# THE NONCOMMUTATIVE CHOQUET BOUNDARY II: HYPERRIGIDITY

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#### ABSTRACT

A (finite or countably infinite) set G of generators of an abstract  $C^*$ algebra A is called **hyperrigid** if for every faithful representation of Aon a Hilbert space  $A \subseteq \mathcal{B}(H)$  and every sequence of unital completely positive linear maps  $\phi_1, \phi_2, \ldots$  from  $\mathcal{B}(H)$  to itself,

 $\lim_{n \to \infty} \|\phi_n(g) - g\| = 0, \forall g \in G \implies \lim_{n \to \infty} \|\phi_n(a) - a\| = 0, \forall a \in A.$ 

We show that one can determine whether a given set G of generators is hyperrigid by examining the noncommutative Choquet boundary of the operator space spanned by  $G \cup G^*$ . We present a variety of concrete applications and discuss prospects for further development.

# 1. Introduction

In a previous paper [Arv08] it was shown that every separable operator system has sufficiently many boundary representations, thereby providing a noncommutative counterpart of the function-theoretic fact that the closure of the Choquet boundary is the Silov boundary. Considering the central position of the latter in both potential theory and approximation theory, it is natural to expect corresponding applications of the noncommutative Choquet boundary to the theory of operator spaces. In this paper we initiate a study of what might be called noncommutative approximation theory, focusing on the question: How does one determine whether a set of generators of a  $C^*$ -algebra is hyperrigid?

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Definition 1.1: A finite or countably infinite set G of generators of a  $C^*$ -algebra A is said to be **hyperrigid** if for every faithful representation  $A \subseteq \mathcal{B}(H)$  of A on a Hilbert space and every sequence of unit-preserving completely positive (UCP) maps  $\phi_n : \mathcal{B}(H) \to \mathcal{B}(H), n = 1, 2, ...,$ 

(1.1) 
$$\lim_{n \to \infty} \|\phi_n(g) - g\| = 0, \ \forall g \in G \implies \lim_{n \to \infty} \|\phi_n(a) - a\| = 0, \ \forall a \in A$$

We have lightened notation in this definition by identifying A with its image  $\pi(A)$  in a faithful nondegenerate representation  $\pi : A \to \mathcal{B}(H)$  on a Hilbert space H. Significantly, hyperrigidity of a set G of operators on a Hilbert space H implies not only that (1.1) should hold for sequences of UCP maps  $\phi_n$  defined on  $\mathcal{B}(H)$ , but also that the property should persist for every other faithful representation of A. Note too that a set G is hyperrigid iff the linear span of  $G \cup G^*$  is hyperrigid, so that hyperrigidity is properly thought of as a property of self-adjoint operator subspaces of a  $C^*$ -algebra. In principle, one could adjoin the identity to  $G \cup G^*$  as well, but for many examples—especially those involving sets of compact operators—it is best not to adjoin the identity operator to G. Hence we allow that a set G of operators, its operator space and its generated  $C^*$ -algebra may not contain a unit.

The general characterization of hyperrigid generators given in Theorem 2.1 provides the following criterion: A separable operator system S that generates a  $C^*$ -algebra A is hyperrigid iff every representation  $\pi : A \to \mathcal{B}(H)$  on a separable Hilbert space H has the unique extension property in the sense that the only unital completely positive (UCP) map  $\phi : A \to \mathcal{B}(H)$  that satisfies  $\phi \upharpoonright_S = \pi \upharpoonright_S$  is  $\phi = \pi$  itself.

The simplest examples of hyperrigid generators G are obtained by a direct application of this criterion. These examples are associated with "extremal" properties of the operators in G which force the unique extension property (and therefore hyperrigidity) through a direct application of the Schwarz inequality and the Stinespring representation of UCP maps. The following two results illustrate the point: They are proved in Section 3.

THEOREM 1.2: Let  $X \in \mathcal{B}(H)$  be a self-adjoint operator and let  $\mathcal{A}$  be the  $C^*$ -algebra generated by X. Then  $G = \{X, X^2\}$  is a hyperrigid generator for  $\mathcal{A}$ .

THEOREM 1.3: Let  $V_1, \ldots, V_n \in \mathcal{B}(H)$  be an arbitrary set of isometries that generates a  $C^*$ -algebra  $\mathcal{A}$ . Then  $G = \{V_1, \ldots, V_n, V_1V_1^* + \cdots + V_nV_n^*\}$  is a hyperrigid generator for  $\mathcal{A}$ .

Remark 1.4: Theorem 1.2 can be viewed as a noncommutative strengthening of a classic approximation-theoretic result of Korovkin: see Remark 1.8 for further discussion. The referee has pointed out that Theorem 1.2 can be formulated in terms of the multiplicative domains of certain UCP maps, and after that reformulation, Lemma 3.1 of [JOR03] gives a norm estimate that leads to an alternate proof of Theorem 1.2 when applied to the function system  $\{1, x, x^2\} \subseteq$ C[a, b], where a and b are appropriate bounds on the spectrum of the operator X.

Finally, note that Theorem 1.3 implies that for every  $n \ge 2$ , the standard set of generators  $G = \{V_1, \ldots, V_n\}$  of the Cuntz  $C^*$ -algebra  $\mathcal{O}_n$  is hyperrigid. The referee has also pointed out a related result of Neshveyev and Størmer (Theorem 6.2.6 of [NS06]), concerning generating sets of unitary operators.

On the other hand, we emphasize that most hyperrigid operator systems  $S \subseteq C^*(S)$  do not share the conspicuous extremal properties associated with Theorems 1.2 and 1.3, and one cannot establish hyperrigidity of the more subtle examples by such direct methods. The purpose of this paper is to identify the obstruction to hyperrigidity *in general* in terms of the noncommutative Choquet boundary. We conjecture that this is the only obstruction in Section 4. While we are unable to establish the conjecture in general, we do prove it when  $C^*(S)$  has countable spectrum, and that leads to a variety of hyperrigidity results with distinctly new features. We now describe two more subtle examples which are concrete special cases of more general results that are proved in Sections 5 through 9.

POSITIVE LINEAR MAPS OF MATRIX ALGEBRAS. Building on work of Chandler Davis [Dav57], Choi showed in [Cho74] that for a unit-preserving positive linear map  $\phi$  of unital  $C^*$ -algebras, the inequality

(1.2) 
$$f(\phi(A)) \le \phi(f(A))$$

holds for every function  $f : (a, b) \to \mathbb{R}$  that is **operator convex** in the sense of Bendat–Sherman [BS55] and every self-adjoint operator A with spectrum in (a, b). Note that the spectrum of  $\phi(A)$  is also contained in (a, b), so that one

can form both f(A) and  $f(\phi(A))$  by way of the functional calculus. In [Pet86], Petz asked when *equality* can hold in (1.2), and showed that if  $f:(a,b) \to \mathbb{R}$  is an operator convex function that is *not* of the form f(x) = ax + b and equality holds in (1.2), then the restriction of  $\phi$  to the algebra of polynomials in A is multiplicative.

We want to broaden Petz' question in the following way. Fix a real-valued continuous function  $f : [a, b] \to \mathbb{R}$  defined on a compact interval. We say that f is **rigid** if for every every self-adjoint operator A in a unital  $C^*$ -algebra  $\mathcal{A}$  whose spectrum is contained in [a, b] and every unit preserving positive linear map  $\phi : \mathcal{A} \to \mathcal{B}$  into another unital  $C^*$ -algebra, one has

$$f(\phi(A)) = \phi(f(A)) \implies \phi(A^n) = \phi(A)^n, \quad n = 1, 2, \dots$$

The following result—a consequence of Theorem 9.4 below—characterizes the rigid functions with respect to maps on matrix algebras, that is, the  $C^*$ -algebras  $\mathcal{A} = \mathcal{B}(H)$  with finite-dimensional H:

THEOREM 1.5: For every real-valued function  $f \in C[a, b]$ , the following are equivalent:

(i) For every unital positive linear map of matrix algebras  $\phi : \mathcal{M} \to \mathcal{N}$  and every self-adjoint operator  $A \in \mathcal{M}$  having spectrum in [a, b],

$$\phi(f(A)) = f(\phi(A)) \implies \phi(A^n) = \phi(A)^n, \quad \forall n = 1, 2, \dots$$

(ii) f is either strictly convex or strictly concave.

Recall that a real function  $f \in C[a, b]$  is said to be **strictly convex** if for any two distinct points  $x \neq y$  in [a, b] and every  $t \in (0, 1)$ ,

$$f(t \cdot x + (1-t) \cdot y) < t \cdot f(x) + (1-t) \cdot f(y).$$

f is said to be strictly concave when -f is strictly convex.

Remark 1.6 (Relation to Petz' theorem): It follows from the characterization of [BS55] that an operator convex function f that is not an affine function must be real-analytic with f'' > 0 throughout (a, b). Since such functions are strictly convex, Theorem 1.5 implies Petz' result for maps on matrix algebras. Since most continuous strictly convex functions are not operator convex, this is a significant extension of the result of [Pet86].

It is natural to ask if Theorem 1.5 holds for unital positive linear maps of more general unital  $C^*$ -algebras; indeed, we will show in Section 9 that the

implication (i)  $\implies$  (ii) holds in that generality. While we conjecture that the opposite implication (ii)  $\implies$  (i) holds as well, that has not been proved (see Section 9 for further discussion).

HYPERRIGID GENERATORS OF  $\mathcal{K}$ . We conclude with a fourth hyperrigidity result about a familiar—perhaps the *most* familiar—compact operator.

THEOREM 1.7: Consider the Volterra integration operator V acting on the Hilbert space  $H = L^2[0, 1]$ ,

$$Vf(x) = \int_0^x f(t) dt, \quad f \in L^2[0, 1].$$

It is well-known that V is irreducible, generating the  $C^*$ -algebra  $\mathcal{K}$  of all compact operators. This operator has the following additional properties:

(i)  $G = \{V, V^2\}$  is hyperrigid; and in particular, for every sequence of unital completely positive maps  $\phi_n : \mathcal{B}(H) \to \mathcal{B}(H)$  for which

$$\lim_{n \to \infty} \|\phi_n(V) - V\| = \lim_{n \to \infty} \|\phi_n(V^2) - V^2\| = 0,$$

one has

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$$\lim_{n \to \infty} \|\phi_n(K) - K\| = 0$$

for every compact operator  $K \in \mathcal{B}(H)$ .

(ii) The smaller generating set  $G_0 = \{V\}$  of  $\mathcal{K}$  is not hyperrigid.

While the hyperrigidity property (i) of  $\{V, V^2\}$  formally resembles the hyperrigidity property of  $\{X, X^2\}$  in Theorem 1.2, the two settings are fundamentally different because V is not a self-adjoint operator. Indeed, while Theorem 1.2 is more or less a direct consequence of the Schwarz inequality and Stinespring's theorem, the proof of Theorem 1.7 will make essential use of the noncommutative Choquet boundary (see Corollary 8.3, a consequence of the more general Theorem 8.1).

The paper is organized into three parts. Part 1 is relatively short and contains the basic characterization hyperrigid operator systems. Using that characterization, we discuss two of the simplest examples of hyperrigid generators and prove Theorems 1.2 and 1.3.

In order to deal with the more subtle aspects of hyperrigid generators it is necessary to bring in the noncommutative Choquet boundary, and Part 2 is devoted to those issues. We show how boundary representations are involved in the obstruction to hyperrigidity in Corollary 4.2, and following that, we

conjecture that this is the *only* obstruction in general. We are unable to prove the conjecture in general, but we do prove it for generators of  $C^*$ -algebras that have countable spectra (Theorem 5.1). When the generated  $C^*$ -algebra is not unital, there is an additional obstruction associated with the "point at infinity" and we identify that obstruction in concrete operator-theoretic terms in Theorem 6.1. In Section 7 we introduce the noncommutative counterparts of peak points and show how one uses them to identify boundary representations for examples involving compact operators in Theorem 7.2.

When a set G of operators generates a commutative  $C^*$ -algebra  $\cong C(X)$ , it is possible to formulate a "localized" version of Conjecture 4.3. Part 3is devoted to a discussion of this kind of localization, and in Theorem 11.1 we prove an appropriate local version of Conjecture 4.3.

We work extensively with representations  $\pi : A \to \mathcal{B}(H)$  of  $C^*$ -algebras A on Hilbert spaces H throughout this paper, and we require that all representations should be nondegenerate. Thus, H should be the closed linear span of the set of vectors  $\{\pi(a)\xi : a \in A, \xi \in H\}$ ; and if A has a unit **1** then this entails  $\pi(\mathbf{1}) = \mathbf{1}_H$ .

Finally, a word about notation. When dealing with abstract  $C^*$ -algebras A, it is customary to refer to elements of A with lowercase letters  $a \in A$ , while when dealing with  $C^*$ -algebras of operators  $\mathcal{A} \subseteq \mathcal{B}(H)$  it seems more appropriate to refer to operators with uppercase letters  $A \in \mathcal{A}$ , as we have already done in the introduction. Of course, the two usages are inconsistent. But it seems punctilious to insist on referring to an operator on a Hilbert space H with  $a \in \mathcal{B}(H)$ , and we revert at times (in Sections 8 and 9) to more traditional operator-theoretic notation. Hopefully, this will not cause problems for the reader.

Remark 1.8 (Quantizing Korovkin's theorem): When specialized appropriately, Theorem 1.2 provides a noncommutative strengthening of a classical theorem of approximation theory. To review that briefly, a seminal theorem of P. P. Korovkin [Kor53], [Kor60] makes the following assertion: If a sequence of positive linear maps  $\phi_1, \phi_2, \ldots : C[0, 1] \to C[0, 1]$  has the property

$$\lim_{n \to \infty} \|\phi_n(f_k) - f_k\| = 0, \quad k = 0, 1, 2,$$

for the three functions  $f_0(x) = 1$ ,  $f_1(x) = x$ ,  $f_2(x) = x^2$ , then

 $\lim_{n} \|\phi_n(g) - g\| = 0, \quad \forall \ g \in C[0, 1].$ 

Korovkin's theorem generated considerable activity among researchers in approximation theory, and far-reaching generalizations were discovered during the 1960s, following the realization that the fundamental principle underlying it is that every point of the unit interval is a peak point for the 3-dimensional function system  $[1, x, x^2] \subseteq C[0, 1]$ . The generalizations make essential use of the Choquet boundary, and in one way or another, those we have seen use the fact that the real functions in C(X) form a *lattice*. We will not summarize those developments here, but refer the reader to [Bau61], [BD78], [Phe66], [Phe01], [Šaš67] and the survey [Don82].

Theorem 1.2 strengthens Korovkin's theorem in a nontrivial way. To see that in concrete terms, consider the multiplication operator X on  $L^2[0, 1]$ ,

$$(X\xi)(t) = t \cdot \xi(t), \quad t \in [0,1], \quad \xi \in L^2[0,1].$$

For every sequence of UCP maps  $\phi_1, \phi_2, \ldots : \mathcal{B}(L^2[0,1]) \to \mathcal{B}(L^2[0,1])$  that satisfies

(1.3) 
$$\lim_{n \to \infty} \|\phi_n(X) - X\| = \lim_{n \to \infty} \|\phi_n(X^2) - X^2\| = 0,$$

Theorem 1.2 implies that  $\phi_n(Y)$  converges in norm to Y for every multiplication operator  $Y = M_f$  with  $f \in C[0, 1]$ . Of course, if each of the given maps  $\phi_n$ leaves the commutative  $C^*$ -algebra  $\mathcal{A} = \{M_f : f \in C[0, 1]\}$  invariant, then this would follow from Korovkin's theorem. However, we do not assume that; indeed, the spaces  $\phi_n(\mathcal{A})$  need not commute with X or with each other. If one attempts to use the methods of classical approximation theory to prove this operator-theoretic result, one finds that the argument breaks down precisely because a pair of self-adjoint operators A, B acting on a Hilbert space need not have a least upper bound or greatest lower bound, even when AB = BA.

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### Part 1. Basic results

### 2. Characterization of hyperrigidity

We now prove the basic characterization of hyperrigid operator systems.

THEOREM 2.1: For every separable operator system S that generates a  $C^*$ -algebra A, the following are equivalent:

- (i) S is hyperrigid.
- (ii) For every nondegenerate representation  $\pi : A \to \mathcal{B}(H)$  on a separable Hilbert space and every sequence  $\phi_n : A \to \mathcal{B}(H)$  of UCP maps,

 $\lim_{n \to \infty} \|\phi_n(s) - \pi(s)\| = 0 \ \forall s \in S \implies \lim_{n \to \infty} \|\phi_n(a) - \pi(a)\| = 0 \ \forall a \in A.$ 

- (iii) For every nondegenerate representation  $\pi : A \to \mathcal{B}(H)$  on a separable Hilbert space,  $\pi \upharpoonright_S$  has the unique extension property.
- (iv) For every unital  $C^*$ -algebra B, every unital homomorphism of  $C^*$ algebras  $\theta : A \to B$  and every UCP map  $\phi : B \to B$ ,

$$\phi(x) = x \; \forall x \in \theta(S) \implies \phi(x) = x \; \forall x \in \theta(A).$$

*Proof.* Since the implication (ii)  $\implies$  (iii) is trivial, we prove (i)  $\implies$  (ii) and (iii)  $\implies$  (iv)  $\implies$  (i).

(i)  $\implies$  (ii): Let  $\pi : A \to \mathcal{B}(H)$  be a nondegenerate representation on a separable Hilbert space and let  $\phi_n : A \to \mathcal{B}(H)$  be a sequence of UCP maps such that  $\|\phi_n(s) - \pi(s)\| \to 0$  for all  $s \in S$ .

Let  $\sigma : A \to \mathcal{B}(K)$  be a faithful representation of A on another separable space K. Then  $\sigma \oplus \pi : A \to \mathcal{B}(K \oplus H)$  is a faithful representation, so that each of the linear maps  $\omega_n : (\sigma \oplus \pi)(A) \to \mathcal{B}(K \oplus H)$ 

$$\omega_n: \sigma(a) \oplus \pi(a) \mapsto \sigma(a) \oplus \phi_n(a), \quad a \in A,$$

is unit preserving and completely positive. By the extension theorem of [Arv69],  $\omega_n$  can be extended to a UCP map  $\tilde{\omega}_n : \mathcal{B}(K \oplus H) \to \mathcal{B}(K \oplus H)$ . Since  $\phi_n \upharpoonright_S$ converges to  $\pi \upharpoonright_S$  point-norm,  $\tilde{\omega}_n$  converges point-norm to the identity map on  $(\sigma \oplus \pi)(S)$ . So by hypothesis (i),  $\tilde{\omega}_n$  must converge point-norm to the identity map on  $(\sigma \oplus \pi)(A)$ . We conclude that for every  $a \in A$ ,

$$\begin{split} \limsup_{n \to \infty} \|\phi_n(a) - \pi(a)\| &\leq \limsup_{n \to \infty} \|\sigma(a) \oplus \phi_n(a) - \sigma(a) \oplus \pi(a)\| \\ &= \lim_{n \to \infty} \|\tilde{\omega}_n(\sigma(a) \oplus \pi(a)) - \sigma(a) \oplus \pi(a)\| = 0, \end{split}$$

hence  $\phi_n$  converges point-norm to  $\pi$  on A.

(iii)  $\implies$  (iv): Let  $\theta : A \to B$  be a unit preserving \*-homomorphism of  $C^*$ algebras, and let  $\phi : B \to B$  be a UCP map that satisfies  $\phi(\theta(s)) = \theta(s), s \in S$ . Vol. 184, 2011

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We have to show that

(2.1) 
$$\phi(\theta(a)) = \theta(a), \quad a \in A.$$

For that, let  $B_0 = \theta(A)$  and, assuming that separable  $C^*$ -algebras  $B_k$  have been defined for  $0 \le k \le n$  that satisfy  $B_k \cup \phi(B_k) \subseteq B_{k+1}$  for k < n, let  $B_{n+1}$ be be the  $C^*$ -subalgebra of B generated by

$$B_n \cup \phi(B_n) \cup \phi^2(B_n) \cup \cdots$$
.

It is clear from its construction that the norm-closure  $B_{\infty}$  of  $B_0 \cup B_1 \cup B_2 \cup \cdots$ is a separable  $C^*$ -subalgebra of B satisfying  $\phi(B_{\infty}) \subseteq B_{\infty}$ , and hence it has a faithful representation on some separable Hilbert space H. After making the obvious identification we may assume that  $B_{\infty} \subseteq \mathcal{B}(H)$ .

By the extension theorem of [Arv69], there is a UCP map  $\tilde{\phi}: \mathcal{B}(H) \to \mathcal{B}(H)$ that restricts to  $\phi$  on  $B_{\infty}$ , and in particular  $\tilde{\phi}(\theta(s)) = \theta(s)$  for  $s \in S$ . Since  $a \in A \mapsto \theta(a) \in \mathcal{B}(H)$  is a representation on a separable Hilbert space, hypothesis (iii) implies that  $\tilde{\phi}$  must fix  $\theta(A)$  elementwise. We conclude that  $\phi(\theta(a)) = \tilde{\phi}(\theta(a)) = \theta(a), a \in A$ , and (2.1) is proved.

(iv)  $\implies$  (i): Suppose that  $A \subseteq \mathcal{B}(H)$  is faithfully represented on some Hilbert space H, and  $\phi_1, \phi_2, \ldots : \mathcal{B}(H) \to \mathcal{B}(H)$  is a sequence of UCP maps satisfying  $\lim_n \|\phi_n(s) - s\| = 0$  for all  $s \in S$ . We have to prove

(2.2) 
$$\lim_{n \to \infty} \|\phi_n(a) - a\| = 0, \quad \forall \ a \in A.$$

To that end, write  $B = \mathcal{B}(H)$ , let  $\ell^{\infty}(B)$  be the  $C^*$ -algebra of all bounded sequences with components in B and let  $c_0(B)$  be the ideal of all sequences in  $\ell^{\infty}(B)$  that converge to zero in norm.

Consider the UCP map  $\phi_0: \ell^\infty(B) \to \ell^\infty(B)$  defined by

$$\phi_0(b_1, b_2, b_3, \dots) = (\phi_1(b_1), \phi_2(b_2), \phi_3(b_3), \dots).$$

This map carries the ideal  $c_0(B)$  into itself, hence it promotes to a UCP map of the quotient  $\phi: \ell^{\infty}(B)/c_0(B) \to \ell^{\infty}(B)/c_0(B)$  by way of

$$\phi(x + c_0(B)) = \phi_0(x) + c_0(B), \quad x \in \ell^{\infty}(B).$$

Now consider the natural embedding  $\theta: A \to \ell^{\infty}(B)/c_0(B)$ ,

$$\theta(a) = (a, a, a, \dots) + c_0(B).$$

By hypothesis,  $\|\phi_n(s) - s\| \to 0$  as  $n \to \infty$  for  $s \in S$ , and therefore

$$\phi(\theta(s)) = (\phi_1(s), \phi_2(s), \dots) + c_0(B) = (s, s, \dots) + c_0(B) = \theta(s).$$

Hence  $\phi$  restricts to the identity map on  $\theta(S)$ .

Applying hypothesis (iv) to the inclusions

 $\theta(S) \subseteq \theta(A) \subseteq \ell^{\infty}(B)/c_0(B)$ 

and the UCP map  $\phi : \ell^{\infty}(B)/c_0(B) \to \ell^{\infty}(B)/c_0(B)$ , we conclude that  $\phi$  must fix every element of  $\theta(A)$ . Since  $\theta(a) = (a, a, ...) + c_0(B)$  and

$$\phi(\theta(a)) = (\phi_1(a), \phi_2(a), \dots) + c_0(B),$$

we must have  $(\phi_1(a) - a, \phi_2(a) - a, \dots) \in c_0(B)$ , and (2.2) follows.

It is significant that hyperrigidity is preserved under passage to quotients:

COROLLARY 2.2: Let S be a hyperrigid separable operator system with generated C<sup>\*</sup>-algebra A, let K be an ideal in A and let  $a \in A \mapsto \dot{a} \in A/K$  be the quotient map. Then  $\dot{S}$  is a hyperrigid operator system in A/K.

*Proof.* An immediate consequence of property (ii) of Theorem 2.1.

### 3. Applications I: Two basic examples

THEOREM 3.1: Let  $x \in \mathcal{B}(H)$  be a self-adjoint operator with at least 3 points in its spectrum and let A be the C<sup>\*</sup>-algebra generated by x and 1. Then

- (i)  $G = \{\mathbf{1}, x, x^2\}$  is a hyperrigid generator for A, while
- (ii)  $G_0 = \{\mathbf{1}, x\}$  is not a hyperrigid generator for A.

Proof. (i): By Theorem 2.1, it suffices to show that every nondegenerate representation  $\pi : C^*(x) \to \mathcal{B}(K)$  has the unique extension property. To prove that, let  $\phi : A \to \mathcal{B}(K)$  be a UCP map that satisfies  $\phi(x) = \pi(x)$  and  $\phi(x^2) = \pi(x^2)$ . We have to show that  $\phi$  is multiplicative on A.

For that, Stinespring's theorem implies that there is a Hilbert space L containing K and a representation  $\sigma : A \to \mathcal{B}(L)$  such that  $\phi(a) = P\sigma(a) \upharpoonright_K$ ,  $a \in A$ , where  $P \in \mathcal{B}(L)$  is the projection onto K. We have

$$P\sigma(x)(1-P)\sigma(x)P = P\sigma(x^{2})P - P\sigma(x)P\sigma(x)P = \phi(x^{2})P - \phi(x)^{2}P$$
  
=  $\pi(x^{2})P - \pi(x)^{2}P = 0.$ 

This implies that  $|(\mathbf{1} - P(\sigma(x)P)|^2 = 0)$ , hence  $(\mathbf{1} - P)\sigma(x)P = 0$ , i.e.,  $\sigma(x)$  leaves H invariant. Since A is the norm-closed algebra generated by  $\mathbf{1}$  and x,

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it follows that  $\sigma(A)$  leaves H invariant, and consequently  $\phi(a) = P\sigma(a) \upharpoonright_K$  is a multiplicative linear map.

(ii): Choose points  $\lambda_1 < \lambda_2 < \lambda_3$  in the spectrum  $\Sigma$  of x. Then  $\lambda_2$  is a convex combination of  $\lambda_1$  and  $\lambda_3$ . For k = 1, 2, 3, let  $\rho_k$  be the state of A defined by

$$\rho_k(f(x)) = f(\lambda_k), \quad f \in C(\Sigma).$$

Each  $\rho_k$  is an irreducible representation of A, and by the preceding remark, the restriction of  $\rho_2$  to the function system  $S = \text{span}\{\mathbf{1}, x\}$  is a convex combination of  $\rho_1 \upharpoonright_S$  and  $\rho_2 \upharpoonright_S$ . Since  $\rho_1 \neq \rho_3$ ,  $\rho_2 \upharpoonright_S$  fails to have the unique extension property, and Theorem 2.1 implies that S is not hyperrigid.

Note that the hypothesis on the cardinality of the spectrum of x was not used in the proof of item (i) of Theorem 3.1.

Remark 3.2 (Other hyperrigid generators): Let I = [a, b] be a compact real interval and let  $f : I \to \mathbb{R}$  be a continuous function and let  $A \in \mathcal{B}(H)$  be a self-adjoint operator with spectrum in [a, b]. One can ask: Is  $\{\mathbf{1}, A, f(A)\}$ a hyperrigid generator of  $C^*(A)$ ? Theorem 3.1 answers affirmatively for the particular function  $f(t) = t^2$ ; but the proof of Theorem 3.1 is tailored to this particular function. In general, there is a stringent constraint: If the answer to the above question is yes, then f must be either strictly convex or strictly concave. This is a consequence of results of Section 9 (see Proposition 9.3).

Conversely, if f is strictly convex or strictly concave, then for every selfadjoint operator A with *discrete* spectrum in I,  $\{\mathbf{1}, A, f(A)\}$  is a hyperrigid generator. This can be established by making use of Proposition 4.4 at the appropriate place in the proof of Theorem 9.4 below. We believe that the same is true without the discrete spectrum hypothesis, but that depends on the validity of the commutative case of Conjecture 4.3 (see Remark 9.5).

We now discuss a class of highly noncommutative examples. Let  $u_1, \ldots, u_n$  be an arbitrary set of isometries that act on some Hilbert space. The "defect operator"  $D = u_1 u_1^* + \cdots + u_n u_n^*$  is positive and its norm satisfies  $1 \leq ||D|| \leq n$ , with many possibilities for D depending on how the  $u_k$  are chosen. In this section we exhibit a hyperrigid generator for the  $C^*$ -algebra generated by  $u_1, \ldots, u_n$ , assuming nothing about the structure of the defect operator or relations that may exist between the various  $u_k$ .

THEOREM 3.3: Let  $u_1, \ldots, u_n$  be a set of isometries that generate a  $C^*$ -algebra A and let

(3.1) 
$$G = \{u_1, \dots, u_n, u_1 u_1^* + \dots + u_n u_n^*\}.$$

Then G is a hyperrigid generator for A.

*Proof.* Let S be the operator system spanned by  $G \cup G^*$  and the identity. By item (iii) of Theorem 2.1, it suffices to show that for every nondegenerate representation  $\pi$  of A,  $\pi \upharpoonright_S$  has the unique extension property.

To prove that, fix a representation  $\pi : A \to \mathcal{B}(H)$  and let  $v_1, \ldots, v_n$  be the isometries  $v_k = \pi(u_k)$ ,  $k = 1, \ldots, n$ . Let  $\phi : A \to \mathcal{B}(H)$  be a UCP map satisfying  $\phi(u_k) = v_k$ ,  $1 \le k \le n$ , and  $\phi(u_1u_1^* + \cdots + u_nu_n^*) = v_1v_1^* + \cdots + v_nv_n^*$ . We have to show that  $\phi = \pi$ .

For that, we use Stinespring's theorem to express  $\phi$  in the form

$$\phi(x) = V^* \sigma(x) V, \quad x \in A,$$

where  $\sigma$  is a representation of A on a Hilbert space  $K, V : H \to K$  is an isometry, and which is minimal in the sense that  $\sigma(A)VH$  spans K.

We claim first that  $\sigma(u_k)V = Vv_k$ ,  $1 \le k \le n$ . Indeed, for k = 1, ..., n we have

$$V^*\sigma(u_k)^*VV^*\sigma(u_k)V = \phi(u_k)^*\phi(u_k) = u_k^*u_k = \mathbf{1}_H,$$

hence  $V^* \sigma(u_k) (\mathbf{1} - VV^*) \sigma(u_k) V = 0$ , so that  $\sigma(u_k)$  leaves VH invariant. The claim follows because  $\sigma(u_k) V = VV^* \sigma(u_k) V = V \phi(u_k) = V v_k$ .

Note next that since  $\sum_k v_k v_k^* = \pi(\sum_k u_k u_k^*) = \phi(\sum_k u_k u_k^*)$ , we have

$$\sum_{k=1}^{n} \sigma(u_{k}) V V^{*} \sigma(u_{k})^{*} = \sum_{k=1}^{n} V v_{k} v_{k}^{*} V^{*} = V \phi(\sum_{k=1}^{n} u_{k} u_{k}^{*}) V$$
$$= V V^{*} \sum_{k=1}^{n} \sigma(u_{k} u_{k}^{*}) V V^{*}$$
$$= \sum_{k=1}^{n} V V^{*} \sigma(u_{k}) \sigma(u_{k}^{*}) V V^{*}$$

and since  $\sigma(u_k)V = VV^*\sigma(u_k)V$  for all k, subtracting the left side from the right leads to

$$\sum_{k=1}^{n} VV^* \sigma(u_k) (\mathbf{1}_K - VV^*) \sigma(u_k)^* VV^* = 0,$$

and hence  $(\mathbf{1}_K - VV^*)\sigma(u_k)^*VV^* = 0$  for all k. We conclude that VH is invariant under both  $\sigma(u_k)$  and  $\sigma(u_k)^*$  for all k; and since A is generated by the  $u_k$  it follows that  $\sigma(A)VH \subseteq VH$ . By minimality, we must have VH = K, which implies that V is unitary and therefore  $\phi(x) = V^{-1}\sigma(x)V$  is a representation. Since  $\phi$  agrees with  $\pi$  on a generating set, the desired conclusion  $\phi = \pi$  follows.

Since the Cuntz algebras  $\mathcal{O}_n$  are generated by sets of isometries  $u_1, \ldots, u_n$ satisfying the single condition  $u_1u_1^* + \cdots + u_nu_n^* = \mathbf{1}$ , we can discard the identity operator from the generating set G of (3.1) to conclude:

COROLLARY 3.4: The set  $G = \{u_1, \ldots, u_n\}$  of generators of the Cuntz algebra  $\mathcal{O}_n$  is hyperrigid.

### Part 2. Role of the noncommutative Choquet boundary

### 4. Obstruction to hyperrigidity

An **operator system** is a self-adjoint linear subspace of a unital  $C^*$ -algebra A that contains the unit of A, and the  $C^*$ -subalgebra of A generated by S is denoted  $C^*(S)$ . Given a unital completely positive (UCP) map  $\phi$  from an operator system S to a unital  $C^*$ -algebra B, we say that  $\phi$  has the **unique extension** property if it has a unique UCP extension  $\tilde{\phi} : C^*(S) \to B$ , and moreover this extension is multiplicative  $\tilde{\phi}(xy) = \tilde{\phi}(x)\tilde{\phi}(y), x, y \in C^*(S)$ . By a **boundary** representation for S we mean an *irreducible* representation  $\pi : C^*(S) \to \mathcal{B}(H)$  such that  $\pi \upharpoonright_S$  has the unique extension property. There is a more intrinsic characterization of the unique extension property that we do not require here (see Proposition 2.4 of [Arv08]). Much of the discussion to follow rests on a result of [Arv08], which we repeat here for reference:

THEOREM 4.1: Every separable operator system  $S \subseteq C^*(S)$  has sufficiently many boundary representations in the sense that for every  $n \ge 1$  and every  $n \times n$  matrix  $(s_{ij})$  with components  $s_{ij} \in S$ , one has

$$||(s_{ij})|| = \sup_{\pi} ||(\pi(s_{ij}))||,$$

the supremum on the right taken over all boundary representations  $\pi$  for S.

Let X be a compact metrizable space and let  $S \subseteq C(X)$  be a function system, namely a linear subspace of C(X) that is closed under complex conjugation and

contains the constant functions. There is no essential loss if one assumes that S separates points of X. Let p be a point of X; by a **representing measure** for p one means a (Borel) probability measure  $\mu$  on X satisfying

$$\int_X f(x) \, d\mu(x) = f(p), \quad f \in S.$$

The set  $K_p$  of all representing measures for p is a weak\*-compact convex subset of the dual of C(X), and it contains the point mass  $\delta_p$  concentrated at p. If  $K_p = \{\delta_p\}$ , then p is said to belong to the Choquet boundary of X (relative to S), sometimes written  $\partial_S(X)$ . It is not obvious that the Choquet boundary is nonempty; but it is always nonempty when S separates points, and in fact its closure is the Silov boundary—the smallest closed set  $K \subseteq X$  with the property that every function in S achieves its maximum value on K (see Proposition 6.4 of [Phe01]). The following comments show that Theorem 4.1 generalizes this fact to noncommutative operator systems.

For every operator system  $S \subseteq A = C^*(S)$ , there is a largest (closed twosided) ideal  $K \subseteq A$  such that the quotient map  $a \in A \mapsto \dot{a} \in A/K$  is completely isometric on S. The quotient  $C^*$ -algebra A/K is called the  $C^*$ -envelope of S. The  $C^*$ -envelope of S depends only on the internal structure of S and not on the embedding of S in its generated  $C^*$ -algebra. This ideal was introduced and shown to exist for a variety of examples in [Arv69], where it was called the **Silov boundary ideal** since it is the noncommutative counterpart of the Silov boundary of a function system. The existence of the Silov boundary ideal in general was left open, and the issue was later settled affirmatively by Hamana [Ham79a], [Ham79b] as a consequence of his work on injective envelopes. More recently, Dritschel and McCullough [DM05] gave a second proof of the existence of this ideal in general that is independent of the theory of injective envelopes. During the past decade or so, the terminology for the ideal K has been contracted to **Silov ideal** for S. On the other hand, in the noncommutative context it seems more appropriate to refer to K simply as the **boundary ideal** for S, as we shall do throughout this paper.

It was shown in Theorem 2.2.3 of [Arv69] that for every operator system that has sufficiently many boundary representations (in the sense of Theorem 4.1), the boundary ideal is the intersection of the kernels of all boundary representations. Note that the existence of sufficiently many boundary representations in general was left open in [Arv69] and [Arv72], and was not addressed in Hamana's

work on injectivity. Since Theorem 4.1 establishes that property for separable operator systems, it provides a third proof of the existence of the boundary ideal in such cases. Of course, this is the noncommutative counterpart of the fact that the closure of the Choquet boundary of a function system is the Silov boundary.

We deduce the following necessary conditions for hyperrigidity:

COROLLARY 4.2: Let S be a separable operator system generating a  $C^*$ -algebra A. If S is hyperrigid, then every irreducible representation of A is a boundary representation for S. In particular, the boundary ideal of a hyperrigid operator system must be  $\{0\}$ .

*Proof.* The first assertion is an immediate consequence of condition (ii) of Theorem 2.1. The second follows from it, together with Theorem 4.1, which implies that the boundary ideal is the intersection of the kernels of all boundary representations for S.

We now conjecture that the obstructions described in Corollary 4.2 are the only obstructions to hyperrigidity. Indeed, we will prove that conjecture for classes of examples in Section 5:

CONJECTURE 4.3: If every irreducible representation of A is a boundary representation for a separable operator system  $S \subseteq A$ , then S is hyperrigid.

It is known that a direct sum of UCP maps with the unique extension property has the unique extension property (see [DM05]). For completeness, we conclude the section by proving that result in the form we require.

PROPOSITION 4.4: Let  $S \subseteq A = C^*(S)$  be an operator system, and for each *i* in an index set *I*, let  $\pi_i : A \to \mathcal{B}(H_i)$  be a representation such that  $\pi_i \upharpoonright_S$  has the unique extension property. Then the direct sum of UCP maps

$$\bigoplus_{i\in I} \pi_i \upharpoonright_S : S \to \mathcal{B}\left(\bigoplus_{i\in I} H_i\right)$$

has the unique extension property.

Proof. Let  $\phi: A \to \mathcal{B}(\bigoplus_{i \in I} H_i)$  be an extension of  $\pi$  to a UCP map from A to  $\mathcal{B}(\bigoplus_{i \in I} H_i)$ , and for each  $i \in I$ , let  $\phi_i: A \to \mathcal{B}(H_i)$  be the UCP map

$$\phi_i(a) = P_i \phi(a) \upharpoonright_{H_i}, \quad a \in A,$$

where  $P_i$  is the projection on  $H_i$ . Since  $\phi_i$  restricts to  $\pi_i$  on S, the unique extension property of  $\pi_i \upharpoonright_S$  implies that  $\phi_i(a) = \pi_i(a)$  for all  $a \in A$ , or equivalently,  $P_i\phi(a)P_i = \pi(a)P_i$ . By the Schwarz inequality applied to  $\phi$ ,

$$P_{i}\phi(a)^{*}(1 - P_{i})\phi(a)P_{i} = P_{i}\phi(a)^{*}\phi(a)P_{i} - P_{i}\phi(a)^{*}P_{i}\phi(a)P_{i}$$
$$\leq P_{i}\phi(a^{*}a)P_{i} - P_{i}\phi(a)^{*}P_{i}\phi(a)P_{i}$$
$$= \pi(a^{*}a)P_{i} - \pi(a)^{*}\pi(a)P_{i} = 0.$$

Hence  $|(\mathbf{1} - P_i)\phi(a)P_i|^2 = 0$ , and it follows that  $P_i$  commutes with the selfadjoint family of operators  $\phi(A)$ . So for every  $a \in A$  we have

$$\phi(a) = \sum_{i \in I} \phi(a) P_i = \sum_{i \in I} P_i \phi(a) P_i = \sum_{i \in I} \pi(a) P_i = \pi(a)$$

as asserted.

# 5. Countable spectrum

Let A be a separable  $C^*$ -algebra. By the **spectrum** of A we mean the set  $\hat{A}$  of unitary equivalence classes of irreducible representations of A. In general,  $\hat{A}$  carries a natural Borel structure that separates points of  $\hat{A}$ , and it is well-known that A is type I iff the Borel structure of  $\hat{A}$  is countably separated. In this section we prove Conjecture 4.3 for operator systems S whose generated  $C^*$ -algebra has countable spectrum. This class of  $C^*$ -algebras includes those generated by sets of compact operators (and the identity) as well as many others. It is closed under most of the natural ways of forming new  $C^*$ -algebras from given ones (countable direct sums, quotients, ideals, extensions, crossed products with compact Lie groups), but of course it fails to contain most commutative  $C^*$ -algebras.

THEOREM 5.1: Let S be a separable operator system whose generated  $C^*$ -algebra A has countable spectrum, such that every irreducible representation of A is a boundary representation for S. Then S is hyperrigid.

Proof. By item (iii) of Theorem 2.1, it suffices to show that for every representation  $\pi: A \to \mathcal{B}(H)$  of A on a separable Hilbert space, the UCP map  $\pi \upharpoonright_S$  has the unique extension property. Since the spectrum of A is countable, A is a type I  $C^*$ -algebra, hence  $\pi$  decomposes uniquely into a direct integral of mutually disjoint type I factor representations. Using countability of  $\hat{A}$  again, the direct integral must in fact be a countable direct sum. Hence  $\pi$  can be decomposed Vol. 184, 2011

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into a direct sum of subrepresentations  $\pi_n : A \to \mathcal{B}(H_n)$ 

(5.1) 
$$H = H_1 \oplus H_2 \oplus \cdots, \quad \pi = \pi_1 \oplus \pi_2 \oplus \cdots$$

with the property that each  $\pi_n$  is unitarily equivalent to a finite or countable direct sum of copies of a single irreducible representation  $\sigma_n : A \to \mathcal{B}(K_n)$ .

By hypothesis, each UCP map  $\sigma_n \upharpoonright_S$  has the unique extension property. Hence the above decomposition expresses  $\pi \upharpoonright_S$  as a (double) direct sum of UCP maps with the unique extension property. By Proposition 4.4, it follows that  $\pi \upharpoonright_S$  has the unique extension property.

# 6. Generators of nonunital C\*-algebras

In this section we discuss sets G of operators that generate a nonunital  $C^*$ algebra A—for example, sets of compact operators on an infinite-dimensional Hilbert space. One can adjoin the identity operator to obtain a unital  $C^*$ algebra  $\tilde{A} = A + \mathbb{C} \cdot \mathbf{1}$ , at the cost of introducing an additional one-dimensional irreducible representation  $\pi_{\infty} : \tilde{A} \to \mathbb{C}$  that represents "evaluation at  $\infty$ "

(6.1) 
$$\pi_{\infty}(a+\lambda\cdot\mathbf{1})=\lambda, \quad a\in A, \quad \lambda\in\mathbb{C}.$$

It is a fact that  $\pi_{\infty}$  may or may not be a boundary representation for the operator system  $\tilde{S}$  spanned by  $G \cup G^* \cup \{1\}$ ; and when it is not a boundary representation, G cannot be hyperrigid. The purpose of this section is to identify this obstruction to hyperrigidity in concrete operator-theoretic terms. We will show that  $\pi_{\infty}$  is a boundary representation for  $\tilde{S}$  iff the original (nonunital) space S spanned by  $G \cup G^*$  "almost contains" strictly positive operators.

A self-adjoint operator  $x \in A$  is said to be **almost dominated** by S if there is a sequence of self-adjoint operators  $s_n \in S$  such that

$$s_n + \frac{1}{n} \cdot \mathbf{1} \ge x, \quad n = 1, 2, \dots$$

A more familiar notion is strict positivity: A positive operator  $p \in A$  is called **strictly positive** if for every positive linear functional  $\phi \in A'$ ,

$$\phi(p) = 0 \implies \phi = 0$$

It is well-known that separable  $C^*$ -algebras contain many strictly positive operators; for example, if  $e_1 \leq e_2 \leq \cdots$  is a countable approximate unit for A,

then for every sequence of positive numbers  $c_1, c_2, \ldots$  with finite sum,

$$p = c_1 \cdot e_1 + c_2 \cdot e_2 + \cdots$$

is a strictly positive operator in A.

THEOREM 6.1: Let S be a self-adjoint operator space that generates a nonunital  $C^*$ -algebra A, let  $\tilde{A} = A + \mathbb{C} \cdot \mathbf{1}$ ,  $\tilde{S} = S + \mathbb{C} \cdot \mathbf{1}$ , and let  $\pi_{\infty} : \tilde{A} \to \mathbb{C}$  be the representation at  $\infty$ . The following are equivalent.

- (i)  $\pi_{\infty}$  is a boundary representation for  $\tilde{S}$ .
- (ii) A contains a strictly positive operator that is almost dominated by S.
- (iii) Every self-adjoint operator  $x \in A$  is almost dominated by S.

Our proof of Theorem 6.1 requires an operator-algebraic variation of a classic minimax principle—a consequence of Krein's extension theorem for positive linear functionals. While the result is known in one form or another to specialists, we lack a specific reference and include a proof for completeness. Let S be an operator system and let B be the (unital)  $C^*$ -algebra generated by S. A state of S is a positive linear functional  $\phi$  on S such that  $\phi(\mathbf{1}) = 1$ . Krein's extension theorem implies that every state of S can be extended to a state of B, and we write  $E_{\phi}$  for the weak\*-compact convex set of all extensions of  $\phi$  to a state of B.

PROPOSITION 6.2: Let S be an operator system that generates a  $C^*$ -algebra B. For every state  $\phi$  of S and every self-adjoint operator  $x \in B$ ,

$$\sup\{\phi(s) : s = s^* \in S, \ s \le x\} = \min\{\rho(x) : \rho \in E_{\phi}\},\ \inf\{\phi(s) : s = s^* \in S, \ s > x\} = \max\{\rho(x) : \rho \in E_{\phi}\}.$$

Proof of Proposition 6.2. We prove the first formula; the second one follows from it by replacing x with -x. If  $\rho \in E_{\phi}$  and  $s = s^* \leq x$ , then

$$\phi(s) = \rho(s) \le \rho(x)$$

and one obtains  $\leq$  after taking the sup over s and the inf over  $\rho$ .

For the inequality  $\geq$ , let *L* be the left hand side. We claim that there is a  $\rho \in E_{\phi}$  with  $L = \rho(x)$ . For the proof, we may assume that  $x \notin S$ , and consider the linear functional defined on the operator system  $S + \mathbb{C} \cdot x$  by

$$\phi(s + \lambda x) = \phi(s) + \lambda L, \quad s \in S, \ \lambda \in \mathbb{C}.$$

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We claim that  $\hat{\phi}$  is a state of  $S + \mathbb{C} \cdot x$ . Since  $\hat{\phi}(\mathbf{1}) = 1$ , after rescaling, this reduces to checking  $s + x \ge 0 \implies s + L \ge 0$  and  $s - x \ge 0 \implies \phi(s) - L \ge 0$ , where in both cases s is a self-adjoint element of S.

If  $s + x \ge 0$ , then  $x \ge -s$  so that  $-\phi(s) = \phi(-s) \le L$ , hence  $\phi(s) + L \ge 0$ . If  $s - x \ge 0$ , then for every  $t = t^* \in S$  satisfying  $t \le x \le s$  we have  $t \le s$ , hence  $\phi(t) \le \phi(s)$  and therefore  $L \le \phi(s)$  by the arbitrariness of t. The desired inequality  $\phi(s) - L \ge 0$  follows.

By Krein's extension theorem,  $\phi$  can be extended to a state  $\rho$  of B, and such an extension satisfies  $\rho \in E_{\phi}$  and  $L = \rho(x)$ .

Proof of Theorem 6.1. Since A must contain strictly positive elements, the implication (iii)  $\implies$  (ii) is trivial. We prove (i)  $\implies$  (iii) and (ii)  $\implies$  (i).

(i)  $\implies$  (iii): Let x be a self-adjoint element of A. Applying the second formula of Proposition 6.2 to the operator system  $\tilde{S}$  and its state  $\phi = \pi_{\infty} \upharpoonright_{\tilde{S}}$  and noting that  $E_{\phi} = \{\pi_{\infty}\}$  by hypothesis (i), we find that

$$\inf\{\lambda \in \mathbb{R} : \exists s = s^* \in S, \ s + \lambda \cdot \mathbf{1} \ge x\} = \pi_{\infty}(x) = 0.$$

It follows that there is a sequence  $s_n = s_n^* \in S$  such that  $s_n + \frac{1}{n} \cdot \mathbf{1} \ge x$ , hence x is almost dominated by S.

(ii)  $\Longrightarrow$  (i): Assuming (ii), let  $\rho$  be a state of  $\tilde{A}$  that satisfies  $\rho \upharpoonright_{\tilde{S}} = \pi_{\infty} \upharpoonright_{\tilde{S}}$ . We have to show that  $\rho = \pi_{\infty}$ . To that end, choose a strictly positive element  $p \in A$  that is almost dominated by S, and consider the positive linear functional  $\sigma \in A'$  defined by  $\sigma = \rho \upharpoonright_A$ . By the hypothesis on p there is a sequence  $s_n = s_n^* \in S$  such that  $s_n + \frac{1}{n} \cdot \mathbf{1} \geq p$  for  $n = 1, 2, \ldots$  Applying  $\rho$  to this inequality and using  $\rho(s_n) = \pi_{\infty}(s_n) = 0$ , we conclude that

$$\frac{1}{n} \ge \sigma(p) \ge 0, \quad n = 1, 2, \dots,$$

hence  $\sigma(p) = 0$ . It follows that  $\sigma = 0$  by strict positivity of p, which implies the desired conclusion  $\rho = \pi_{\infty}$ .

The following sufficient condition is easy to check for many examples.

COROLLARY 6.3: Let  $S \subseteq A$  be as in Theorem 6.1. If S contains a strictly positive operator of A, then  $\pi_{\infty} : \tilde{A} \to \mathbb{C}$  is a boundary representation for  $\tilde{S}$ .

*Proof.* If S itself contains a strictly positive operator p, then condition (ii) of Theorem 6.1 is satisfied.

### 7. Peaking representations

Given an exact sequence of  $C^*$ -algebras

$$0 \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow 0$$

in which  $a \in A \mapsto \dot{a} \in A/K = B$  is the natural quotient map, recall that every nondegenerate representation

$$\pi: A \to \mathcal{B}(H)$$

of A decomposes uniquely into a central direct sum of representations

$$\pi = \pi_K \oplus \pi_B$$

where  $\pi_K$  is the unique extension to A of a nondegenerate representation of the ideal K, and where  $\pi_B$  is a nondegenerate representation of A that annihilates K. When  $\pi = \pi_K$  we say that  $\pi$  **lives on** K. In an obvious sense, the spectrum of A decomposes into a disjoint union

(7.1) 
$$\hat{A} = \hat{K} \cup \hat{B}.$$

Now let  $S \subseteq \mathcal{B}(H)$  be a concrete operator system that generates a  $C^*$ -algebra A. In general, the set K of all compact operators in A is a closed two-sided ideal. In this section we address the problem of identifying the points of  $\hat{K}$  that correspond to boundary representations for S in cases where  $K \neq \{0\}$ , and we show how one can identify the boundary representations of  $\hat{K}$  as noncommutative counterparts of peak points of function systems.

Definition 7.1: Let S be a separable operator system that generates a  $C^*$ -algebra A. An irreducible representation  $\pi : A \to \mathcal{B}(H)$  is said to be **peaking** for S if there is an  $n \geq 1$  and an  $n \times n$  matrix  $(s_{ij})$  over S such that

(7.2) 
$$\|(\pi(s_{ij}))\| > \|(\sigma(s_{ij}))\|$$

for every irreducible representation  $\sigma$  inequivalent to  $\pi$ , written  $\sigma \nsim \pi$ .  $\pi$  is said to be **strongly peaking** if there is an  $n \ge 1$  and an  $n \times n$  matrix  $(s_{ij})$ over S such that

(7.3) 
$$\|(\pi(s_{ij}))\| > \sup_{\sigma \not\sim \pi} \|(\sigma(x_{ij}))\|.$$

An  $n \times n$  matrix  $(s_{ij})$  satisfying (7.2) (resp. (7.3)) is called a **peaking op**erator (resp. strong peaking operator) for  $\pi$ . Strongly peaking irreducible representations correspond to isolated points of  $\hat{A}$ , and they arise naturally

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when compact operators are present—such as in the setting of Theorem 7.2 below. We shall have nothing more to say about peaking representations that are not strongly peaking in this paper.

The following characterization of boundary representations generalizes the Boundary Theorem of [Arv72], and provides the basis for more concrete results on hyperrigid sets of compact operators such as Corollary 7.3 and Theorem 8.1.

THEOREM 7.2: Let  $S \subseteq \mathcal{B}(H)$  be a separable concrete operator system and let A be the  $C^*$ -algebra generated by S. Let K be the ideal of all compact operators in A, assume that  $K \neq \{0\}$ , and let  $\hat{K}$  be the set of unitary equivalence classes of irreducible representations of A that live on K.

Then  $\hat{K}$  contains boundary representations for S iff the quotient map

$$x \in A \mapsto \dot{x} \in A/K$$

is not completely isometric on S. Assuming that is the case, then among the irreducible representations of  $\hat{K}$ , the boundary representations for S are precisely the strongly peaking ones.

Proof. If  $\hat{K}$  contains no boundary representations, then because of the dichotomy (7.1), every boundary representation must annihilate K, and consequently it factors through the quotient map  $a \in A \mapsto \dot{a} \in A/K$ . By Theorem 4.1, there are sufficiently many boundary representations  $\pi_i$ ,  $i \in I$ , for S so that

$$\|(\dot{s}_{ij})\| \le \|(s_{ij})\| = \sup_{i \in I} \|(\pi_i(s_{ij}))\| \le \|(\dot{s}_{ij})\|$$

for every  $n \times n$  matrix  $(s_{ij})$  over S and every  $n \ge 1$ . Hence the quotient map is completely isometric on S. Conversely, if the quotient map is completely isometric on S, then we claim that no  $\pi \in \hat{K}$  can be a boundary representation. Indeed, for every irreducible representation  $\pi : A \to \mathcal{B}(H_{\pi})$  that lives in K, the hypothesis implies that the map

$$\dot{s} \in \dot{S} \subseteq A/K \mapsto \pi(s)$$

is completely positive, and hence can be extended to a completely positive linear map  $\phi : A/K \to \mathcal{B}(H_{\pi})$ . The map  $a \in A \mapsto \phi(\dot{a})$  is therefore a completely positive linear map that restricts to  $\pi$  on S, and which annihilates K. This map differs from  $\pi$  because  $\pi$  lives in K, hence  $\pi$  does not have the unique extension property.

Turning now to the proof of the last sentence, enumerate the distinct elements of  $\hat{K}$  as  $\{\pi_1, \pi_2, \ldots\}$ , and view each  $\pi_k$  as an irreducible subrepresentation of the identity representation of A, so that  $\pi_k(a) = a \upharpoonright_{H_k}, a \in A$ , where  $H_1, H_2, \ldots \subseteq H$  are mutually orthogonal reducing subspaces for A.

Assuming first that  $\pi_1$ , say, is a boundary representation for S, we claim that  $\pi_1$  is strongly peaking for S. Indeed, if  $\pi_1$  were not strongly peaking, then for every  $n \ge 1$  and every  $n \times n$  matrix  $(s_{ij})$  over S we would have

$$\|(\pi_1(s_{ij})\| \le \sup_{\sigma} \max(\|(\sigma(s_{ij}))\|, \|(\pi_2(s_{ij}))\|, \|(\pi_3(s_{ij}))\|, \dots),$$

where  $\sigma$  ranges over all irreducible representations of A that annihilate K. Let  $\rho : A/K \to \mathcal{B}(L)$  be a faithful representation of A/K and consider the representation  $\tilde{\rho}$  of A defined by

$$\tilde{
ho}(a) = 
ho(\dot{a}) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \cdots$$

The preceding inequalities imply that the map

$$\tilde{\rho}(s) \mapsto \pi_1(s), \quad s \in S,$$

is completely contractive. Since it is also unit-preserving, it must be completely positive, and hence by the extension theorem of [Arv69] there is a UCP map  $\phi : \tilde{\rho}(A) \to \mathcal{B}(H_{\pi_1})$  such that

$$\phi(\tilde{\rho}(s)) = \pi_1(s), \quad s \in S.$$

Since the UCP map  $\phi \circ \tilde{\rho} : A \to \mathcal{B}(H_{\pi_1})$  extends  $\pi \upharpoonright_S$  and  $\pi_1$  is assumed to be a boundary representation for S, it follows that  $\phi \circ \tilde{\rho} = \pi$  on A and, in particular,

$$\phi(\tilde{\rho}(k)) = \pi_1(k), \quad k \in K.$$

Noting that for  $k \in K$ ,

$$\tilde{\rho}(k) = 0 \oplus \pi_2(k) \oplus \pi_3(k) \oplus \cdots,$$

it follows that the map  $\pi_2(k) \oplus \pi_3(k) \oplus \cdots \mapsto \pi_1(k)$  is completely contractive, or equivalently, that the map

$$k \upharpoonright_{H_2 \oplus H_3 \oplus \dots} \mapsto \pi_1(k), \quad k \in K,$$

defines an irreducible representation of the  $C^*$ -algebra  $K_0 = K \upharpoonright_{H_2 \oplus H_3 \oplus \cdots}$ . Since  $K_0$  is a  $C^*$ -algebra of compact operators,  $\pi_1$  must be unitarily equivalent to one of the irreducible subrepresentations of the identity representation of  $K_0$ , namely  $\pi_2, \pi_3, \ldots$ , say  $\pi_1 \sim \pi_r$ , for some  $r \geq 2$ . It follows that  $\pi_1$  is equivalent

to  $\pi_r$ , and we have arrived at a contradiction. Hence  $\pi_1$  must have been strongly peaking for S.

Conversely, assume that one of the elements of  $\hat{K}$ , say  $\pi_1$ , is strongly peaking. Let  $\{\sigma_i : i \in I\}$  be a complete set of mutually inequivalent boundary representations for S. We claim that  $\pi_1$  is equivalent to some  $\sigma_i$ , and is therefore a boundary representation. Indeed, if that were not the case, then by definition of strong peaking representation (7.3), there would be an  $n \geq 1$  and an  $n \times n$ matrix  $(s_{ij})$  over S such that

(7.4) 
$$\|(\pi_1(s_{ij}))\| > \sup_{i \in I} \|(\sigma_i(s_{ij}))\|.$$

On the other hand, since the list  $\{\sigma_i : i \in I\}$  contains all boundary representations up to equivalence, Theorem 4.1 implies that the right side of (7.4) is  $||(s_{ij})||$ . We conclude that  $||(\pi_1(s_{ij}))|| > ||(s_{ij})||$ , and hence the completely bounded norm of  $\pi_1 \upharpoonright_S$  is > 1. But representations are completely contractive, hence the assumption that  $\pi_1 \approx \sigma_i$  for all  $i \in I$  was false.

The following result provides concrete criteria for checking hyperrigidity for generators of  $C^*$ -algebras of compact operators. See Section 8 for specific examples of how one makes use of it.

COROLLARY 7.3: Let  $G \subseteq \mathcal{B}(H)$  be a finite or countably infinite set of compact operators on an infinite-dimensional Hilbert space, let S be the linear span of  $G \cup G^*$  and let A be the C<sup>\*</sup>-algebra generated by G. Then G is hyperrigid iff:

- (a) Every irreducible subrepresentation of the identity representation of A is strongly peaking for the operator system S + C ⋅ 1, and
- (b) S almost dominates some strictly positive operator in A.

Proof. Let  $\tilde{A} = A + \mathbb{C} \cdot \mathbf{1}$  be the unitalization of A and let  $\tilde{S}$  be the operator system spanned by  $G \cup G^*$  and the identity operator. The irreducible representations of  $\tilde{A}$  are the irreducible subrepresentations  $\pi_1, \pi_2, \ldots$  of the identity representation of  $\tilde{A}$ , together with the one-dimensional representation  $\pi_{\infty}(a + \lambda \cdot \mathbf{1}) = \lambda, a \in A, \lambda \in \mathbb{C}$ . By Theorem 7.2, (a) is equivalent to the assertion that every irreducible subrepresentation of the identity representation is a boundary representation for  $\tilde{S}$ , and by Theorem 6.1, (b) is equivalent to the assertion that  $\pi_{\infty}$  is a boundary representation for  $\tilde{S}$ . Since the spectrum of  $\tilde{A}$  is countable, Corollary 4.2 and Theorem 5.1 show that these assertions are equivalent to the hyperrigidity of G.

### 8. Applications II: Volterra type operators

In this section we identify a broad class of irreducible compact operators that includes the Volterra integration operator on  $L^2[0, 1]$ , we show that for such operators  $V, \mathcal{G} = \{V, V^2\}$  is a hyperrigid generator for the  $C^*$ -algebra of compact operators, but that the smaller generator  $\mathcal{G}_0 = \{V\}$  is not hyperrigid.

By standard spectral theory, every self-adjoint operator B decomposes uniquely into a difference  $B = B_+ - B_-$ , where  $B_{\pm} \ge 0$  and  $B_+B_- = 0$ . A self-adjoint operator  $B \in \mathcal{B}(H)$  is said to be **essential** if its positive and negative parts  $B_+$  and  $B_-$  both have infinite rank. A straightforward argument shows that if B is essential and F is a self-adjoint finite rank operator, then B + F is also essential.

THEOREM 8.1: Let  $V \in \mathcal{B}(H)$  be an irreducible compact operator with cartesian decomposition V = A + iB, where A is a finite rank positive operator and B is essential with ker  $B = \{0\}$ . Then:

(i) G = {V, V<sup>2</sup>} is a hyperrigid generator for the C\*-algebra K of compact operators. In particular, for every sequence of unital completely positive maps φ<sub>n</sub> : B(H) → B(H) that satisfies

$$\lim_{n \to \infty} \|\phi_n(V) - V\| = \lim_{n \to \infty} \|\phi_n(V^2) - V^2\| = 0$$

one has

 $\lim_{n \to \infty} \|\phi_n(K) - K\| = 0$ 

for every compact operator K.

(ii) The subset  $\mathcal{G}_0 = \{V\}$  is not a hyperrigid generator.

Proof. Note first that V must generate the full  $C^*$ -algebra  $\mathcal{K}$  of compact operators, since  $\mathcal{K}$  contains no proper irreducible  $C^*$ -subalgebras. Let  $\mathcal{S}$  be the linear span of  $V, V^*, V^2, V^{2*}$  and let  $\tilde{\mathcal{S}} = \mathcal{S} + \mathbb{C} \cdot \mathbf{1}$ . Then  $\tilde{\mathcal{S}}$  is an operator system generating the  $C^*$ -algebra  $\mathcal{K} + \mathbb{C} \cdot \mathbf{1}$ , whose irreducible representations are  $\pi_{\infty}$ and, up to equivalence, the identity representation.

(i): The cartesian decomposition of  $V^2 = (A + iB)(A + iB)$  is

$$V^2 = (A^2 - B^2) + i(AB + BA).$$

Since A is a positive finite rank operator, we can find a c > 0 so that  $A^2 \leq c \cdot A$ , hence

$$-c \cdot A + (A^2 - B^2) \le -B^2 < 0$$

is a strictly negative operator in S. Corollary 6.3 implies that  $\pi_{\infty}$  is a boundary representation for  $\tilde{S}$ . The other irreducible representation of  $C^*(\tilde{S})$  is equivalent to the identity representation, and obviously V is itself a peaking operator for the identity representation restricted to S. Theorem 7.2 implies that the identity representation is a boundary representation for  $\tilde{S}$ , so by Corollary 7.3,  $\mathcal{G} = \{V, V^2\}$  is a hyperrigid generator for  $\mathcal{K}$ .

(ii): Consider the operator space  $S_0 = \text{span}\{A, B\}$  and let Q be the projection on  $AH^{\perp}$ . Since Q is of finite codimension, QBQ is also essential, and using the spectral theorem we can write

$$QBQ = C_+ - C_-$$

where  $C_{\pm} \ge 0$  and  $C_{+}C_{-} = 0$ , both nonzero. Choose vectors  $\xi_{\pm} \in C_{\pm}H$  such that

$$\langle C_+\xi_+,\xi_+\rangle = \langle C_-\xi_-,\xi_-\rangle > 0,$$

and let  $\rho = \omega_{\xi_+} + \omega_{\xi_-}$ ;  $\rho$  is a nonzero positive normal functional that satisfies  $\rho(B) = \rho(QBQ) = 0$  and  $\rho(A) = 0$  because  $\rho$  lives in  $AH^{\perp}$ . Hence  $\rho(S_0) = \{0\}$ , and it follows after normalization that  $\rho$  is a normal state other than  $\pi_{\infty}$  that agrees with  $\pi_{\infty}$  on the span of  $\{1, V, V^*\}$ . Therefore  $\pi_{\infty}$  is not a boundary representation, so by Corollary 4.2,  $\{V\}$  is not a hyperrigid generator of  $\mathcal{K}$ .

Now let V be the standard Volterra operator acting on  $L^2[0, 1]$ ,

(8.1) 
$$Vf(x) = \int_0^x f(t) \, dt, \quad f \in L^2[0,1]$$

LEMMA 8.2: The real part of V is  $\frac{1}{2}E$ , where E is the projection on the onedimensional space of constant functions. The imaginary part of V is unitarily equivalent to the following diagonal operator D on  $\ell^2(\mathbb{Z})$ :

(8.2) 
$$(Du)(n) = \frac{-1}{(2n+1)\pi}u(n), \quad n \in \mathbb{Z}, \ u \in \ell^2(\mathbb{Z}).$$

In particular, V belongs to the Schatten class  $\mathcal{L}^p$  iff p > 1.

*Proof.* This result is surely known, and we merely sketch the argument. The adjoint of V is the operator

$$V^*f(x) = \int_x^1 f(t) \, dt$$

It follows that  $V + V^*$  is the projection on the space of constants, and moreover  $V^* = E - V$ , so that  $i\Im V$  is the skew adjoint compact operator

$$A = \frac{1}{2}(V - V^*) = V - \frac{1}{2}E.$$

To solve the eigenvalue problem for A one sets  $Af = \lambda f$  and differentiates (in the sense of distributions) to obtain  $f = \lambda f'$ . There are no nonzero solutions f when  $\lambda = 0$ , and for  $\lambda \neq 0$  we must have  $f(x) = C \cdot e^{\omega x}$  for some imaginary  $\omega \in \mathbb{C}$ . Substitution of the latter expression for f in the equation  $Af = \lambda f$ leads to a solution iff  $\omega = (2n+1)\pi i$  for some  $n \in \mathbb{Z}$ , and the possible values of  $\lambda$  are

$$\lambda_n = \frac{1}{\omega_n} = \frac{1}{(2n+1)\pi i}, \quad n \in \mathbb{Z},$$

with corresponding eigenfunctions  $f_n(x) = e^{(2n+1)\pi ix}$ ,  $n \in \mathbb{Z}$ . In particular, the asserted form (8.2) for the imaginary part of V follows.

We conclude:

COROLLARY 8.3: The Volterra operator V of (8.1) satisfies the hypotheses of Theorem 8.1 above, and therefore its conclusion as well.

### 9. Applications III: Positive maps on matrix algebras

Let  $f : [a, b] \to \mathbb{R}$  be a continuous function defined on a compact real interval. Notice that if  $\phi : \mathcal{A} \to \mathcal{B}$  is a unit-preserving positive linear map of unital  $C^*$ -algebras and A is a self-adjoint operator in  $\mathcal{A}$  with spectrum in [a, b], then  $\phi(A)$  is an operator in  $\mathcal{B}$  with similar properties. We will say that f is **rigid** if for every UCP map of unital  $C^*$ -algebras  $\phi : \mathcal{A} \to \mathcal{B}$  and every self-adjoint operator  $A \in \mathcal{A}$  with  $\sigma(A) \subseteq [a, b]$ , one has

(9.1) 
$$\phi(f(A)) = f(\phi(A)) \implies \phi(A^n) = \phi(A)^n, \quad \forall n = 1, 2, \dots$$

The purpose of this section is to identify rigid functions in the following sense. We show that rigid functions must be either strictly convex or strictly concave in Proposition 9.3. In this commutative context, Conjecture 4.3 would imply the converse, namely that every strictly convex function is rigid. While we are unable to prove that assertion, we do prove it for operators  $A \in \mathcal{B}(H)$  and maps  $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$  when dim  $H < \infty$  in Theorem 9.4.

Fix a real-valued function  $f \in C[a, b]$  and let  $u \in C[a, b]$  be the coordinate function  $u(x) = x, x \in [a, b]$ . In order to determine whether f is rigid, we must

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PROPOSITION 9.1: Let  $S \subseteq C[a, b]$  be the function system spanned by the three functions  $f, u, \mathbf{1}$  and let

$$\Gamma = \{ (x, f(x) : x \in [a, b] \} \subseteq \mathbb{R}^2$$

be the graph of f and let  $\operatorname{conv} \Gamma \subseteq \mathbb{R}^2$  be its (necessarily compact) convex hull. The Choquet boundary of S is the set of points  $x \in [a, b]$  such that (x, f(x)) is an extreme point of  $\operatorname{conv} \Gamma$ .

**Proof.** The proof, an exercise in elementary convexity theory, is a consequence of the following three observations. First, the state space of S is naturally identified with the space of all probability measures on the compact convex set  $K = \operatorname{conv} \Gamma$ . Second, the extreme points of K are the points  $k \in K$  for which the point mass  $\delta_k$  is the unique probability measure on K having k as its barycenter; and such points must belong to  $\Gamma$ . Third, the Choquet boundary of S is identified with the points of  $\Gamma$  that have the extremal property of the preceding sentence.

Proposition 9.1 identifies the Choquet boundary  $\partial_S[a, b]$  of the function system  $S \subseteq C[a, b]$  spanned by  $f, u, \mathbf{1}$ . If one combines that with the following result, one identifies the functions f for which  $\partial_S[a, b] = [a, b]$  as precisely those which are either strictly convex or strictly concave.

PROPOSITION 9.2: For every continuous function  $f : [a, b] \to \mathbb{R}$ , the following are equivalent:

- (i) Every point of the graph  $\Gamma = \{(x, f(x)) : x \in [a, b]\}$  of f is an extreme point of the convex hull of  $\Gamma$ .
- (ii) f is either strictly convex or strictly concave.

*Proof.* (i)  $\Longrightarrow$  (ii): Assuming that (i) holds, we claim first that if  $x_1, \ldots, x_n$  are points of I = [a, b] and  $t_1, \ldots, t_n$  are positive numbers with sum 1, then

$$(9.2) \quad f(t_1x_1 + \dots + t_nx_n) = t_1f(x_1) + \dots + t_nf(x_n) \implies x_1 = \dots = x_n$$

Indeed, the left side of the implication implies that for  $x_0 = t_1 x_1 + \cdots + t_n x_n$ ,

$$(x_0, f(x_0)) = t_1 \cdot (x_1, f(x_1)) + \dots + t_n \cdot (x_n, f(x_n)),$$

which by (i) implies  $x_1 = \cdots = x_n = x_0$ . Next, we claim that for any two pairs of distinct points  $x \neq y, u \neq v$  in I and 0 < s, t < 1, the two inequalities

$$\begin{split} f(sx + (1-s)y) &< sf(x) + (1-s)f(y), \\ f(tu + (1-t)v) &> tf(u) + (1-t)f(v) \end{split}$$

cannot both hold. For if they do, then the function  $F: [0,1] \to \mathbb{R}$ 

$$F(\lambda) = f(\lambda(sx + (1 - s)y)) + (1 - \lambda)(tu + (1 - t)v)) -\lambda(sf(x) + (1 - s)f(y)) - (1 - \lambda)(tf(u) + (1 - t)f(v))$$

is continuous, positive at  $\lambda = 0$  and negative at  $\lambda = 1$ , so by the intermediate value theorem, there is a  $\lambda \in (0, 1)$  for which  $F(\lambda) = 0$ , which contradicts (9.2) for  $x_1 = x, x_2 = y, x_3 = u, x_4 = v$ . Hence one or the other inequalities must be satisfied throughout, so that f is either strictly convex or strictly concave. The proof of (ii)  $\Longrightarrow$  (i) is straightforward.

PROPOSITION 9.3: A rigid function  $f \in C[a, b]$  is either strictly convex or strictly concave.

Proof of Proposition 9.3. We actually prove a somewhat stronger version of Proposition 9.3 in its contrapositive formulation: If f is neither strictly convex nor strictly concave, then there is a finite-dimensional Hilbert space H, a self-adjoint operator  $A \in \mathcal{B}(H)$  with spectrum in [a,b], and a unital completely positive map  $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$  such that  $\phi(A) = A$ ,  $\phi(f(A)) = f(\phi(A))$ , but  $\phi$ is not multiplicative on the algebra of polynomials in A. In particular, f is not rigid.

Indeed, let  $f : [a,b] \to \mathbb{R}$  be a continuous function that is neither strictly convex nor strictly concave. By Proposition 9.2, the graph  $\Gamma$  of f must contain some point  $(x_0, f(x_0))$  that is not an extreme point of its convex hull, and hence can be written as a nontrivial convex combination of two distinct points of the convex hull of  $\Gamma$ . Since  $\Gamma \subseteq \mathbb{R}^2$ , every point of the convex hull of  $\Gamma$  is a convex combination of at most 3 points of  $\Gamma$ , and we conclude that  $(x_0, f(x_0))$  can be written as a convex combination of at most 6 points of  $\Gamma$ ,  $(x_i, f(x_i)), x_i \neq x_0$ ,  $i = 1, \ldots, n \leq 6$ . By discarding some of the points  $x_1, \ldots, x_n$  and reducing n if necessary, we can assume that the n + 1 points  $x_0, x_1, \ldots, x_n \in [a, b]$  are distinct. By the choice of  $x_1, \ldots, x_n$ , there are numbers  $t_1, \ldots, t_n \in [0, 1]$  such Vol. 184, 2011

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that

(9.3) 
$$x_0 = \sum_{k=1}^n t_k x_k \text{ and } f(x_0) = \sum_{k=1}^n t_k \cdot f(x_k),$$

and consider the positive linear map  $\phi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$  defined by

$$\phi(\lambda_0, \lambda_1, \dots, \lambda_n) = (t_1\lambda_1 + \dots + t_n\lambda_n, \lambda_1, \dots, \lambda_n), \quad \lambda_k \in \mathbb{C}.$$

Viewing  $\mathbb{C}^{n+1}$  as the algebra of all diagonal matrices in  $M_{n+1} = M_{n+1}(\mathbb{C})$ ,  $\phi$ becomes a unit-preserving positive (and hence completely positive) linear map. We may extend  $\phi$  to a UCP map  $\tilde{\phi} : M_{n+1} \to M_{n+1}$  in many ways, for example, by composing it with the trace-preserving conditional expectation of  $M_{n+1}$  onto the diagonal subalgebra. In order to conserve notation, we continue to write  $\phi$  for an extension of the original map on diagonal matrices to a completely positive map of  $M_{n+1}$  into itself.

Consider the diagonal operator

$$A = (x_0, x_1, \dots, x_n).$$

The conditions (9.3) on  $t_1, \ldots, t_n$  imply the two operator formulas

$$\phi(A) = A, \qquad \phi(f(A)) = f(A).$$

Finally, since  $x_i \neq x_j$  for  $i \neq j$ , the algebra generated by A is all diagonal sequences in  $\mathbb{C}^{n+1}$ , and obviously  $\phi$  does not fix all diagonal sequences. Together, these properties imply that the restriction of  $\phi$  to the algebra of polynomials in A is not multiplicative.

THEOREM 9.4: Let [a, b] be a compact real interval and let  $f \in C[a, b]$  be real valued. If f is strictly convex, then for every pair H, K of finite-dimensional Hilbert spaces, every self-adjoint operator  $A \in \mathcal{B}(H)$  having spectrum in [a, b], and every UCP map  $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$  satisfying

(9.4) 
$$\phi(f(A)) = f(\phi(A)),$$

the restriction of  $\phi$  to the algebra of polynomials in A is multiplicative.

Conversely, if neither f nor -f is strictly convex, then there is a Hilbert space H of dimension at most 7, a self-adjoint operator  $A \in \mathcal{B}(H)$  with spectrum in [a, b], and a UCP map  $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$  such that

$$\phi(A) = A, \quad \phi(f(A)) = f(A),$$

and which is not multiplicative on the algebra of polynomials in A.

Proof. To prove the first paragraph, let  $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$  be a UCP map, let  $A = A^* \in \mathcal{B}(H)$  have its spectrum in I and satisfy (9.2), and assume that f is strictly convex. Let  $B = \phi(A) \in \mathcal{B}(K)$ . Consider the representations  $\pi : C[a, b] \to \mathcal{B}(H)$  and  $\sigma : C[a, b] \to \mathcal{B}(K)$  defined by

$$\pi(g) = g(A), \quad \sigma(g) = g(B), \qquad g \in C(X).$$

Let  $u(x) = x, x \in X$ , be the coordinate function and let S be the 2- or 3dimensional function system spanned by u, f and the constants. We have arranged that  $\phi(\pi(u)) = \sigma(u)$ , and  $\phi(\pi(f)) = \sigma(f)$ , hence

$$(9.5) \qquad \qquad \phi \circ \pi \restriction_S = \sigma \restriction_S .$$

Since K is finite-dimensional,  $\sigma$  is a finite direct sum of (one-dimensional) irreducible representations of C[a, b], and such representations correspond to points of [a, b]. Since f is assumed to be strictly convex, Proposition 9.2 implies that every point of the graph  $\Gamma$  of f is an extreme point of the convex hull of  $\Gamma$ ; and Proposition 9.1 implies that every point of [a, b] belongs to the Choquet boundary of [a, b] relative to S. Hence  $\sigma$  is a direct sum of one dimensional representations with the unique extension property. By Proposition 4.4,  $\sigma$  itself has the unique extension property; and since  $\phi \circ \pi$  restricts to  $\sigma$  on S, it follows that  $\phi \circ \pi = \sigma$ . Hence the restriction of  $\phi$  to  $\pi(C[a, b])$  is multiplicative.

The assertion of the second paragraph follows from the proof of Proposition 9.2. Indeed, the construction in that proof exhibits a Hilbert space H of dimension at most 7, an operator  $A = A^* \in \mathcal{B}(H)$  and a unital completely positive map  $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$  with the stated properties.

Remark 9.5 (Infinite-dimensional generalizations): Naturally, one would hope that the second paragraph of Theorem 9.4 remains true if one drops the hypothesis of finite dimensionality of H; but that has not been proved. Note that it would be enough to prove Conjecture 4.3 for commutative  $C^*$ -algebras. In turn, that would provide a generalization of Theorem 3.1 to cases in which  $G = \{\mathbf{1}, x, x^2\}$  is replaced with  $G = \{\mathbf{1}, x, f(x)\}$  for any continuous strictly convex function f and any self-adjoint operator x.

#### Part 3. A local version of Conjecture 4.3

It is conceivable that Conjecture 4.3 might fail for reasons yet unknown; and in that event one needs to know what *can* be proved. In the remaining sections

we take up this issue in the commutative case of function systems  $S \subseteq C(X)$ , where X is a compact metric space, and we show that function systems satisfy a "localized" version of Conjecture 4.3.

More precisely, let  $S \subseteq C(X)$  be a linear space of continuous functions that separates points, contains the constants, is closed under complex conjugation, and assume that every point  $p \in X$  has a unique representing measure in the sense that the only probability measure  $\mu$  on X satisfying

$$f(p) = \int_X f \, d\mu, \quad f \in S$$

is the point mass  $\mu = \delta_p$ . By Theorem 2.1, to prove Conjecture 4.3 it is enough to prove the following assertion: For every separably-acting representation  $\pi : C(X) \to \mathcal{B}(H)$  and every positive linear map  $\phi : C(X) \to \mathcal{B}(H)$  such that  $\phi \upharpoonright_S = \pi \upharpoonright_S$ , one has

(9.6) 
$$\phi(f) = \pi(f), \quad f \in C(X).$$

Let E be the spectral measure of  $\pi$ —namely the projection valued measure on the  $\sigma$ -algebra of Borel subsets of X that satisfies

$$\pi(f) = \int_X f(x) \, dE(x), \quad f \in C(X).$$

We will show that (9.6) is true *locally* in the following sense: For every positive linear map  $\phi : C(X) \to \mathcal{B}(H)$  that restricts to  $\pi$  on S and for every point  $p \in X$ ,

(9.7) 
$$\lim_{\epsilon \to 0} \|(\phi(f) - \pi(f))E(B_{\epsilon}(p))\| = 0, \quad f \in C(X),$$

where  $B_{\epsilon}(p) = \{x \in X : d(x, p) \le \epsilon\}$  is the ball of radius  $\epsilon > 0$  about p. Indeed, the limit (9.7) is zero uniformly in p (see Theorem 11.1).

### 10. The local $C^*$ -algebra of a representation of C(X)

Throughout this section, X will denote a compact metric space with metric  $d: X \times X \to [0, \infty)$ . Every representation  $\pi: C(X) \to \mathcal{B}(H)$  gives rise to a spectral measure  $F \to E(F)$  on the Borel subsets  $F \subseteq X$ , and which is uniquely defined by

$$\langle \pi(f)\xi,\xi\rangle = \int_X f(x)\langle E(dx)\xi,\xi\rangle, \quad \xi\in H, \ f\in C(X).$$

We say that  $\pi : C(X) \to \mathcal{B}(H)$  is a **separable** representation if the space H on which it acts is a separable Hilbert space. All representations  $\pi$  are assumed to be nondegenerate, so that  $\pi(\mathbf{1}) = \mathbf{1}$ .

Let  $\pi : C(X) \to \mathcal{B}(H)$  be a representation and let  $p \in X$ . An operator  $A \in \mathcal{B}(H)$  is said to be **locally null at** p if for every  $\epsilon > 0$  there is an open neighborhood U of p such that  $||AE(U)|| \le \epsilon$  and  $||A^*E(U)|| \le \epsilon$ .

PROPOSITION 10.1: Let  $\pi : C(X) \to \mathcal{B}(H)$  be a representation. Then for every operator  $A \in \mathcal{B}(H)$  the following are equivalent:

- (i) A is locally null at every point of X.
- (ii) A is uniformly locally null in the following sense: Letting  $B_{\delta}(p) = \{q \in X : d(p,q) < \epsilon\}$  be the  $\delta$ -ball about a point  $p \in X$ , we have

(10.1) 
$$\sup_{p \in X} (\|AE(B_{\delta}(p))\| + \|A^*E(B_{\delta}(p))\|) \to 0 \quad \text{as } \delta \to 0 + .$$

*Proof.* (i)  $\implies$  (ii): It suffices to show that for every operator  $A \in \mathcal{B}(H)$ ,

(10.2) 
$$\lim_{\delta \to 0} \|AE(B_{\delta}(p))\| = 0 \ \forall p \in X \implies \lim_{\delta \to 0} \sup_{p \in X} \|AE(B_{\delta}(p))\| = 0.$$

Contrapositively, let  $\delta_n > 0$  be a sequence tending to 0 such that

(10.3) 
$$||AE(B_{\delta}(p_n))|| \ge \alpha > 0, \quad n = 1, 2, \dots$$

By compactness,  $\{p_n\}$  has a convergent subsequence, and by passing to that subsequence we may assume that  $p_n \to p \in X$  as  $n \to \infty$ . For every  $\delta > 0$  we will have  $B_{\delta}(p) \supseteq B_{\delta_n}(p_n)$  for sufficiently large n, and for such n, (10.3) implies

 $\|AE(B_{\delta}(p))\| \ge \|AE(B_{\delta}(p_n))\| \ge \alpha,$ 

from which we conclude

$$\inf_{\delta>0} \|AE(B_{\delta}(p))\| \ge \alpha,$$

contradicting item (i) at the point p. (ii)  $\implies$  (i) is trivial.

Definition 10.2: Let  $\pi : C(X) \to \mathcal{B}(H)$  be a representation. An operator  $A \in \mathcal{B}(H)$  is said to be **locally null** (relative to  $\pi$ ) if it satisfies the equivalent conditions of Proposition 10.1.  $\mathcal{N}_{\pi}$  will denote the set of all operators that are locally null with respect to  $\pi$ .

Remark 10.3 (Structure of  $\mathcal{N}_{\pi}$ ): Consider the linear space of operators

$$\mathcal{L}_{\pi} = \{ A \in \mathcal{B}(H) : \lim_{\delta \to 0} \|AE(B_{\delta}(p))\| = 0, \quad \forall p \in X \}.$$

Obviously,  $\mathcal{L}_{\pi}$  is a norm-closed left ideal in  $\mathcal{B}(H)$  for which  $\mathcal{N}_{\pi} = \mathcal{L}_{\pi} \cap \mathcal{L}_{\pi}^{*}$ . Moreover, the norm-closed linear span of  $\mathcal{L}_{\pi} \cdot \mathcal{L}_{\pi}^{*}$  is a two-sided ideal in  $\mathcal{B}(H)$ , which when nonzero can only be the  $C^{*}$ -algebra  $\mathcal{K}$  of all compact operators on H or all of  $\mathcal{B}(H)$ . We conclude that either (a)  $\mathcal{N}_{\pi} = \{0\}$ , or (b)  $\mathcal{N}_{\pi} = \mathcal{K}$ , or (c)  $\mathcal{N}_{\pi}$  contains  $\mathcal{K}$  together with some noncompact operators, in which case it is strongly Morita equivalent to  $\mathcal{B}(H)$ .

PROPOSITION 10.4: If  $\pi : C(X) \to \mathcal{B}(H)$  is a separable representation with no point spectrum, then  $\mathcal{N}_{\pi}$  contains the C<sup>\*</sup>-algebra  $\mathcal{K}$  of compact operators.

Proof. We claim first that  $\mathcal{N}_{\pi}$  contains every rank one projection  $A \in \mathcal{K}$ . Indeed, let  $A\xi = \langle \xi, f \rangle f$ , where f is a unit vector in H. Then for every  $p \in X$ and  $\delta > 0$ ,  $AE(B_{\delta}(p))$  is a rank one operator with

$$||AE(B_{\delta}(p))||^{2} = ||E(B_{\delta}(p))f||^{2} = \langle E(B_{\delta}(p))f, f \rangle,$$

and the latter tends to zero as  $\delta \downarrow 0$  because the hypothesis on  $\pi$  implies that the probability measure defined on X by  $\mu(S) = \langle E(S)f, f \rangle$  is nonatomic. Hence  $A \in \mathcal{N}_{\pi}$ . The spectral theorem implies every self-adjoint compact operator can be norm approximated by linear combinations of rank one projections, hence  $\mathcal{N}_{\pi} \supseteq \mathcal{K}$ .

The basic facts that connect  $\mathcal{N}_{\pi}$  to the structure of X are as follows:

PROPOSITION 10.5: If X is countable, then  $\mathcal{N}_{\pi} = \{0\}$  for every separable representation  $\pi : C(X) \to \mathcal{B}(H)$ . If X is uncountable, then there is a separable representation  $\pi$  of C(X) such that  $\mathcal{N}_{\pi}$  contains non-compact operators, and in fact  $\mathcal{N}_{\pi}$  is strongly Morita equivalent to  $\mathcal{B}(H)$ .

Proof. Assume that X is countable and let  $\pi : C(X) \to \mathcal{B}(H)$  be a separably acting representation. The set of factor representations of C(X) being countable  $(\cong X)$ , reduction theory shows that  $\pi$  decomposes into a direct sum of disjoint factor representations, which in this simple context means

$$\pi(f) = \sum_{n \ge 1}^{\oplus} f(p_n) E_n$$

where the  $E_k$  are a sequence of mutually orthogonal projections with sum **1** and  $p_1, p_2, \ldots$  is a (finite or infinite) sequence of distinct points of X. Hence the spectral measure of  $\pi$  is atomic and is concentrated on  $\{p_1, p_2, \ldots\}$ . It follows

that for every operator  $A \in \mathcal{N}_{\pi}$  we must have

$$||AE_n|| = \inf_{\delta > 0} ||AE(B_{\delta}(p_n))|| = 0, \quad n = 1, 2, \dots,$$

hence  $A = \sum_{n} AE_n = 0.$ 

Assume now that X is uncountable. Since X can be viewed as a standard Borel space, it contains a Borel subset that is isomorphic to the unit interval [0,1], and hence X supports a nonatomic Borel probability measure  $\mu$ . Let  $H = L^2(X,\mu)$  and let  $\pi$  be the usual representation of C(X) on  $L^2(X,\mu)$  in which  $\pi(f)$  acts as multiplication by f.

By Proposition 10.4,  $\mathcal{N}_{\pi}$  contains all compact operators on H. Now let  $\infty \cdot \pi$  be the direct sum of a countably infinite number of copies of  $\pi$ . For every compact operator  $K \in \mathcal{B}(H)$ , the direct sum  $\infty \cdot K = K \oplus K \oplus \cdots$  of copies of K must belong to  $\mathcal{N}_{\infty \cdot \pi}$ . Since none of the operators  $\infty \cdot K$  is compact when  $K \neq 0$ , Remark 10.3 implies that  $\infty \cdot \pi$  is a representation of C(X) with the stated properties.

# 11. Local uniqueness of UCP extensions

Continuing our discussion of function systems  $S \subseteq C(X)$  on compact metric spaces X, in this section we prove:

THEOREM 11.1: Given a separable representation  $\pi : C(X) \to \mathcal{B}(H)$ , let  $\phi : C(X) \to \mathcal{B}(H)$  be a UCP map such that  $\phi(s) - \pi(s) \in \mathcal{N}_{\pi}$  for every  $s \in S$ . If every point of X belongs to the Choquet boundary  $\partial_S X$ , then

$$\phi(f) - \pi(f) \in \mathcal{N}_{\pi}, \quad \forall \ f \in C(X).$$

The proof of Theorem 11.1 requires the following estimate:

PROPOSITION 11.2: Let  $S \subseteq C(X)$  be an arbitrary function system and let  $\phi: C(X) \to \mathcal{B}(H)$  be a UCP map with the property

$$\phi(g) - \pi(g) \in \mathcal{N}_{\pi}, \quad \forall g \in S.$$

Then for  $p \in X$  and every  $f \in C(X)$  we have

(11.1) 
$$\limsup_{n \to \infty} \|\phi(f) E(B_{1/n}(p))\|^2 \le \inf\{s(p) : s \in S, \ s \ge |f|^2\}.$$

Proof. For each n = 1, 2, ... choose a unit vector  $\xi_n \in E(B_n(p))H$  such that

(11.2) 
$$\|\phi(f)E(B_{1/n}(p))\|^2 \le \|\phi(f)\xi_n\|^2 + 1/n,$$

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and fix a function  $s \in S$  satisfying  $s \ge |f|^2$ . Then

$$\|\phi(f)\xi_n\|^2 = \langle \phi(f)^*\phi(f)\xi_n, \xi_n \rangle \le \langle \phi(|f|^2)\xi_n, \xi_n \rangle \le \langle \phi(s)\xi_n, \xi_n \rangle.$$

Now fix  $\epsilon > 0$ . Since  $\xi_n$  is a unit vector in  $E(B_{1/n}(p))H$ , it follows from the hypothesis  $\phi(s) - \pi(s) \in \mathcal{N}_{\pi}$  that for sufficiently large n we will have

$$|\langle (\phi(s) - \pi(s))\xi_n, \xi_n \rangle| \le \|(\phi(f) - \pi(f))E(B_{1/n}(p))\| \le \epsilon$$

and therefore

$$\|\phi(f)\xi_n\|^2 \le \langle \phi(s)\xi_n, \xi_n \rangle \le \langle \pi(s)\xi_n, \xi_n \rangle + \epsilon = \int_X s(x)\langle E(dx)\xi_n, \xi_n \rangle + \epsilon.$$

Since  $\xi_n \in B_{1/n}(p)$ , the measure  $\langle E(\cdot)\xi_n, \xi_n \rangle$  is supported on the closure of  $B_{1/n}(p)$ . Hence the term on the right is dominated by

$$\sup\{s(x): d(x,p) \le 1/n\} + \epsilon$$

which, by continuity of s at p, is in turn dominated by  $s(p) + 2\epsilon$  for sufficiently large n. Finally, since  $\epsilon$  can be arbitrarily small, we obtain

$$\limsup_{n \to \infty} \|\phi(f)\xi_n\|^2 \le s(p).$$

From (11.2) we conclude that

$$\limsup_{n \to \infty} \|\phi(f)E_n\|^2 \le s(p),$$

and the estimate (11.1) follows after taking the infimum over s.

We will also make use of the following property of points with unique representing measures, a consequence of a more general minimax principle based on the Hahn–Banach theorem (see formula (1.2) of [Gli67]):

LEMMA 11.3: Let u be a real function in C(X) and let p be a point in the Choquet boundary of X relative to S. Then

$$u(p) = \inf\{s(p) : s \in S, s \ge u\}.$$

Proof of Theorem 11.1. Fix  $f \in C(X)$  and let  $A = \phi(f) - \pi(f)$ . We have to show that for every point  $p \in X$ 

(11.3) 
$$\lim_{n \to \infty} \|AE(B_{1/n}(p))\| = 0.$$

Fixing p, note that by replacing f with  $f - f(p)\mathbf{1}$ , it suffices to prove (11.3) for functions f that vanish at p. For such a function f we claim first that

(11.4) 
$$\lim_{n \to \infty} \|\pi(f)E(B_{1/n}(p))\| = 0.$$

Indeed, we have

$$\|\pi(f)E(B_{1/n}(p))\| = \left\| \int_{B_{1/n}(p)} f(x) E(dx) \right\| \le \sup_{x \in B_{1/n}(p)} |f(x)|,$$

and the term on the right tends to |f(p)| = 0 as  $n \to \infty$ .

So to prove (11.3), we have to show that  $\|\phi(f)E(B_{1/n}((p))\|$  tends to zero as  $n \to \infty$ . To see that, note that Proposition 11.2 implies

(11.5) 
$$\limsup_{n \to \infty} \|\phi(f) E(B_{1/n}(p))\| \le \inf\{s(p) : s \in S, \ s \ge |f|^2\}.$$

Since p belongs to the Choquet boundary, Lemma 11.3 implies that the right side of (11.5) is  $|f(p)|^2 = 0$ . Thus (11.3) is proved.

### References

- [Arv69] W. Arveson, Subalgebras of C<sup>\*</sup>-algebras, Acta Mathematica **123** (1969), 141–224.
- [Arv72] W. Arveson, Subalgebras of C\*-algebras II, Acta Mathematica 128 (1972), 271– 308.
- [Arv08] W. Arveson, The noncommutative Choquet boundary, Journal of the American Mathematical Society 21 (2008), 1065–1084. arXiv:OA/0701329v4.
- [Bau61] H. Bauer, Silovscher rand und Dirichletsches problem, Université de Grenoble. Annales de l'Institut Fourier **11** (1961), 89–136.
- [BD78] H. Bauer and K. Donner, Korovkin approximation in  $C_0(X)$ , Mathematische Annalen **236** (1978), 225–237.
- [BS55] J. Bendat and S. Sherman, Monotone and convex operator functions, Transactions of the American Mathematical Society 79 (1955), 58–71.
- [Cho74] M.-D. Choi, A Schwarz inequality for positive linear maps on C\*-algebras Illinois Journal of Mathematics 18 (1974), 565–574.
- [Dav57] C. Davis, A schwarz inequality for convex operator functions, Proceedings of the American Mathematical Society 8 (1957), 42–44.
- [DM05] M. Dritschel and S. McCullough, Boundary representations for families of representations of operator algebras and spaces, Journal of Operator Theory 53 (2005), 159–167.
- [Don82] K. Donner, Extension of positive operators and Korovkin theorems, Lecture Notes in Mathematics 904, Springer-Verlag, Berlin, 1982.
- [Gli67] I. Glicksberg, The abstract F. and M. Riesz theorem, Journal of Functional Analysis 1 (1967), 109–122.

- [Ham79a] M. Hamana, Injective envelopes of C\*-algebras, Journal of the Mathematical Society of Japan 31 (1979), 181–197.
- [Ham79b] M. Hamana, Injective envelopes of operator systems Kyoto University. Research Institute for Mathematical Sciences. Publications 15 (1979), 773–785.
- [JOR03] M. Junge, N. Ozawa, and Z. Ruan, On  $\mathcal{OL}^{\infty}$  structures of nuclear C<sup>\*</sup>-algebras, Mathematische Annalen **325** (2003), 449–483.
- [Kor53] P. P. Korovkin, On convergence of linear positive operators in the space of continuous functions, Doklady Akademii Nauk SSSR (N.S.) 90 (1953), 961–964.
- [Kor60] P. P. Korovkin, Linear Operators and Approximation Theory, Hindustan publishing Corp., Delhi, 1960.
- [NS06] S. Neshveyev and E. Størmer, Dynamical Entropy in Operator Algebras, Ergebnisse der mathematik und ihrer grenzgebiete, Springer, Hiedelberg, 2006.
- [Pet86] D. Petz, On the equality in Jensen's inequality for operator convex functions, Integral Equations and Operator Theory 9 (1986), 744–747.
- [Phe66] R. R. Phelps, Lectures on Choquet's Theorem, Volume 7 of Van Nostrand Mathematical Studies, Van Nostrand, Princeton, 1966.
- [Phe01] R. R. Phelps, Lectures on Choquet's Theorem, Lecture Notes in Mathematics 1757, second edition, Springer-Verlag, Berlin, 2001.
- [Šaš67] Yu. A. Šaškin, The Milman-Choquet boundary and approximation theory, Functional Analysis and its Applications 1 (1967), 170–171.