JOURNAL OF FUNCTIONAL ANALYSIS 19, 302–317 (1975)

Normal Weights on W^*-Algebras

UFFE HAAGERUP

Institute of Mathematics, University of Copenhagen, 2100 Copenhagen Ø, Denmark

Communicated by the Editors

Received August 5, 1974

Let φ be a weight on a W^*-algebra. If φ is normal, in the sense that it respects monotone increasing limits, then φ is the sum of positive normal functionals. This provides the complete solution to a problem raised by J. Dixmier.

INTRODUCTION

A weight on a C^*-algebra A is a function φ: $A_+ \to [0, \infty]$ with the properties:

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$, $x, y \in A_+$,
(ii) $\varphi(\lambda x) = \lambda\varphi(x)$, $x \in A_+$, $\lambda \geqslant 0$,

using the convention $0 \cdot \infty = 0$.

Let φ be a weight on a W^*-algebra M. We say that

(1) φ is completely additive if $\varphi(\sum x_i) = \sum \varphi(x_i)$ for any set $\{x_i\}$ of positive elements for which $\sum x_i$ is defined;

(2) φ is normal if $\varphi(\text{l.u.b. } x_i) = \text{l.u.b. } \varphi(x_i)$ for any uniformly bounded increasing set $\{x_i\}$ of positive elements.

The main result in this paper is that, for any weight φ on a W^*-algebra M, the following conditions are equivalent.

(1) φ is completely additive;

(2) φ is normal;

(3) φ is σ-weakly lower semicontinuous;

(4) $\varphi(x) = \sup_{\omega \in F} \omega(x)$, $\forall x \in M_+$, where F is a set of positive normal functionals;

(5) $\varphi(x) = \sum_{i \in I} \varphi_i(x)$, $x \in M_+$, where $\{\varphi_i\}$ is a set of positive normal functionals.

The implications (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial. In [1] Combes has made a partial solution to (3) $\Rightarrow$ (4). He proved that if φ is a σ-weakly, lower semicontinuous weight, then there exists a set F of positive normal functionals, such that $\varphi(x) = \sup_{\omega \in F} \omega(x)$ whenever $\varphi(x) < \infty$. The implication (4) $\Rightarrow$ (5) has been proved by Pedersen and Takesaki [5] by use of left Hilbert algebra theory. The reader should be aware that Takesaki has introduced condition (4) as a definition of normality.

This paper consists of two sections. In Section 1 the equivalence of (1), (2), and (3) are shown. In Section 2 we prove the implication (3) $\Rightarrow$ (4) in a slightly more general situation, namely for a "subadditive weight" i.e. a function $\varphi\colon M_+ \to [0, \infty]$ with the properties

(i) $x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y)$, $x, y \in M_+$,

(ii) $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$, $x, y \in M_+$,

(iii) $\varphi(\lambda x) = \lambda\varphi(x)$, $x \in M_+$, $\lambda \geqslant 0$.

The equivalence of (2) and (5) gives the complete solution to a problem posed by Dixmier [2, Chap. I, Sect. 4, p. 52].

1

We will first recall the construction of the representation induced by a weight (cf. [1, Sect. 2]). Let φ be a weight on a C^*-algebra A. Put

$$n_\varphi = \{x \in A \mid \varphi(x^*x) < \infty\},$$

$$m_\varphi = n_\varphi{}^* n_\varphi = \operatorname{span}\{y^*x \mid x, y \in n_\varphi\},$$

$$N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}.$$

Let H denote the completion of the pre-Hilbert space n_φ/N_φ with inner product $(\alpha(x) \mid \alpha(y)) = \varphi(y^*x)$, where α is the quotient map $n_\varphi \to n_\varphi/N_\varphi$. It is easily seen that there is a unique *-homomorphism $\pi\colon A \to B(H)$ so that

$$\pi(a)\,\alpha(x) = \alpha(ax), \qquad a \in A, \quad x \in n_\varphi .$$

(π, H) is called the representation induced by φ.

LEMMA 1.1. *Let φ be a weight on a W^*-algebra M and let (π, H) be* *the representation induced by φ. There is a unique linear map β of m_φ into the predual of $\pi(M)'$ satisfying*

$$\beta(y^*x)(a') = (a'\alpha(x) \mid \alpha(y)), \qquad a' \in \pi(M)', \quad x, y \in n_\varphi .$$

Proof. The uniqueness of β follows from $m_\varphi = \operatorname{span}\{y^*x \mid x, y \in n_\varphi\}$. Let $x, y \in n_\varphi$, and assume that $x^*x = y^*y$. There is a partial isometry $u \in M$ so that $x = uy$ and $y = u^*x$. Thus $\forall a' \in \pi(M)'$:

$$(a'\alpha(x) \mid \alpha(x)) = (a'\alpha(x) \mid \pi(u)\,\alpha(y)) = (a'\pi(u^*)\,\alpha(x) \mid \alpha(y)) = (a'\alpha(y) \mid \alpha(y)).$$

Therefore the map

$$\beta_0 \colon x^*x \to \omega'_{\alpha(x)}$$

is a well-defined map of m_φ^+ into $\pi(M)'_*$. Obviously $\beta_0(\lambda x) = \lambda\beta_0(x)$ $\forall\lambda \geqslant 0$. We will show that β_0 is additive:

Let $x, y \in m_\varphi^+$ and put $z = x + y$. There exist operators $s, t \in M$ so that $x^{1/2} = sz^{1/2}$, $y^{1/2} = tz^{1/2}$ and $s^*s + t^*t$ is the support projection of z. Hence $\forall a' \in \pi(M)'$:

$$\begin{aligned}(a'\alpha(z^{1/2}) \mid \alpha(z^{1/2})) &= (a'\pi(s^*s + t^*t)\,\alpha(z^{1/2}) \mid \alpha(z^{1/2}))\\ &= (a'\pi(s)\,\alpha(z^{1/2}) \mid \pi(s)\,\alpha(z^{1/2})) + (a'\pi(t)\,\alpha(z^{1/2}) \mid \pi(t)\,\alpha(z^{1/2}))\\ &= (a'\alpha(x^{1/2}) \mid \alpha(x^{1/2})) + (a'\alpha(x^{1/2}) \mid \alpha(x^{1/2})).\end{aligned}$$

Thus $\beta_0(x + y) = \beta_0(x) + \beta_0(y)$.

Since m_φ is spanned by m_φ^+, β_0 has a linear extension β to m_φ. Using the identity $y^*x = \sum_{k=0}^{3} i^{-k}(x + i^ky)^*(x + i^ky)$ we find that

$$\beta(y^*x)(a') = (a'\alpha(x) \mid \alpha(y)), \qquad a' \in \pi(M)', \quad x, y \in n_\varphi .$$

In the following, α and β denote the maps in Lemma 1.1.

LEMMA 1.2. *For any self-adjoint element x in m_φ we have*

$$\| \beta(x)\| = \inf\{\varphi(a) + \varphi(b) \mid x = a - b,\ a, b \in m_\varphi^+\}.$$

Proof. The function ρ on $(m_\varphi)_{\mathrm{s.a.}}$ defined by

$$\rho(x) = \inf\{\varphi(a) + \varphi(b) \mid x = a - b,\ a, b \in m_\varphi^+\}$$

is a seminorm on $(m_\varphi)_{\mathrm{s.a.}}$. It is easily seen that $\rho(x) = \varphi(x)$, $\forall x \in m_\varphi^+$. If $x = a - b$, $a, b \in m_\varphi^+$, then

$$\| \beta(x)\| \leqslant \| \beta(a)\| + \| \beta(b)\| = \varphi(a) + \varphi(b).$$

Hence $\| \beta(x)\| \leqslant \rho(x)$, $x \in (m_\varphi)_{\mathrm{s.a.}}$.

To show the converse inequality let $x_0 \in (m_\varphi)_{s.a.}$. By the Hahn–Banach theorem we can find a functional μ on $(m_\varphi)_{s.a.}$ so that $\mu(x_0) = \rho(x_0)$ and $|\mu(x)| \leqslant \rho(x)$, $\forall x \in (m_\varphi)_{s.a.}$. μ can be extended to a self-adjoint functional on m_φ. The extension will also be denoted μ. Since

$$-\varphi(x^*x) \leqslant \mu(x^*x) \leqslant \varphi(x^*x), \qquad x \in n_\varphi,$$

we get

$$\begin{aligned} |\mu(y^*x)| &\leqslant \tfrac{1}{2}[|(\varphi+\mu)(y^*x)| + |(\varphi-\mu)(y^*x)|] \\ &\leqslant \tfrac{1}{2}[(\varphi+\mu)(x^*x)^{1/2}(\varphi+\mu)(y^*y)^{1/2} \\ &\qquad + (\varphi-\mu)(x^*x)^{1/2}(\varphi-\mu)(y^*y)^{1/2}] \\ &\leqslant \tfrac{1}{2}[(\varphi+\mu)(x^*x) + (\varphi-\mu)(x^*x)]^{1/2} \\ &\qquad \times [(\varphi+\mu)(y^*y) + (\varphi-\mu)(y^*y)]^{1/2} \\ &= \varphi(x^*x)^{1/2}\,\varphi(y^*y)^{1/2}, \qquad x, y \in n_\varphi. \end{aligned}$$

Hence there exists a bounded operator T in $B(H)$ so that $\|T\| \leqslant 1$ and $\mu(y^*x) = (T\alpha(x) \mid \alpha(y))$, $x, y \in n_\varphi$. Furthermore $\forall a \in M$, $\forall x$, $y \in n_\varphi$:

$$\begin{aligned} (T\pi(a)\,\alpha(x) \mid \alpha(y)) &= (T\alpha(ax) \mid \alpha(y)) = \mu(y^*ax) \\ &= (T\alpha(x) \mid \alpha(a^*y)) = (\pi(a)\,T\alpha(x) \mid \alpha(y)). \end{aligned}$$

Thus $T \in \pi(M)'$. Hence $\mu(z) = \beta(z)(T)$ for any $z \in m_\varphi$, and therefore $\rho(x_0) = |\mu(x_0)| \leqslant \|\beta(x_0)\|\,\|T\| \leqslant \|\beta(x_0)\|$.

This completes the proof.

LEMMA 1.3. *Let φ be a normal weight on a W^*-algebra M and let x_n be a bounded sequence of elements in $m_\varphi{}^+$.*

(1) *If $x_n \to^{\sigma-s} x \in M$ and if $\beta(x_n)$ is convergent, then $x \in m_\varphi{}^+$.*

(2) *If $x_n \to^{\sigma-s} 0$ and if $\beta(x_n)$ is convergent, then $\beta(x_n) \to 0$. ($\sigma - s$ denotes the σ-strong topology. On $\pi(M)'_*$ we use the norm topology.)*

Proof. (1) Let ϵ be a positive number. Put $\psi = \lim \beta(x_n)$. We can choose a subsequence y_n of x_n so that $\|\beta(y_n) - \psi\| \leqslant \epsilon \cdot 2^{-n}$, $n \in \tilde{N}$. Thus $\|\beta(y_{n+1} - y_n)\| \leqslant (3/2)\epsilon \cdot 2^{-n}$.

By Lemma 1.2 there exists a_n, $b_n \in m_\varphi{}^+$ so that

$$y_{n+1} - y_n = a_n - b_n \qquad \text{and} \qquad \varphi(a_n) + \varphi(b_n) \leqslant 2\epsilon \cdot 2^{-n}.$$

Thus

$$y_{n+1} \leqslant y_1 + \sum_{k=1}^{n} a_n .$$

Put $f_\nu(t) = t(1 + \nu t)^{-1}$, $\nu > 0$. f_ν is operator monotone in the sense that $-(1/\nu) < x \leqslant y$ implies that $f_\nu(x) \leqslant f_\nu(y)$. Hence $f_\nu(y_{n+1}) \leqslant f_\nu(y_1 + \sum_{k=1}^{n} a_k)$. $f_\nu(y_1 + \sum_{k=1}^{n} a_k)$ is a bounded increasing sequence and therefore has a least upper bound.

Since $y_n \to^{\sigma-s} x$ we get $f_\nu(y_n) \to^{\sigma-s} f_\nu(x)$ (cf. [4]). Thus

$$f_\nu(x) \leqslant \text{l.u.b.}\, f_\nu\left(y_1 + \sum_{k=1}^{n} a_k\right)$$

and

$$\varphi(f_\nu(x)) \leqslant \text{l.u.b.}\, \varphi\left(f_\nu\left(y_1 + \sum_{k=1}^{n} a_k\right)\right) \leqslant \text{l.u.b.}\, \varphi\left(y_1 + \sum_{k=1}^{n} a_k\right)$$

$$\leqslant \varphi(y_1) + \sum_{k=1}^{\infty} (2\epsilon)2^{-k} = \varphi(y_1) + 2\epsilon.$$

Since $f_\nu(x) \nearrow x$ for $\nu \to 0$ we get that $\varphi(x) \leqslant \varphi(y_1) + 2\epsilon < \infty$. This proves (1).

(2) Let ϵ, ψ, a_n, b_n be as in (1). We then have

$$y_1 - y_{n+1} \leqslant \sum_{k=1}^{n} b_k .$$

Put $K = \sup \| x_n \|$. Then $y_1 - y_{n+1} \geqslant -K$, $\forall n$. For any $\nu < 1/K$ we get

$$f_\nu(y_1 - y_{n+1}) \leqslant \text{l.u.b.}\, f_\nu\left(\sum_{k=1}^{n} b_k\right).$$

By the assumptions $y_n \to^{\sigma-s} 0$ for $n \to \infty$. Therefore

$$f_\nu(y_1 - y_{n+1}) \xrightarrow{\sigma-s} f_\nu(y_1).$$

Hence

$$f_\nu(y_1) \leqslant \text{l.u.b.}\, f_\nu\left(\sum_{k=1}^{n} b_k\right).$$

Thus

$$\varphi(f_\nu(y_1)) \leqslant \text{l.u.b.}\ \varphi\left(f_\nu\left(\sum_{k=1}^{n} b_k\right)\right)$$

$$\leqslant \text{l.u.b.}\ \varphi\left(\sum_{k=1}^{n} b_k\right) \leqslant \sum_{k=1}^{\infty} 2\epsilon \cdot 2^{-k} = 2\epsilon.$$

Using $f_\nu(y_1) \nearrow y_1$ for $\nu \to 0$ we get $\varphi(y_1) \leqslant 2\epsilon$. Hence $\| \psi \| \leqslant \| \psi - \beta(y_1)\| + \| \beta(y_1)\| \leqslant (1/2)\epsilon + 2\epsilon = (5/2)\epsilon$. This is valid for any $\epsilon > 0$. Thus $\psi = \lim \beta(x_n) = 0$.

If A is a Banach space, we let A_r denote the set $\{x \in A \mid \| x \| \leqslant r\}$.

LEMMA 1.4. *Let φ be a normal weight on a σ-finite W^*-algebra M and let $G(\alpha) = \{(x, \alpha(x)) \mid x \in n_\varphi\}$ be the graph of α. Then $G(\alpha) \cap (M_r \times H_t)$ is $\sigma(M \times H, M_* \times H^*)$-compact for any $r, t > 0$.*

Proof. Since H is a reflexive Banach space and $M = (M_*)^*$, $M \times H$ is the dual of the Banach space $M_* \times H^*$ with norm $\|(\varphi, \xi^*)\| = \| \varphi \| + \| \xi^* \|$. The dual norm on $M \times H$ is $\|(x, \xi)\| = \max\{\| x \|, \| \xi \|\}$. Since $G(\alpha) \cap (M_r \times H_t)$ is convex it is $\sigma(M \times H, M_* \times H^*)$-closed iff it is closed in any topology compatible with the duality between $M \times H$ and $M_* \times H^*$. Hence it is enough to show that $G(\alpha) \cap (M_r \times H_t)$ is closed in the product of σ-strong* topology on M and norm topology on H.

Let (x, ξ) be in the (σ-strong*) $\times$ norm closure of $G(\alpha) \cap (M_r \times H_t)$. Since M is σ-finite, M_r is metrisable in the $\sigma - s^*$-topology. Hence there exists a sequence $\{x_n\} \subseteq M_r$ so that $x_n \to^{\sigma-s^*} x$ and $\alpha(x_n) \to \xi$, $\| \alpha(x_n)\| \leqslant t$. Thus $x_n{}^*x_n \to^{\sigma-s} x^*x$ and $\beta(x_n{}^*x_n) = \omega'_{\alpha(x_n)} \to \omega_\xi'$. By Lemma 1.3(1), $x^*x \in m_\varphi{}^+$. Thus $x \in n_\varphi$. Hence $(x_n - x)^*(x_n - x) \to^{\sigma-s} 0$ and $\beta((x_n - x)^*(x_n - x)) = \omega'_{\alpha(x_n)-\alpha(x)} \to \omega'_{\xi-\alpha(x)}$. By Lemma 1.3(2), $\omega'_{\xi-\alpha(x)} = 0$ and therefore $\xi = \alpha(x)$. Hence $(x, \xi) \in G(\alpha)$. This completes the proof.

LEMMA 1.5. *Let φ be a weight on a σ-finite W^*-algebra. The following conditions are equivalent.*

(1) *φ is completely additive,*

(2) *φ is normal,*

(3) *φ is σ-weakly lower semicontinuous.*

Proof. (3) $\Rightarrow$ (1): trivial. (1) $\Rightarrow$ (2): Let x_i be a bounded increasing set of operators in M_+ with $x_i \nearrow x$. Since bounded subsets of M is metrisable in the σ-strong topology, we can find a sequence $\{y_n\} \subseteq \{x_i\}$ so that $y_n \nearrow x$ and l.u.b. $\varphi(x_i) =$ l.u.b. $\varphi(y_n)$.

Put $z_n = y_{n+1} - y_n$. Then $x = y_1 + \sum_{n=1}^{\infty} z_n$. Thus

$$\varphi(x) = \varphi(y_1) + \sum_{n=1}^{\infty} \varphi(z_n) = \lim_{n\to\infty} \varphi\left(y_1 + \sum_{k=1}^{n} z_k\right) = \text{l.u.b. } \varphi(y_n).$$

Hence φ is normal.

(2) $\Rightarrow$ (3): By Lemma 1.4, $G(\alpha) \cap (M_r \times H_t)$ is $\sigma(M \times H, M_* \times H^*)$-compact. $\{x \in M_r \mid \varphi(x^*x) \leqslant t^2\}$ is the range of $G(\alpha) \cap (M_r \times H_t)$ by the projection $(x, \xi) \to x$. Hence $\{x \in M_r \mid \varphi(x^*x) \leqslant t^2\}$ is σ-weakly compact. Thus by [2, Chap. 1, Sect. 3, Theorem 1(iv)], $\{x \in M \mid \varphi(x^*x) \leqslant t^2\}$ is σ-weakly closed. Now let x be in the σ-weak closure of $\{a \in M_+ \mid \varphi(a) \leqslant t^2\}$. Since the set is convex, there exists a net $\{x_i\} \subseteq M_+$, $\varphi(x_i) \leqslant t^2$ so that $x_i \to^{\sigma-s} x$. Then $x_i^{1/2} \to^{\sigma-s} x^{1/2}$ (cf. [4]). Thus $\varphi(x) = \varphi(x^{1/2}x^{1/2}) \leqslant t^2$. This completes the proof.

DEFINITION 1.6. Let A be a partially ordered vector space. A subset E of $A_+ = \{x \in A \mid x \geqslant 0\}$ is called hereditary if $x \in E$ and $0 \leqslant y \leqslant x$ implies that $y \in E$.

LEMMA 1.7. *Let M be a W^*-algebra. Put $M_0 = \bigcup_{p\in\Sigma} pMp$ where Σ is the set of σ-finite projections in M. Let E be a convex, hereditary subset of $M_0{}^+$. Then E is σ-weakly closed relative to M_0 iff $E \cap pMp$ is σ-weakly closed for any $p \in \Sigma$.*

Proof. It is easily seen that if E is σ-weakly closed relative to M_0 then $E \cap pMp$ is σ-weakly closed for any $p \in \Sigma$. To show the opposite note first that if $p \in \Sigma$, then the unit ball in the left ideal Mp is metrisable in the σ-strong topology. Namely, let μ be a positive normal functional with support p, then the seminorm $x \to \mu(x^*x)^{1/2}$ induce the σ-strong topology on $(Mp)_1$. Let E be a convex hereditary subset of M_0 so that $M \cap pMp$ is σ-weakly closed for any $p \in \Sigma$.

Put $F = \{x \in M \mid x^*x \in E\}$. F is a convex subset of M. Let namely $x, y \in F$ and $\lambda \in [0, 1]$, then

$$\begin{aligned}
&(\lambda x + (1-\lambda)y)^*(\lambda x + (1-\lambda)y) \\
&\quad = \lambda^2 x^*x + (1-\lambda)^2 y^*y + \lambda(1-\lambda)(x^*y + y^*x) \\
&\quad \leqslant \lambda^2 x^*x + (1-\lambda)^2 y^*y + \lambda(1-\lambda)(x^*x + y^*y) \\
&\quad = \lambda x^*x + (1-\lambda) y^*y.
\end{aligned}$$

Hence $\lambda x + (1-\lambda) y \in F$. Furthermore, if $s \in M$, $\|s\| \leqslant 1$, then $sF \subseteq F$ because $(sx)^* sx = x^*s^*sx \leqslant x^*x$.

We will show that pF is σ-weakly closed for any $p \in \Sigma$, or equivalently that F^*p is σ-weakly closed. Using [2, Chap. 1, Sect. 3, Theorem 1(iv)], it is enough to show that $F^*p \cap M_r$ is σ-strongly closed for any $r > 0$.

Choose x so that x^* belongs to the σ-strong closure of $F^*p \cap M_r$. Since $Mp \cap M_r$ is metrisable in σ-strong topology, we can find a sequence $\{x_n\} \subseteq pF$, $\| x_n \| \leqslant r$ so that $x_n^* \to^{\sigma-s} x^*$. Since the support and range projections of x_n are σ-finite, and since the least upper bound of a countable set of σ-finite projections again is a σ-finite projection there exists $q \in \Sigma$ so that $x_n \in qMq$, $\forall n \in \tilde{N}$. We have

$$x_n \in F \cap qMq = \{x \in qMq \mid x^*x \in E \cap qMq\}.$$

Since $E \cap qMq$ is σ-weakly closed $F \cap qMq$ is σ-strongly closed and thus σ-weakly closed. Hence $x \in F \cap qMq$. Obviously $px = x$. Hence $x \in pF$. This shows that pF is σ-weakly closed for any $p \in \Sigma$.

Now let $y \in \bar{E}^{\sigma-w} \cap M_0$. Then there exists a net $\{y_i\} \subseteq E$ so that $y_i \to^{\sigma-s} y$. Let p be the support projection of y. We have $py_i^{1/2} \to^{\sigma-s} py^{1/2} = y^{1/2}$. Since p is σ-finite pF is closed. Thus $y^{1/2} \in pF \subseteq F$ and therefore $y \in E$. Hence E is σ-weakly closed relative to M_0.

THEOREM 1.8. *Let φ be a weight on a W^*-algebra M. The following three conditions are equivalent.*

(1) *φ is completely additive,*

(2) *φ is normal,*

(3) *φ is σ-weakly lower semicontinuous.*

Proof. It is easily seen that (3) $\Rightarrow$ (2) $\Rightarrow$ (1). (1) $\Rightarrow$ (3): Put $E = \{x \in M_+ \mid \varphi(x) \leqslant 1\}$. By Lemma 1.5, $E \cap pMp$ is σ-weakly closed for any σ-finite projection p. Thus by Lemma 1.7, $E \cap M_0$ is σ-weakly closed relative to M_0. Let $(p_i)_{i \in I}$ be a maximal set of orthogonal σ-finite projections. Then $\sum_{i \in I} p_i = 1$. For any finite subset J of I let p_J denote the projection $\sum_{i \in J} p_i$. Obviously $p_J \nearrow 1$. Now let $x \in \bar{E}^{\sigma-w}$. Then there exists a net $\{x_\alpha\} \subseteq E$ so that $x_\alpha \to^{\sigma-s} x$. Thus $x^{1/2} \to^{\sigma-s} x^{1/2}$ and $x_\alpha^{1/2} p_J x_\alpha^{1/2} \to^{\sigma-w} x^{1/2} p_J x^{1/2}$ for any finite subset J of I. Since M_0 is a two-sided ideal in M we have $x_\alpha^{1/2} p_J x_\alpha^{1/2} \in M_0$ and $x^{1/2} p_J x^{1/2} \in M_0$. Since $x_\alpha^{1/2} p_J x_\alpha^{1/2} \in E \cap M_0$ we get $x^{1/2} p_J x^{1/2} \in E \cap M_0$, $\forall J \subseteq I$, J finite. Using $x = \sum_{i \in I} x^{1/2} p_i x^{1/2}$ we get

$$\varphi(x) = \sum_{i \in I} \varphi(x^{1/2} p_i x^{1/2}) = \lim \varphi(x^{1/2} p_J x^{1/2}).$$

This shows that $\varphi(x) \leqslant 1$. Hence E is σ-weakly closed.

As a special case of Theorem 1.8 we get the following well-known theorem (cf. [2, Chap. 1, Sect. 4, Theorem 1; and 3]).

COROLLARY 1.9. *Let φ be a positive functional on a W^*-algebra M. The following conditions are equivalent.*

(1) *φ is completely additive,*

(2) *φ is normal,*

(3) *φ is σ-weakly continuous.*

Proof. It is only left to show that if φ is σ-weakly lower semicontinuous on M_+, then φ is σ-weakly continuous on M. Let $(x_i)_{i\in I}$ be a net on $M_{\text{s.a.}}$ so that $\| x_i \| \leqslant 1$ and $x_i \to^{\sigma-w} x$. Then

$$\liminf \varphi(x_i) = \liminf \varphi(1 + x_i) - \varphi(1) \geqslant \varphi(1 + x) - \varphi(1) = \varphi(x),$$
$$\limsup \varphi(x_i) = \varphi(1) - \liminf \varphi(1 - x_i) \leqslant \varphi(1) - \varphi(1 - x) = \varphi(x).$$

Hence the restriction of φ to $(M_1)_{\text{s.a.}}$ is σ-weakly continuous. Using [2, Chap. 1, Sect. 3, Theorem 1(ii)] we get the required result.

PROBLEM 1.10. Let M be a W^*-algebra and φ a function $M_+ \to [0, \infty]$ with the properties

(i) $x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y)$, $x, y \in M_+$;

(ii) $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$, $x, y \in M_+$;

(iii) $\varphi(\lambda x) = \lambda\varphi(x)$, $x \in M_+$, $\lambda \geqslant 0$.

(iv) $\varphi(\text{l.u.b. } x_i) = \text{l.u.b. } \varphi(x_i)$ for any uniformly bounded increasing set $\{x_i\}$ of positive elements.

Is φ σ-weakly lower semicontinuous?

PROBLEM 1.11. Let φ be a weight on a W^*-algebra M, and assume that the restriction of φ to any commutative σ-weakly closed subalgebra is normal.

Is φ normal?

Remark 1.12. J. Dixmier has shown that if φ is a positive functional on a W^*-algebra M with the property that $\varphi(\sum p_i) = \sum \varphi(p_i)$ for any set $\{p_i\}$ of mutually orthogonal projections in M, then φ is normal (cf. [3]). This cannot be generalized to weights. Consider the W^*-algebra $L^\infty(\dot{N})$ of all bounded sequences. The weight defined by

$$\varphi((a_n)_{n\in\dot{N}}) = \begin{cases} \sum a_n, & \text{if } (a_n)_{n\in\dot{N}} \text{ has finite support,} \\ \infty, & \text{otherwise,} \end{cases}$$

is completely additive on the projections, but φ is not normal.

2

Consider a locally convex Hausdorff vector space A over the scalar field R, which has a partial ordering defined by a closed convex cone A_+, satisfying $A_+ \cap (-A_+) = \{0\}$ and $(A_+ - A_+)^- = A$. Let A' be the topological dual space of A. The dual cone of A_+:

$$A_+' = \{\varphi \in A' \mid \varphi(x) \geqslant 0, \forall x \in A_+\}$$

defines a partial ordering of A'.

We recall that a subset E of A_+ is called hereditary if $x \in E$ and $0 \leqslant y \leqslant x$ implies that $y \in E$.

For $E \subseteq A$ we put $E^0 = \{\varphi \in A' \mid \varphi(x) \geqslant -1, \forall x \in E\}$.

For $E \subseteq A_+$ we put $E^\wedge = \{\varphi \in A_+' \mid \varphi(x) \leqslant 1, \forall x \in E\}$.

If $F \subseteq A'$ (resp. A_+') F^0 and $F^\wedge$ is defined symmetrically.

PROPOSITION 2.1. *In the above situation the following three conditions are equivalent.*

(1) *For any convex, closed, hereditary subset* E *of* A_+ *is* $E = (E - A_+)^- \cap A_+$.

(2) *For any convex, closed, hereditary subset* E *of* A_+ *is* $E = E^{\wedge\wedge}$.

(3) *Any lower semicontinuous function* $\varphi: A_+ \to [0, \infty]$ *satisfying*

(i) $x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y)$, $x, y \in A_+$,

(ii) $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$, $x, y \in A_+$,

(iii) $\varphi(\lambda x) = \lambda\varphi(x)$, $x \in A_+$, $\lambda \geqslant 0$,

has the form

$$\varphi(x) = \sup\{\omega(x) \mid \omega \in F\},$$

where $F = \{\omega \in A_+' \mid \omega(x) \leqslant \varphi(x), \forall x \in A_+\}$.

Proof. (1) $\Rightarrow$ (2): Put $F = E^\wedge$. We will first show that $F' = \{\omega \in A' \mid \omega(x) \leqslant 1, \forall x \in E - A_+\}$ is equal to F. Obviously $F \subseteq F'$. Let $w \in F'$. Then $\forall x \in A_+$, $\forall \lambda \geqslant 0$: $\omega(-\lambda x) \leqslant 1$. Thus $\forall x \in A_+$: $\omega(x) \geqslant 0$. Therefore $\omega \geqslant 0$. Hence

$$F' \subseteq \{\omega \in A_+' \mid \omega(x) \leqslant 1, \forall x \in E - A_+\} \subseteq F.$$

Now using the bipolar theorem we get

$$(E - A_+)^- = (E - A_+)^{00} = (-F)^0 = \{x \in A \mid \omega(x) \leqslant 1, \forall \omega \in F\}.$$

Thus by (1):

$$E = (E - A_+)^- \cap A_+ = \{x \in A_+ \mid \omega(x) \leqslant 1, \forall \omega \in F\} = E^{\wedge\wedge}.$$

(2) $\Rightarrow$ (3): Let φ be a lower semicontinuous function on A_+ satisfying the conditions of (3), then $E = \{x \in A_+ \mid \varphi(x) \leqslant 1\}$ is a convex, closed, hereditary subset of A_+.

Put $F = E^\wedge = \{\omega \in A_+' \mid \omega(x) \leqslant 1, \forall x \in E\}$. Note that if $\omega \in A_+'$, then

$$\omega \leqslant \varphi \Leftrightarrow \{y \in A_+ \mid \varphi(y) \leqslant 1\} \subseteq \{y \in A_+ \mid \omega(y) \leqslant 1\}.$$

Hence $F = \{\omega \in A_+' \mid \omega \leqslant \varphi\}$.

Put $\psi(x) = \sup_{\omega \in F} \omega(x)$, $x \in A_+$. Obviously $\psi \leqslant \varphi$. Assume now that there exists $x_0 \in A_+$ so that $\psi(x_0) < \varphi(x_0)$. By multiplying x_0 with a suitable scalar we can assume that $\psi(x_0) < 1 < \varphi(x_0)$ ($\varphi(x_0)$ might be $+\infty$). By the definition of E we have $x_0 \notin E$. By (2):

$$\begin{aligned} E &= \{x \in A_+ \mid \omega(x) \leqslant 1, \forall \omega \in F\} \\ &= \{x \in A_+ \mid \psi(x) \leqslant 1\}. \end{aligned}$$

Hence $x_0 \in E$, which gives a contradiction.

(3) $\Rightarrow$ (1): Let E be a convex, closed, hereditary subset of A_+. Put

$$\varphi(x) = \inf\{\lambda > 0 \mid x \in \lambda E\}, \qquad x \in A_+.$$

In particular $\varphi(x) = +\infty$ iff $x \notin \bigcup_{\lambda > 0} (\lambda E)$. It is easy to check that φ satisfies (i), (ii), and (iii). Furthermore,

$$\lambda E = \{x \in A_+ \mid \varphi(x) \leqslant \lambda\}.$$

Hence φ is lower semicontinuous. Thus by (3),

$$\varphi(x) = \sup_{\omega \in F} \omega(x) \qquad \text{where} \quad F = \{\omega \in A_+' \mid \omega \leqslant \varphi\}.$$

Therefore $E - A_+ \subseteq \{x \in A \mid \omega(x) \leqslant 1, \forall \omega \in F\}$. Since the latter is closed we get

$$(E - A_+)^- \cap A_+ \subseteq \{x \in A_+ \mid \omega(x) \leqslant 1, \forall \omega \in F\} \subseteq E.$$

Hence $E = (E - A_+)^- \cap A_+$.

THEOREM 2.2. *The self-adjoint part $M_{\text{s.a.}}$ of a W^*-algebra M equipped with σ-weak topology satisfies the conditions of Proposition* 2.1.

Proof. We will prove (1) in Proposition 2.1. We shall use some properties of the functions

$$f_\alpha\colon]-1/\alpha, \infty[\to]-\infty, 1/\alpha[, \qquad \alpha > 0,$$

defined by $f_\alpha(t) = t(1 + \alpha t)^{-1}$, namely:

(a) $\alpha \leqslant \beta \Rightarrow f_\alpha(t) \geqslant f_\beta(t)$, $t \in]-1/\beta, \infty[$;

(b) $f_\alpha(t) \nearrow t$ when $\alpha \to 0$ and $-1/\alpha < t$;

(c) $f_{\alpha+\beta}(t) = f_\alpha(f_\beta(t))$, $t > -1/(\alpha + \beta)$;

(d) f_α is operator monotone in the sense that

$$-1/\alpha < x \leqslant y \Rightarrow f_\alpha(x) \leqslant f_\alpha(y), \qquad x, y \in M_{\text{s.a.}}\,.$$

For $x \in M_{\text{s.a.}}$ we put $\alpha_x = \sup\{\alpha > 0 \mid -1/\alpha \leqslant x\}$. Let E be a convex σ-weakly closed hereditary subset of M_+. Put

$$G = \{x \in M_{\text{s.a.}} \mid f_\alpha(x) \in E - M_+\,, \forall \alpha \in]0, \alpha_x[\}.$$

We will show that

(1) $G \cap M_r$ is σ-strongly closed,

(2) $G \cap M_r$ is convex.

(1): Let $x \in (G \cap M_r)^{-\sigma-s}$. Then there exists a net $(x_i)_{i\in I}$ so that $x_i \in G$, $\| x_i \| \leqslant r$, and $x_i \to^{\sigma-s} x$. For any $\alpha \in]0, 1/2r[$ we have $f_\alpha(x_i) \in E - M_+$. Hence we can for each $i \in I$ find $y_i \in E$ so that $f_\alpha(x_i) \leqslant y_i$. Now using that f_α is operator monotone we get

$$f_{2\alpha}(x_i) = f_\alpha(f_\alpha(x_i)) \leqslant f_\alpha(y_i).$$

Since $f_{2\alpha}$ is continuous on $[-r, r]$, $\alpha \in]0, 1/2r[$ we get $f_{2\alpha}(x_i) \to^{\sigma-s} f_{2\alpha}(x)$. Since $0 \leqslant f_\alpha(y_i) \leqslant 1/\alpha$ there exists a subnet $f_\alpha(y_{i_k})$ so that $f_\alpha(y_{i_k}) \to y_\alpha'$ σ-weakly. $f_\alpha(y_i) \in E$ because $0 \leqslant f_\alpha(y_i) \leqslant y_i$. Then using that E is σ-weakly closed we find $y_\alpha' \in E$. Furthermore:

$$y_\alpha' - f_{2\alpha}(x) = \lim_{\sigma-w} (f_\alpha(y_{i_k}) - f_{2\alpha}(x_{i_k})) \geqslant 0.$$

Hence $f_{2\alpha}(x) \in E - M_+$, $\forall \alpha \in]0, 1/2r[$ or equivalently $f_\alpha(x) \in E - M_+$, $\forall \alpha \in]0, 1/r[$. Using that $\beta \geqslant \alpha \Rightarrow f_\beta \leqslant f_\alpha$ we find that $f_\beta(x) \in (E - M_+) - M_+ = E - M_+$, $\forall \beta \in [1/r, \alpha_x[$. Hence $x \in G$.

(2): The convexity will follow if we show that

$$G \cap M_r = ((E - M_+) \cap M_t)^{-\sigma-s} \cap M_r \qquad \text{for} \quad t > r.$$

$\subseteq$: Let $x \in G \cap M_r$. Then $f_\alpha(x) \in E - M_+$, $\alpha \in]0, \alpha_x[$. Since $f_\alpha(x) \in M_t$ for sufficiently small α, and since $f_\alpha(x) \nearrow x$ we get $G \cap M_r \subseteq ((E - M_+) \cap M_t)^{-\sigma-s}$.

$\supseteq$: Since $f_\alpha(x) \leqslant x$ for $\alpha \in]0, \alpha_x[$ we have $G \supseteq E - M_+$, and thus $G \cap M_t \supseteq (E - M_+) \cap M_t$. Then by (1) $G \cap M_t \supseteq ((E - M_+) \cap M_t)^{-\sigma-s}$. Hence

$$G \cap M_r = (G \cap M_t) \cap M_r \supseteq ((E - M_+) \cap M_t)^{-\sigma-s} \cap M_r .$$

Since $G \cap M_r$ is convex for any $r > 0$, G is also convex. Then using [2, Chap. 1, Sect. 3, Theorem 1(iv)], G is σ-weakly closed. Note that

$$E - M_+ \subseteq G \subseteq (E - M_+)^{-\sigma-w}.$$

The last inclusion follows from $f_\alpha(x) \nearrow x$ for $\alpha \to 0$. Hence $G = (E - M_+)^{-\sigma-w}$.

Now let $x \in (E - M_+)^{-\sigma-w} \cap M_+ = G \cap M_+$. Then $f_\alpha(x) \in E - M_+$, $\forall \alpha > 0$. Since E is hereditary, $f_\alpha(x) \in E$. Then using $f_\alpha(x) \nearrow x$ we find $x \in E$. Hence $(E - M_+)^{-\sigma-w} \cap E \subseteq E$. The converse inclusion is trivial. This completes the proof.

As an application of Theorem 2.2 we get a new proof of the corresponding result for C^*-algebras (cf. [1, Proposition 1.7, and Remark 1.6]).

COROLLARY 2.3. *The self-adjoint part of a C^*-algebra A, equipped with norm topology, satisfies the conditions in Proposition* 2.1.

Proof. We will prove (1) in Proposition 2.1. We can imbed A in its second dual A^{**}, which is a W^*-algebra (cf. [6]). Let E be a convex, uniformly closed, hereditary subset of A_+, and put $E' = \bar{E}^{\sigma-w}$ (closure in $\sigma(A^{**}, A^*)$-topology). Since E is convex and uniformly closed it is also $\sigma(A, A^*)$-closed. Hence $E = E' \cap A$.

We will now show that E' is a hereditary subset of A_+^{**}. Let $x \in E'$ and $y \in A^{**}$ with $0 \leqslant y \leqslant x$. There exists $s \in A^{**}$, $\| s \| \leqslant 1$, so that $y^{1/2} = sx^{1/2}$. By Kaplansky's density theorem there exists $s_j \in A$, $\| s_j \| \leqslant 1$ so that $s_j \to^{\sigma-s} s$. Furthermore there exists a net $(x_i)_{i \in I}$, $x_i \in E$, so that $x_i \to^{\sigma-s} x$. This implies that $x_i^{1/2} \to^{\sigma-s} x^{1/2}$. Hence $s_j x_i^{1/2} \to^{\sigma-s} sx^{1/2}$ and then

$$(s_j x_i^{1/2})^*(s_j x_i^{1/2}) \xrightarrow{\sigma-w} (sx^{1/2})^*(sx^{1/2}) = y.$$

Since $0 \leqslant (s_j x_i^{1/2})^*(s_j x_i^{1/2}) \leqslant x_i$ we get that $(s_j x_i^{1/2})^*(s_j x_i^{1/2}) \in E$. Hence $y \in E'$. This shows that E' is hereditary.

By Theorem 2.2 $(E' - A_+^{**})^{-\sigma-w} \cap A_+^{**} = E'$. Hence

$$(E - A_+)^- \cap A_+ \subseteq (E' - A_+^{**})^{-\sigma-w} \cap A_+^{**} \cap A = E' \cap A = E.$$

The converse inclusion is trivial.

For the sake of completeness we will show that if M is a W^*-algebra, then the self-adjoint part of the predual M_* satisfies the conditions in Proposition 2.1.

LEMMA 2.4. *Let M be a W^*-algebra and μ a positive normal functional on M. Let (π, H) be the representation induced by μ and let ξ_0 be the range of* 1 *by the quotient map $M \to M/N_\mu \subseteq H$ (cf. Sect.* 1). *Then the linear map Φ: $\pi(M)' \to M_*$ defined by*

$$\Phi(a')(x) = (a'\pi(x)\xi_0 \mid \xi_0), \qquad a' \in \pi(M)', \quad x \in M,$$

has the following properties.

(1) *Φ is an order isomorphism of $\pi(M)'_{\text{s.a.}}$ on $\Phi(\pi(M)'_{\text{s.a.}})$.*

(2) $\Phi(\pi(M)'_{\text{s.a.}}) = \{\varphi \in (M_*)_{\text{s.a.}} \mid \exists k > 0\colon -k\mu \leqslant \varphi \leqslant k\mu\}$.

(3) *Φ is $\sigma(\pi(M)', \pi(M)'_*) - \sigma(M_*, M)$ continuous.*

(4) *Φ is a homeomorphism of $\{x \in \pi(M)'_{\text{s.a.}} \mid -1 \leqslant x \leqslant 1\}$ on $\{\varphi \in (M_*)_{\text{s.a.}} \mid -\mu \leqslant \varphi \leqslant \mu\}$ with respect to the topologies in* (3).

Proof. Note first that ξ_0 is cyclic for $\pi(M)$ and that

$$\mu(x) = (\pi(x)\xi_0 \mid \xi_0), \qquad x \in M.$$

(1): Obviously $a' \geqslant 0$ implies $\Phi(a') \geqslant 0$. Assume that $\Phi(a') \geqslant 0$; then $(a'\pi(x)\xi_0 \mid \pi(x)\xi_0) \geqslant 0$, $\forall x \in M$, which implies that $a' \geqslant 0$. Hence Φ is an order isomorphism of $\pi(M)'_{\text{s.a.}}$ on $\Phi(\pi(M)'_{\text{s.a.}})$. In particular Φ is injective.

(2): Using that Φ preserves order and that $\Phi(1) = \mu$ we find that $\Phi(\pi(M)'_{\text{s.a.}}) \subseteq \{\varphi \in M_* \mid \exists k > 0\colon -k\mu \leqslant \varphi \leqslant k\mu\}$. To show the converse inclusion, let $\varphi \in M_*$ and assume that $-k\mu \leqslant \varphi \leqslant k\mu$ for some $k > 0$. Then

$$|\varphi(x^*x)| \leqslant k(\pi(x)\xi_0 \mid \pi(x)\xi_0), \qquad x \in M.$$

Hence there exists a bounded self-adjoint operator T in $B(H)$ so that $\varphi(y^*x) = (T\pi(x)\xi_0 \mid \pi(y)\xi_0)$ and $\| T \| \leqslant k$. As in the proof of Lemma 1.2 we see that $T \in \pi(M)'$. It is easy to check that $\varphi = \Phi(T)$.

(3): Trivial.

(4): This is a consequence of the fact that a continuous, injective map of a compact set into a Hausdorff space is a homeomorphism on its range.

PROPOSITION 2.5. *Let M be a W^*-algebra. The self-adjoint part of the predual M_* equipped with norm topology satisfies the conditions in Proposition* 2.1.

Proof. We will show (1) in Proposition 2.1. Let E be a convex, hereditary (norm)closed subset of $M_*{}^+$ and let $\varphi \in (E - M_*{}^+)^- \cap M_*{}^+$. There exists a sequence $\{\varphi_n\} \subseteq E - M_*{}^+$, so that $\| \varphi_n - \varphi \| \leqslant 2^{-n}$. For each $n \in \dot{N}$ we can choose a $\psi_n \in E$ so that $\varphi_n \leqslant \psi_n$. Put

$$\mu = \varphi + \sum_{n=1}^{\infty} | \varphi_n - \varphi | + \sum_{n=1}^{\infty} 2^{-n} \psi_n / \| \psi_n \|.$$

Since $\sum_{n=1}^{\infty} \| \varphi_n - \varphi \| < \infty$ and $\sum_{n=1}^{\infty} 2^{-n} < \infty$, μ is a well-defined functional in $M_*{}^+$. Furthermore,

$$-\mu \leqslant \varphi_n \leqslant \mu, \qquad n \in \dot{N};$$
$$0 \leqslant \psi_n \leqslant 2^n \| \psi_n \| \mu, \qquad n \in \dot{N}.$$

Let (π, H) be the representation of M defined by μ and let Φ be the map in Lemma 2.4. Put $E_1 = \Phi^{-1}(E)$. Using Lemma 2.4(1) we see that E_1 is a convex, hereditary subset of $\pi(M)'$. Furthermore, E_1 is σ-weakly closed because Φ is $\sigma(\pi(M)', \pi(M)'_*) - \sigma(M_*, M)$-continuous. ($E$ is a convex, norm-closed subset of M_*, and therefore closed in any topology compatible with the duality of M_* and $(M_*)^* = M$). By Theorem 2.2, $(E_1 - \pi(M)'_+)^{-\sigma-w} \cap \pi(M)'_+ = E_1$. Put $x_n = \Phi^{-1}(\varphi_n)$, $x = \Phi^{-1}(\varphi)$, and $y_n = \Phi^{-1}(\psi_n)$ (cf. Lemma 2.4(2)). Since $x_n \leqslant y_n$ we have $x_n \in E_1 - \pi(M)'_+$. Using Lemma 2.4(4) we find that $x_n \to^{\sigma-w} x$. Hence $x \in (E_1 - \pi(M)'_+)^{-\sigma-w} \cap \pi(M)'_+ = E_1$. Thus $\varphi = \Phi(x) \in E$. This completes the proof.

Remark 2.6. Let M be a W^*-algebra. Since both $M_{s.a.}$ and $(M_*)_{s.a.}$ satisfy Proposition 2.1(2), the map $E \to E^\wedge$ is a bijective map of the σ-weakly closed, convex, hereditary subsets of M_+ onto the norm-closed, convex, hereditary subsets of $M_*{}^+$, and the inverse map is $F \to F^\wedge$.

PROBLEM 2.7. Let A be a C^*-algebra, and let A^* be the dual space equipped with $\sigma(A^*, A)$-topology. Does $A^*_{s.a.}$ satisfy the conditions in Proposition 2.1 ? (It is not difficult to show that the answer is affirmative, if A is commutative.)

ACKNOWLEDGMENT

I thank Gert K. Pedersen for many fruitful conversations and for his lectures on the use of operator monotone functions in C^*-algebra theory.

REFERENCES

1. F. COMBES, Poids sur une C^*-algèbre, *J. Math. Pures Appl.* **47** (1968), 57–100.
2. J. DIXMIER, "Les Algèbres d'Opérateurs dans l'Espace Hilbertien," Gauthier–Villars, Paris, 1969.
3. J. DIXMIER, Formes linéaires sur un anneau d'opérateurs, *Bull. Soc. Math. France* **81** (1953), 9–39.
4. R. V. KADISON, Strong continuouity of operator functions, *Pacific J. Math.* **28** (1968), 121–129.
5. G. K. PEDERSEN AND M. TAKESAKI, The Radon–Nikodym theorem for von Neumann algebras, *Acta Math.* **130** (1973), 53–87.
6. S. SAKAI, C^*-Algebras and W^*-Algebras, Springer–Verlag, Berlin, 1971.

Printed in Belgium