Injectivity and decomposition of completely bounded maps

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Introduction

A linear map S from a C*-algebra A into a C*-algebra B is completely positive if

$$S \otimes i_m : A \otimes M_m \to B \otimes M_m$$

is positive for all m. Here M_m is the algebra of complex $m \times m$ matrices and i_m is the identity on M_m. Moreover a linear map T from A to B is completely bounded if

$$\sup_{m \in \mathbb{N}} \| T \otimes i_m \| < \infty .$$

The supremum is called the completely bounded norm of T and is denoted $\|T\|_{cb}$.

In 1979 Wittstock proved the striking result that any completely bounded map from a C*-algebra A into an injective C*-algebra B is a linear combination of completely positive maps from A to B. More specifically he proved that if $T : A \to B$ is a completely bounded selfadjoint map (i.e. $T(x^*) = T(x)^*$, $x \in A$), then there exist completely positive maps T_1, T_2 from A to B, such that

$$T = T_1 - T_2 \quad \text{and} \quad \|T_1 + T_2\| \leq \|T\|_{cb}$$

(cf. [27, Satz 4.5]). Later Paulsen found a simpler proof of Wittstock's result based on Arveson's extension theorem (cf. [15, Cor. 2.6] and [2, Thm. 1.2.9]). He also proved that for any (not necessarily selfadjoint) completely bounded linear map T from a C*-algebra A into an injective C*-algebra B, there exist completely positive maps S_1, S_2 from A to B, such that $\|S_i\| \leq \|T\|_{cb}$ i=1,2, and such that

$$x \to \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix}$$

is a completely positive map from A to $B \otimes M_2$. (This follows from [16, thm. 2.5]).

In the following we let CP(A,B) (resp. CB(A,B)) denote the set of completely positive (resp. completely bounded) maps from a C*-algebra A to a C*-algebra B. The main result of this paper is the following converse to Wittstock's theorem:

Let N _be a non-injective von Neumann algebra, then for every infinite dimensional C*-algebra_ A, _there exists a completely bounded map_ T : A → N, _which is not a linear combination of completely positive maps. In particular a von Neumann algebra_ N _is injective if and only if_ CB(N,N) = span CP(N,N). (cf. Theorem 2.6 and corollary 2.8).

It is essential that N is a von Neumann algebra, because Huruya has recently given an example of a non-injective C*-algebra B, such that CB(A,B) = span CP(A,B) for all C*-algebras A (cf. [10]). Smith proved in [20, example 2.1] that for the abelian C*-algebra A = C([0,1]), one has

$$\text{span } CP(A,A) \subsetneqq CB(A,A).$$

The first example of a von Neumann algebra N for which

$$\text{span } CP(A,N) \subsetneqq CB(A,N)$$

for some C*-algebra A was given by Huruya and Tomiyama (cf. [11, example 12]).

We apply our result to show that for every infinite dimensional C*-algebra A, there exists a completely bounded map T of A into some quotient C*-algebra B/J, which has no completely bounded lifting $\tilde{T}$ from A to B

$$\begin{array}{ccc} & \overset{\tilde{T}}{\nearrow} & B \\ & & \downarrow \\ A & \xrightarrow[T]{} & B/J \end{array}$$

(cf. corollary 2.9). Hence the Choi-Effros lifting theorem for completely positive maps [4] fails for completely bounded maps, even if A is abelian. If $\dim(A) < \infty$, T has of course always a linear lifting. However, we show that for a particular choice of B and J, we can find completely bounded maps T_n from $M_n = M_n(\mathbb{C})$, $n \geqq 3$ to B/J, such that

$$\|\tilde{T}_n\|_{cb} \geqq \frac{n}{2\sqrt{n-1}} \|T_n\|_{cb}$$

for any linear lifting $\tilde{T}_n$ of T_n. (cf. prop. 3.2). This gives the negative answer to a problem posed by Paulsen [17].

To prove the above mentioned results, it is convenient to introduce a norm $\| \ \|_{dec}$ on span CP(A,B) for arbitrary C*-algebras A and B. For $T \in$ span CP(A,B), we let $\|T\|_{dec}$ denote the infimum of those $\lambda \geqq 0$, for which there exist $S_1, S_2 \in CP(A,B)$, such that

$$x \to \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix}$$

is a completely positive map from A to $B \otimes M_2$. If T is self-adjoint, $\|T\|_{dec}$ is simply

$$\|T\|_{dec} = \inf \{ \|T_1 + T_2\| \mid T = T_1 - T_2 \ , \ T_1, T_2 \in CP(A,B) \}$$

(cf. def. 1.1. and prop. 1.3). We show that the inequality

$$\|T\|_{cb} \leqq \|T\|_{dec}$$

always holds, so by Wittstock's and Paulsen's results

$$\|T\|_{cb} = \|T\|_{dec}$$

whenever B is injective. Our main result (theorem 2.6) is a relative easy consequence of the following characterization of injective von Neumann algebras, which we prove in theorem 2.1:

A von Neumann algebra N is injective if and only if there exists $c \in \mathbb{R}_+$, such that for all linear maps T from ℓ_n^∞ to N,

$$\|T\|_{dec} \leqq c\|T\|_{cb} .$$

Here ℓ_n^∞ denotes n-dimensional abelian C*-algebra $\ell^\infty\{1,\ldots,n\}$. The starting point in the proof of theorem 2.1 is that the hyperfinite II_1-factor R can be characterized among all factors on a separable Hilbert space by the property that

$$\|\sum_{i=1}^{n} u_i \otimes u_i^c\|_{H\otimes H^c} = n$$

for any finite set $u_1,\ldots,u_n$ of unitaries in R. This was proved by Connes as an offshoot of his work on injective factors (cf. [6, Remark 5.29]). Thus if N is a non-injective finite factor (on a separable Hilbert space) one can choose unitaries $u_1,\ldots,u_n \in N$ such that

$$\frac{1}{n}\|\sum_{i=1}^{n} u_i \otimes u_i^c\|_{H\otimes H^c} < 1 .$$

By considering the r'th power of $\sum_{i=1}^{n} u_i \otimes u_i^c$, we can obtain $m = n^r$

unitaries $v_1,\dots,v_m \in N$, such that

$$\frac{1}{m} \| \sum_{i=1}^{m} v_i \otimes v_i^c \|_{H \otimes H^c}$$

is smaller than any given constant γ . Now if one define $T : \ell_m^\infty \to N$ by

$$T(c_1,\dots,c_m) = \sum_{i=1}^{m} c_i v_i$$

it turns out that $\|T\|_{dec} > \gamma^{-\frac{1}{2}} \|T\|_{cb}$, which proves theorem 2.1 in the case of II_1-factors on a separable Hilbert space. The general case is obtained by extending Connes' result to finite von Neumann algebras with a non-trivial center (lemma 2.2) and by using Takesaki's decomposition of a type III von Neumann algebra as a crossed product of a semifinite algebra with a one-parameter group of automorphisms.

In section 3 we give concrete examples of linear maps T_n from ℓ_n^∞ to the von Neumann algebra $\mathcal{M}(\mathbb{F}_2)$ associated with the regular representation of the free group on two generators, such that $\|T_n\|_{dec} > \|T_n\|_{cb}$ for $n \geqq 3$, and

$$\|T_n\|_{dec} / \|T_n\|_{cb} \to \infty \quad \text{for} \quad n \to \infty$$

(cf. example 3.1). On the other hand, we prove in prop. 3.4 that for any linear map T from ℓ_2^∞ to a von Neumann algebra N ,

$$\|T\| = \|T\|_{cb} = \|T\|_{dec} .$$

§1.

Decomposable linear maps between C*-algebras.

Let A,B be C*-algebras. We will call a bounded linear map from A to B decomposable if it is a linear combination of completely positive maps from A to B. Note first that a bounded linear map T from A to B is decomposable if and only if there exist $S_1, S_2 \in CP(A,B)$, such that

$$(*) \qquad R(x) = \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix}$$

defines a completely positive map from A to $B \otimes M_2$. Assume namely that $T = \sum_{i=1}^{n} c_i T_i$, $c_i \in \mathbb{C}$ and $T_i \in CP(A,B)$. Then clearly $S_1 = S_2 = \sum_{i=1}^{n} |c_i| T_i$ can be used. Conversely if $T \in B(A,B)$ and there exist $S_1, S_2 \in CP(A,B)$ such that (*) defines a completely positive map R from A to $B \otimes M_2$, one checks easily that

$$T = (T_1 - T_2) + i(T_3 - T_4)$$

where

$$T_1 = \tfrac{1}{4}(S_1 + S_2 + T + T^*) \ , \quad T_2 = \tfrac{1}{4}(S_1 + S_2 - T - T^*) \ ,$$

$$T_3 = \tfrac{1}{4}(S_1 + S_2 - iT + iT^*) \ , \quad T_4 = \tfrac{1}{4}(S_1 + S_2 + iT - iT^*)$$

are four completely positive maps from A to B. (T^* is the linear map given by $T^*(x) = T(x^*)^*$, $x \in A$).

For two linear maps R_1, R_2 from A to B we write

$$R_1 \underset{cp}{\leq} R_2$$

if $R_2 - R_1$ is completely positive.

Definition 1.1

Let A and B be C*-algebras and let $T : A \to B$ be a bounded linear map. If T is decomposable we let $\|T\|_{dec}$ denote the infimum of those $\lambda \geqq 0$ for which there exist $S_1, S_2 \in CP(A,B)$, such that $\|S_i\| \leqq \lambda$, $i = 1,2$, and

$$R(x) = \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix}$$

is a completely positive map from A to $B \otimes M_2$. If T is not decomposable, we put $\|T\|_{dec} = +\infty$.

Remark 1.2

We could equivalently have defined $\|T\|_{dec}$ as the infimum of those $\lambda \geqq 0$ for which there exist $S_1, S_2 \in CP(A,B)$, such that $\|S_i\| \leqq \lambda$, $i = 1,2$, and

$$\tilde{R}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} S_1(x_{11}) & T^*(x_{12}) \\ T(x_{21}) & S_2(x_{22}) \end{pmatrix}$$

is a completely bounded map from $A \otimes M_2$ to $B \otimes M_2$. Indeed if $\tilde{R}$ is completely positive, so is R, because

$$\tilde{R} = R \circ P$$

where P is the completely positive map from A to $A \otimes M_2$ given by

$$P(x) = \begin{pmatrix} x & x \\ x & x \end{pmatrix} .$$

To prove the converse, let $(e_{ij})_{i=1,2}$ be the matrix units of M_2, and let $Q : M_2 \otimes M_2 \to M_2$ be the linear map defined by

$$Q(e_{ij} \otimes e_{k\ell}) = \begin{cases} e_{ij} & \text{for } i=k \text{ and } j=\ell \\ 0 & \text{otherwise.} \end{cases}$$

One checks easily that Q is completely positive (Q can be written as $Q = Q_2 \circ Q_1$ where $Q_1(x) = exe$, $e = e_{11} \otimes e_{11} + e_{22} \otimes e_{22}$, and Q_2 is a $*$-isomorphism of $e(M_2 \otimes M_2)e$ onto M_2). Since

$$\tilde{R} = (i_B \otimes Q) \circ (R \otimes i_2)$$

it follows that $\tilde{R}$ is completely positive whenever R is.

<u>Proposition 1.3</u>

Let A and B be C*-algebras.

(1) If $T \in B(A,B)$ is a selfadjoint decomposable linear map, then

$$\|T\|_{dec} = \inf\{\|S\| \mid S \in CP(A,B),\ -S \underset{cp}{\leq} T \underset{cp}{\leq} S\}$$

$$= \inf\{\|T_1 + T_2\| \mid T_1, T_2 \in CP(A,B),\ T = T_1 - T_2\}.$$

(2) Let $T \in B(A,B)$ and let $\tilde{T} \in B(A, B \otimes M_2)$ be the selfadjoint linear map given by

$$\tilde{T}(x) = \begin{pmatrix} 0 & T(x^*)^* \\ T(x) & 0 \end{pmatrix}$$

then

$$\|T\|_{dec} = \|\tilde{T}\|_{dec}.$$

(3) Any decomposable map T from A to B is completely bounded and

$$\|T\|_{cb} \leq \|T\|_{dec}.$$

(4) If T is a completely positive map from A to B, then

$$\|T\|_{dec} = \|T\|_{cb} = \|T\|.$$

(5) If C is a third C*-algebra, and $T_1 \in B(A,B)$, $T_2 \in B(B,C)$ are two decomposable linear maps, then $T_2 \circ T_1$ is a decomposable map from A to C, and

$$\|T_2 \circ T_1\|_{dec} \leqq \|T_2\|_{dec} \|T_1\|_{dec}$$

proof

(1) If x,y are selfadjoint elements in a C*-algebra D, then

$$-y \leqq x \leqq y \Rightarrow \begin{pmatrix} y & x \\ x & y \end{pmatrix} \geqq 0 .$$

Moreover, if x,y,z are selfadjoint elements in D, then

$$\begin{pmatrix} y_1 & x \\ x & y_2 \end{pmatrix} \geqq 0 \Rightarrow -\tfrac{1}{2}(y_1 + y_2) \leqq x \leqq \tfrac{1}{2}(y_1 + y_2) .$$

Applying this to elements in $B \otimes M_m$, it follows that if $T, S \in B(A,B)$ are selfadjoint maps, then

$$-S \underset{cp}{\leqq} T \underset{cp}{\leqq} S \Rightarrow \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in CP(A, B \otimes M_2)$$

and if $T, S_1, S_2 \in B(A,B)$ are selfadjoint maps, then

$$\begin{pmatrix} S_1 & T \\ T & S_2 \end{pmatrix} \in CP(A, B \otimes M_2) \Rightarrow -\tfrac{1}{2}(S_1 + S_2) \underset{cp}{\leqq} T \underset{cp}{\leqq} \tfrac{1}{2}(S_1 + S_2).$$

This proves the first equality in (1). To prove the second equality in (1), assume that $T \in B(A,B)$, $S \in CP(A,B)$ and

$$-S \underset{cp}{\leqq} T \underset{cp}{\leqq} S$$

Then $T_1 - T_2$ where $T_1 = \tfrac{1}{2}(S+T)$, $T_2 = \tfrac{1}{2}(S-T)$ are completely positive and $T_1 + T_2 = S$. Conversely if $T = T_1 - T_2$, where $T_1, T_2 \in CP(A,B)$, then

$$-(T_1 + T_2) \underset{cp}{\leqq} T \underset{cp}{\leqq} (T_1 + T_2)$$

This proves the second equality.

(2) We prove first that $\|\tilde{T}\|_{dec} \leqq \|T\|_{dec}$. Clearly we can assume that $\|T\|_{dec} < \infty$. Let $\varepsilon > 0$. There exist $S_1, S_2 \in CP(A,B)$ such that

$$R(x) = \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix} , \quad x \in A$$

is completely positive, and $\|S_i\| \leqq \|T\|_{dec} + \varepsilon$, $i = 1,2$.
We put

$$\tilde{S}(x) = \begin{pmatrix} S_1(x) & 0 \\ 0 & S_2(x) \end{pmatrix} , \quad x \in A .$$

Then clearly $\tilde{S} \in CP(A, B \otimes M_2)$, $\|\tilde{S}\| \leqq \|T\|_{dec} + \varepsilon$ and

$$-\tilde{S} \underset{cp}{\leqq} \tilde{T} \underset{cp}{\leqq} \tilde{S} .$$

Since ε is arbitrary we have $\|\tilde{T}\|_{dec} \leqq \|T\|_{dec}$. We prove next that $\|T\|_{dec} \leqq \|\tilde{T}\|_{dec}$. We can assume that $\|\tilde{T}\|_{dec} < \infty$. Let $\varepsilon > 0$. By (1) there exists $\tilde{S} \in CP(A , B \otimes M_2)$, such that

$$-\tilde{S} \underset{cp}{\leqq} \tilde{T} \underset{cp}{\leqq} \tilde{S} .$$

and $\|\tilde{S}\| \leqq \|T\|$.

We have

$$\tilde{S}(x) = \begin{pmatrix} S_{11}(x) & S_{12}(x) \\ S_{21}(x) & S_{22}(x) \end{pmatrix} , \quad x \in A$$

where $S_{11}, S_{22} \in CP(A,B)$, $S_{21}, S_{12} \in B(A,B)$ and $S_{12} = S_{21}^*$.

Let $u \in B \otimes M_2$ be the unitary

$$u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Then

$$u\,\tilde{S}(x)u^* = \begin{pmatrix} S_{11}(x) & -S_{12}(x) \\ -S_{21}(x) & S_{22}(x) \end{pmatrix} , \quad x \in A$$

and

$$u\,\tilde{T}(x)u^* = -\tilde{T}(x) \qquad x \in A .$$

Therefore

$$-ad(u) \circ \tilde{S} \underset{cp}{\leqq} -\tilde{T} \underset{cp}{\leqq} ad(u) \circ \tilde{S} .$$

In particular

$$\mathrm{ad}(u)\circ\tilde{S} + \tilde{T} \underset{cp}{\geqq} 0 \quad .$$

Put

$$R(x) = \begin{pmatrix} S_{11}(x) & T(x^*)^* \\ T(x) & S_{22}(x) \end{pmatrix} \qquad x \in A \ .$$

Then R is completely positive, because

$$R(x) = \tfrac{1}{2}(\tilde{S} + \tilde{T}) + \tfrac{1}{2}(\mathrm{ad}(u)\circ\tilde{S} + \tilde{T}) \ .$$

Moreover

$$\max\{\|S_{11}\|, \|S_{22}\|\} = \|\tilde{S}\| < \|\tilde{T}\|_{dec} + \varepsilon \ .$$

This proves that $\|T\|_{dec} \leqq \|\tilde{T}\|_{dec}$.

(3) It is clear that any linear combination of completely positive maps is completely bounded. Let $T \in B(A,B)$ be a decomposable map, and assume first that $T = T^*$. Let $\varepsilon > 0$. By (1) there exist $T_1, T_2 \in CP(A,B)$, such that $T = T_1 - T_2$ and

$$\|T_1 + T_2\| < \|T\|_{dec} + \varepsilon \ .$$

For $R \in B(A,B)$, be put $R^{(m)} = R \otimes i_m$, where i_m is the identity on the m×m-matrices M_m . For $x \in (A \otimes M_m)_{s.a.}$ we have

$$\begin{aligned} T^{(m)}(x) &= T_1^{(m)}(x) - T_2^{(m)}(x) \\ &\leqq T_1^{(m)}(|x|) + T_2^{(m)}(|x|) \\ &= (T_1 + T_2)^{(m)}(|x|) \end{aligned}$$

and similarly

$$-T^{(m)}(x) \leqq (T_1 + T_2)^{(m)}(|x|) \ .$$

Since $T_1 + T_2$ is completely positive,

$$\|T_1 + T_2\|_{cb} = \|T_1 + T_2\| \ .$$

Thus

$$\|T^{(m)}(x)\| \leqq \|T_1 + T_2\| \; \|x\| \quad .$$

If $x \in A \otimes M_m$ is not selfadjoint, then

$$y = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \in (A \otimes M_{2m})_{s.a.}$$

Since $(T^{(m)})^* = T^{(m)}$ we have

$$T^{(2m)}(y) = \begin{pmatrix} 0 & T^{(m)}(x)^* \\ T^{(m)}(x) & 0 \end{pmatrix} \in (B \otimes M_{2m})_{s.a.}$$

Hence

$$\|T^{(m)}(x)\| = \|T^{(2m)}(y)\| \leqq \|T_1 + T_2\| \; \|y\| = \|T_1 + T_2\| \; \|x\| \quad .$$

This shows that $\|T\|_{cb} \leqq \|T\|_{dec} + \varepsilon$.

(4) It is well known that $\|T\|_{cb} = \|T\|$ for any completely positive map. The equality $\|T\|_{dec} = \|T\|_{cb}$ follows from (1) and (3).

(5) It is clear that $T_2 \circ T_1 \in \text{span}\, CP(A,C)$. Choose

$$S_1^{(1)}, \; S_1^{(2)} \in CP(A,B) \quad \text{and} \quad S_2^{(1)}, \; S_2^{(2)} \in CP(B,C)$$

such that

$$R_i(x) = \begin{pmatrix} S_i^{(1)}(x) & T_i^*(x) \\ T_i(x) & S_i^{(2)}(x) \end{pmatrix} \quad , \quad i = 1,2$$

defines completely positive maps $R_1 \in CP(A, B \otimes M_2)$ and $R_2 \in CP(B, C \otimes M_2)$, such that

$$\max \{\|S_i^{(1)}\| \, , \, \|S_i^{(2)}\|\} \leqq \|T_i\|_{dec} + \varepsilon \quad .$$

By remark 1.2 the map $\tilde{R}_2 \in B(B \otimes M_2 \, , \, C \otimes M_2)$ given by

$$\tilde{R}_2 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} S_2^{(1)}(x_{11}) & T_2^*(x_{12}) \\ T_2(x_{21}) & S_2^{(2)}(x_{22}) \end{pmatrix}$$

is completely positive. Hence $\tilde{R}_2 \circ R_1 \in CP(A, C \otimes M_2)$.

For $x \in A$,
$$\tilde{R}_2 \circ R_1(x) = \begin{pmatrix} S_2^{(1)} \circ S_1^{(1)}(x) & T_2^* \circ T_1^*(x) \\ T_2 \circ T_1(x) & S_2 \circ S_1(x) \end{pmatrix} .$$

Therefore
$$\|T_2 \circ T_1\|_{dec} \leqq \max\{\|S_2^{(1)} \circ S_1^{(1)}\| , \|S_2^{(2)} \circ S_1^{(2)}\|\}$$
$$\leqq (\|T_2\|_{dec} + \varepsilon)(\|T_1\|_{dec} + \varepsilon)$$

This proves (5).

Proposition 1.4

Let A and B be C*-algebras.

(1) The decomposable maps from A to B form a Banach space with norm $\| \ \|_{dec}$.

(2) If every completely bounded map from A to B is decomposable, then there exists a constant $c < \infty$, such that
$$\|T\|_{dec} \leqq c\|T\|_{cb}$$
for all $T \in CB(A,B)$.

proof

(1) Put $V(A,B) = \text{span}\, CP(A,B)$. It is clear that $\| \ \|_{dec}$ is a norm on $V(A,B)$. Since $\|T^*\|_{dec} = \|T\|_{dec}$ for all $T \in V(A,B)$ it is sufficient to prove that the selfadjoint part of $(V(A,B), \| \ \|_{dec})$ is complete. This follows in fact from [20, Remark p. 159], but since no proof is given there, we will include a proof: Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of selfadjoint linear maps from A to B , such that
$$\sum_{n=1}^{\infty} \|T_n\|_{dec} < +\infty .$$

Since $B(A,B)$ is a Banach space, there exists an operator $T \in B(A,B)$ such that
$$\lim_{p\to\infty} \|\sum_{n=1}^{p} T_n - T\| = 0 .$$

By prop. 1.3(2), there exists $S_n \in CP(A,B)$, such that

$$-S_n \underset{cp}{\leq} T_n \underset{cp}{\leq} S_n$$

and $\|S_n\| \leq 2\|T_n\|_{dec}$. In particular

$$\sum_{n=1}^{\infty} \|S_n\| < \infty \ .$$

Therefore we can define $R_p \in B(A,B)$, by

$$R_p = \sum_{n=p+1}^{\infty} S_n \ , \quad p = 1,2,3,\ldots$$

Each R_p is completely positive. Since the cone $CP(A,B)$ is closed in $B(A,B)$ one gets

$$-R_1 \underset{cp}{\leq} T \underset{cp}{\leq} R_1 \ .$$

Thus $T \in V(A,B)$. Moreover for all $p \in \mathbb{N}$,

$$-R_p \underset{cp}{\leq} T - \sum_{n=1}^{p} T_n \leq R_p \ .$$

This implies that

$$\|T - \sum_{n=1}^{p} T_n\|_{dec} \leq \|R_p\| \leq 2 \sum_{n=p+1}^{\infty} \|T\|_{dec} \ .$$

Therefore

$$\lim_{p\to\infty} \|T - \sum_{n=1}^{p} T_n\|_{dec} = 0 \ .$$

This proves that the selfadjoint part of $V(A,B)$ is complete in the $\| \ \|_{dec}$-norm (cf. f.inst. [12, lemma 1.5.2]).

(2) Follows from (1) by applying the open mapping theorem to the identity map from

$$(V(A,B) \ , \ \| \ \|_{dec}) \ \text{ to } \ (CB(A,B) \ , \ \| \ \|_{cb}) .$$

Remark 1.5

We do not know whether the infimum in the definition of $\|T\|_{dec}$

is actually a minimum i.e. whether S_1, S_2 in definition 1.1 can be chosen such that

$$\max\{\|S_1\|, \|S_2\|\} = \|T\|_{dec} .$$

However, this is true in two important cases, namely if B is a von Neumann algebra or if B is an injective C*-algebra. More generally it is true whenever there exists a conditional expectation ε from B** to B : Assume namely that $T \in B(A,B)$ is decomposable. By a simple compactness argument one can find $S_1, S_2 \in CP(A,B^{**})$, such that

$$R(x) = \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix}$$

is a completely positive map from A to $B^{**} \otimes M_2$ and $\max\{\|S_1\|, \|S_2\|\} \leqq \|T\|_{dec}$.

Then

$$R'(x) = \begin{pmatrix} \varepsilon\circ S_1(x) & T(x^*)^* \\ T(x) & \varepsilon\circ S_2(x) \end{pmatrix}$$

defines a completely positive map from A to $B \otimes M_2$, and

$$\max\{\|\varepsilon\circ S_1\|, \|\varepsilon\circ S_2\|\} \leqq \|T\|_{dec} .$$

The converse inequality is trivial.

Clearly, under the same condition on B , one gets also that the two infima in Prop. 1.2(1) are actually minima.

Having remark 1.2 and remark 1.5 in mind Wittstock's and Paulsen's theorems [27, Satz 4.5] and [16, theorem 2.5] can be reformulated in the following way:

Theorem 1.6 (Wittstock, Paulsen).

Let T be a completely bounded linear map from a C*-algebra A into an injective C*-algebra B , then T is decomposable and

$$\|T\|_{dec} = \|T\|_{cb} .$$

§2.

The main results.

For $n \in \mathbb{N}$, we let ℓ_n^∞ denote the n-dimensional abelian C*-algebra $\ell^\infty\{1,\dots,n\}$.

Theorem 2.1

Let N be a von Neumann algebra. Then the following four conditions are equivalent

(1) N is injective .

(2) For every C*-algebra A and every completely bounded map T from A to N , $\|T\|_{dec} = \|T\|_{cb}$.

(3) For every $n \in \mathbb{N}$, and for every linear map T from ℓ_n^∞ to N , $\|T\|_{dec} = \|T\|_{cb}$.

(4) There exists a constant $c \in \mathbb{R}_+$, such that for every $n \in \mathbb{N}$ and for every linear map T from ℓ_n^∞ to N , $\|T\|_{dec} \leq c\|T\|_{cb}$.

Note that (1) => (2) is Wittstock's and Paulsen's result, and that (2) => (3) => (4) is trivial, so we have to prove (4) => (1).

For any complex linear space E we let E^c denote the conjugate space i.e. the set E equipped with the same addition as before, but where the scalar multiplication is given by

$$(c,x) \to \bar{c}x \quad , \quad c \in \mathbb{C} , x \in E.$$

For $x \in E$, we let x^c denote the corresponding element in E^c. If A is an algebra, we consider A^c as an algebra with unchanged multiplication i.e.

$$(ab)^c = a^c b^c \ , \quad a,b \in A \ .$$

In [6, Remark 5.29] Connes proved that for a factor N op type II_1 acting on a separable Hilbert space H , the following two conditions are equivalent

(i) N is injective .

(ii) For any finite set $u_1,\dots,u_n$ of unitaries in N

$$\|\sum_{i=1}^{n} u_i \otimes u_i^c\|_{H\otimes H^c} = n \ .$$

The key step in the proof of (4) => (1) is the following extension of Connes' result:

Lemma 2.2

Let N be a von Neumann algebra acting on a Hilbert space H. The following two conditions are equivalent:

(i) N is finite and injective .

(ii) For any finite set $u_1,\dots,u_n$ of unitaries in N and for any non-zero central projection p in N ,

$$\|\sum_{i=1}^{n} p\,u_i \otimes (pu_i)^c\|_{H\otimes H^c} = n \ .$$

proof

(i) => (ii) : Assume that N is finite and injective. Since any non-zero central projection in N dominates a σ-finite non-zero central projection it is sufficient to prove (2) when p is σ-finite. By passing to the reduced algebra pN , it is sufficient to consider the case, where N itself is σ-finite and p = 1. Let τ be a normal faithful tracial state on N. For $a \in N$ we let L_a (resp. R_a) denote the multiplication with a from left (resp. from right) on $L^2(N,\tau)$. Since any injective von Neumann algebra is semidiscrete (cf. [26] and [7]),

$$\|\sum_{i=1}^{m} L_{a_i} R_{b_i^*}\| \leq \|\sum_{i=1}^{m} a_i \otimes b_i^c\|_{H \otimes H^c}$$

for every $m \in \mathbb{N}$ and every $a_1,\dots,a_m$, $b_1,\dots,b_m \in N$. In particular, for any finite set of unitaries $u_1,\dots,u_n$ in N

$$\|\sum_{i=1}^{n} u_i \otimes u_i^c\|_{H \otimes H^c} \geq \|\sum_{i=1}^{n} L_{u_i} R_{u_i^*}\|$$

$$\geq \|\sum_{i=1}^{n} u_i\ 1 \circ u_i^*\|_2 = n .$$

This proves that (ii) => (i). For the proof of (ii) => (i) we shall need the notion of <u>hypertraces</u> introduced by Connes [6, Remark 5.34]. A state ω on $B(H)$ is called a hypertrace for N if for all $x \in B(H)$ and all $a \in N$,

$$\omega(ax) = \omega(xa).$$

Consider now the following two conditions on a von Neumann algebra N:

(iii) For every non-zero central projection p in N , there exists a hypertrace ω for N , such that $\omega(1-p) = 0$.

(iv) For every state ω_o on $Z(N)$ (the center of N), there exists a hypertrace ω for N , such that $\omega(z) = \omega_o(z)$ for all $z \in Z(N)$.

We will prove that (ii) => (iii) => (iv) => (i). Assume that N satisfies (ii). Let $HS(H)$ denote the space of Hilbert-Schmidt operators on H and let $\|\ \|_{HS}$ be the Hilbert-Schmidt norm. Since $HS(H)$ can be identified in a natural way with $H \otimes H^c$, one gets that for $a_1,\dots,a_n$, $b_1,\dots,b_n \in B(H)$,

$$\|\sum_{i=1}^{n} a_i \otimes b_i^c\|_{H \otimes H^c} = \sup\{\|\Sigma a_i x b_i^*\|_{HS} \mid x \in HS(H) ,\ \|x\|_{HS} \leq 1\}.$$

Let p be a non-zero central projection in N. Let $\mathcal{F}$ be the family of sets

$$F = (u_1, u_2, \dots, u_n, \varepsilon)$$

where $n \in \mathbb{N}$, $u_1, \dots, u_n$ are distinct unitaries in N, and $\varepsilon > 0$. Let $F = (u_1, \dots, u_n, \varepsilon) \in \mathcal{F}$. By (ii)

$$\| p \otimes p^c + \sum_{i=1}^{n} (pu_i) \otimes (pu_i)^c \| = n + 1 \ .$$

Therefore we can choose $x_F \in HS(H)$, such that $\|x_F\|_{HS} \leqq 1$, and

$$\| p\, x_F\, p + \sum_{i=1}^{n} pu_i\, x_F\, pu_i^* \| > (n+1) - \varepsilon \ .$$

By exchanging x_F with px_Fp, we have still $\|x_F\|_{HS} \leqq 1$. Moreover

$$px_F = x_Fp = x_F$$

and

$$\|x_F + \sum_{i=1}^{n} u_i\, x_F\, u_i^* \|_{HS} > (n+1) - \varepsilon \ .$$

Since for $k=1,\dots,n$ we have

$$\| \sum_{i \neq k} u_i x_F u_i^* \|_{HS} \leqq n - 1$$

it follows that

$$\|x_F + u_k x_F u_k^* \|_{HS} > 2 - \varepsilon \quad , \quad k=1,\dots,n \ .$$

So, by the parallelogramidentity

$$\begin{aligned} \|x_F - u_k\, x_F\, u_k^* \|_{HS}^2 &\leqq 2\|x_F\|_{HS}^2 + 2\|u_k\, x_F\, u_k^*\|_{HS}^2 - (2-\varepsilon)^2 \\ &\leqq 4 - (2-\varepsilon)^2 \\ &< 4\varepsilon \ . \end{aligned}$$

Since $\|x_F\|_{HS} = \|u_k\, x_F\, u_k^*\|_{HS}$ we have also

$$\|x_F\|_{HS} > 1 - \tfrac{1}{2}\varepsilon \ .$$

Define a positive functional ω_F on N by

$$\omega_F(a) = (a\,x_F\,,\,x_F)_{HS} = T_r(a\,x_F\,x_F^*).$$

For $a \in N$, and $x,y \in HS(H)$,

$$|(ax,x)_{HS} - (ay,y)_{HS}| = \tfrac{1}{2}|(a(x+y),\,(x-y))_{HS} + (a(x-y),\,(x+y))_{HS}|$$

$$\leqq \|a\|\ \|x-y\|_{HS}\,\|x+y\|_{HS}\,.$$

Hence for $a \in N$ and $i=1,\dots,n$.

$$\omega_F(a - u_i a u^*_i) \leqq \|a\|\ \|x_F - u_i x_F u_i^*\|_{HS}\|x_F + u_i x_F u_i^*\|_{HS}$$

$$\leqq 4\varepsilon^{\frac{1}{2}}\|a\|.$$

Also $\omega_F(1-p) = 0$, and $\omega_F(1) = \|x_F\|^2_{HS} > 1-\varepsilon$.

The set $\mathfrak{F}$ is directed with the ordering $\leqq$ given by

$$(u_1,\dots,u_n,\varepsilon) \leqq (v_1,\dots,v_m,\delta)$$

if $\{v_1,\dots,v_m\}$ contains the set $\{u_1,\dots,u_n\}$ and $\delta \leqq \varepsilon$. Let $\omega \in B(H)^*$ be a $\sigma(B(H)^*, B(H))$ cluster point for the net $(\omega_F)_{F\in\mathfrak{F}}$. Clearly ω is a state on $B(H)$,

$$\omega(uxu^*) = \omega(x)\ ,\quad x \in B(H)\ ,\quad u \in U(N)$$

i.e. ω is a hypertrace for N. Moreover $\omega(1-p) = 0$. Hence we have proved that (ii) => (iii).

(iii) => (iv): Let ω_o be a state on $Z(N)$, and let

$$P = \{p_1,\dots,p_r\}$$

be a "partition of the unity" in $Z(N)$, i.e. $r \in \mathbb{N}$ and $p_1,\dots,p_r$ are non-zero pairwise orthogonal projections in $Z(N)$ with sum 1. If N satisfies (iii) we can choose hypertraces $\omega_1,\dots,\omega_r \in B(H)^*$ for N, such that $\omega_k(1-p_k) = 0$. Put now

$$\omega_P = \sum_{k=1}^{r} \omega_o(p_k)\omega_k .$$

Then ω_P is a hypertrace on N , and

$$\omega_P(p_k) = \omega_o(p_k).$$

The set $\mathcal{P}$ of partition of the unity in $Z(N)$ is directed by the ordering $\leqq$, where $P \leqq Q$ means that each projection in P can be written as a sum of projections in Q . Let now ω be a $\sigma(B(H)^*, B(H))$-cluster point for the net $(\omega_P)_{P\in\mathcal{P}}$. Then ω is a hypertrace for N , and ω coincides with ω_o on every central projection. Hence

$$\omega(x) = \omega_o(x) \quad , \quad x \in Z(N).$$

(iv) => (i) : Assume that N satisfies (iv). We prove first that N is finite: Let $e \in Z(N)$ be the largest finite projection in $Z(N)$. If $1-e \neq 0$, we can choose a state ω_o on $Z(N)$, such that $\omega_o(1-e) = 1$. By (iv) there exists a hypertrace $\omega \in B(H)^*$ for N such that $\omega(1-e) = 1$. The restriction of ω to $(1-e)N$ is a tracial state. This gives a contradiction, because $(1-e)N$ is properly infinite. Hence $e = 1$ and N is finite. Since any finite von Neumann algebra is a direct sum of σ-finite, finite algebras, we can in the rest of the proof of (3) => (1) assume that N itself is σ-finite and finite. Let ω_o be a normal faithful state on $Z(N)$ and let $\omega \in B(H)^*$ be a hypertrace for N that extends ω_o . The restriction τ of ω to N is a trace on N. Let T be the central-valued trace on N , then

$$\tau = \tau \circ T = \omega_o \circ T .$$

This shows that τ is a normal, faithful tracial state on N.

For $x \in B(H)$, we let φ_x be the functional on N given by

$$\varphi_x(a) = \omega(ax) = \omega(xa) \quad , \quad a \in N$$

In particular $\varphi_1(a) = \tau(a)$.

If $0 \leqq x \leqq 1$, then for all $a \in N_+$,

$$\varphi_x(a) = \omega(ax) = \omega(a^{\frac{1}{2}}xa^{\frac{1}{2}}) \geqq 0$$

and

$$\varphi_x(a) = \tau(a) - \omega(a^{\frac{1}{2}}(1-x)a^{\frac{1}{2}}) \leqq \tau(a).$$

Therefore $0 \leqq \varphi_x \leqq \tau$. Hence there is a unique $b_x \in N_+$, $0 \leqq b_x \leqq 1$, such that

$$\varphi_x(a) = \tau(b_x a) .$$

Since N is spanned by the positive elements in N of norm $\leqq 1$, the map $x \to b_x$ can be extended to a linear map $E : B(H) \to N$, such that

$$\tau(E(x)a) = \varphi_x(a) = \omega(xa) \quad , \quad x \in B(H) \quad , \quad a \in N.$$

Clearly, E is positive, $E(1) = 1$. Moreover for $a_1, a_2 \in N$

$$\begin{aligned}\tau(E(a_1xa_2)b) &= \omega(a_1xa_2b) = \omega(xa_2ba_1)\\ &= \tau(E(x)a_2ba_1) = \tau(a_1E(x)a_2b)\end{aligned}$$

for every $b \in N$. This shows that $E(a_1xa_2) = a_1E(x)a_2$ i.e. E is a conditional expectation of $B(H)$ onto N. Hence N is injective. This completes the proof of lemma 2.2.

Lemma 2.3

Let N be a von Neumann algebra on a Hilbert space H. The following two conditions are equivalent

(i) N is finite and injective .

(ii') There exists a constant $\gamma > 0$, such that for every finite set $u_1,\dots,u_n$ of unitaries in N and any non-zero central projection p in N,

$$\|\sum_{i=1}^{n} pu_i \otimes (pu_i)^c\|_{H\otimes H^c} \geq \gamma n .$$

<u>proof</u>

(i) => (ii') follows from lemma 2.2. To prove (ii') => (i) assume that N satisfies (ii') with $\gamma = \gamma_o > 0$, but that N does not satisfy (i). By lemma 2.2 we can choose a central projection p and unitaries $u_1,\dots,u_n$ in N, such that

$$\|\sum_{i=1}^{n} pu_i \otimes (pu_i)^c\|_{H\otimes H^c} < n .$$

Put

$$\alpha = \frac{1}{n}\|\sum_{i=1}^{n} pu_i \otimes (pu_i)^c\|_{H\otimes H^c} .$$

Since $\alpha < 1$, we can choose $r \in \mathbb{N}$, such that $\alpha^r < \gamma_o$. Put $\Lambda = \{1,\dots,n\}^r$. Note that Λ is a finite set with n^r elements. For $\lambda = (i_1,\dots,i_r) \in \Lambda$, put

$$v_\lambda = u_{i_1}u_{i_2}\dots u_{i_r} .$$

Then

$$\sum_{\lambda\in\Lambda} pv_\lambda \otimes (pv_\lambda)^c = \left(\sum_{i=1}^{p} pu_i \otimes (pu_i)^c\right)^r$$

and therefore

$$\|\sum_{\lambda\in\Lambda} pv_\lambda \otimes (pv_\lambda)^c\| \leq (\alpha n)^r < \gamma_o n^r .$$

This contradicts that N satisfies (ii') with $\gamma = \gamma_o$. Hence (ii') => (i).

Lemma 2.4

Let H and K be Hilbert spaces and let $a_1,\dots,a_n \in B(H)$, $b_1,\dots,b_n \in B(K)$. Then

$$\|\sum_{i=1}^{n} a_i \otimes b_i^c\|_{H\otimes K^c} \leqq \|\sum_{i=1}^{n} a_i \otimes a_i^c\|^{\frac{1}{2}}_{H\otimes H^c} \cdot \|\sum_{i=1}^{n} b_i \otimes b_i^c\|^{\frac{1}{2}}_{K\otimes K^c} .$$

proof

Assume first that $H = K$. By the usual identification of $H\otimes H^c$ with the Hilbert-Schmidt operators $HS(H)$ on H, we have

$$\|\sum_{i=1}^{n} a_i \otimes b_i^c\|_{H\otimes H^c} = \sup\{\|\sum_{i=1}^{n} a_i x b_i^*\|_{HS} \mid \|x\|_{HS} \leqq 1\}$$

$$= \sup\{\mathrm{Tr}(\sum_{i=1}^{n} a_i x b_i^* y^*) \mid \|x\|_{HS} \leqq 1,\ \|y\|_{HS} \leqq 1\}.$$

Let $x,y \in HS(H)$, $\|x\|_{HS} \leqq 1$, $\|y\|_{HS} \leqq 1$, and let $x = u|x|$, $y = v|y|$ be the polardecompositions of x and y. Put

$$x_1 = u|x|^{\frac{1}{2}}, \quad x_2 = |x|^{\frac{1}{2}}$$
$$y_1 = v|y|^{\frac{1}{2}}, \quad y_2 = |y|^{\frac{1}{2}} .$$

Then

$$x = x_1x_2, \quad y = y_1y_2$$
$$|x| = x_2^*x_2, \quad |y| = y_2^*y_2$$
$$|x^*| = x_1x_1^*, \quad |y^*| = y_1y_1^* .$$

Therefore

$$\sum_{k=1}^{n} \mathrm{Tr}(y^*a_kxb_k^*) = \sum_{k=1}^{n} \mathrm{Tr}(y_1^* a_k x_1x_2b_k^*y_2^*)$$

$$\leqq \sum_{k=1}^{n} \mathrm{Tr}(y_1^* a_k x_1 (y_1^* a_k x_1)^*)^{\frac{1}{2}} \mathrm{Tr}((x_2b_k^*y_2^*)^*(x_2b_k^*y_2^*))^{\frac{1}{2}}$$

$$\leqq \left(\sum_{k=1}^{n} \mathrm{Tr}(y_1^*a_kx_1x_1^*a_k^*y_1)\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} \mathrm{Tr}(y_2b_kx_2^*x_2b_k^*y_2^*)\right)^{\frac{1}{2}}$$

$$= \left(\sum_{k=1}^{n} \mathrm{Tr}(|y^*|a_k|x^*|a_k^*)\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \mathrm{Tr}(|y|b_k|x|b_k^*)\right)^{\frac{1}{2}}$$

$$\leqq \left\|\sum_{k=1}^{n} a_k \otimes a_k^c\right\| \left\|\sum_{k=1}^{n} b_k \otimes b_k^c\right\| .$$

Here we have used that

$\| |x| \|_{HS} = \| |x^*| \|_{HS} = \|x\|_{HS} \leqq 1$ and $\| |y| \|_{HS} = \| |y^*| \|_{HS} = \|y\|_{HS} \leqq 1$.

This completes the proof in the case $H = K$. The general case can be reduced to this case if one puts

$$\tilde{H} = H \otimes K$$

and considers the operators $\tilde{a}_1,\dots,\tilde{a}_n$, $\tilde{b}_1,\dots,\tilde{b}_n \in B(\tilde{H})$ given by

$$\tilde{a}_k(\xi,\eta) = (a_k\xi, 0)$$

$$\tilde{b}_k(\xi,\eta) = (0, b_k\eta)$$

for $\xi \in H$ and $\eta \in K$.

Lemma 2.5

Let $u_1,\dots,u_n$ be n unitaries in a finite von Neumann algebra N , let p be a non-zero central projection in N , and let T be the linear map from ℓ_n^∞ to N given by

$$T(c_1,\dots,c_n) = p\left(\sum_{i=1}^{n} c_i u_i\right).$$

Then

a) $\|T\|_{cb} \leqq n^{\frac{1}{2}} \left\|\sum_{i=1}^{n} pu_i \otimes (pu_i)^c\right\|^{\frac{1}{2}}$

b) $\|T\|_{dec} = n$.

proof

a) Let $m \in \mathbb{N}$, and put $T^{(m)} = T \otimes i_m$, where i_m is the identity on M_m. An element x in the unitball of $\ell_n^\infty \otimes M_m$ is given by a set $(x_1,\dots,x_n)$ of n elements in the unitball of M_m . We have

$$T^{(m)}(x) = \sum_{k=1}^{n} pu_k \otimes x_k .$$

We have $M_m \cong B(K)$, where $\dim K = n$. Hence by lemma 2.4:

$$\|T^{(m)}(x)\| \leqq \|\sum_{i=1}^{n} pu_k \otimes (pu_k)^c\|^{\frac{1}{2}}_{H\otimes H^c} \cdot \|\sum_{k=1}^{n} x_k^c \otimes x_k\|_{K^c \otimes K} .$$

$$\leqq \|\sum_{i=1}^{n} pu_k \otimes (pu_k)^c\|^{\frac{1}{2}}_{H\otimes H^c} \cdot n^{\frac{1}{2}} .$$

This proves a).

b) Since $\|T\|_{dec} = \|\tilde{T}\|_{dec}$, where $\tilde{T} : A \to B \otimes M_2$ is defined by,

$$\tilde{T}(c_1,\dots,c_n) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \cdot \sum_{k=1}^{n} c_k \begin{pmatrix} 0 & u_k^* \\ u_k & 0 \end{pmatrix} \in N \otimes M_2 ,$$

(cf. prop. 1.3(2)) it is sufficient to consider the case where $u_1,\dots,u_n$ are selfadjoint unitaries. Put

$$S(c_1,\dots,c_n) = \left(\sum_{k=1}^{n} c_k\right) p .$$

Then S is a positive map freom ℓ_n^∞ to N and

$$-S \leqq T \leqq S .$$

However, by [21, thm. 4] a positive map from ℓ_n^∞ to N is automaticly completely positive.

Therefore

$$\|T\|_{dec} \leqq \|S(1)\| = n\|p\| = n.$$

Let now τ be a normal tracial state on N for which $\tau(1-p) = 0$, and let $e \leqq p$ be the support projection of τ. It is well known that

$$\|x\|_1 = \tau(|x|), \quad x \in M$$

is a norm on eN, and since

$$\|x\|_1 = \|ex\|_1$$

for all $x \in N$, $\|\cdot\|_1$ is a seminorm on N. Assume that $R : \ell_n^\infty \to N$ is a completely positive map, such that

$$-R \underset{cp}{\leqq} T \underset{cp}{\leqq} R \quad .$$

Put $x_k = R(p_k)$, where $p_1,\dots,p_n$ are the minimal projections in ℓ_n^∞ . Then

$$-x_k \leqq pu_k \leqq x_k \qquad k = 1,\dots,n\,.$$

Therefore

$$\tau(x_k) = \|\tfrac{1}{2}(x_k + pu_k)\|_1 + \|\tfrac{1}{2}(x_k - pu_k)\|_1$$

$$\geqq \|\tfrac{1}{2}(x_k + pu_k) - \tfrac{1}{2}(x_k - pu_k)\|_1 = \|pu_k\|_1 = \tau(p) = 1.$$

Hence

$$\|R(1)\| = \|\sum_{k=1}^n x_k\| \geqq \sum_{k=1}^n \tau(x_k) \geqq n \; .$$

This shows that $\|T\|_{dec} \geqq n$.

<u>proof of theorem 2.1</u>

It remains to be proved that (4) => (1). Assume first that N is finite. Let $u_1,\dots,u_n$ be n unitaries in N , let p be a non-zero central projection in N , and let $T : \ell_n^\infty \to N$ be the linear map

$$T(c_1,\dots,c_n) = p(\sum_{i=1}^n c_i u_i) \; .$$

By lemma 2.5

$$\|T\|_{cb} \leqq n^{\frac{1}{2}} \|\sum_{i=1}^n pu_i \otimes (pu_i)^c\|^{\frac{1}{2}}$$

and

$$\|T\|_{dec} = n.$$

Thus, if $\|T\|_{dec} \leqq c\|T\|_{cb}$, we get that

$$\left\| \sum_{i=1}^{n} pu_i \otimes (pu_i)^c \right\| \geq n/c^2 \ .$$

Hence, if N satisfies condition (4), it follows from lemma 2.3 that N is injective. This proves (4) => (1) for N finite.

To prove (4) => (1) for a general von Neumamn algebra, we show first that if a von Neumann algebra M satisfies condition (4) in theorem 2.1, then

(a) Any reduced algebra $N = pMp$ of M satisfies condition (4) in theorem 2.1 .

(b) Any sub von Neumann algebra of N which is the range of a conditional expectation $\varepsilon : M \to N$ satisfies condition (4) in theorem 2.1.

Let namely $T : \ell_n^\infty \to N$ be a linear map. Since in both cases (a) and (b), $N \subseteq M$, where M satisfies (4) with $c = c_o$, there exist completely positive maps S_1, S_2 from ℓ_n^∞ to M, such that $\|S_i\| \leq c_o \|T\|_{cb}$, $i = 1,2$ and

$$R(x) = \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix} , \quad x \in \ell_n^\infty$$

is a completely positive map from ℓ_n^∞ to $M \otimes M_2$ (cf. def. 1.1 and remark 1.4). By putting $S_i' = pS_i(\cdot)p$ in case a) and $S_i' = \varepsilon \circ S$ in case b) one gets completely positive maps S_1', S_2' from ℓ_n^∞ to N, such that

$$R'(x) = \begin{pmatrix} S_1'(x) & T(x^*)^* \\ T(x) & S_2'(x) \end{pmatrix} , \quad x \in \ell_n^\infty$$

defines a completely positive map from ℓ_n^∞ to $N \otimes M_2$. Hence

$$\|T\|_{dec} \leq \max\{\|S_1'\|, \|S_2'\|\} \leq c_o \|T\|_{cb} \ .$$

This proves (a) and (b) above.

Let now N be a semifinite von Neumann algebra that satisfies condition (4) in theorem 2.1. By [23, Chap. 5, prop. 1.40], N can be written in the form

$$N = \bigoplus_{i\in I} (N_i \hat{\otimes} B(H_i))$$

where $(H_i)_{i\in I}$ is a family of Hilbert spaces and $(N_i)_{i\in I}$ is a family of finite von Neumann algebras. By (a) above each N_i satisfies condition (4). Thus by the first part of the proof each N_i is injective, which implies that N itself is injective completing the proof of (4) => (1) for semifinite algebras.

Assume next that N is a von Neumann algebra of type III that satisfies condition (4). By [22] N is the crossed product of a semifinite Neumann algebra M and a one-parameter group of automorphisms (θ_s) on M

$$N = M \times_\theta \mathbb{R} .$$

Let $\hat{\theta}$ be the dual action of $\mathbb{R}$ on N (cf. [22, Def. 4.1]), and let m be a left invariant mean on $\mathbb{R}$. Then

$$x \to \int_{-\infty}^{\infty} \hat{\theta}_s(x) \, dm(s).$$

defines a conditional expectation ε of N onto the fixed point algebra $N_{\hat{\theta}}$ for $\hat{\theta}$. By [22, thm. 6.1] $N_{\hat{\theta}}$ is isomorphic to M. Thus by (b) above, M also satisfies condition (4), and hence M is injective by the first part of the proof. But the crossed product of an injective von Neumann algebra by an abelian group is again injective (cf. [6, prop. 6.8]). Hence (4) => (1) for von Neumann algebras of type III. Since a general von Neumann algebra is the direct sum of a semifinite algebra and a type III-algebra, we are done.

Theorem 2.6

Let N be a non-injective von Neumann algebra.

a) For every infinite dimensional C*-algebra A , there exists a map $T \in CB(A,N)$, which is not a linear combination of completely positive maps from A to N.

b) For every infinite dimensional von Neumann algebra M , there exists a normal map $T \in CB(M,N)$ which is not a linear combination of completely positive maps from M to N .

For the proof of theorem 2.6 we shall need

Lemma 2.7

Let A be an infinite dimensional C*-algebra. For each $n \in \mathbb{N}$, there exist completely positive maps

$$R_n : \ell_n^\infty \to A \quad , \quad S_n : A \to \ell_n^\infty$$

such that $\|R_n\| \leqq 1 \quad \|S_n\| \leqq 1$, and

$$S_n \circ R_n(x) = x \ , \quad x \in \ell_n^\infty \ .$$

If A is a von Neumann algebra R_n and S_n can be chosen normal and unitpreserving.

proof

Let B be a maximal abelian *-subalgebra of A. Since B is infinite dimensional (cf. : [12, exercise 4.6.12]), the spectrum $\hat{B}$ of B is infinite. Let $n \in \mathbb{N}$. We can choose n distinct characters

$$\omega_1, \dots, \omega_n \in \hat{B} \ .$$

Moreover, since B is isomorphic to $C_o(\hat{B})$, we can choose n positive selfadjoint elements

$$b_1, \ldots, b_n \in B$$

such that $\|b_i\| \leqq 1$, $\omega_i(b_i) = 1$ for $i=1,\ldots,n$ and such that the corresponding functions on $C_o(\hat{B})$ have disjoint supports. Let $\varphi_1,\ldots,\varphi_n$ be extensions of $\omega_1,\ldots,\omega_n$ to states on A. Put

$$R_n(c_1,\ldots,c_n) = \sum_{i=1}^{n} c_i b_i \qquad c_i \in \mathbb{C}$$

and

$$S_n(a) = (\varphi_1(a),\ldots,\varphi_n(a)) \qquad a \in A \ .$$

Since a positive map from a C*-algebra to another C*-algebra is automaticly completely positive if one of the algebras is abelian (cf. [21, thm. 4] and [2, prop. 1.2.2])., R_n and S_n are completely positive. Moreover one gets easily that $\|R_n\| \leqq 1$, $\|S_n\| \leqq 1$ and $S_n \circ R_n(x) = x$ for $x \in \ell_n^\infty$.

If A is an infinite dimensional von Neumann algebra, let instead $c_1,\ldots,c_n$ be n non-zero orthogonal projections with sum 1, let $\varphi_1,\ldots,\varphi_n$ be normal states on A , such that the support projection of φ_i is less or equal to c_i , $i=1,\ldots,n$, and define R_n and S_n by the above formulas. Then R_n, S_n satisfy all the conditions stated in the second part of lemma 2.7.

<u>proof of theorem 2.6</u>

a) Let N be a von Neumann algebra, and let A be any infinite dimensional C*-algebra. Assume that every completely bounded map from A to N is decomposable. By prop. 1.5, there exists a constant $c \in \mathbb{R}_+$, such that

$$\|T'\| \leqq c\|T'\|_{cb}$$

for all $T' \in CB(A,N)$. For every $n \in \mathbb{N}$ we can choose completely bounded maps $R_n : \ell_n^\infty \to A$ and $T_n : A \to \ell_n^\infty$ which satisfy the conditions of lemma 2.7. Let T be a linear map from ℓ_n^∞ to N.

Since

$$T = (T \circ S_n) \circ R_n$$

we get from prop. 1.3(4)(5) that

$$\|T\|_{dec} \leqq \|T \circ S_n\|_{dec} .$$

Therefore

$$\|T\|_{dec} \leqq c\|T \circ S_n\|_{cb} \leqq c\|T\|_{cb} .$$

Hence N satisfies the condition (4) in theorem 2.1, i.e. N is injective.

b) Let M,N be von Neumann algebras, $\dim M = +\infty$, and assume that any normal map $T \in CB(M,N)$ is decomposable. Since

$$V_n(M,N) = \{T \in \text{span } CP(M,N) \mid T \text{ normal}\}$$

is a closed subspace of the Banach space

$$(\text{span } CP(M,N) , \| \ \|_{dec})$$

it follows as in the proof of prop. 1.4 that there exists $c \in \mathbb{R}_+$, such that

$$\|T'\|_{dec} \leqq c\|T'\|_{cb}$$

for all normal maps $T' \in CB(M,N)$. Hence, as in the proof of a) we can conclude that N is injective. This proves theorem 2.6.

If M and N are two von Neumann algebras, we let $CP_n(M,N)$ (resp. $CB_n(M,N)$) denote the set of normal completely positive (resp. normal completely bounded) maps from M to N.

<u>Corollary 2.8</u>

Let N be a von Neumann algebra. The following three conditions are equivalent

(1) N is injective.

(2) $CB(N,N) = \text{span } CP(N,N)$.

(3) $CB_n(N,N) = \text{span } CP_n(N,N)$.

proof

From theorem 2.6 it follows that (1) <=> (2) <=> (3'), where (3') is the condition

$$(3') \quad CB_n(N,N) \subseteqq \text{span } CP(N,N) .$$

However, if a normal map T from N to N is a linear combination of completely positive maps $T_1,\dots,T_n$ from N to N

$$T = \sum_{i=1}^{n} c_i T_i$$

then also

$$T = \sum_{i=1}^{n} c_i T_i^{(n)}$$

where $T_i^{(n)},\dots,T_n^{(n)}$ are the normal parts of $T_1,\dots,T_n$ (cf. [23, def. 2.15]). Therefore (3) <=> (3').

Corollary 2.9

Let R be the hyperfinite II_1-factor with tracial state τ, and let ω be a free ultrafilter on R,

$$R^{\omega} = \ell^{\infty}(\mathbb{N},R)/I_{\omega}$$

where I_{ω} is the ideal in $\ell^{\infty}(\mathbb{N},R)$ consisting of those bounded sequences (x_n) in R for which

$$\lim_{n\to\omega} \tau(x_n^* x_n) = 0.$$

Then for every infinite dimensional C*-algebra A, there exists a completely bounded map T from A to R^{ω}, such that T has no completely bounded lifting $\tilde{T} : A \to \ell^{\infty}(\mathbb{N},R)$.

proof

It is well known that R^{ω} is a II_1-factor with tracial state

τ_ω given by

$$\tau_\omega(x) = \lim_{n\to\omega} \tau(x_n) ,$$

where $(x_n)_{n\in\mathbb{N}}$ is a representing sequence for $x \in R^\omega$ (cf. [19, Chap. II, sects. 6,7] and [14, p. 451]. Moreover by an argument due to Wassermann R^ω is not injective: Let $\mathbb{F}_2$ be the free group on two generators, then by [25, p. 244], there exists a sequence of representations $(\pi_n)_{n\in\mathbb{N}}$ of finite $\mathbb{F}_2$ into finite dimensional subfactors F_n of R such that

$$\lim_{n\to\infty} \tau(\pi_n(g)) = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases}$$

where τ is the normalized trace. Hence as in [25, page 245] one sees that R^ω contains a subfactor isomorphic to $\mathcal{M}(\mathbb{F}_2)$, the von Neumann algebra associated with the regular representation of $\mathbb{F}_2$, which implies that R^ω is not injective (cf. proof of [25, prop. 1.7]).

Let now A be any infinite dimensional C*-algebra. By theorem 2.6 there exists a completely bounded map $T : A \to R^\omega$, which is not decomposable. Assume that $\tilde{T} : A \to \ell^\infty(\mathbb{N},R)$ is a completely bounded lifting of T. Since R is injective, $\ell^\infty(\mathbb{N},R)$ is also an injective von Neumann algebra. Thus by prop. 1.6 , $\tilde{T}$ is a linear combination of completely positive maps. But since, $T = \rho \circ \tilde{T}$, where $\rho : \ell^\infty(\mathbb{N},R) \to R^\omega$ is the quotient map, T is also a linear combination of completely positive maps, which gives a contradiction. Hence T has no completely bounded lifting.

§3.

Examples and complements.

Example 3.1

Let $\mathbb{F}_2$ be the free group on two generators a and b, and let λ be the left regular representation of $\mathbb{F}_2$. Choose a free, infinite set $\{x_1, x_2, \ldots\}$ in $\mathbb{F}_2$, f.inst.

$$x_n = b^n a b^{-n}, \quad n \in \mathbb{N}$$

and define a linear map T_n from ℓ_n^∞ to $\mathcal{M}(\mathbb{F}_2) = \lambda(\mathbb{F}_2)''$ by

$$T_n(c_1, \ldots, c_n) = \frac{1}{2\sqrt{n-1}} \sum_{i=1}^{n} c_i \lambda(x_i) \qquad (n \geqq 2).$$

We will show that

$$\|T_n\| = \|T_n\|_{cb} = 1$$

while

$$\|T_n\|_{dec} = \frac{n}{2\sqrt{n-1}}.$$

In [1], Akemann and Ostrand proved that

$$\Big\| \sum_{i=1}^{n} \lambda(x_i) \Big\| = 2\sqrt{n-1}, \quad n \geqq 2.$$

They also proved ([1], Theorem III F) that, for $c_1, \ldots, c_n \in \mathbb{C}$,

$$\Big\| \sum_{i=1}^{n} c_i \lambda(x_i) \Big\| = \Big\| \sum_{i=1}^{n} |c_i| \lambda(x_i) \Big\|.$$

In particular,

$$\Big\| \sum_{i=1}^{n} c_i \lambda(x_i) \Big\| = 2\sqrt{n-1}$$

for $n \geqq 2$ and $|c_1| = |c_2| = \cdots = |c_n| = 1$.

Hence $\|T_n(u)\| = 1$ for every unitary operator $u \in \ell_n^\infty$, and since the unit ball in any finite dimensional C*-algebra is the convex hull of the unitary operators, we conclude that $\|T_n\| = 1$.

Let $m \in \mathbb{N}$, and put $T^{(m)} = T \otimes i_m$, where i_m is the identity on M_m. Every unitary operator $u \in \ell_n^\infty \otimes M_m$ is of the form

$$u = (u_1, \ldots, u_n)$$

where $u_1, \ldots, u_n$ are unitary m×m-matrices. Clearly,

$$T_n^{(m)}(u) = \frac{1}{2\sqrt{n-1}} \sum_{i=1}^{n} \lambda(x_i) \otimes u_i \ .$$

We can identify the subgroup of $\mathbb{F}_2$ generated by $\{x_1, x_2, \ldots\}$ with the free group $\mathbb{F}_\infty$ on infinite (countable) many generators. The restriction λ' of λ to $\mathbb{F}_\infty$ is just a multiple of the left regular representation λ_∞ of $\mathbb{F}_\infty$. Therefore,

$$\|T_n^{(m)}(u)\| = \frac{1}{2\sqrt{n-1}} \left\| \sum_{i=1}^{n} \lambda_\infty(x_i) \otimes u_i \right\| \ .$$

Let π be the unitary representation of $\mathbb{F}_\infty$ on the m-dimensional Hilbert space $\mathbb{C}^m$ for which

$$\pi(x_i) = u_i \ , \quad i \in \mathbb{N} \ .$$

Then, by [8, Addendum 13.11.3], $\lambda \otimes \pi$ is unitary equivalent to $\lambda \otimes \tau_o$, where τ_o is the trivial representation of $\mathbb{F}_\infty$ on $\mathbb{C}_m$.

Hence,

$$\|T_n^{(m)}(u)\| = \frac{1}{2\sqrt{n-1}} \left\| \sum_{i=1}^{n} \lambda_\infty(x_i) \right\|$$

$$= \frac{1}{2\sqrt{n-1}} \left\| \sum_{i=1}^{n} \lambda(x_i) \right\| = 1 \ ,$$

which proves that $\|T_n^{(m)}\| = 1$ for all m . Hence $\|T_n\|_{cb} = 1$.
Finally, by Lemma 2.5 (b), we have

$$\|T_n\|_{dec} = \frac{n}{2\sqrt{n-1}} .$$

From Example 3.1 and the proof of Corollary 2.8, we get:

Proposition 3.2

Let R be the hyperfinite factor, let ω be a free ultrafilter on $\mathbb{N}$, and let

$$R^{\omega} = \ell^{\infty}(\mathbb{N},R)/I_{\omega}$$

as in Corollary 2.8.

(1) For $n \in \mathbb{N}$, $n \geq 3$, there exists a linear map,

$$T : \ell_n^{\infty} \to R^{\omega} ,$$

such that, for any lifting of T to a linear map $\tilde{T}$ from ℓ_n^{∞} to $\ell^{\infty}(\mathbb{N},R)$,

$$\|\tilde{T}\|_{cb} \geq \frac{n}{2\sqrt{n-1}} \|T\|_{cb} .$$

(2) For $n \in \mathbb{N}$, $n \geq 3$, there exists a linear map,

$$T : M_n \to R^{\omega} ,$$

such that, for every linear lifting of T to a map $\tilde{T}$ from M_n to $\ell^{\infty}(\mathbb{N},R)$,

$$\|\tilde{T}\|_{cb} \geq \frac{n}{2\sqrt{n-1}} \|T\|_{cb} .$$

Proof

(1) By the proof of Corollary 2.8 we can identify $\mathcal{M}(\mathbb{F}_2)$ with a subfactor of R^ω. Let $n \geq 3$, and let $T : \ell_n^\infty \to R^\omega$ be the map obtained by composing T_n from Example 3.1 with the inclusion map. Then $\|T\|_{cb} = 1$, and by Lemma 2.5 (b), we have still

$$\|T\|_{dec} = \frac{n}{2\sqrt{n-1}} .$$

Let $\rho : \ell^\infty(\mathbb{N},R) \to R^\omega$ be the quotient map. If $\tilde{T}$ is a linear lifting of T, then clearly

$$\|\tilde{T}\|_{dec} \geq \|\rho\circ\tilde{T}\|_{dec} = n/2\sqrt{n-1} ,$$

and since $\ell^\infty(\mathbb{N},R)$ is injective, we have $\|\tilde{T}\|_{cb} = \|T\|_{dec}$. This proves (1).

(2) Let $n \geq 3$, and let $(e_{ij})_{i,j=1,\dots,n}$ be the matrix units in M_n. Define a linear map R from ℓ_n^∞ to M_n and a linear map S from M_n to ℓ_n^∞ by

$$R(c_1,\dots,c_n) = \sum_{i=1}^{n} c_i e_{ii}$$

$$S(\Sigma\, a_{ij}\, e_{ij}) = (a_{11},\dots,a_{nn}) .$$

Then R,S are completely positive,

$$R(1) = 1 , \quad S(1) = 1$$

and

$$(S\circ R)(x) = x , \quad x \in \ell_n^\infty .$$

Let $T : \ell_n^\infty \to R^\omega$ be chosen as in (1) and define $T' \in B(M_n,R^\omega)$ by

$$T' = T\circ S .$$

Then

$$T = T' \circ R \quad .$$

From these two equalities we get

$$\|T'\|_{cb} = \|T\|_{cb} \quad \text{and} \quad \|T'\|_{dec} = \|T\|_{dec}$$

(cf. Proposition 1.3 (4) and (5)). If $\tilde{T}'$ is any linear lifting of T', then, as in (1), we get

$$\|\tilde{T}'\|_{cb} = \|\tilde{T}'\|_{dec} \geqq \|T'\|_{dec} = \|T\|_{dec} = \frac{n}{2\sqrt{n-1}}$$

while $\|T'\|_{cb} = \|T\|_{cb} = 1$. This proves (2).

It is worthwhile to compare Example 3.1 with an example due to Landford, which has been discussed in papers of Loebl [13, Lemma 2.1], Tsui [24, Lemma 3.2], and Huruya and Tomiyama [11, Lemma 1]. We present the example in an updated version:

Example 3.3 (Landford)

Let B be the C*-algebra generated by a sequence $(u_n)_{n\in\mathbb{N}}$ of selfadjoint anticommuting operators:

$$u_k = u_k^* \ , \quad u_k^2 = 1 \ , \quad u_k u_\ell + u_\ell u_k = 0 \ , \quad k \neq \ell \ .$$

From the theory of Clifford algebras it follows that $u_1, u_2, \ldots, u_{2n}$ generates a finite dimensional factor of type $I_{(2^n)}$. Therefore B is isomorphic on the infinite tensorproduct of a sequence of 2×2-matrices. In particular, B has a unique tracial state τ . We will consider B in the representation induced by τ . Thus the weak closure of B is the hyperfinite II_1-factor R .

Consider now the linear map T from ℓ_n^∞ to R given by

$$T_n(c_1, \ldots, c_n) = \frac{1}{\sqrt{2n}} \sum_{k=1}^{n} c_k u_k \quad .$$

Based on computations made in [13] and [24], it was showed in [11, Lemma 1] that $\|T\| \leqq 1$ and $\|T\|_{cb} \geqq \sqrt{n/2}$. In fact, it is not hard to show that

$$\|T_n\| = 1 \quad \text{and} \quad \|T_n\|_{dec} = \|T_n\|_{cb} = \sqrt{n/2} \ .$$

To prove the first equality, put

$$c_k = e^{ik\pi/n}, \quad k=1,\dots,n \ ,$$

and let a_k and b_k be the real and imaginary parts of c_k. Since $|c_k| = 1$ and since

$$\sum_{k=1}^{n} c_k^2 = 0$$

we have

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} b_k^2 = \frac{n}{2} \quad \text{and} \quad \sum_{k=1}^{n} a_k b_k = 0 \ .$$

Let A and B be the self-adjoint operators defined by

$$A = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} a_k u_k \ , \quad B = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} b_k u_k \ .$$

A straightforward computation shows that

$$A^2 = B^2 = 1 \quad \text{and} \quad AB+BA = 0 \ ,$$

from which it follows that

$$(A+iB)(A+iB)^*(A+iB) = 4(A+iB) \ .$$

Therefore $\frac{1}{2}(A+iB)$ is a partial isometry, and since $\frac{1}{2}(A+iB) \neq 0$, we get $\|\frac{1}{2}(A+iB)\| = 1$. Using that

$$T_n(c_1,\dots,c_n) = \tfrac{1}{2}(A+iB) \ ,$$

we conclude that $\|T_n\| \geqq 1$. Hence $\|T_n\| = 1$. From Lemma 2.5 (b) we have $\|T_n\|_{dec} = \sqrt{n/2}$, and since R is injective, also $\|T_n\|_{cb} = \sqrt{n/2}$.

In Example 3.1, $\|T_n\|_{cb} < \|T_n\|_{dec}$ for $n \geqq 3$ and in Example 3.3, $\|T_n\| < \|T_n\|_{cb}$ for $n \geqq 3$. However, in both cases

$$\|T_2\| = \|T_2\|_{cb} = \|T_2\|_{dec} .$$

This turns out to be true in general:

Proposition 3.4

For every von Neumann algebra N and every linear map T from ℓ_2^∞ to N,

$$\|T\| = \|T\|_{cb} = \|T\|_{dec} .$$

The proof of Proposition 3.4 is based on the following lemma:

Lemma 3.5

Let N be a von Neumann algebra with a separating vector. Let $x_1,\dots,x_n \in N$ and let $T : \ell_n^\infty \to N$ be given by

$$T(c_1,\dots,c_n) = \sum_{i=1}^{n} c_i x_i , \quad c_i \in \mathbb{C} .$$

Then

$$\|T\|_{dec} = \sup\{ \|\sum_{i=1}^{n} x_i v_i\| \mid v_i \in N', \|v_i\| \leqq 1\} ,$$

where N' is the commutant of N.

Proof

We prove first the inequality $\geqq$. We may assume that $\|T\|_{dec} = 1$. Using Remark 1.3, we can choose completely positive maps S_1, S_2 from ℓ_n^∞ to N, such that $\|S_i\| \leqq 1$, $i=1,2$, and such that

$$R(x) = \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix}, \quad x \in \ell_n^\infty$$

defines a completely positive map from ℓ_n^∞ to $N \otimes M_2$. Let $p_1, \ldots, p_n$ be the minimal projections in ℓ_n^∞ , and put

$$y_i = S_1(p_i), \quad z_i = S_2(p_i), \quad i=1,\ldots,n .$$

Then $y_i \geqq 0$, $z_i \geqq 0$, $\sum_{i=1}^{n} y_i \leqq 1$, $\sum_{i=1}^{n} z_i \leqq 1$, and

$$\begin{pmatrix} y_i & x_i^* \\ x_i & z_i \end{pmatrix} \geqq 0, \quad i=1,\ldots,n .$$

Let $u_1, \ldots, u_n$ be unitaries in N' , and put

$$a = \sum_{i=1}^{n} x_i u_i \quad .$$

Then

$$\begin{pmatrix} 1 & a^* \\ a & 1 \end{pmatrix} \geqq \sum_{i=1}^{n} \begin{pmatrix} 1 & 0 \\ 0 & u_i \end{pmatrix} \begin{pmatrix} y_i & x_i^* \\ x_i & z_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u_i^* \end{pmatrix} \geqq 0$$

which implies that $\|a\| \leqq 1$. By the Russo-Dye Theorem [18, Thm. 1], the unit ball of N' is the norm closed convex hull of the unitary operators in N' . Hence

$$\sup\{\|\sum_{i=1}^{n} x_i v_i\| \mid v_i \in N', \|v_i\| \leqq 1\} \leqq 1 = \|T\|_{dec} \quad .$$

To prove next the inequality $\leqq$ in Lemma 3.5, we can assume that

$$(*) \qquad \sup\{\|\sum_{i=1}^{n} x_i v_i\| \mid v_i \in N', \|v_i\| \leqq 1\} = 1 .$$

Let E be the subspace of $(N' \otimes \ell_n^\infty) \otimes M_2$ of operators of the form

$$\begin{pmatrix} a\otimes 1 & w \\ v & b\otimes 1 \end{pmatrix}$$

where $a,b \in N'$ and $v,w \in N'\otimes\ell_n^\infty$. Then E is a self-adjoint set of operators and $1 \in E$, i.e. E is an operator-system in the sense of Choi and Effros [4, p. 162]. Let ξ_o be a separating unit vector for N and let ω be the linear functional on E given by

$$\omega\begin{pmatrix} a\otimes 1 & w \\ v & b\otimes 1\end{pmatrix} = ((a + b + \sum_{i=1}^{n} (x_i v_i + x_i^* v_i^*))\xi_o, \xi_o)$$

where

$$v = (v_1,\dots,v_n) \ , \quad w = (w_1,\dots,w_n) \ , \quad v_i, w_i \in N' \ .$$

We will prove that ω is a positive functional on E . Assume that

$$x = \begin{pmatrix} a\otimes 1 & w \\ v & b\otimes 1\end{pmatrix} \in E_+ \quad .$$

Then clearly $w = v^*$ and $a,b \in N'_+$. For $\varepsilon>0$, put $a_\varepsilon = a+\varepsilon 1$ and $b_\varepsilon = b+\varepsilon 1$. Then

$$\begin{pmatrix} 1 & (a_\varepsilon\otimes 1)^{-\frac{1}{2}} v^* (b_\varepsilon\otimes 1)^{-\frac{1}{2}} \\ (b_\varepsilon\otimes 1)^{-\frac{1}{2}} v (a_\varepsilon\otimes 1)^{-\frac{1}{2}} & 1\end{pmatrix}$$

is a positive operator, because it is equal to

$$\begin{pmatrix} a_\varepsilon\otimes 1 & 0 \\ 0 & b_\varepsilon\otimes 1\end{pmatrix}^{-\frac{1}{2}} (x+\varepsilon 1) \begin{pmatrix} a_\varepsilon\otimes 1 & 0 \\ 0 & b_\varepsilon\otimes 1\end{pmatrix}^{-\frac{1}{2}} \quad .$$

Hence $\| (b_\varepsilon\otimes 1)^{-\frac{1}{2}} v (a_\varepsilon\otimes 1)^{-\frac{1}{2}} \| \leqq 1$, or equivalently

$$\| b_\varepsilon^{-\frac{1}{2}} v_i a_\varepsilon^{-\frac{1}{2}} \| \leqq 1 \ , \quad i=1,\dots,n \ .$$

Therefore, by the assumption (*)

$$\| \sum_{i=1}^{n} x_i b_\varepsilon^{-\frac{1}{2}} v_i a_\varepsilon^{-\frac{1}{2}} \| \leqq 1 \ .$$

Since $x_i \in N$ and $v_i, a_\varepsilon, b_\varepsilon \in N'$, we get that

$$-\sum_{i=1}^{n}((x_i v_i + x_i^* v_i^*)\xi_o, \xi_o) = -2\,\mathrm{Re}((\sum_{i=1}^{n} x_i v_i)\xi_o, \xi_o)$$

$$= -2\,\mathrm{Re}((\sum_{i=1}^{n} x_i b_\varepsilon^{-\frac{1}{2}} v_i a_\varepsilon^{-\frac{1}{2}}) a_\varepsilon^{\frac{1}{2}}\xi_o, b_\varepsilon^{\frac{1}{2}}\xi_o)$$

$$\leqq 2\,\|a_\varepsilon^{\frac{1}{2}}\xi_o\|\ \|b_\varepsilon^{\frac{1}{2}}\xi_o\|$$

$$\leqq (a_\varepsilon\xi_o, \xi_o) + (b_\varepsilon\xi_o, \xi_o)$$

$$= ((a+b)\xi_o, \xi_o) + 2\varepsilon \quad .$$

Since ε was arbitrary, we conclude that ω is positive. Hence

$$\|\omega\| = \omega(1) = 2 \ .$$

(The fact that $\|\omega\| = \omega(1)$ for positive functionals on operator systems can be proved as for C*-algebras, cf. proof of [12, Theorem 4.3.2].) Let $\tilde{\omega}$ be a Hahn-Banach extension of ω to $N' \otimes \ell_n^\infty \otimes M_2$. Then

$$\|\tilde{\omega}\| = \tilde{\omega}(1) = 2$$

so $\tilde{\omega}$ is a positive functional on $N' \otimes \ell_n^\infty \otimes M_2$.

Let $p_1, \ldots, p_n$ be the minimal projections in ℓ_n^∞. Put

$$\varphi_i(a) = \tilde{\omega}\begin{pmatrix} a\otimes p_i & 0 \\ 0 & 0 \end{pmatrix}$$

$$\psi_i(b) = \tilde{\omega}\begin{pmatrix} 0 & 0 \\ 0 & b\otimes p_i \end{pmatrix}$$

for $a,b \in N'$ and $i=1,\dots,n$. By the definition of ω

$$\sum_{i=1}^{n} \varphi_i(a) = \omega\begin{pmatrix} a\otimes 1 & 0 \\ 0 & 0 \end{pmatrix} = (a\xi_o,\xi_o)$$

and

$$\sum_{i=1}^{n} \psi_i(b) = \omega\begin{pmatrix} 0 & 0 \\ 0 & b\otimes 1 \end{pmatrix} = (b\xi_o,\xi_o)$$

for $a,b \in N'$. From [9, Part I, Chap. 4, Lemma 1] there exist positive operators $y_1,\dots,y_n,z_1,\dots,z_n \in N$, such that

$$\varphi_i(a) = (ay_i\xi_o,\xi_o) \ , \quad a \in N'$$

$$\psi_i(b) = (bz_i\xi_o,\xi_o) \ , \quad b \in N' \ .$$

Note that $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i = 1$, because ξ_o is cyclic for N' and for all $a,b \in N'$:

$$\sum_{i=1}^{n} (y_ia\xi_o,b\xi_o) = \sum_{i=1}^{n} \varphi_i(b^*a) = (a\xi_o,b\xi_o)$$

$$\sum_{i=1}^{n} (z_ia\xi_o,b\xi_o) = \sum_{i=1}^{n} \psi_i(b^*a) = (a\xi_o,b\xi_o) \ .$$

Let $a,b \in N'$. By the Cauchy-Schwartz inequality for positive functionals, we have

$$\begin{aligned}(x_ia\xi_o,b\xi_o) &= (x_ib^*a\xi_o,\xi_o) \\ &= \omega\begin{pmatrix} 0 & 0 \\ b^*a\otimes p_i & 0 \end{pmatrix} \\ &= \omega\left(\begin{pmatrix} 0 & 0 \\ 0 & b\otimes p_i \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ a\otimes p_i & 0 \end{pmatrix}\right) \\ &\leq \tilde{\omega}\begin{pmatrix} a^*a\otimes p_i & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{2}} \tilde{\omega}\begin{pmatrix} 0 & 0 \\ 0 & b^*b\otimes p_i \end{pmatrix}^{\frac{1}{2}} \\ &= (y_ia\xi_o,a\xi_o)^{\frac{1}{2}}(z_ib\xi_o,b\xi_o)^{\frac{1}{2}} \ .\end{aligned}$$

Since ξ_o is cyclic for N', we conclude that

$$\begin{pmatrix} y_i & x_i^* \\ x_i & z_i \end{pmatrix} \geq 0 , \quad i=1,\dots,n .$$

Define now $S_1, S_2 : \ell_n^\infty \to N$ by

$$S_1(c_1,\dots,c_n) = \sum_{i=1}^{n} c_i y_i$$

$$S_2(c_1,\dots,c_n) = \sum_{i=1}^{n} c_i z_i .$$

Then

$$R(x) = \begin{pmatrix} S_1(x) & T(x^*)^* \\ T(x) & S_2(x) \end{pmatrix} , \quad x \in \ell_n^\infty$$

is clearly a positive map from ℓ_n^∞ to $N \otimes M_2$, and since ℓ_n^∞ is abelian, it is also completely positive. Since $S_1(1) = S_2(1) = 1$ we have

$$\|T\|_{dec} \leqq 1 = \sup\{ \|\sum_{i=1}^{n} x_i v_i\| \mid v_i \in N', \|v_i\| \leqq 1\} .$$

This completes the proof of Lemma 3.5.

<u>Proof of Proposition 3.4</u>

Let T be a linear map from ℓ_2^∞ into a von Neumann algebra N. Since

$$\|T\| \leqq \|T\|_{cb} \leqq \|T\|_{dec} ,$$

it is sufficient to prove that $\|T\|_{dec} \leqq \|T\|$. Let p_1, p_2 be the two minimal projections in ℓ_2^∞ and put $x_i = T(p_i)$, $i=1,2$. Since the extreme points of the unit ball of ℓ_n^∞ are of the form

$$(c_1, c_2) , \quad c_1, c_2 \in \mathbb{C} , \quad |c_1| = |c_2| = 1 ,$$

we have

$$\|T\| = \sup\{\|x_1+cx_2\| \mid c\in\mathbb{C},\ |c|=1\} .$$

Assume first that N is σ-finite. Then, via the G.N.S.-representation, we can obtain that N acts on a Hilbert space H with a cyclic and separating vector ξ_o . By Lemma 3.5,

$$\|T\|_{dec} = \sup\{\|x_1v_1+x_2v_2\| \mid v_i\in N',\ \|v_i\| \leqq 1\} .$$

By the Russo-Dye theorem, it is sufficient to consider unitary operators v_1,v_2 in N' . In this case,

$$\|x_1v_1+x_2v_2\| = \|x_1+x_2v_2v_1^*\| .$$

Therefore

$$\|T\|_{dec} = \sup\{\|x_1+x_2u\| \mid u\in N',\ u \text{ unitary}\} .$$

If u has finite spectrum, then

$$u = \sum_{i=1}^{r} \lambda_i p_i ,$$

where $\lambda_i \in sp(u)$ and p_i are orthogonal projections in N' with sum 1 . Since the subspaces $p_i(H)$, $i=1,\dots,r$ are invariant under x_1 and x_2 , we get in this case

$$\|x_1+x_2u\| = \sup\{\|x_1+\lambda x_2\| \mid \lambda\in sp(u)\} .$$

Since every unitary in N' can be approximated in norm by unitaries with finite spectrum,

$$\|T\|_{dec} \leqq \sup\{\|x_1+cx_2\| \mid c\in\mathbb{C},\ |c|=1\} = \|T\| .$$

If N is not σ-finite, we can choose a net (p_λ) of σ-finite projections in N, such that $p_\lambda \to 1$ strongly. Using the first part of the proof on the map $T_\lambda : \ell_2^\infty \to p_\lambda M p_\lambda$ given by

$$T_\lambda(x) = p_\lambda x p_\lambda \ , \quad x \in \ell_2^\infty \ ,$$

we find completely positive maps $S_\lambda^{(1)}$, $S_\lambda^{(2)}$ from ℓ_2^∞ to $p_\lambda M p_\lambda \subseteqq M$, such that $\|S_\lambda^{(i)}\| \leqq \|T\|$, $i=1,2$, and such that

$$R_\lambda(x) = \begin{pmatrix} S_\lambda^{(1)}(x) & T_\lambda(x^*)^* \\ T_\lambda(x) & S_\lambda^{(2)}(x) \end{pmatrix} \ , \quad x \in \ell_2^\infty$$

is a completely positive map from ℓ_2^∞ to $N \otimes M_2$. Let $R : \ell_2^\infty \to N \otimes M_2$ be a clusterpoint for the net (R_λ) in the topology of pointwise σ-weak convergence on $B(\ell_2^\infty, N \otimes M_2)$. Then R is a completely positive map of the form

$$R(x) = \begin{pmatrix} S^{(1)}(x) & T(x^*)^* \\ T(x) & S^{(2)}(x) \end{pmatrix} \ , \quad x \in \ell_2^\infty \ ,$$

where $S^{(1)}, S^{(2)} : \ell_2^\infty \to N$ are completely positive and $\|S^{(i)}\| \leqq \|T\|$. Hence $\|T\|_{dec} \leqq \|T\|$.

<u>Corollary 3.6</u>

Let N be a von Neumann algebra, and let $x \in N$. The following two conditions are equivalent

(i) There exists $a \in N_{s.a.}$, $0 \leqq a \leqq 1$, such that

$$\begin{pmatrix} a & x^* \\ x & 1-a \end{pmatrix} \geqq 0$$

(ii) $w(x) \leqq \frac{1}{2}$, where $w(x)$ is the numerical radius of x.

<u>Proof</u>

Recall that the numerical range $W(x)$ of an operator $x \in B(H)$ is

$$\{(x\xi, \xi) \mid \xi \in H, \ \|\xi\| = 1\} \ ,$$

and the numerical radius $w(x)$ of x is

$$w(x) = \sup\{|\lambda| \mid \lambda \in W(x)\}$$

$$= \sup\{|(x\xi,\xi)| \mid \xi \in H, \|\xi\| = 1\}$$

(cf. [3, pp. 1-2]). To prove (i) ⇒ (ii), let $\xi \in H$ be a unit vector and let $c \in \mathbb{C}$, $|c|=1$. Put $\xi' = (\xi, c\xi) \in H \oplus H$ and put

$$b = \begin{pmatrix} a & x^* \\ x & 1-a \end{pmatrix} .$$

If $b \geqq 0$, then $(b\xi', \xi') \geqq 0$. Thus

$$1 + 2\,\mathrm{Re}(c(x\xi,\xi)) \geqq 0 ,$$

so by choosing c, such that $c(x\xi,\xi) = -|(x\xi,\xi)|$, we get

$$|(x\xi,\xi)| \leqq \tfrac{1}{2} .$$

Conversely, if $w(x) \leqq \frac{1}{2}$, then for $c \in \mathbb{C}$, $|c|=1$,

$$\begin{aligned} \|cx+\bar{c}x^*\| &= \sup\{|((cx+\bar{c}x^*)\xi,\xi)| \mid \xi \in H, \|\xi\| = 1\} \\ &= 2\sup\{|\mathrm{Re}(c(x\xi,\xi))| \mid \xi \in H, \|\xi\| = 1\} \\ &\leqq 2w(x) \\ &\leqq 1 . \end{aligned}$$

Hence also $\|x+\bar{c}^2x^*\| \leqq 1$. Consider now the map $T : \ell_2^\infty \to N$ given by

$$T(c_1,c_2) = c_1x+c_2x^* .$$

Clearly,

$$\|T\| = \sup\{\|x+\gamma x^*\| \mid \gamma \in \mathbb{C}, |\gamma|=1\} \leqq 1 .$$

Hence, by Prop. 3.4, $\|T\|_{dec} \leqq 1$. Thus there exist $y_1, y_2, z_1, z_2 \in N_+$, such that $y_1+y_2 \leqq 1$, $z_1+z_2 \leqq 1$, and

$$\begin{pmatrix} y_1 & x^* \\ x & y_2 \end{pmatrix} \geqq 0 , \quad \begin{pmatrix} z_1 & x \\ x^* & z_2 \end{pmatrix} \geqq 0 .$$

Hence also

$$\begin{pmatrix} y_1+z_2 & x^* \\ x & y_2+z_1 \end{pmatrix} \geqq 0 .$$

Put $a = y_1+z_2$. Then $1-a \geqq y_2+z_1$. This proves (i).

Remark 3.7

In [4, Thm. 3.4], Choi and Effros proved that a von Neumann algebra N is injective if and only if for $n \in \mathbb{N}$, $n \geqq 2$, any unit preserving, completely positive map T from an operator system $E \subseteq M_n$ of codimension 1 into N can be extended to a completely positive map $\tilde{T}$ from M_n to N . It is somewhat surprising that for $n=2$ such an extension exists, even if N is not injective. This follows easily from Corollary 3.6:

Let E be any three-dimensional operator system in M_2 , then, by a change of basis, we can obtain that

$$E = \left\{ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \middle| \; c_{11} = c_{22} \right\} .$$

Let $T : E \to N$ be completely positive and unit preserving, and put

$$x = T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Since

$$1+cx+\bar{c}x^* = T\begin{pmatrix} 1 & \bar{c} \\ c & 1 \end{pmatrix} \geq 0$$

whenever $|c| = 1$, it follows that $w(x) \leq \frac{1}{2}$. Hence, by Cor. 3.6, there exists $a \in N_+$, such that

$$\begin{pmatrix} a & x^* \\ x & 1-a \end{pmatrix} \geq 0 .$$

Therefore,

$$\tilde{T}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = c_{11}a + c_{22}(1-a) + c_{21}x + c_{12}x^*$$

defines a complete positive extension $\tilde{T} : M_2 \to N$ of T (use [4, Lemma 2.1]).

Problem 3.8

Let N be a von Neumann algebra, such that

$$\|T\|_{cb} = \|T\|_{dec}$$

for every linear map T from ℓ_3^∞ to N. Is N injective?

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