|\eta\|.$$

Consequently, to prove the theorem, it will suffice to show, for any g and h in $L^2(G)$ such that $\bar{e}_\xi * g = g$ and $\bar{e}_\eta * h = h$, that

$$|(\lambda(x) g, h)| \leqq (d_\xi d_\eta)^{\frac{1}{2}} \|g\|_2 \, \|h\|_2 \, \Xi(x) \qquad \forall x \in G. \tag{18}$$

Next, we define left-K-invariant functions $\tilde{g}$ and $\tilde{h}$ on G by the formulae

$$\tilde{g}(x) = \sup_{k \in K} |g(kx)| \qquad \forall x \in G,$$

$$\tilde{h}(x) = \sup_{k \in K} |h(kx)| \qquad \forall x \in G.$$

Clearly,

$$\begin{aligned} |(\lambda(x) g, h)| &= |\int_G g(x^{-1} y)\, h(y)\, dy| \\ &\leqq \int_G \tilde{g}(x^{-1} y)\, \tilde{h}(y)\, dy \\ &= (\lambda(x)\tilde{g}, \tilde{h}) \end{aligned}$$

and $\lambda(k)\tilde{g} = \tilde{g}$ and $\lambda(k)\tilde{h} = \tilde{h}$ for all k in K. If $g = \bar{e}_\xi * g$, then

$$g(k' x) = \bar{e}_\xi * g(k' x)$$

whence

$$\begin{aligned} |g(k' x)| &\leqq \|e_\xi\|_2 (\int_K |g(k^{-1} k' x)|^2\, dk)^{\frac{1}{2}} \\ &= d_\xi^{\frac{1}{2}} (\int_K |g(kx)|^2\, dk)^{\frac{1}{2}}, \end{aligned}$$

and

$$\tilde{g}(x) \leqq d_\xi^{\frac{1}{2}} (\int_K |g(kx)|^2\, dk)^{\frac{1}{2}}.$$

Also,

$$\begin{aligned} (\int_G |\tilde{g}(x)|^2\, dx)^{\frac{1}{2}} &\leqq d_\xi^{\frac{1}{2}} (\int_G \int_K |g(kx)|^2\, dk\, dg)^{\frac{1}{2}} \\ &= d_\xi^{\frac{1}{2}} \|g\|_2. \end{aligned}$$

Analogous considerations apply to h and $\tilde{h}$, so in order to prove (18), it will suffice to show that, if $\tilde{g}, \tilde{h} \in L^2(G)$ and $\lambda(k)\tilde{g} = \tilde{g}$ and $\lambda(k)\tilde{h} = \tilde{h}$ for all k in K, then

$$(19) \qquad |(\lambda(x)\tilde{g}, \tilde{h})| \leqq \|\tilde{g}\|_2\, \|\tilde{h}\|_2\, \Xi(x) \qquad \forall x \in G.$$

The last step of the proof is to show (19) holds. We recall (4) that $G = KP$ and (5) that

$$\int_G f(x)\, dx = \int_K \int_P f(kp)\, \delta(p)\, dp\, dk \qquad \forall f \in C_c(G).$$

It follows that, for $\tilde{g}$ and $\tilde{h}$ as above, that

$$(\lambda(x)\tilde{g}, \tilde{h}) = \int_K \int_P \tilde{g}(x^{-1} kp)\, \tilde{h}(kp)\, \delta(p)\, dp\, dk.$$

By the Schwartz inequality,

$$|(\lambda(x)\tilde{g}, \tilde{h})| \leqq \int_K \left(\int_P |\tilde{g}(x^{-1}kp)|^2 \delta(p)\,dp\right)^{\frac{1}{2}} \left(\int_P |\tilde{h}(kp)|^2 \delta(p)\,dp\right)^{\frac{1}{2}} dk. \tag{20}$$

Since $\tilde{h}$ is left-K-invariant, for any k in K,

$$\|\tilde{h}\|_2 = \left(\int_K \int_P |\tilde{h}(kp)|^2 \delta(p)\,dp\,dk\right)^{\frac{1}{2}} = \left(\int_P |\tilde{h}(kp)|^2 \delta(p)\,dp\right)^{\frac{1}{2}}. \tag{21}$$

From the Iwasawa decomposition (4), for fixed x in G and k in K, we may write $x^{-1}k$ as $k_0 p_0$. By using the left-K-invariance of $\tilde{g}$, we see

$$\begin{aligned}\left(\int_P |\tilde{g}(x^{-1}kp)|^2 \delta(p)\,dp\right)^{\frac{1}{2}} &= \left(\int_P |\tilde{g}(k_0 p_0 p)|^2 \delta(p)\,dp\right)^{\frac{1}{2}} \\ &= \left(\int_P |\tilde{g}(p_0 p)|^2 \delta(p_0 p)\,dp\right)^{\frac{1}{2}} \delta(p_0)^{-\frac{1}{2}} \\ &= \left(\int_P |\tilde{g}(p)|^2 \delta(p)\,dp\right)^{\frac{1}{2}} \delta(p_0)^{-\frac{1}{2}} \\ &= \|\tilde{g}\|_2\, \delta(p_0)^{-\frac{1}{2}}.\end{aligned} \tag{22}$$

Finally, recalling that δ was extended to G so that $\delta(k_0 p_0) = \delta(p_0)$, we see that $\delta(p_0)^{-\frac{1}{2}} = \delta(x^{-1}k)$, and from (20), (21) and (22) we conclude that

$$\begin{aligned}|(\lambda(x)\tilde{g}, \tilde{h})| &\leqq \int_K \|\tilde{g}\|_2\, \|\tilde{h}\|_2\, \delta(x^{-1}k)^{-\frac{1}{2}} dk \\ &= \|\tilde{g}\|_2\, \|\tilde{h}\|_2\, \Xi(x),\end{aligned}$$

by definition (7) of Ξ. This proves (19) and thereby the theorem. ∎

Remark. This theorem owes much to Herz [Hz], who showed that, if $\phi(x) = (\lambda(x)g, h)$, with g and h in $L^2(G)$, then, by Minkowski's integral inequality,

$$\begin{aligned}\left(\int_K \int_K |\phi(k'xk)|^2\, dk\,dk'\right)^{\frac{1}{2}} &= \left(\int_K \int_K \left|\int_G g(k^{-1}x^{-1}y)\,\bar{h}(k'y)\,dy\right|^2 dk\,dk'\right)^{\frac{1}{2}} \\ &\leqq \int_G g^{\#}(x^{-1}y)\, h^{\#}(y)\,dy \\ &= (\lambda(x)g^{\#}, h^{\#}),\end{aligned}$$

where

$$g^{\#}(x) = \left(\int_K |g(kx)|^2\, dk\right)^{\frac{1}{2}} \qquad \forall x \in G$$

and

$$h^{\#}(x) = \left(\int_K |h(kx)|^2\, dk\right)^{\frac{1}{2}} \qquad \forall x \in G$$

and then, using the fact that $\lambda(k)g^{\#} = g^{\#}$ and $\lambda(k)h^{\#} = h^{\#}$ for all k in K, deduced that

$$(\lambda(x)g^{\#}, h^{\#}) \leqq \|g\|_2\, \|h\|_2\, \Xi(x) \qquad \forall x \in G. \tag{23}$$

Our third step is just Herz' proof of (23). Dual to Herz' result is the inequality

$$\|\lambda(f)\| \leqq \int_G \Big(\int_K \int_K |f(k'xk)|^2 \, dk\, dk'\Big)^{\frac{1}{2}} \Xi(x)\, dx$$

for f for which the right hand side makes sense. This is closely related to work of the second named author on harmonic analysis on free groups [Haa].

Applications

These two results imply that, for a semisimple algebraic group G, a unitary representation is weakly contained in the regular representation if and only if (a dense set of) its matrix coefficients lie in $L^{2+\varepsilon}(G)$ for each ε in $\mathbb{R}^+$. However, for a noncompact abelian group G, the irreducible unitary representations are one-dimensional and their matrix coefficients are just (multiples of) the group characters, which never lie in $L^2(G)$ if $q < \infty$. Theorem 2 provides a clear indication of the remarkable difference between harmonic analysis on semisimple and on abelian groups. Further, Theorem 1 and 2 together provide an abstract rationale for the centrality of the notion of "temperedness" in the theory of Harish-Chandra ([HC3], [HC4], [HC7]). Here is another application of Theorem 1.

Corollary. *Suppose that σ is an irreducible unitary representation of the semisimple group G, and that for some vector ξ in $\mathbf{H}_\sigma$, the diagonal matrix coefficient $\phi: x \longrightarrow (\sigma(x)\xi, \xi)$ lies in $L^{2k+\varepsilon}(G)$ for all ε in $\mathbb{R}^+$, where k is a positive integer. Then*

(a) *all matrix coefficients of σ lie in $L^{2k+\varepsilon}(G)$ for all ε in $\mathbb{R}^+$;*

(b) *if $\xi, \eta \in \mathbf{H}_\sigma$ are K-finite, then*

$$|(\sigma(x)\xi, \eta)| \leqq (\dim \mathbf{H}_\xi \dim \mathbf{H}_\eta)^{\frac{1}{2}} \|\xi\| \, \|\eta\| \, \Xi^{\frac{1}{k}}(x) \qquad \forall x \in G; \tag{24}$$

(c) *if $(\hat{G})_p$ denotes the subset of the unitary dual $\hat{G}$ of G consisting of representations which have a dense set of matrix coefficients in $L^p(G)$, then $\bigcap_{p>2k} (\hat{G})_p$ is a closed subset of $\hat{G}$.*

Proof. (a) By arguing as in (12), we see there is a dense subspace $\mathbf{H}$ of $\mathbf{H}_\sigma$ so that if $\theta, \zeta \in \mathbf{H}$, then $\psi: x \longrightarrow (\sigma(x)\theta, \zeta)$ lies in $L^{2k+\varepsilon}(G)$. Consider the k-fold tensor product $\mathbf{H}_\sigma \otimes \mathbf{H}_\sigma \otimes \cdots \otimes \mathbf{H}_\sigma = \mathbf{H}_\sigma^{\otimes k}$. Let $\sigma^{\otimes k}$ denote the action of G on $\mathbf{H}_\sigma^{\otimes k}$ via the k-fold tensor product of σ. If $\theta_i, \zeta_i \in \mathbf{H}$, and $\theta' = \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_k$, $\zeta' = \zeta_1 \otimes \cdots \otimes \zeta_k$, then

$$\psi'(x) = (\sigma^{\otimes k}(x)\theta', \zeta')' = \prod_{i=1}^{k} (\sigma(x)\theta_i, \zeta_i)$$

where $(,)'$ indicates the inner product in $\mathbf{H}_\sigma^{\otimes k}$. Hence $\psi'(x)$ lies in $L^{2+\varepsilon}(G)$ for all ε in $\mathbb{R}^+$. By Theorem 1, we see that $\sigma^{\otimes k}$ is weakly contained in the regular representation, and by the Kunze-Stein phenomenon [Co], (the same proof works for p-adic groups) all coefficients of $\sigma^{\otimes k}$ lie in $L^{2+\varepsilon}(G)$. In particular, the matrix coefficients $x \longrightarrow (\sigma(x)\theta, \zeta)^k$, $\theta, \zeta \in \mathbf{H}_\sigma$, lie in $L^{2+\varepsilon}(G)$, whence $x \longrightarrow (\sigma(x)\theta, \zeta)$ lies in $L^{2k+\varepsilon}(G)$.

(b) If $\xi \in \mathbf{H}_\sigma$ is K-finite, then $\xi^{\otimes k} = \xi \otimes \xi \otimes \cdots \otimes \xi$ in $\mathbf{H}_\sigma^{\otimes k}$ is also K-finite. More precisely, if $\mathbf{H}_{(\xi^{\otimes k})}$ is the subspace of $\mathbf{H}_\sigma^{\otimes k}$ generated by $\sigma^{\otimes k}(K)\,\xi^{\otimes k}$, then

$$\mathbf{H}_{(\xi^{\otimes k})} \subseteqq (\mathbf{H}_\xi)^{\otimes k}.$$

In particular $\dim \mathbf{H}_{(\xi^{\otimes k})} \leqq (\dim \mathbf{H}_\xi)^k$. Hence, applying estimate (8) to the coefficient $x \to (\sigma(x), \xi, \eta)^k$ of $\mathbf{H}_\sigma^{\otimes k}$, and taking k-th roots yields (24).

(c) This statement follows directly from (b) since (24) is a pointwise estimate and therefore is valid on closed subsets of $\hat{G}$. ▮

There are singular representations of semisimple groups, such as the oscillator representation of $\mathrm{Sp}_{2n}(\mathbb{R})$, which have coefficients which belong to $L^{2k+\varepsilon}(G)$ for some k in $\mathbb{N}$ and all ε in $\mathbb{R}^+$, but which do not belong to $L^{2k}(G)$ [Hw]. Thus part (a) of the corollary is a substantial improvement of Lemma 7. 3 of [Co]. Similarly parts (b) and (c) improve on Corollaries 7. 2 and 7. 3 of [Hw].

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