

# The Probability of Entanglement

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**Abstract:** We show that states on tensor products of matrix algebras whose ranks are relatively small are *almost surely* entangled, but that states of maximum rank are not. More precisely, let  $M = M_m(\mathbb{C})$  and  $N = M_n(\mathbb{C})$  be full matrix algebras with  $m \geq n$ , fix an arbitrary state  $\omega$  of  $N$ , and let  $E(\omega)$  be the set of all states of  $M \otimes N$  that extend  $\omega$ . The space  $E(\omega)$  contains states of rank  $r$  for every  $r = 1, 2, \dots, m \cdot \text{rank } \omega$ , and it has a filtration into compact subspaces

$$E^1(\omega) \subseteq E^2(\omega) \subseteq \dots \subseteq E^{m \cdot \text{rank } \omega} = E(\omega),$$

where  $E^r(\omega)$  is the set of all states of  $E(\omega)$  having rank  $\leq r$ .

We show first that for every  $r$ , there is a real-analytic manifold  $V^r$ , homogeneous under a transitive action of a compact group  $G^r$ , which parameterizes  $E^r(\omega)$ . The unique  $G^r$ -invariant probability measure on  $V^r$  promotes to a probability measure  $P^{r,\omega}$  on  $E^r(\omega)$ , and  $P^{r,\omega}$  assigns probability 1 to states of rank  $r$ . The resulting probability space  $(E^r(\omega), P^{r,\omega})$  represents “choosing a rank  $r$  extension of  $\omega$  at random”.

Main result: For every  $r = 1, 2, \dots, [\text{rank } \omega/2]$ , states of  $(E^r(\omega), P^{r,\omega})$  are almost surely entangled.

## 1. Introduction

In the literature of physics and quantum information theory, a state  $\rho$  of the tensor product of two matrix algebras  $M \otimes N$  is said to be *separable* (or *classically correlated*) if it is a convex combination of product states

$$\rho = t_1 \cdot \sigma_1 \otimes \tau_1 + t_2 \cdot \sigma_2 \otimes \tau_2 + \dots + t_r \cdot \sigma_r \otimes \tau_r,$$

where the coefficients  $t_k$  are nonnegative and sum to 1, and where  $\sigma_k, \tau_k$  are states of  $M$  and  $N$  respectively [Wer89]. Remark 7.5 below implies that the set of separable states is a compact convex subset of the state space of  $M \otimes N$ . A state that is not separable is

said to be *entangled*. The so-called *separability problem* of determining whether a given state of  $M \otimes N$  is entangled is a subject of current research [HHHH07]. It is considered difficult, and computationally, has been shown to be NP-hard. The purpose of this paper is to show that *almost surely, a state of  $M \otimes N$  of relatively small rank is entangled*.

The set  $E(\omega)$  of all extensions of a fixed state  $\omega$  of  $N$  to a state of  $M \otimes N$  is a compact convex subspace of the state space of  $M \otimes N$ , and it admits a filtration into compact subspaces

$$E^1(\omega) \subseteq E^2(\omega) \subseteq \dots \subseteq E^{m \cdot \text{rank } \omega}(\omega) = E(\omega),$$

where  $E^r(\omega)$  is the space of all extensions  $\rho$  of  $\omega$  satisfying  $\text{rank } \rho \leq r$ . In Sects. 2 through 6 we show that for each  $r$  there is a uniquely determined unbiased probability measure  $P^{r,\omega}$  on  $E^r(\omega)$ , and that  $P^{r,\omega}$  is concentrated on the set of states of rank  $= r$ . Hence the probability space  $(E^r(\omega), P^{r,\omega})$  represents “choosing a rank  $r$  extension of  $\omega$  at random”. The main result below is an assertion about the probability of entanglement in the various probability spaces  $(E^r(\omega), P^{r,\omega})$ , namely that the probability of entanglement is 1 when  $r$  is relatively small (see Theorem 9.1 and Remark 9.2). We also point out in Theorem 10.1 that this behavior does not persist through large values of  $r$ , since for  $r = m \cdot \text{rank } \omega$ , the probability  $p$  of entanglement is shown to be positive (and  $< 1$ ).

*Remark 1.1.* (Terminology and conventions). Let  $H$  be a finite dimensional Hilbert space. A state  $\rho$  of  $\mathcal{B}(H)$  has an associated density operator  $A \in \mathcal{B}(H)$ , defined by  $\rho(X) = \text{trace}(AX)$ ,  $X \in \mathcal{B}(H)$ . In the literature of quantum information theory, the operation of restricting  $\rho$  to a subfactor  $\mathcal{N} \subseteq \mathcal{B}(H)$  corresponds to a “partial tracing” operation on its density operator, in which  $A \in \mathcal{B}(H)$  is mapped to the operator  $\bar{A} \in \mathcal{N}$  that is defined uniquely by

$$\rho(Y) = \text{trace}_{\mathcal{N}}(\bar{A}Y), \quad Y \in \mathcal{N}, \quad (1.1)$$

where  $\text{trace}_{\mathcal{N}}$  denotes the trace of  $\mathcal{N}$  normalized so that it takes the value 1 on minimal projections of  $\mathcal{N}$ . In more operator-algebraic terms, the partial trace of  $A$  is  $\bar{A} = \mu \cdot E(A)$ , where  $E : \mathcal{B}(H) \rightarrow \mathcal{N}$  is the conditional expectation defined by the trace of  $\mathcal{B}(H)$  (with any normalization) and  $\mu$  is the multiplicity of the representation of  $\mathcal{N}$  associated with the inclusion  $\mathcal{N} \subseteq \mathcal{B}(H)$ . The constant  $\mu$  is forced on the formula  $\bar{A} = \mu \cdot E(A)$  by the normalization specified for  $\text{trace}_{\mathcal{N}}$  in (1.1), and this non-invariant feature of (1.1) leads to a problem if one attempts to interpret it for more general  $*$ -subalgebras  $\mathcal{N} \subseteq \mathcal{B}(H)$ . More significantly, the right side of (1.1) loses all meaning for type III subfactors  $\mathcal{N} \subseteq \mathcal{B}(H)$  when  $H$  is infinite dimensional - a situation of some importance for algebraic quantum field theory. We choose to avoid such issues by dealing with restrictions and extensions of states rather than partial traces of operators and their inverse images.

*Remark 1.2.* (Literature and related results). A significant part of the literature of physics and quantum information theory makes some connection with probabilistic aspects of entanglement. The following papers (and references therein) represent a sample. The papers [Sza04, AS06] concern Hilbert spaces  $H_N = (\mathbb{C}^2)^{\otimes N}$  for large  $N$ , and sharp estimates are obtained for the smallness of the ratio of the volume of separable states to the volume of all states. In [Par04], the maximal dimension of a linear subspace of  $H_1 \otimes \dots \otimes H_N$  that contains no nonzero product vectors is calculated, and in [HLW06] it is shown that random subspaces of  $H \otimes K$  are likely to contain only near-maximally entangled vectors. [Loc00] discusses “minimal” decompositions for separable states into convex combinations of pure product states (also see [Uhl98, STV98]). The survey

[PR02] also deserves mention. For early results on the existence of a separable ball in the state space see [BCJ<sup>+</sup>99]. A probabilistic study of separable states is carried out in [ZHS98], where lower and upper bounds are obtained for the probability of the set of separable states. Those authors make use of a rather different probability space, and there appears to be negligible overlap between [ZHS98] and this paper. Finally, the paper [PGWP<sup>+</sup>08] concerning maximal violations of Bell's inequalities for tripartite systems certainly bears on issues of entanglement. Most of the above developments focus on obtaining asymptotic estimates of the volume of separable states. The following results differ in that they make concrete assertions in *all* dimensions.

In order to prove Theorem 9.1, we introduce a numerical invariant of states of tensor products of matrix algebras - called the wedge invariant - that can detect entanglement. We now give a precise definition of the wedge invariant, deferring proofs to later sections, and follow that with some general remarks on how the wedge invariant enters into the proof of Theorem 9.1.

Its definition requires that we work with operators rather than matrices, hence we shift attention to states  $\rho$  defined on concrete operator algebras  $\mathcal{B}(K) \otimes \mathcal{B}(H) \cong \mathcal{B}(K \otimes H)$ , where  $H$  and  $K$  are finite dimensional Hilbert spaces. Fix a state  $\rho$  of  $\mathcal{B}(K \otimes H)$ , let  $r$  be the rank of its density operator, and choose vectors  $\zeta_1, \dots, \zeta_r \in K \otimes H$  such that

$$\rho(x) = \sum_{k=1}^r \langle x \zeta_k, \zeta_k \rangle, \quad x \in \mathcal{B}(K \otimes H). \quad (1.2)$$

The vectors  $\zeta_k$  need not be eigenvectors of the density operator of  $\rho$ , but necessarily they are linearly independent. Let  $\omega$  be the state of  $\mathcal{B}(H)$  defined by restriction

$$\omega(x) = \rho(\mathbf{1}_K \otimes x), \quad x \in \mathcal{B}(H). \quad (1.3)$$

The rank of  $\omega$  depends on  $\rho$ , and can be any integer from 1 to  $n = \dim H$ . Fix a Hilbert space  $K_0$  of dimension rank  $\omega$ , such as  $K_0 = \mathbb{C}^{\text{rank } \omega}$ . The basic GNS construction applied to  $\omega$ , together with the representation theory of matrix algebras, leads to the existence of a unit vector  $\xi \in K_0 \otimes H$  that is cyclic for the algebra  $\mathbf{1}_{K_0} \otimes \mathcal{B}(H)$ , and has the property

$$\omega(x) = \langle (\mathbf{1}_{K_0} \otimes x) \xi, \xi \rangle, \quad x \in \mathcal{B}(H). \quad (1.4)$$

We point out that this procedure of passing from  $\omega$  to the vector state defined by  $\xi$  is known as *purification* in the physics literature.

Fixing such a unit vector  $\xi$ , we define an  $r$ -tuple of operators  $v_1, \dots, v_r$  as follows. Because of (1.3) and (1.4), one can show that for each  $k = 1, \dots, r$  there is a unique operator  $v_k : K_0 \rightarrow K$  such that

$$(v_k \otimes x) \xi = (\mathbf{1}_K \otimes x) \zeta_k, \quad x \in \mathcal{B}(H),$$

and one finds that  $v_1, \dots, v_r \in \mathcal{B}(K_0, K)$  satisfies  $v_1^* v_1 + \dots + v_r^* v_r = \mathbf{1}_{K_0}$ . The  $r$ -tuple  $(v_1, \dots, v_r)$  depends on the choice of  $\zeta_1, \dots, \zeta_r$  as well as the choice of  $\xi \in K_0 \otimes H$ . But it is also a fact that if  $\zeta'_1, \dots, \zeta'_r$  is another set of  $r$  vectors that satisfies (1.2) and  $\xi'$  is

another cyclic vector satisfying (1.4), then the resulting  $r$ -tuple of operators  $(v'_1, \dots, v'_r)$  is related to  $(v_1, \dots, v_r)$  as follows:

$$v'_i = \sum_{j=1}^r \lambda_{ij} v_j w, \quad 1 \leq i \leq r, \quad (1.5)$$

where  $(\lambda_{ij})$  is a unitary  $r \times r$  matrix of scalars and  $w$  is a unitary operator in  $\mathcal{B}(K_0)$  (see Sect. 8).

For every choice of integers  $i_1, \dots, i_r$  with  $1 \leq i_1, \dots, i_r \leq r$  the tensor product of operators  $v_{i_1} \otimes \dots \otimes v_{i_r}$  belongs to  $\mathcal{B}(K_0^{\otimes r}, K^{\otimes r})$ . Hence we can define an operator  $v_1 \wedge \dots \wedge v_r \in \mathcal{B}(K_0^{\otimes r}, K^{\otimes r})$  as the alternating average

$$v_1 \wedge \dots \wedge v_r = \frac{1}{|G|} \sum_{\pi \in G} (-1)^\pi v_{\pi(1)} \otimes \dots \otimes v_{\pi(r)}, \quad (1.6)$$

the sum extended over the group  $G$  of all permutations  $\pi$  of  $\{1, \dots, r\}$ . The permutation group  $G$  acts naturally as unitary operators on both  $K_0^{\otimes r}$  and  $K^{\otimes r}$ , and we may form their symmetric and antisymmetric subspaces. For example, in terms of the unitary representation  $\pi \mapsto U_\pi$  of  $G$  on  $K^{\otimes r}$ ,

$$\begin{aligned} K_+^{\otimes r} &= \{\zeta \in K^{\otimes r} : U_\pi \zeta = \zeta, \quad \pi \in G\}, \\ K_-^{\otimes r} &= \{\zeta \in K^{\otimes r} : U_\pi \zeta = (-1)^\pi \zeta, \quad \pi \in G\}. \end{aligned}$$

The operator  $v_1 \wedge \dots \wedge v_r$  maps the symmetric subspace of  $K_0^{\otimes r}$  to the antisymmetric subspace of  $K^{\otimes r}$ , hence its restriction to  $K_{0+}^{\otimes r}$  is an operator in  $\mathcal{B}(K_{0+}^{\otimes r}, K_-^{\otimes r})$ . This operator also depends on the choice of  $\xi, \eta_1, \dots, \eta_r$ . However, because of (1.5), the rank of  $v_1 \wedge \dots \wedge v_r \upharpoonright_{K_{0+}^{\otimes r}}$  is a well-defined nonnegative integer that we associate with the state  $\rho$ ,

$$w(\rho) = \text{rank}(v_1 \wedge \dots \wedge v_r \upharpoonright_{K_{0+}^{\otimes r}}).$$

In a similar way, we may form the wedge product of the  $r$ -tuple of adjoints  $v_k^* : K \rightarrow K_0$  to obtain an operator  $v_1^* \wedge \dots \wedge v_r^* \in \mathcal{B}(K^{\otimes r}, K_0^{\otimes r})$ , and restrict it to the symmetric subspace  $K_+^{\otimes r} \subseteq K^{\otimes r}$  to obtain a second integer  $w^*(\rho) = \text{rank}(v_1^* \wedge \dots \wedge v_r^* \upharpoonright_{K_+^{\otimes r}})$ . Thus we can make the following

**Definition 1.3.** *The wedge invariant of a state  $\rho$  of  $\mathcal{B}(K \otimes H)$  is defined as the pair of nonnegative integers  $(w(\rho), w^*(\rho))$ , where*

$$w(\rho) = \text{rank}(v_1 \wedge \dots \wedge v_r \upharpoonright_{K_{0+}^{\otimes r}}), \quad w^*(\rho) = \text{rank}(v_1^* \wedge \dots \wedge v_r^* \upharpoonright_{K_+^{\otimes r}}).$$

The wedge invariant has two principal features. First, it is capable of detecting entanglement because of the following result of Sect. 8:

**Theorem 1.4.** *If  $\rho$  is a separable state of  $\mathcal{B}(K \otimes H)$ , then  $w(\rho) \leq 1$  and  $w^*(\rho) \leq 1$ .*

This separability criterion differs fundamentally from others that involve positive linear maps (see [Per96, Sto07], or the survey [HHHH07]).

The second feature of the wedge invariant is that it is associated with subvarieties of the real algebraic varieties that will be used to parameterize states in the following sections. To illustrate that geometric feature in broad terms, let  $Y$  and  $Z$  be finite-dimensional complex vector spaces, let  $\mathcal{B}(Y, Z)$  be the space of all linear operators from  $Y$  to  $Z$ , and consider the set  $\mathcal{B}(Y, Z)^r$  of all  $r$ -tuples  $v = (v_1, \dots, v_r)$  with components  $v_k \in \mathcal{B}(Y, Z)$ . Then for every  $k = 1, 2, \dots$ , the set of  $r$ -tuples

$$W^r(k) = \{v = (v_1, \dots, v_r) \in \mathcal{B}(Y, Z)^r : \text{rank}(v_1 \wedge \dots \wedge v_r \upharpoonright_{Y_+^{\otimes r}}) \leq k\}$$

is an algebraic set - namely the set of common zeros of a finite set  $f_1, \dots, f_p$  of real-homogeneous multivariate polynomials  $f_k : \mathcal{B}(Y, Z)^r \rightarrow \mathbb{R}$ . This leads to the following fact that provides a key step in the proof of Theorem 9.1 below: *Let  $r = 1, 2, \dots$  and let  $M$  be a  $d$ -dimensional connected real-analytic submanifold of  $\mathcal{B}(Y, Z)^r$  that contains a point  $(v_1, \dots, v_r) \in M$  for which*

$$\text{rank}(v_1 \wedge \dots \wedge v_r \upharpoonright_{Y_+^{\otimes r}}) > k \quad (1.7)$$

*for some  $k \geq 1$ . Then (1.7) is generic in the sense that for every relatively open subset  $U \subseteq M$  endowed with real-analytic coordinates,  $U \cap W^r(k)$  is a set of  $d$ -dimensional Lebesgue measure zero.*

The methods we use are a mix of matrix/operator theory, convexity, and basic real algebraic geometry. In Sect. 11, we offer some general remarks that address the broader issue of whether one can expect an effective “real-analytic” characterization of entanglement in general. For the reader’s convenience we have included two appendices containing formulations of some known results about real-analytic varieties of matrices that are fundamental for our main results. Finally, a significant part of the background material of Sects. 4 and 5 can be found scattered throughout the literature of operator algebras or quantum information theory (references to some of it can be found in the survey [HHHH07]). We have included proofs of everything we need for readability, and to set the point of view that we take for the main results later on.

We also point out that further applications to completely positive maps on matrix algebras are developed in a sequel to this paper [Arv08].

## 2. The Noncommutative Spheres $V^r(n, m)$

Let  $m, n$  be positive integers with  $m \geq n$ . For every  $r = 1, 2, \dots$ , we work with the space  $V^r(n, m)$  of all  $r$ -tuples  $v = (v_1, \dots, v_r)$  of complex  $m \times n$  matrices  $v_k$  such that

$$v_1^* v_1 + \dots + v_r^* v_r = \mathbf{1}_n. \quad (2.1)$$

There is a natural left action of the unitary group  $U(rm)$  on  $V^r(n, m)$ , defined as follows. An element of  $U(rm)$  can be viewed as a unitary  $r \times r$  matrix  $w = (w_{ij})$  with entries  $w_{ij}$  in the matrix algebra  $M_m(\mathbb{C})$ , and it acts on an element  $v = (v_1, \dots, v_r) \in V^r(n, m)$  by way of  $w \cdot v = v'$ , where

$$v'_i = \sum_{j=1}^r w_{ij} v_j, \quad 1 \leq i \leq r. \quad (2.2)$$

There is also a right action of the unitary group  $U(n)$  on  $V^r(n, m)$ , in which  $u \in U(n)$  acts on  $v \in V^r(n, m)$  by  $(v_1, \dots, v_r) \cdot u = (v_1 u, \dots, v_r u)$ . Both actions are better understood in terms of operators, after the identifications of the following paragraph have been made.

**2.1. The varieties  $V^r(H, K)$ .** Note that  $n$  precedes  $m$  in the notation for  $V^r(n, m)$ . This convention arises from the interpretation of  $V^r(n, m)$  as a space of operators rather than matrices. If  $H$  and  $K$  are complex Hilbert spaces of respective dimensions  $n$  and  $m$ , then the space  $V^r(H, K)$  of all  $r$ -tuples of operators  $v = (v_1, \dots, v_r)$  with components  $v_k \in \mathcal{B}(H, K)$  that satisfy the counterpart of (2.1),

$$v_1^* v_1 + \dots + v_r^* v_r = \mathbf{1}_H, \quad (2.3)$$

can be identified with  $V^r(n, m)$  after making a choice of orthonormal bases for both  $H$  and  $K$ , and all statements about  $V^r(n, m)$  have appropriate counterparts in the more coordinate-free context of the spaces  $V^r(H, K)$ . Throughout this paper, it will serve our purposes better to interpret  $V^r(n, m)$  as the space of  $r$ -tuples of operators  $V^r(H, K)$ .

$V^r(H, K)$  is a compact subspace of the complex vector space  $\mathcal{B}(H, K)^r$  of all  $r$ -tuples of operators  $v = (v_1, \dots, v_r)$  with components in  $\mathcal{B}(H, K)$ , on which the unitary group  $U(r \cdot K)$  of the direct sum  $r \cdot K$  of  $r$  copies of  $K$  acts smoothly on the left. Because of the presence of the  $*$ -operation in (2.3), we can also view the ambient space  $\mathcal{B}(H, K)^r$  as a finite dimensional *real* vector space, endowed with the (real) inner product

$$\langle (v_1, \dots, v_r), (w_1, \dots, w_r) \rangle = \Re \sum_{k=1}^r \text{trace } w_k^* v_k, \quad v, w \in \mathcal{B}(H, K)^r. \quad (2.4)$$

The following result summarizes the geometric structure that  $V^r(H, K)$  inherits from its ambient space, when  $H$  and  $K$  are Hilbert spaces satisfying  $n = \dim H \leq m = \dim K < \infty$ .

**Theorem 2.1.** *For every  $r = 1, 2, \dots$ , the space  $V^r(H, K)$  is a compact, connected, real-analytic Riemannian manifold of dimension  $d = n(2rm - n)$ , on which the unitary group  $\mathcal{U}(r \cdot K)$  acts as a transitive group of isometries. In particular, the natural measure associated with its Riemannian metric is proportional to the unique probability measure on  $V^r(H, K)$  that is invariant under the transitive  $\mathcal{U}(r \cdot K)$ -action.*

*Proof.* We identify the space  $\mathcal{B}(H, K)^r$  of  $r$ -tuples of operators as the space  $\mathcal{B}(H, r \cdot K)$  of all operators from  $H$  into the direct sum  $r \cdot K$  of  $r$  copies of  $K$ , in which an  $r$ -tuple  $v = (v_1, \dots, v_r)$  of operators in  $\mathcal{B}(H, K)$  is identified with the single operator  $\tilde{v} : H \rightarrow r \cdot K$  defined by

$$\tilde{v}\xi = (v_1\xi, \dots, v_r\xi), \quad \xi \in H.$$

After this identification,  $V^r(H, K)$  becomes the space of all isometries in  $\mathcal{B}(H, r \cdot K)$ , and Theorem A.2 implies that  $V^r(H, K)$  inherits the structure of a connected real-analytic submanifold of the ambient real vector space  $\mathcal{B}(H, r \cdot K) \cong \mathcal{B}(H, K)^r$  in which it is embedded, and that the unitary group  $\mathcal{U}(r \cdot K)$  acts transitively on it by left multiplication.

The inner product (2.4) on  $\mathcal{B}(H, K)^r$  restricts so as to give a Riemannian metric on the tangent bundle of  $V^r(H, K)$ , thereby making it into a compact Riemannian manifold.

Notice that the action of  $\mathcal{U}(r \cdot K)$  is actually defined on the larger inner product space  $\mathcal{B}(H, K)^r$ , and its action on  $\mathcal{B}(H, K)^r$  is by isometries. Indeed, let  $u \in \mathcal{U}(r \cdot K)$ , and view  $u$  as an  $r \times r$  matrix  $(u_{ij})$  of operators  $u_{ij}$  in  $\mathcal{B}(K)$ . Choosing  $v, w \in \mathcal{B}(H, K)^r$  and setting  $v' = u \cdot v$  and  $w' = u \cdot w$  as in (2.2), then  $\sum_k u_{ki}^* u_{kj} = \delta_{ij} \mathbf{1}_K$  because  $u = (u_{ij})$  is unitary, hence

$$\begin{aligned} \langle v', w' \rangle &= \Re \sum_{k=1}^r \text{trace}(w_k'^* v_k') = \Re \sum_{i,j,k=1}^r \text{trace}(w_i^* u_{ki}^* u_{kj} v_j) \\ &= \Re \sum_{i=1}^r \text{trace}(w_i^* v_i) = \langle v, w \rangle. \end{aligned}$$

Hence  $\mathcal{U}(r \cdot K)$  acts as isometries on the Riemannian submanifold  $V^r(H, K)$ .

Finally, the dimension calculation amounts to little more than subtracting the number of real equations appearing in the matrix equation (2.1) from the real dimension  $\dim_{\mathbb{R}}(\mathcal{B}(H, K)^r)$  of the vector space  $\mathcal{B}(H, K)^r$ .  $\square$

*Remark 2.2.* [Right action of  $\mathcal{U}(H)$  on  $V^r(H, K)$ ]. The right action of the unitary group  $\mathcal{U}(H)$  on  $r$ -tuples of operators in  $\mathcal{B}(H, K)^r$  is defined by

$$(v, w) \in \mathcal{B}(H, K)^r \times \mathcal{U}(H) \mapsto v \cdot w = (v_1 w, \dots, v_r w).$$

This action of  $\mathcal{U}(H)$  commutes with the left action of  $\mathcal{U}(r \cdot K)$  and it preserves the inner product of  $\mathcal{B}(H, K)^r$ . Hence it restricts to a right action of  $\mathcal{U}(H)$  on  $V^r(H, K)$  that commutes with the transitive left action, and which also acts as isometries relative to the Riemannian structure of  $V^r(H, K)$ .

*Remark 2.3.* [The invariant measure class of  $V^r(H, K)$ ]. Perhaps it is unnecessary to point out that the natural measure class of  $V^r(H, K)$  is that of Lebesgue measure in local coordinates; more precisely, *relative to real-analytic local coordinates on an open subset of  $V^r(H, K)$ , the measure  $\mu$  associated with the Riemannian metric is mutually absolutely continuous with the transplant of Lebesgue measure to that chart.*

**2.2. Subvarieties of  $V^r(H, K)$ .** There is an intrinsic notion of real-analytic function  $f : V^r(H, K) \rightarrow \mathbb{R}$ , namely a function such that for every real-analytic isomorphism  $u : D \rightarrow U$  of an open ball  $D \subseteq \mathbb{R}^d$  onto an open set  $U \subseteq V^r(H, K)$ ,  $f \circ u$  is a real-analytic function on  $D$  (see Appendix 12). Similarly, given a finite dimensional real vector space  $W$ , one can speak of real-analytic functions

$$F : V^r(H, K) \rightarrow W, \tag{2.5}$$

and though it is rarely necessary to do so, one can reduce the analysis of such vector functions to that of  $k$ -tuples of real-valued analytic functions by composing  $F$  with a basis of linear functionals  $\rho_1, \dots, \rho_k$  for the dual of  $W$ .

**Remark 2.4.** (Homogeneous polynomials). Virtually all of the analytic functions (2.5) that we will encounter are obtained by restricting homogeneous polynomials defined on the ambient space  $\mathcal{B}(H, K)^r$  to  $V^r(H, K)$ . Let  $V$  and  $W$  be finite dimensional real vector spaces. A map  $F : V \rightarrow W$  is said to be a real homogeneous polynomial (of degree  $k$ ) if it has the form  $F(v) = G(v, v, \dots, v)$ , where  $G$  is a real multilinear mapping  $G : V^k \rightarrow W$  in  $k$  variables. Though this terminology is slightly abusive in that the zero function qualifies as a homogeneous polynomial of every positive degree, it will not cause problems in this paper. A function  $F : V \rightarrow W$  is a homogeneous polynomial of degree  $k$  iff  $\rho \circ F$  is a scalar-valued homogeneous polynomial of degree  $k$  for every linear functional  $\rho : W \rightarrow \mathbb{R}$ .

**Definition 2.5.** By a **subvariety** of  $V^r(H, K)$  we mean a subspace  $Z$  of  $V^r(H, K)$  of the form

$$Z = \{v \in V^r(H, K) : F(v) = 0\},$$

where  $F : V^r(H, K) \rightarrow W$  is a real-analytic function taking values in some finite-dimensional real vector space  $W$ .

Subvarieties are obviously compact. As a concrete example, the set

$$Z = \{v = (v_1, \dots, v_r) \in V^r(H, K) : \text{rank } v_1 \leq 2\}$$

is the zero subvariety associated with the restriction to  $V^r(H, K)$  of the cubic homogeneous polynomial  $F : \mathcal{B}(H, K)^r \rightarrow \mathcal{B}(\wedge^3 H, \wedge^3 K)$ , where

$$F(v) = (v_1 \otimes v_1 \otimes v_1) \upharpoonright_{H \wedge H \wedge H}.$$

**Proposition 2.6.** Let  $Z$  be a subvariety of  $V^r(H, K)$  and let  $\mu$  be the natural measure of  $V^r(H, K)$ . If  $Z \neq V^r(H, K)$ , then  $\mu(Z) = 0$ .

*Proof.* Let  $F : V^r(H, K) \rightarrow W$  be a real-analytic function taking values in a finite dimensional real vector space such that

$$Z = \{v \in V^r(H, K) : F(v) = 0\}.$$

$F$  cannot vanish identically because  $Z \neq V^r(H, K)$ ; and since  $V^r(H, K)$  is connected and  $F$  is real-analytic, it cannot vanish identically on any nonempty open subset of  $V^r(H, K)$ .

Let  $d = \dim(V^r(H, K))$  and let  $\mu$  be the natural measure of  $V^r(H, K)$  associated with its Riemannian metric. To show that  $\mu(Z) = 0$ , it suffices to show that every point of  $V^r(H, K)$  has a neighborhood  $U$  such that  $\mu(U \cap Z) = 0$ . To prove that, fix a point  $v \in V^r(H, K)$  and choose an open neighborhood  $U$  of  $v$  that can be coordinatized by the open unit ball  $B \subseteq \mathbb{R}^d$  by way of a real-analytic isomorphism  $u : B \rightarrow U$  (see Appendix 12). The composition  $F \circ u : B \rightarrow W$  is a real-analytic mapping that does not vanish identically on  $B$ , hence there is a real-linear functional  $\rho : W \rightarrow \mathbb{R}$  such that  $\rho \circ F \circ u$  does not vanish identically on  $B$ . Since  $\rho \circ F \circ u$  is a real-valued analytic function of its variables, Proposition B.1 implies that the set  $\tilde{Z}$  of its zeros has Lebesgue measure zero. It follows that  $u(\tilde{Z}) \subseteq U$  is a set of  $\mu$ -measure zero that contains  $U \cap Z$ , hence  $\mu(U \cap Z) = 0$ .  $\square$



### 3. The Unbiased Probability Spaces $(X^r, P^r)$

Let  $H, K$  be Hilbert spaces, with  $n = \dim H \leq m = \dim K < \infty$ . In Sect. 6, we will show that the spaces  $V^r(H, K)$  can be used to parameterize states of  $\mathcal{B}(K \otimes H)$ . The parameterizing map is not injective, but it promotes naturally to an injective map of a quotient  $X^r$  of  $V^r(H, K)$ . We now introduce these spaces  $X^r$  and we show that each of them carries a unique unbiased probability measure  $P^r$ , so that  $(X^r, P^r)$  becomes a topological probability space that serves to parameterize states faithfully. In this section we summarize the basic properties of these probability spaces and discuss some of the random variables that will enter into the analysis of states later on.

The group  $U(r)$  of all scalar  $r \times r$  unitary matrices in  $M_r(\mathbb{C})$  is identified with a subgroup of  $\mathcal{U}(r \cdot K)$ —consisting of unitary operator matrices with components in  $\mathbb{C} \cdot \mathbf{1}_K$ , hence it acts naturally on  $V^r(H, K)$ , in which  $\lambda = (\lambda_{ij}) \in U(r)$  acts on  $v = (v_1, \dots, v_r) \in V^r(H, K)$  by way of  $\lambda \cdot v = v'$ , where

$$v'_i = \sum_{j=1}^r \lambda_{ij} v_j, \quad i = 1, 2, \dots, r. \quad (3.1)$$

Since  $U(r)$  is compact and acts smoothly on  $V^r(H, K)$ , its orbit space is a compact metrizable space  $X^r$ . Moreover, the natural projection

$$v \in V^r(H, K) \mapsto \dot{v} \in X^r$$

is a continuous surjection with the following universal property that we will use repeatedly: *For every topological space  $Y$  and every continuous function  $f : V^r(H, K) \rightarrow Y$  satisfying  $f(\lambda \cdot v) = f(v)$  for  $\lambda \in U(r)$ ,  $v \in V^r(H, K)$ , there is a unique continuous function  $\hat{f} : X^r \rightarrow Y$  such that  $\hat{f}(\dot{v}) = f(v)$ ,  $v \in V^r(H, K)$ .* Note too that the commutative  $C^*$ -algebra  $C(X^r)$  is isomorphic to the  $C^*$ -subalgebra  $A \subseteq C(V^r(H, K))$  of functions  $f \in C(V^r(H, K))$  that satisfy  $f(\lambda \cdot v) = f(v)$  for  $\lambda \in U(r)$ ,  $v \in V^r(H, K)$ .

It follows that the quotient space  $X^r$  carries a unique unbiased probability measure  $P^r$  that is defined on Borel subsets  $E$  by promoting the unique invariant probability measure  $\mu$  of  $V^r(H, K)$ ,

$$P^r(E) = \mu\{v \in V^r(H, K); \dot{v} \in E\}, \quad E \subseteq X^r.$$

Equivalently, in terms of the identification  $C(X^r) \cong A \subseteq C(V^r(H, K))$  of the previous paragraph,  $P^r$  is the measure on the Gelfand spectrum  $X^r$  of  $A$  that the Riesz-Markov theorem associates with the state

$$\rho(f) = \int_{V^r(H, K)} f(v) d\mu(v), \quad f \in A.$$

In this way we obtain a compact metrizable probability space  $(X^r, P^r)$ . Notice that  $(X^r, P^r)$  depends not only on  $r$ , but also  $H$  and  $K$  - or at least on their dimensions  $n$  and  $m$ . However, since  $H$  and  $K$  will be fixed throughout the discussions to follow, we can safely lighten notation by omitting reference to these extra parameters.

*Remark 3.1.* (Right action of  $\mathcal{U}(H)$  on  $X^r$ ). Note that while the left action of the larger group  $\mathcal{U}(r \cdot K)$  acts transitively on  $V^r(H, K)$ , that symmetry is lost when one passes to the orbit space  $X^r$  because  $U(r)$  is not a normal subgroup of  $\mathcal{U}(r \cdot K)$ . On the other

hand, the right action of  $\mathcal{U}(H)$  on  $V^r(H, K)$  does promote naturally to a right action on  $X^r$ . Moreover, since the right action on  $V^r(H, K)$  preserves the Riemannian metric, it also preserves the natural measure  $\mu$  of  $V^r(H, K)$ . We conclude: *The right action of the unitary group  $\mathcal{U}(H)$  on  $X^r$  gives rise to a compact group of measure-preserving homeomorphisms of the topological probability space  $(X^r, P^r)$ .*

**Remark 3.2.** (The rank variable). We begin by defining a random variable

$$\text{rank} : X^r \rightarrow \{1, 2, \dots, r\}.$$

For  $v = (v_1, \dots, v_r) \in V^r(H, K)$ , let  $\mathcal{S}_v = \text{span}\{v_1, \dots, v_r\}$  be the complex linear subspace of  $\mathcal{B}(H, K)$  spanned by its component operators. Elementary linear algebra shows that  $\mathcal{S}_{\lambda \cdot v} = \mathcal{S}_v$  for every  $\lambda = (\lambda_{ij}) \in U(r)$ , and in particular the dimension of  $\mathcal{S}_v$  depends only on the image  $\dot{v}$  of  $v$  in  $X^r$ . Hence we can define a function  $\text{rank} : X^r \rightarrow \{1, 2, \dots, r\}$  by

$$\text{rank}(\dot{v}) = \dim \mathcal{S}_v, \quad v \in V^r(H, K). \quad (3.2)$$

Since the function  $v \mapsto \dim \mathcal{S}_v$  is lower semicontinuous in the sense that  $\{v \in V^r(H, K) : \dim \mathcal{S}_v \leq k\}$  is closed for every  $k$ , it follows that the rank function is Borel-measurable, and hence defines a random variable. Moreover, since  $\dim \mathcal{S}_{v \cdot w} = \dim \mathcal{S}_v$  for every  $w \in \mathcal{U}(H)$ , the rank variable is invariant under the right action of  $\mathcal{U}(H)$  on  $X^r$ .

Significantly, rank is almost surely constant throughout  $X^r$ :

**Theorem 3.3.** *For every  $r = 1, 2, \dots, mn$ ,  $P^r\{x \in X^r : \text{rank}(x) \neq r\} = 0$ .*

The proof of Theorem 3.3 requires:

**Lemma 3.4.** *For every  $r = 1, 2, \dots, mn$ ,  $V^r(H, K)$  contains an  $r$ -tuple  $v = (v_1, \dots, v_r)$  with linearly independent component operators  $v_1, \dots, v_r$ .*

*Proof.* Fixing  $r$ ,  $1 \leq r \leq mn$ , we claim first that there is a linearly independent set of operators  $a_1, \dots, a_r : H \rightarrow K$  such that

$$\ker a_1 \cap \dots \cap \ker a_r = \{0\}. \quad (3.3)$$

Indeed, since  $\dim \mathcal{B}(H, K) = mn \geq r$ , we can find a linearly independent subset  $b_1, \dots, b_r \in \mathcal{B}(H, K)$ . Set  $H_0 = \ker b_1 \cap \dots \cap \ker b_r$  and let  $r \cdot K$  be the direct sum of  $r$  copies of  $K$ . The linear operator  $B : \xi \in H \mapsto (b_1\xi, \dots, b_r\xi) \in r \cdot K$  has kernel  $H_0$ , hence  $\dim BH + \dim H_0 = n \leq m = \dim K \leq \dim(r \cdot K)$ , and therefore  $\dim H_0 \leq \dim(r \cdot K) - \dim BH$ . Hence there is a partial isometry  $B'$  in  $\mathcal{B}(H, r \cdot K)$  with initial space  $H_0$  and final space contained in  $BH^\perp$ . Writing  $B'\xi = (b'_1\xi, \dots, b'_r\xi)$  with  $b'_k \in \mathcal{B}(H, K)$ , we set

$$a_1 = b_1 + b'_1, \quad a_2 = b_2 + b'_2, \dots, \quad a_r = b_r + b'_r.$$

These operators restrict to a linearly independent set of operators from  $H_0^\perp$  into  $K$ , hence they are a linearly independent subset of  $\mathcal{B}(H, K)$ ; and since the operator  $B + B' \in \mathcal{B}(H, r \cdot K)$  has trivial kernel, (3.3) follows.

Fix such an  $r$ -tuple  $a_1, \dots, a_r$ . Then  $a_1^*a_1 + \dots + a_r^*a_r$  is an invertible operator in  $\mathcal{B}(H)$ , and we can define a new  $r$ -tuple  $v_1, \dots, v_r$  in  $\mathcal{B}(H, K)$  by

$$v_k = a_k(a_1^*a_1 + \dots + a_r^*a_r)^{-1/2}, \quad k = 1, \dots, r.$$

The operators  $v_k$  are also linearly independent, and by its construction, the  $r$ -tuple  $v = (v_1, \dots, v_r)$  belongs to  $V^r(H, K)$ .  $\square$

*Proof of Theorem 3.3.* Consider the function  $F : V^r(H, K) \rightarrow \wedge^r \mathcal{B}(H, K)$  obtained by restricting the homogeneous polynomial defined on  $\mathcal{B}(H, K)^r$ ,

$$F(v) = v_1 \wedge \cdots \wedge v_r, \quad v = (v_1, \dots, v_r) \in \mathcal{B}(H, K)^r,$$

to the submanifold  $V^r(H, K)$ . Obviously,  $F$  is real-analytic, and elementary multilinear algebra implies that for every  $v = (v_1, \dots, v_r) \in V^r(H, K)$ ,

$$\{v_1, \dots, v_r\} \text{ is linearly dependent } \iff v_1 \wedge \cdots \wedge v_r = 0.$$

Hence  $\dim \mathcal{S}_v < r \iff F(v) = 0$ . It follows from Lemma 3.4 that the polynomial  $F$  does not vanish identically on  $V^r(H, K)$ , so by Proposition 2.6, its zero variety  $Z = \{v \in V^r(H, K) : F(v) = 0\}$  is a closed subset of  $V^r(H, K)$  of  $\mu$ -measure zero. Moreover,  $Z$  is invariant under the left action of  $U(r)$  on  $V^r(H, K)$  because for  $\lambda \in U(r)$ ,  $v = (v_1, \dots, v_r) \in V^r(H, K)$  and  $\lambda \cdot v = (v'_1, \dots, v'_r)$  as in (3.1), we have

$$F(\lambda \cdot v) = v'_1 \wedge \cdots \wedge v'_r = \det(\lambda_{ij}) \cdot v_1 \wedge \cdots \wedge v_r = \det(\lambda_{ij}) \cdot F(v).$$

It follows that  $\dot{Z}$  is a closed set of probability zero in  $X^r$ ,

$$P^r(\{x \in X^r : \text{rank}(x) < r\}) = P^r(\dot{Z}) = \mu(Z) = 0,$$

and Theorem 3.3 follows.  $\square$

#### 4. Operators Associated with Extensions of States

Let  $H_0$  be a finite dimensional Hilbert space and let  $N \subseteq \mathcal{B}(H_0)$  be a subfactor - a  $*$ -subalgebra with trivial center that contains the identity operator. Every state  $\omega$  of  $N$  can be extended in many ways to a state of  $\mathcal{B}(H_0)$ . In this section we show that the range of the density operator of every extension  $\rho$  is linearly isomorphic to a certain operator space associated with the pair  $(\rho, \omega)$ . While this identification is technically straightforward, it seems not to be part of the lore of matrix algebras. The details follow.

For every state  $\omega$  of  $N$ , the set  $E(\omega)$  of all extensions of  $\omega$  to a state of  $\mathcal{B}(H_0)$  is a compact convex subset of the state space of  $\mathcal{B}(H_0)$ . We begin with some elementary observations that relate properties of  $\omega$  to properties of the various states in  $E(\omega)$ . The *support projection* of a state  $\rho$  of  $\mathcal{B}(H_0)$  is defined as the smallest projection  $p \in \mathcal{B}(H_0)$  such that  $\rho(p) = 1$ ; the range  $pH_0$  of the support projection of  $\rho$  is the same as the range of its density operator, and the dimension of that space is called the *rank* of  $\rho$ .

**Lemma 4.1.** *Let  $N \subseteq \mathcal{B}(H_0)$  be a subfactor, let  $\omega$  be a state of  $N$ , and let  $p$  be the smallest projection in  $N$  satisfying  $\omega(p) = 1$ . Then the range of the density operator of every state in  $E(\omega)$  is contained in  $pH_0$ .*

*Proof.* Choose  $\rho \in E(\omega)$ . Since  $p \in N$ , we have  $\rho(p) = \omega(p) = 1$ . It follows that the support projection  $q \in \mathcal{B}(H_0)$  of  $\rho$  satisfies  $q \leq p$ .  $\square$

**Remark 4.2.** (Extensions of faithful states). It is significant that for purposes of analyzing the structure of  $E(\omega)$ , one can restrict attention to extensions of faithful states  $\omega$ . Indeed, letting  $p$  be as in Lemma 4.1, we see that since every state in  $E(\omega)$  is supported in  $pH_0$ , it can be viewed as a state of  $\mathcal{B}(pH_0) = p\mathcal{B}(H_0)p$  that extends the faithful state defined by restricting  $\omega$  to the corner  $pNp \subseteq N$ . Since  $pNp$  is also a subfactor of  $\mathcal{B}(pH_0)$ , the asserted reduction is apparent.

**Remark 4.3.** (Commutants and tensor products). Let  $M = N'$  be the commutant of  $N$  in  $\mathcal{B}(H_0)$ .  $M$  is also a subfactor, and we can identify the  $C^*$ -algebra  $\mathcal{B}(H_0)$  with  $M \otimes N$ . Since we intend to discuss entanglement among the states of  $E(\omega)$ , it is better to view  $E(\omega)$  as the set of states  $\rho$  on the tensor product  $M \otimes N$  that satisfy

$$\rho(b) = \omega(\mathbf{1}_M \otimes b), \quad b \in N.$$

Having made these identifications, we are free to introduce new “coordinates” that realize  $M$  as  $\mathcal{B}(K)$ ,  $N$  as  $\mathcal{B}(H)$ , and  $M \otimes N$  as  $\mathcal{B}(K \otimes H)$ .

**Remark 4.4.** (Mixed states of  $N$ ). Since every extension of a pure state  $\omega$  of  $N$  to  $M \otimes N$  is easily seen to be separable, the separability problem has content only for extensions to  $M \otimes N$  of *mixed* states  $\omega$ . In view of Remark 4.2, we should analyze extensions of faithful states of  $N$  to  $M \otimes N$  in cases where  $N = \mathcal{B}(H)$  and  $\dim H \geq 2$ .

We collect the following elementary fact – a textbook exercise on the GNS construction and the representation theory of matrix algebras.

**Lemma 4.5.** *Let  $H$  be a finite-dimensional Hilbert space and let  $\omega$  be a state of  $\mathcal{B}(H)$  of rank  $r$ . Then there is a unit vector  $\xi_\omega \in \mathbb{C}^r \otimes H$  such that  $\omega(b) = \langle (\mathbf{1}_{\mathbb{C}^r} \otimes b)\xi_\omega, \xi_\omega \rangle$ ,  $b \in \mathcal{B}(H)$ , and  $\xi_\omega$  is a cyclic vector for the algebra  $\mathbf{1}_{\mathbb{C}^r} \otimes \mathcal{B}(H)$ . If  $\xi'_\omega$  is another vector in  $\mathbb{C}^r \otimes H$  with the same property, then there is a unique unitary operator  $w \in \mathcal{B}(\mathbb{C}^r)$  such that  $\xi'_\omega = (w \otimes \mathbf{1}_H)\xi_\omega$ .*

**Proposition 4.6.** *Let  $\omega$  be a state of  $\mathcal{B}(H)$ , let  $K_0$  be a Hilbert space of dimension  $\text{rank } \omega$ , and let*

$$\omega(b) = \langle (\mathbf{1}_{K_0} \otimes b)\xi_\omega, \xi_\omega \rangle, \quad b \in \mathcal{B}(H)$$

*be a representation of  $\omega$  with the properties of Lemma 4.5.*

*For every state  $\rho$  of  $\mathcal{B}(K \otimes H)$  that restricts to  $\omega$*

$$\rho(\mathbf{1}_K \otimes b) = \omega(b), \quad b \in \mathcal{B}(H),$$

*and for every vector  $\zeta$  in the range  $R$  of the density operator of  $\rho$ , there is a unique operator  $v \in \mathcal{B}(K_0, K)$  such that  $(v \otimes \mathbf{1}_H)\xi_\omega = \zeta$ . Moreover, the natural map  $v \mapsto (v \otimes \mathbf{1}_H)\xi_\omega$  from the operator space*

$$\mathcal{S} = \{v \in \mathcal{B}(K_0, K) : (v \otimes \mathbf{1}_H)\xi_\omega \in R\}$$

*to  $R$  defines an isomorphism of complex vector spaces  $\mathcal{S} \cong R$ . In particular,  $\text{rank } \rho = \dim \mathcal{S}$ .*

*Proof.* For existence of the operator  $v$ , we claim first that for every  $b \in \mathcal{B}(H)$ ,

$$(\mathbf{1}_{K_0} \otimes b)\xi_\omega = 0 \implies (\mathbf{1}_K \otimes b)\zeta = 0.$$

Indeed, if  $(\mathbf{1}_{K_0} \otimes b)\xi_\omega = 0$  then  $\omega(b^*b) = \|(\mathbf{1}_{K_0} \otimes b)\xi_\omega\|^2 = 0$ , so that  $bp = 0$ ,  $p$  being the support projection of  $\omega$ . Since  $\zeta$  belongs to the range of the support projection  $q$  of  $\rho$  and since  $q \leq \mathbf{1}_K \otimes p$  by Lemma 4.1, it follows that  $(\mathbf{1}_K \otimes b)\zeta = (\mathbf{1}_K \otimes b)(\mathbf{1}_K \otimes p)\zeta = (\mathbf{1}_K \otimes bp)\zeta = 0$ .

Hence we can define an operator  $\tilde{v} : K_0 \otimes H \rightarrow K \otimes H$  by

$$\tilde{v}(\mathbf{1}_{K_0} \otimes b)\xi_\omega = (\mathbf{1}_K \otimes b)\zeta, \quad b \in \mathcal{B}(H).$$

It is clear from its definition that  $\tilde{v}(\mathbf{1}_{K_0} \otimes b) = (\mathbf{1}_K \otimes b)\tilde{v}$  for  $b \in \mathcal{B}(H)$ , so that  $\tilde{v}$  admits a unique factorization  $\tilde{v} = v \otimes \mathbf{1}_H$  with  $v \in \mathcal{B}(K_0, K)$ , in the sense that  $\tilde{v}(\xi \otimes \eta) = v\xi \otimes \eta$ , for  $\xi \in K_0, \eta \in H$ .

Uniqueness of  $v$  is a straightforward consequence of the fact that  $\xi_\omega$  is cyclic for the algebra  $\mathbf{1}_{K_0} \otimes \mathcal{B}(H)$ . Finally, the last sentence is apparent from these assertions, since  $v \mapsto (v \otimes \mathbf{1}_H)\xi_\omega \in K \otimes H$  is a linear map.  $\square$

Proposition 4.6 leads to the following operator-theoretic criterion for separability. While it does not characterize the property, we will give an operator-theoretic characterization of separability later in Proposition 7.7. It has been pointed out that the following result can be read out of the assertions of Theorem 4 of [HSR03].

**Corollary 4.7.** *Let  $\omega, \xi_\omega, \rho, R$  and*

$$\mathcal{S} = \{v \in \mathcal{B}(K_0, K) : (v \otimes \mathbf{1}_H)\xi_\omega \in R\}$$

*be as in Proposition 4.6. Let  $v \in \mathcal{S}$  and let  $\zeta = (v \otimes \mathbf{1})\xi_\omega$ . Then  $\zeta$  has the form  $\zeta = \xi \otimes \eta$  for vectors  $\xi \in K, \eta \in H$  iff  $\text{rank}(v) \leq 1$ . If  $\rho$  is a separable state, then the operator space  $\mathcal{S}$  has a basis consisting of rank-one operators.*

*Proof.* Fix  $v \in \mathcal{S}$  and assume first that  $(v \otimes \mathbf{1})\xi_\omega$  decomposes into a tensor product  $\xi \otimes \eta$  for vectors  $\xi \in K, \eta \in H$ . We use the fact that  $\xi_\omega$  is cyclic for  $\mathbf{1}_{K_0} \otimes \mathcal{B}(H)$  to write

$$\begin{aligned} vK_0 \otimes H &= (v \otimes \mathbf{1}_H)(\mathbf{1}_{K_0} \otimes \mathcal{B}(H))\xi_\omega = (\mathbf{1}_K \otimes \mathcal{B}(H))(v \otimes \mathbf{1}_H)\xi_\omega \\ &= \xi \otimes \mathcal{B}(H)\eta = \xi \otimes H. \end{aligned}$$

It follows that  $vK_0 = \mathbb{C} \cdot \xi$ , as asserted. Conversely, if  $vK_0 = \mathbb{C} \cdot \xi$  for some  $\xi \in K$ , then  $(v \otimes \mathbf{1})\xi_\omega \in (v \otimes \mathbf{1})(K \otimes H) \subseteq \xi \otimes H$ , hence there is a vector  $\eta \in H$  such that  $(v \otimes \mathbf{1})\xi_\omega = \xi \otimes \eta$ .

If  $\rho$  is separable, then it can be written as a convex combination of pure separable states of  $\mathcal{B}(K \otimes H)$ , and this implies that  $R$  is spanned by vectors of the form  $\xi \otimes \eta$ , with  $\xi \in K$  and  $\eta \in H$  (this is known as the *range criterion* for separability in the physics literature). Hence there is a linear basis for  $R$  consisting of vectors of the form  $\xi_k \otimes \eta_k$ ,  $k = 1, \dots, r$ . By Proposition 4.6, there are operators  $v_1, \dots, v_r \in \mathcal{B}(K_0, K)$  such that  $(v_k \otimes \mathbf{1}_H)\xi_\omega = \xi_k \otimes \eta_k$ , and Proposition 4.6 also implies that  $v_1, \dots, v_r$  is a linear basis for the operator space  $\mathcal{S}$ . The paragraph above implies  $\text{rank } v_k \leq 1$  for all  $k$ .  $\square$

## 5. Sums of Positive Rank-One Operators

We require the following description of the possible ways a positive finite rank operator  $A$  can be represented as a sum of positive rank one operators

$$A = \xi_1 \otimes \bar{\xi}_1 + \dots + \xi_r \otimes \bar{\xi}_r.$$

Significantly, the vectors  $\xi_1, \dots, \xi_r$  involved in this representation of the operator  $A$  need not be linearly independent - nor even nonzero - and that flexibility is essential for our purposes. For completeness, we include a proof of this bit of the lore of elementary operator theory.

**Proposition 5.1.** *Let  $\xi_1, \dots, \xi_r$  and  $\eta_1, \dots, \eta_r$  be two  $r$ -tuples of vectors in a Hilbert space  $H$ . Then*

$$\xi_1 \otimes \bar{\xi}_1 + \dots + \xi_r \otimes \bar{\xi}_r = \eta_1 \otimes \bar{\eta}_1 + \dots + \eta_r \otimes \bar{\eta}_r, \quad (5.1)$$

*iff there is a unitary  $r \times r$  matrix  $(\lambda_{ij})$  of complex numbers such that*

$$\eta_i = \sum_{j=1}^r \lambda_{ij} \xi_j, \quad \xi_i = \sum_{j=1}^r \bar{\lambda}_{ji} \eta_j, \quad 1 \leq i \leq r. \quad (5.2)$$

*Proof.* In the statement of Proposition 5.1, the notation  $\xi \otimes \bar{\xi}$  denotes the operator  $\zeta \mapsto \langle \zeta, \xi \rangle \xi$ . In order to show that (5.1) implies (5.2), consider the two operators  $A, B : \mathbb{C}^r \rightarrow H$  defined by

$$A(\lambda_1, \dots, \lambda_r) = \sum_k \lambda_k \xi_k, \quad B(\lambda_1, \dots, \lambda_r) = \sum_k \lambda_k \eta_k.$$

The adjoint of  $A$  is given by  $A^* \zeta = (\langle \zeta, \xi_1 \rangle, \dots, \langle \zeta, \xi_r \rangle)$ , with a similar formula for  $B^*$ , and the hypothesis (5.1) becomes  $AA^* = BB^*$ . It follows that  $\|A^* \zeta\| = \|B^* \zeta\|$  for all  $\zeta \in H$ , and we can define a partial isometry  $w_0$  with initial space  $A^*H$  and final space  $B^*H$  by setting  $w_0(A^* \zeta) = B^* \zeta$ ,  $\zeta \in H$ . Since  $\mathbb{C}^r$  is finite-dimensional,  $w_0$  can be extended to a unitary operator  $w \in \mathcal{B}(\mathbb{C}^r)$ , and we have  $B = Aw^{-1}$ . Letting  $e_1, \dots, e_r$  be the usual basis for  $\mathbb{C}^r$ , we find that the matrix  $(\lambda_{ij})$  of  $w^{-1}$  relative to  $(e_k)$  satisfies

$$\eta_i = Be_i = Aw^{-1}e_i = \sum_{j=1}^r \lambda_{ij} Ae_j = \sum_{j=1}^r \lambda_{ij} \xi_j.$$

The second formula of (5.2) follows from the line above after substituting these formulas for  $\eta_k$  in  $\sum_k \bar{\lambda}_{ki} \eta_k$  and using unitarity of the matrix  $(\lambda_{ij})$ .

The converse is a straightforward calculation using unitarity of the matrix  $(\lambda_{ij})$  that we omit.  $\square$

## 6. Parameterizing the Extensions of a State

Let  $H, K$  be Hilbert spaces satisfying  $n = \dim H \leq m = \dim K < \infty$ . Given a state  $\omega$  of  $\mathcal{B}(H)$ , we consider the compact convex set  $E(\omega)$  of all extensions of  $\omega$  to a state of  $\mathcal{B}(K \otimes H)$ . Remark 4.2 shows that without loss of generality, we can restrict attention to the case in which  $\omega$  is a *faithful* state of  $\mathcal{B}(H)$ , and we do so.

Consider the filtration of  $E(\omega)$  into compact subspaces

$$E^1(\omega) \subseteq E^2(\omega) \subseteq \dots \subseteq E^{mn}(\omega) = E(\omega),$$

where  $E^r(\omega)$  denotes the space of all states of  $E(\omega)$  satisfying  $\text{rank } \rho \leq r$ . The spaces  $E^r(\omega)$  are no longer convex; but since  $\dim K \geq \dim H$ , one can exhibit pure states in  $E(\omega)$ —for example, the state  $\rho(x) = \langle x\zeta, \zeta \rangle$ , where  $\zeta$  is a unit vector in  $K \otimes H$  of the form

$$\zeta = \sqrt{\lambda_1} \cdot f_1 \otimes e_1 + \dots + \sqrt{\lambda_n} \cdot f_n \otimes e_n, \quad (6.1)$$

where  $e_1, \dots, e_n$  is an orthonormal basis for  $H$  consisting of eigenvectors of the density operator of  $\omega$  with  $\lambda_1, \dots, \lambda_n$  the corresponding eigenvalues, and where  $f_1, \dots, f_n$  is an

arbitrary orthonormal set in  $K$ . In particular, the spaces  $E^r(\omega)$  are nonempty for every  $r \geq 1$ .

Now fix an integer  $r$  in the range  $1 \leq r \leq mn$ . We define a map from the noncommutative sphere  $V^r(H, K)$  to  $E^r(\omega)$  as follows. Since  $\omega$  is faithful, Lemma 4.5 implies that there is a vector  $\xi_\omega \in H \otimes H$  such that

$$\text{span}(\mathbf{1}_H \otimes \mathcal{B}(H))\xi_\omega = H \otimes H, \quad \omega(b) = \langle (\mathbf{1} \otimes b)\xi_\omega, \xi_\omega \rangle, \quad b \in N. \quad (6.2)$$

Choose an  $r$ -tuple  $v = (v_1, \dots, v_r) \in V^r(H, K)$ . Since each  $v_k \otimes \mathbf{1}_H$  maps  $H \otimes H$  to  $K \otimes H$ , we can define a linear functional  $\rho_v$  on  $\mathcal{B}(K \otimes H)$  as follows:

$$\rho_v(x) = \sum_{k=1}^r \langle x(v_k \otimes \mathbf{1}_H)\xi_\omega, (v_k \otimes \mathbf{1}_H)\xi_\omega \rangle, \quad x \in \mathcal{B}(K \otimes H). \quad (6.3)$$

Clearly  $\rho_v$  is positive, and since  $v_1^*v_1 + \dots + v_r^*v_r = \mathbf{1}_H$ , we have

$$\rho_v(\mathbf{1}_K \otimes b) = \sum_{k=1}^r \langle (v_k^*v_k \otimes b)\xi_\omega, \xi_\omega \rangle = \langle (\mathbf{1}_H \otimes b)\xi_\omega, \xi_\omega \rangle = \omega(b),$$

for all  $b \in \mathcal{B}(H)$ . It is obvious that the rank of  $\rho_v$  cannot exceed  $r$ , hence  $\rho_v \in E^r(\omega)$ . The purpose of this section is to prove:

**Theorem 6.1.** *Let  $H, K$  be Hilbert spaces of respective dimensions  $n \leq m$ , let  $\omega$  be a faithful state of  $\mathcal{B}(H)$ , fix a vector  $\xi_\omega \in H \otimes H$  as in (6.2), and define a map*

$$v \in V^r(H, K) \mapsto \rho_v \in E^r(\omega)$$

*as in (6.3). Then  $\rho_v = \rho_{v'}$  iff there is an  $r \times r$  unitary matrix of scalars  $\lambda \in U(r)$  such that  $v' = \lambda \cdot v$ . Moreover, for every  $r = 1, 2, \dots, mn$ , this map is a continuous surjection that maps open subsets of  $V^r(H, K)$  to relatively open subsets of  $E^r(\omega)$ .*

*If  $\xi'_\omega \in H$  is another vector satisfying (6.2), giving rise to another map*

$$v \in V^r(H, K) \mapsto \rho'_v \in E^r(\omega),$$

*then there is a unitary operator  $w \in \mathcal{B}(H)$  satisfying  $\rho'_v = \rho_{v \cdot w}$  for all  $v$ , where  $(v_1, \dots, v_r) \cdot w = (v_1w, \dots, v_rw)$  denotes the right action of  $w \in \mathcal{U}(H)$  on  $v = (v_1, \dots, v_r) \in V^r(H, K)$ .*

*Proof of Theorem 6.1.* Let  $v = (v_1, \dots, v_r)$  and  $v' = (v'_1, \dots, v'_r)$  belong to  $V_r(H, K)$ , and assume first that  $\rho_v = \rho_{v'}$ . Define vectors  $\xi_k, \xi'_k \in K \otimes H$  by  $\xi_k = (v_k \otimes \mathbf{1}_H)\xi_\omega$ ,  $\xi'_k = (v'_k \otimes \mathbf{1}_H)\xi_\omega$ ,  $k = 1, \dots, r$ . The density operators of  $\rho_v$  and  $\rho_{v'}$  are

$$\sum_{k=1}^r \xi_k \otimes \bar{\xi}_k, \quad \text{and} \quad \sum_{k=1}^r \xi'_k \otimes \bar{\xi}'_k$$

respectively, so that the hypothesis  $\rho_v = \rho_{v'}$  is equivalent to the assertion

$$\sum_{k=1}^r \xi_k \otimes \bar{\xi}_k = \sum_{k=1}^r \xi'_k \otimes \bar{\xi}'_k.$$

By Proposition 5.1, there is a unitary  $r \times r$  matrix  $(\lambda_{ij})$  of scalars such that

$$\xi'_i = \sum_{j=1}^r \lambda_{ij} \xi_j, \quad 1 \leq i \leq r.$$

Proposition 4.6 implies that  $v'_i = \sum_j \lambda_{ij} v_j$ ,  $1 \leq i \leq r$ , hence  $v' = \lambda \cdot v$ .

Conversely, suppose there is a unitary matrix  $\lambda = (\lambda_{ij}) \in M_r(\mathbb{C})$  such that  $v' = \lambda \cdot v$ , and consider the vectors in  $K \otimes H$  defined by  $\xi_k = (v_k \otimes \mathbf{1}_H) \xi_\omega$ ,  $\xi'_k = (v'_k \otimes \mathbf{1}_K) \xi_\omega$ ,  $1 \leq k \leq r$ . The relation  $v' = \lambda \cdot v$  implies that

$$\xi'_i = \sum_{j=1}^r \lambda_{ij} \xi_j, \quad (6.4)$$

and the density operators of  $\rho_v$  and  $\rho_{v'}$  are given respectively by

$$\sum_{k=1}^r \xi_k \otimes \bar{\xi}_k, \quad \sum_{k=1}^r \xi'_k \otimes \bar{\xi}'_k.$$

Substitution of (6.4) into the term on the right gives

$$\sum_{k=1}^r \xi'_k \otimes \bar{\xi}'_k = \sum_{k,p,q=1}^r \lambda_{kp} \bar{\lambda}_{kq} \xi_p \otimes \bar{\xi}_q.$$

Since  $(\lambda_{ij})$  is a unitary matrix, this implies  $\sum_k \xi'_k \otimes \bar{\xi}'_k = \sum_p \xi_p \otimes \bar{\xi}_p$ , and  $\rho_{v'} = \rho_v$  follows.

The preceding paragraphs imply that the mapping  $v \mapsto \rho_v$  factors through the quotient  $X^r = V^r(H, K)/U(r)$

$$v \in V^r(H, K) \rightarrow \dot{v} \in X^r \rightarrow \rho_v,$$

and that the second map  $\dot{v} \mapsto \rho_v$  is continuous and injective. Hence it is a homeomorphism of  $X^r$  onto its range, and the composite map  $v \mapsto \rho_v$  is continuous and maps open sets to relatively open subsets of its range.

It remains to show that every state of  $E^r(\omega)$  belongs to the range of  $v \mapsto \rho_v$ . Choose  $\rho \in E^r(\omega)$ . Since the rank of  $\rho$  is at most  $r$  we can write it in the form

$$\rho(x) = \sum_{k=1}^r \langle x \zeta_k, \zeta_k \rangle, \quad x \in \mathcal{B}(K \otimes H), \quad (6.5)$$

where the  $\zeta_k$  are vectors in  $K \otimes H$ , perhaps with some being zero.

By Proposition 4.6, there are operators  $v_1, \dots, v_r \in \mathcal{B}(H, K)$  such that  $\zeta_k = (v_k \otimes \mathbf{1}_H) \xi_\omega$  for each  $k$ , and we claim that  $\sum_k v_k^* v_k = \mathbf{1}_H$ . Indeed, for all  $b_1, b_2 \in \mathcal{B}(H)$  we have

$$\begin{aligned} \langle (\sum_k v_k^* v_k) \otimes b_1 \rangle \xi_\omega, (\mathbf{1}_H \otimes b_2) \xi_\omega \rangle &= \sum_k \langle (v_k \otimes b_2^* b_1) \xi_\omega, (v_k \otimes \mathbf{1}_H) \xi_\omega \rangle \\ &= \sum_k \langle (\mathbf{1}_K \otimes b_2^* b_1) \zeta_k, \zeta_k \rangle = \rho(\mathbf{1}_K \otimes b_2^* b_1) \\ &= \omega(b_2^* b_1) = \langle (\mathbf{1}_H \otimes b_1) \xi_\omega, (\mathbf{1}_H \otimes b_2) \xi_\omega \rangle, \end{aligned}$$

and  $\sum_k v_k^* v_k = \mathbf{1}_H$  follows from cyclicity:  $H \otimes H = (\mathbf{1}_H \otimes \mathcal{B}(H)) \xi_\omega$ .



Substituting back into (6.5), we see that  $v = (v_1, \dots, v_r) \in V^r(H, K)$  has been exhibited with the property  $\rho = \rho_v$ .

To prove the last paragraph, choose another  $\xi'_\omega \in H$  satisfying (6.2). Then we have  $\|(\mathbf{1} \otimes b)\xi_\omega\|^2 = \omega(b^*b) = \|(\mathbf{1} \otimes b)\xi'_\omega\|^2$  for every  $b \in \mathcal{B}(H)$ , hence there is a unique unitary operator in the commutant of  $\mathbf{1} \otimes \mathcal{B}(H)$  that maps  $\xi_\omega$  to  $\xi'_\omega$ . Such an operator has the form  $w \otimes \mathbf{1}$  for a unique unitary operator  $w \in \mathcal{B}(H)$ , hence  $\xi'_\omega = (w \otimes \mathbf{1})\xi_\omega$ . From the definition of the map (6.3), it follows that the corresponding state  $\rho'_v$  is defined on  $x \in \mathcal{B}(K \otimes H)$  by

$$\rho'_v(x) = \sum_{k=1}^r \langle x(v_k \otimes \mathbf{1})\xi'_\omega, (v_k \otimes \mathbf{1})\xi'_\omega \rangle = \sum_{k=1}^r \langle x(v_k w \otimes \mathbf{1})\xi_\omega, (v_k w \otimes \mathbf{1})\xi_\omega \rangle,$$

and the right side is seen to be  $\rho_{v \cdot w}(x)$ .  $\square$

## 7. The Role of $(X^r, P^r)$ in Entanglement

In this section we give an operator-theoretic characterization of separable states and show that the probability of entanglement is positive at all levels (see Theorem 7.9).

Assume that  $n = \dim H \leq m = \dim K < \infty$ , fix  $r = 1, 2, \dots, mn$ , choose a faithful state  $\omega$  of  $\mathcal{B}(H)$ , and choose a vector  $\xi_\omega$  as in (6.2). Theorem 6.1 implies that the parameterizing map  $v \in V^r(H, K) \mapsto \rho_v \in E^r(\omega)$  decomposes naturally into a composition of two maps

$$v \in V^r(H, K) \mapsto \dot{v} \in X^r \mapsto \rho_v \in E^r(\omega). \quad (7.1)$$

We can promote the invariant probability measure  $\mu$  on  $V^r(H, K)$  all the way to  $E^r(\omega)$  by way of the composite map

$$v \in V^r(H, K) \mapsto \rho_v \in E^r(\omega),$$

thereby obtaining a compact metrizable probability space  $(E^r(\omega), P^{r,\omega})$ .

*Remark 7.1.* (Independence of the choice of  $\omega$ ). After noting that the second map of (7.1) implements a measure-preserving homeomorphism of topological probability spaces  $(X^r, P^r) \cong (E^r(\omega), P^{r,\omega})$ , we conclude that *each of the probability spaces  $(E^r(\omega), P^{r,\omega})$  associated with faithful states of  $\mathcal{B}(H)$  is isomorphic to the intrinsic space  $(X^r, P^r)$ , hence they are all isomorphic to each other.*

*Remark 7.2.* (Independence of the choice of  $\xi_\omega$ ). If we choose another vector  $\xi'_\omega \in H$  satisfying (6.2), the resulting parameterization  $v \mapsto \rho'_v$  of  $E^r(\omega)$  differs from that of (7.1), hence the resulting probability measure  $P^{r,\omega'}$  on  $E^r(\omega)$  appears to differ from the one  $P^{r,\omega}$  promoted through the map  $v \mapsto \rho_v$ . However, Theorem 6.1 implies that there is a unitary operator  $w \in \mathcal{U}(H)$  such that  $\rho'_v = \rho_{v \cdot w}$ ,  $v \in V^r(H, K)$ , so that  $P^{r,\omega}$  and  $P^{r,\omega'}$  are respectively promotions (through the same map  $v \mapsto \rho_v$ ) of the measure  $P^r$  and its transform  $P^{r'}$  under the right action of  $w$  on  $X^r$ . Remark 3.1 implies that  $P^{r'} = P^r$ , hence  $P^{r,\omega'} = P^{r,\omega}$ , and therefore  $(E^r(\omega), P^{r,\omega})$  does not depend on the choice of  $\xi_\omega$ .

**Remark 7.3.** (Invariance of rank and separability). It is not obvious that spatial properties of states such as rank and separability are preserved under these identifications. For example, it is not clear that the integer-valued random variable that represents rank on the probability space  $(E^r(\omega), P^{r,\omega})$

$$\rho \in E^r(\omega) \mapsto \text{rank } \rho \in \{1, 2, \dots, r\}$$

is preserved under the isomorphism  $(E^r(\omega_1), P^{r,\omega_1}) \cong (E^r(\omega_2), P^{r,\omega_2})$  for different faithful states  $\omega_1$  and  $\omega_2$ . Similarly, we require that these identifications should preserve separability and entanglement. We establish the invariance of these properties in Propositions 7.4 and 7.7 below by identifying them appropriately in terms of random variables on the intrinsic probability space  $(X^r, P^r)$ .

We first establish the invariance of rank.

**Proposition 7.4.** *Let  $\omega$  be a faithful state of  $\mathcal{B}(H)$ , fix  $r = 1, 2, \dots, mn$  and consider the factorization (7.1) through  $X^r$  of the parameterization map  $v \mapsto \rho_v$ . For every  $v \in V^r(H, K)$ , one has*

$$\text{rank}(\dot{v}) = \text{rank } \rho_v, \quad (7.2)$$

and almost surely, states of  $(E^r(\omega), P^{r,\omega})$  have rank  $r$ .

*Proof.* Formula (7.2) simply restates the last sentence of Proposition 4.6, and the second phrase follows from Theorem 3.3.  $\square$

**Remark 7.5.** (Convex hulls of sets in  $\mathbb{R}^k$ ). We recall some basic lore of convexity theory. A classical result of Carathéodory [Car07, Car11] asserts that every convex combination of points from a subset  $E$  of  $\mathbb{R}^k$  can be written as a convex combination of at most  $k + 1$  points of  $E$ . It follows that the convex hull of a compact subset  $E$  of  $\mathbb{R}^k$  is compact. Since the set of all product states of  $M \otimes N$  is compact, we conclude that the set of separable states of  $M \otimes N$  is compact as well as convex, and the set of entangled states is a relatively open subset of the state space of  $M \otimes N$ .

One can do slightly better for states. Let  $H$  be an  $n$  dimensional Hilbert space. The self-adjoint operators in  $\mathcal{B}(H)$  form a real vector space of dimension  $n^2$ , and the set of self-adjoint operators  $A$  satisfying  $\text{trace } A = 1$  is a hyperplane of dimension  $n^2 - 1$ . So Carathéodory's theorem implies that every state of  $\mathcal{B}(H)$  that belongs to the convex hull of an arbitrary set  $\mathcal{P}$  of states can be written as a convex combination of at most  $n^2$  states of  $\mathcal{P}$ .

These remarks lead to the following known result:

**Lemma 7.6.** *Every separable state of  $\mathcal{B}(K \otimes H)$  is a convex combination of at most  $m^2 n^2$  pure separable states.*

Throughout the remainder of this section, we set  $q = m^2 n^2$  and let  $U(q)$  be the group of all  $q \times q$  unitary matrices  $\mu = (\mu_{ij}) \in M_q(\mathbb{C})$ .

**Proposition 7.7.** *Let  $\omega$  be a faithful state of  $\mathcal{B}(H)$ , let  $\rho \in E^r(\omega)$ , and choose  $v \in V^r(H, K)$  such that  $\rho = \rho_v$ . Then  $\rho$  is separable iff there is a unitary matrix  $\mu = (\mu_{ij})$  in  $U(q)$  such that*

$$\text{rank}\left(\sum_{j=1}^r \mu_{ij} v_j\right) \leq 1, \quad i = 1, 2, \dots, q. \quad (7.3)$$

*Proof.* Assume first that  $\rho$  is separable. By Lemma 7.6, there are vectors  $\xi_i \in K, \eta_i \in H, 1 \leq i \leq q$ , such that

$$\rho(x) = \sum_{i=1}^q \langle x(\xi_i \otimes \eta_i), \xi_i \otimes \eta_i \rangle, \quad x \in \mathcal{B}(K \otimes H).$$

Let  $v'_i = v_i$  if  $1 \leq i \leq r$ , set  $v'_i = 0$  for  $r < i \leq q$  and choose a vector  $\xi_\omega \in H \otimes H$  that represents  $\omega(b) = \langle (\mathbf{1} \otimes b)\xi_\omega, \xi_\omega \rangle$  as in Lemma 4.5. Then the formula  $\rho = \rho_v$  can be rewritten

$$\rho(x) = \sum_{i=1}^q \langle x(v'_i \otimes \mathbf{1})\xi_\omega, (v'_i \otimes \mathbf{1})\xi_\omega \rangle, \quad x \in \mathcal{B}(K \otimes H).$$

By Proposition 5.1, there is a unitary  $q \times q$  matrix  $\lambda = (\lambda_{ij})$  such that

$$\xi_i \otimes \eta_i = \sum_{j=1}^q \lambda_{ij}(v'_j \otimes \mathbf{1})\xi_\omega = \left( \sum_{j=1}^r \lambda_{ij}v_j \otimes \mathbf{1} \right) \xi_\omega, \quad i = 1, \dots, q. \quad (7.4)$$

Proposition 4.6 implies that for every  $i = 1, \dots, q$  there is a unique operator  $w_i : H \rightarrow K$  such that  $(w_i \otimes \mathbf{1})\xi_\omega = \xi_i \otimes \eta_i$ , and (7.4) plus uniqueness implies

$$w_i = \sum_{j=1}^r \lambda_{ij}v_j, \quad i = 1, 2, \dots, q.$$

Finally, Corollary 4.7 implies that  $w_i$  is of rank at most 1, and (7.3) follows.

All of these steps are reversible, and we leave the proof of the converse assertion for the reader.  $\square$

We can now identify the subsets of  $X^r$  that correspond to separable or entangled extensions of faithful states of  $\mathcal{B}(H)$ .

**Proposition 7.8.** *For every  $r = 1, 2, \dots, mn$ , let  $\text{Sep}(V^r(H, K))$  be the subset of  $V^r(H, K)$  defined by the conditions of (7.3),*

$$\text{Sep}(V^r(H, K)) = \{v : \exists \mu \in U(q) \text{ s. t. } \text{rank}\left(\sum_{j=1}^r \mu_{ij}v_j\right) \leq 1, \quad 1 \leq i \leq q\}.$$

*The natural projection  $v \mapsto \dot{v}$  of  $V^r(H, K)$  on  $X^r$  carries  $\text{Sep}(V^r(H, K))$  onto a closed subset  $\text{Sep}(X^r)$  of  $X^r$  that is invariant under the right action of  $\mathcal{U}(H)$ , and which has the following properties: For every faithful state  $\omega$  of  $\mathcal{B}(H)$  and every  $v \in V^r(H, K)$ ,*

- (i)  $\rho_v$  is a separable state of  $E^r(\omega)$  iff  $\dot{v} \in \text{Sep}(X^r)$ ,
- (ii)  $\rho_v$  is an entangled state of  $E^r(\omega)$  iff  $\dot{v} \in X^r \setminus \text{Sep}(X^r)$ .

*Proof.* For a fixed faithful state  $\omega$  of  $\mathcal{B}(H)$ , Proposition 7.7 implies that the homeomorphism  $v \mapsto \rho_v$  maps  $\text{Sep}(X^r)$  onto the space of separable states in  $E^r(\omega)$ . Since the separable states form a closed subset of the state space of  $\mathcal{B}(K \otimes H)$ , it follows that  $\text{Sep}(X^r)$  is closed. Invariance under the right action of  $\mathcal{U}(H)$  on  $X^r$  follows from the fact that for every operator  $v \in \mathcal{B}(H, K)$  and every unitary operator  $w$  on  $H$ ,  $\text{rank}(vw) = \text{rank}(v)$ . Assertion (i) is a restatement of Proposition 7.7, and (ii) follows from (i) since entangled states and separable states are complementary sets.  $\square$

The following result implies that there are plenty of entangled states of all possible ranks. We will obtain sharper results in Sects. 9 and 10.

**Theorem 7.9.** *For every  $r = 1, 2, \dots, mn$ ,  $\text{Sep}(X^r)$  is a proper closed subset of  $X^r$ , and for every faithful state  $\omega$  of  $\mathcal{B}(H)$ , the probability  $p$  of entanglement in  $(E^r(\omega), P^{r,\omega})$  is independent of the choice of  $\omega$  and satisfies*

$$p = 1 - P^r(\text{Sep}(X^r)) = P^r(X^r \setminus \text{Sep}(X^r)) > 0.$$

*Proof.* Fix  $r = 1, 2, \dots, mn$ . We claim first that there is a faithful state  $\omega$  of  $\mathcal{B}(H)$  such that  $E^r(\omega)$  contains an entangled state. To see that, choose an orthonormal basis  $e_1, \dots, e_n$  for  $H$ , an orthonormal set  $f_1, \dots, f_n \in K$ , and let  $\zeta$  be the unit vector

$$\zeta = \frac{1}{\sqrt{n}}(f_1 \otimes e_1 + \dots + f_n \otimes e_n) \in K \otimes H.$$

It is well known that  $\rho(x) = \langle x\zeta, \zeta \rangle$ ,  $x \in \mathcal{B}(K \otimes H)$ , defines a pure entangled state of  $\mathcal{B}(K \otimes H)$  that restricts to the tracial state on  $\mathcal{B}(H)$ .

It is easy to see that there is a self-adjoint operator  $c \in \mathcal{B}(K \otimes H)$  such that  $\rho(c) < 0$  and such that for all states  $\sigma_1$  of  $\mathcal{B}(K)$  and  $\sigma_2$  of  $\mathcal{B}(H)$ , one has

$$(\sigma_1 \otimes \sigma_2)(c) \geq 0, \quad (7.5)$$

e.g., see [HHH96]. We sketch the construction for completeness. Since  $\zeta$  is not a tensor product, we have  $|\langle \xi \otimes \eta, \zeta \rangle| < 1$  for every pair of unit vectors  $\xi \in K$ ,  $\eta \in H$ ; and since the unit spheres of  $K$  and  $H$  are compact, we can choose  $\alpha \in (0, 1)$  such that

$$\max\{|\langle \xi \otimes \eta, \zeta \rangle|^2 : \xi \in K, \eta \in H, \|\xi\| = \|\eta\| = 1\} \leq \alpha < 1.$$

Set  $c = \alpha \cdot \mathbf{1} - \zeta \otimes \bar{\zeta}$ . A calculation shows that  $\rho(c) < 0$ , and by its construction,  $c$  satisfies (7.5) for pure states  $\sigma_1$  and  $\sigma_2$ . Equation (7.5) follows in general, since every state is a convex combination of pure states.

Now choose any projection  $p$  of rank  $r$  in  $\mathcal{B}(K \otimes H)$  whose range contains  $\zeta$ . Then for every  $t \in (0, 1)$ ,

$$\sigma_t(x) = \frac{t}{r} \text{trace}(px) + (1 - t) \cdot \rho(x), \quad x \in \mathcal{B}(K \otimes H)$$

is a state of rank  $r$  that restricts to a faithful state  $\omega_t$  of  $\mathcal{B}(H)$ . Moreover, for sufficiently small  $t$ , we will have  $\sigma_t(c) < 0$ ; and for such  $t$  (7.5) implies that  $\sigma_t$  is not a convex combination of product states, proving the claim.

Choose a faithful state  $\omega$  of  $\mathcal{B}(H)$  such that  $E^r(\omega)$  contains an entangled state  $\rho_0$ . Then the inverse image  $x_0 \in X^r$  of  $\rho_0$  under the map  $\dot{v} \in X^r \mapsto \rho_v \in E^r(\omega)$  is a point in the complement of  $\text{Sep}(X^r)$ , hence  $\text{Sep}(X^r) \neq X^r$ . The set  $X^r \setminus \text{Sep}(X^r)$  is a nonempty open subset of  $X^r$  which therefore has positive  $P^r$ -measure. It follows from Proposition 7.8 that the probability  $p$  of entanglement in  $(E^r, P^{r,\omega})$  satisfies  $p = P^r(X \setminus \text{Sep}(X^r)) > 0$ . Finally, Proposition 7.8 and Remark 7.1 imply that the same assertions are true for the probability space  $(E^r(\omega'), P^{r,\omega'})$  associated with any faithful state  $\omega'$  of  $\mathcal{B}(H)$ , and that the probability of entanglement in  $(E(\omega'), P^{r,\omega'})$  does not depend on the choice of  $\omega'$ .  $\square$

## 8. Properties of the Wedge Invariant

Proposition 7.8 implies that among the states  $\rho_v$  of  $E^r(\omega)$ , the separability property is determined by membership of  $\dot{v}$  in the closed set  $\text{Sep}(X^r)$ . Hence, in order to calculate or estimate the probability of entanglement in the spaces  $(E^r(\omega), P^{r,\omega})$ , one needs to calculate or estimate  $P^r(\text{Sep}(X^r))$ . Writing  $q = m^2 n^2$  as in the preceding section, the set  $\text{Sep}(X^r)$  is identified in Propositions 7.7 and 7.8 as

$$\text{Sep}(X^r) = \bigcup_{\mu \in U(q)} \{ \dot{v} \in X^r : \text{rank}(\sum_{j=1}^r \mu_{ij} v_j) \leq 1, \quad 1 \leq i \leq q \}. \quad (8.1)$$

The set on the right defines an uncountable union of subvarieties of  $V^r(H, K)$ , but it is not a subvariety itself nor even a countable union of subvarieties (see Sect. 11). In this section we reformulate the definition of the wedge invariant (Definition 1.3) as a pair of random variables

$$\dot{w}, \dot{w}^* : X^r \rightarrow \{0, 1, 2, \dots\}.$$

We show that these random variables provide a nontrivial test for separability – i.e., membership in  $\text{Sep}(X^r)$  – and that they define subvarieties

$$A = \{v \in V^r(H, K) : \dot{w}(\dot{v}) \leq 1\}, \quad A^* = \{v \in V^r(H, K) : \dot{w}^*(\dot{v}) \leq 1\},$$

with the property that  $\text{Sep}(X^r) \subseteq \dot{A} \cap \dot{A}^*$ . The latter property is critical for the applications of Sect. 9.

Fix  $r = 1, 2, \dots, mn$  and choose  $v = (v_1, \dots, v_r) \in V^r(H, K)$ . We can form the operator  $v_1 \wedge \dots \wedge v_r \in \mathcal{B}(H^{\otimes r}, K^{\otimes r})$  as in (1.6), and this operator maps the symmetric subspace of  $H^{\otimes r}$  to the antisymmetric subspace of  $K^{\otimes r}$ . If  $v$  and  $v'$  belong to the same  $U(r)$ -orbit, say  $v' = \lambda \cdot v$  with  $\lambda = (\lambda_{ij}) \in U(r)$ , then by elementary multilinear algebra we have

$$v'_1 \wedge \dots \wedge v'_r = \det(\lambda_{ij}) \cdot v_1 \wedge \dots \wedge v_r. \quad (8.2)$$

It follows that  $v'_1 \wedge \dots \wedge v'_r (H_+^{\otimes r}) = v_1 \wedge \dots \wedge v_r (H_+^{\otimes r})$ . Similarly, we can form  $v_1^* \wedge \dots \wedge v_r^* \in \mathcal{B}(K^{\otimes r}, H^{\otimes r})$ , and  $(v_1^* \wedge \dots \wedge v_r^*)(K_+^{\otimes r})$  depends only on the  $U(r)$  orbit of  $v$ . Thus we can define integer-valued random variables  $\dot{w}, \dot{w}^* : X^r \rightarrow \{0, 1, 2, \dots\}$  by

$$\dot{w}(\dot{v}) = \text{rank}(v_1 \wedge \dots \wedge v_r \upharpoonright_{H_+^{\otimes r}}), \quad \dot{w}^*(\dot{v}) = \text{rank}(v_1^* \wedge \dots \wedge v_r^* \upharpoonright_{K_+^{\otimes r}}), \quad (8.3)$$

for  $v \in V^r(H, K)$ . The following result implies that these random variables can detect entanglement. Note too that both random variables  $\dot{w}$  and  $\dot{w}^*$  are invariant under the right action of  $\mathcal{U}(H)$  on  $X^r$ .

**Proposition 8.1.** *For every  $x \in \text{Sep}(X^r)$ , we have  $\dot{w}(x) \leq 1$  and  $\dot{w}^*(x) \leq 1$ .*

*Proof.* We claim that  $\dot{w} \leq 1$  on  $\text{Sep}(X^r)$ . Indeed, every point of  $\text{Sep}(X^r)$  has the form  $x = \dot{v}$ , where  $v = (v_1, \dots, v_r)$  is an  $r$ -tuple in  $V^r(H, K)$  whose associated state  $\rho_v$  is separable. We have to show that the restriction of the operator  $v_1 \wedge \dots \wedge v_r$  to the symmetric subspace  $H_+^{\otimes r}$  has rank  $\leq 1$ .

To see that, note that Corollary 4.7 implies that there is a linearly independent set of operators  $w_1, \dots, w_r \in \mathcal{B}(H, K)$  that has the same linear span as  $v_1, \dots, v_r$ , such

that  $\text{rank } w_k = 1$  for every  $k$ . Since  $v_1, \dots, v_r$  and  $w_1, \dots, w_r$  are linearly independent subsets of  $\mathcal{B}(H, K)$  that have the same linear span  $\mathcal{S}$ , elementary multilinear algebra implies that there is a complex number  $d \neq 0$  such that

$$v_1 \wedge \cdots \wedge v_r = d \cdot w_1 \wedge \cdots \wedge w_r;$$

indeed,  $d$  is the determinant of the linear operator defined on  $\mathcal{S}$  by stipulating that it should carry one basis to the other. Hence it is enough to show that the restriction of  $w_1 \wedge \cdots \wedge w_r$  to  $H_+^{\otimes r}$  has rank at most 1.

For every vector  $\zeta \in H$  we have

$$(w_1 \wedge \cdots \wedge w_r)(\zeta^{\otimes r}) = w_1 \zeta \wedge w_2 \zeta \wedge \cdots \wedge w_r \zeta.$$

Now since each  $w_k$  is of rank at most 1, for every  $k$  there are vectors  $\zeta_k \in H$  and  $\xi_k \in K$  such that  $w_k \zeta_k = \xi_k$  and  $w_k = 0$  on  $\{\zeta_k\}^\perp$ . For each  $k$  we can write  $\zeta = \mu_k \zeta_k + \zeta'_k$ , where  $\mu_k \in \mathbb{C}$  and  $\zeta'_k$  belongs to the kernel of  $w_k$ . Hence the term on the right takes the form

$$w_1(\mu_1 \zeta_1) \wedge w_2(\mu_2 \zeta_2) \wedge \cdots \wedge w_r(\mu_r \zeta_r) = (\mu_1 \mu_2 \cdots \mu_r) \cdot \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_r,$$

so that  $(w_1 \wedge \cdots \wedge w_r)(\zeta^{\otimes r}) \in \mathbb{C} \cdot \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_r$ . Finally, a standard polarization argument shows that the symmetric subspace of  $H^{\otimes r}$  is spanned by vectors of the form  $\zeta^{\otimes r}$  with  $\zeta \in H$ , and the desired assertion

$$(w_1 \wedge \cdots \wedge w_r)(H_+^{\otimes r}) \subseteq \mathbb{C} \cdot \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_r$$

follows.

The proof that

$$\dot{w}^*(\dot{v}) = \text{rank}(v_1^* \wedge \cdots \wedge v_r^* \upharpoonright_{K_+^{\otimes r}}) \leq 1$$

is similar, since the operators  $w_1^*, \dots, w_r^*$  form a basis for the operator space  $\mathcal{S}^*$  consisting of rank-one operators.  $\square$

We have already pointed out that the analysis of states of  $\mathcal{B}(K \otimes H)$  can be reduced to the analysis of states that restrict to faithful states on  $\mathcal{B}(H)$ . Hence the result stated in Theorem 1.4 of the introduction follows from Proposition 8.1 and the fact that for every faithful state  $\omega$  of  $\mathcal{B}(H)$  and every state  $\rho \in E^r(\omega)$  for  $r = 1, 2, \dots, mn$ , we have

$$w(\rho_v) = \dot{w}(\dot{v}), \quad w^*(\rho_v) = \dot{w}^*(\dot{v}), \quad v \in V^r(H, K). \quad (8.4)$$

Most significantly, the wedge invariant is associated with subvarieties:

**Proposition 8.2.** *For every  $r = 1, 2, \dots, mn$ , let*

$$A = \{v \in V^r(H, K) : \dot{w}(\dot{v}) \leq 1\}, \quad A^* = \{v \in V^r(H, K) : \dot{w}^*(\dot{v}) \leq 1\}.$$

*Then both  $A$  and  $A^*$  are subvarieties of  $V^r(H, K)$ .*

*Proof.* The set  $A$  consists of all  $r$ -tuples  $v \in V^r(H, K)$  such that the operator  $G(v) = v_1 \wedge \cdots \wedge v_r \upharpoonright_{H_+^{\otimes r}} \in \mathcal{B}(H_+^{\otimes r}, K_-^{\otimes r})$  satisfies  $\text{rank } G(v) \leq 1$ , or equivalently, that  $G(v) \wedge G(v) = 0$ , where  $G(v) \wedge G(v)$  is now viewed as an operator from  $H_+^{\otimes r} \wedge H_+^{\otimes r}$  to  $K_-^{\otimes r} \wedge K_-^{\otimes r}$ . Hence  $F(v) = G(v) \wedge G(v)$  is a homogeneous polynomial of degree  $2r$  with the property

$$A = \{v \in V^r(H, K) : F(v) = 0\},$$

thereby exhibiting  $A$  as a subvariety. A similar argument with  $v_k^*$  replacing  $v_k$  shows that  $A^*$  is a subvariety.  $\square$

Propositions 8.1 and 8.2 provide no information as to whether the wedge invariant is nontrivial, but the following result does.

**Proposition 8.3.** *Assume that  $\dim K \geq \dim H \geq 2$ . Then for every integer  $r$  satisfying  $1 \leq r \leq \dim H/2$  there is a point  $x \in X^r$  such that  $\text{rank } x = r$  and  $w^*(x) > 1$ , and the following equivalent assertions are true:*

- (i) *The subvariety  $A^*$  of Proposition 8.2 is proper;  $A^* \neq V^r(H, K)$ .*
- (ii) *For every faithful state  $\omega$  of  $\mathcal{B}(H)$  there is a state of rank  $r$  in  $E^r(\omega)$  such that  $w^*(\rho) > 1$ .*

*Proof.* It suffices to exhibit an  $r$ -tuple  $v = (v_1, \dots, v_r) \in V^r(H, K)$  such that  $\text{rank}(v_1^* \wedge \cdots \wedge v_r^* \upharpoonright_{K_+^{\otimes r}}) > 1$ . Since  $v_1^* \wedge \cdots \wedge v_r^* \neq 0$ , the operators  $v_1^*, \dots, v_r^*$  are linearly independent, hence so are  $v_1, \dots, v_r$ . Proposition 4.6 will then imply that the associated state  $\rho_v$  has rank  $r$ , and it will satisfy  $w^*(\rho_v) > 1$  because of the asserted properties of  $v_1, \dots, v_r$ .

We exhibit such operators  $v_1, \dots, v_r$  as follows. Write  $\dim H = 2r + s$  with  $s \geq 0$  and choose an orthonormal basis for  $H$ , enumerated by

$$\{e_1, \dots, e_r, f_1, \dots, f_r\}, \quad \text{or} \quad \{e_1, \dots, e_r, f_1, \dots, f_r, g_1, \dots, g_s\},$$

according to whether  $s = 0$  or  $s > 0$ . Let  $\{e'_i, f'_j, g'_k\}$  be a similarly labelled orthonormal set in  $K$ . For each  $k = 1, \dots, r$ , let  $v_k$  be the unique operator in  $\mathcal{B}(H, K)$  satisfying  $v_k e_i = \delta_{ki} e'_1$  and  $v_k f_i = \delta_{ki} f'_1$  for  $1 \leq i \leq r$  if  $s = 0$ , and otherwise it satisfies the additional conditions  $v_1 g_j = g'_j$  and  $v_2 g_j = \cdots = v_r g_j = 0$  for  $j = 1, \dots, s$  when  $s > 0$ . Each  $v_k$  is a partial isometry whose adjoint  $v_k^*$  maps  $e'_i$  to  $\delta_{ik} e_k$  and  $f'_i$  to  $\delta_{ik} f_k$  for  $1 \leq k \leq r$ . It follows that  $v_1^* v_1 + \cdots + v_r^* v_r = \mathbf{1}_H$ , so that  $v = (v_1, \dots, v_r) \in V^r(H, K)$ .

Now consider the operator  $v_1^* \wedge \cdots \wedge v_r^*$ , restricted to the symmetric subspace  $K_+^{\otimes r}$  of  $K^{\otimes r}$ . We have

$$(v_1^* \wedge \cdots \wedge v_r^*)(e'_1 \otimes \cdots \otimes e'_1) = v_1^* e'_1 \wedge v_2^* e'_1 \wedge \cdots \wedge v_r^* e'_1 = e_1 \wedge e_2 \wedge \cdots \wedge e_r,$$

and similarly  $(v_1^* \wedge \cdots \wedge v_r^*)(f'_1 \otimes \cdots \otimes f'_1) = f_1 \wedge f_2 \wedge \cdots \wedge f_r$ . Since the vectors  $e_1 \wedge e_2 \wedge \cdots \wedge e_r$  and  $f_1 \wedge f_2 \wedge \cdots \wedge f_r$  are mutually orthogonal unit vectors in  $\wedge^r H$ , it follows that  $\text{rank}(v_1^* \wedge \cdots \wedge v_r^* \upharpoonright_{K_+^{\otimes r}}) \geq 2$ .  $\square$

## 9. Entangled States of Small Rank

We now assemble the results of the previous section into a main result. Fix Hilbert spaces  $H, K$  with  $2 \leq n = \dim H \leq m = \dim K < \infty$ .

**Theorem 9.1.** *Let  $r$  be a positive integer satisfying  $1 \leq r \leq n/2$ , let  $\omega$  be a faithful state of  $\mathcal{B}(H)$ , and let  $(E^r(\omega), P^{r,\omega})$  be the probability space of Sect. 7. Then almost every state of  $(E^r(\omega), P^{r,\omega})$  is entangled.*

*Proof.* By Theorem 6.1 and Proposition 8.1, the set of separable states of  $E^r(\omega)$  is a closed subset of

$$\{\rho_v : v \in V^r(H, K), w^*(\rho_v) \leq 1\},$$

hence it suffices to show that the set  $A^* = \{v \in V^r(H, K) : w^*(\rho_v) \leq 1\}$  has  $\mu$ -measure zero. But by Propositions 8.2 and 8.3,  $A^*$  is a proper subvariety of  $V^r(H, K)$ , so that  $\mu(A^*) = 0$  follows from Proposition 2.6.  $\square$

*Remark 9.2.* (The meaning of “relatively small rank”). In somewhat more prosaic terms, Theorem 9.1 has the following consequence. Let  $\rho$  be an arbitrary state of  $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$  and let  $\omega$  be its marginal  $\omega(a) = \rho(\mathbf{1} \otimes a)$ ,  $a \in M_n(\mathbb{C})$ . Then *whenever the inequalities  $2 \cdot \text{rank } \rho \leq \text{rank } \omega \leq m$  are satisfied, one can infer from Theorem 9.1 that  $\rho$  is entangled*, or else one has made a statistically impossible choice of  $\rho$  that cannot be reproduced.

*Remark 9.3.* (States of very small rank). We note that if  $r < \sqrt{n}$  in the hypothesis of Theorem 9.1, then *every* state of  $E^r(\omega)$  is entangled - or equivalently,  $\text{Sep } X^r = \emptyset$ . To sketch the elementary proof of that fact, let  $\rho$  be a separable state of  $\mathcal{B}(K \otimes H)$  such that  $\text{rank } \rho = r$ , with  $n = \dim H \leq \dim K < \infty$ , and let  $R \subseteq K \otimes H$  be the  $r$ -dimensional range of the density operator of  $\rho$ . Since  $\rho$  is separable it has a representation

$$\rho = \sum_{k=1}^s p_k \cdot \omega_k$$

in which the  $p_k$  are positive numbers summing to 1 and the  $\omega_k$  are pure product states of  $\mathcal{B}(K \otimes H)$ . Since each  $p_k > 0$ , the vector  $\xi_k \otimes \eta_k$  associated with each  $\omega_k$  must belong to  $R$ , and we can view the above formula as a relation between states of  $\mathcal{B}(R)$ . At this point, Caratheodory's theorem (see Remark 7.5) implies that there is a subset  $S \subseteq \{\xi_1 \otimes \eta_1, \dots, \xi_s \otimes \eta_s\} \subseteq R$  containing at most  $r^2$  vectors such that  $\rho$  can be written

$$\rho = \sum_{k=1}^{r^2} p'_k \cdot \omega'_k,$$

where the  $p'_k$  are nonnegative numbers with sum 1 and the  $\omega'_k$  are pure product states associated with vectors in  $S$ . Assuming now that  $\rho \in E^r(\omega)$ , then  $\rho$  restricts to a *faithful* state of  $\mathcal{B}(H)$  and hence  $r^2 \geq n$ . It follows that  $E^r(\omega)$  *contains no separable states when  $r < \sqrt{n}$* . I am indebted to an anonymous referee for pointing out the idea behind this observation.



## 10. Entangled States of Large Rank

Let  $H, K$  be Hilbert spaces with  $n = \dim H \leq m = \dim K < \infty$ . We conclude with an observation showing that the behavior of Theorem 9.1 does not persist through states of large rank. While the first sentence of Theorem 10.1 is essentially known (for example, see [GB02, GB05]), we sketch a proof for completeness.

**Theorem 10.1.** *The set of separable states of  $\mathcal{B}(K \otimes H)$  of rank  $mn$  contains a nonempty relatively open subset of the state space of  $\mathcal{B}(K \otimes H)$ .*

*Moreover, for every faithful state  $\omega$  of  $\mathcal{B}(H)$ , the set of entangled states of  $E^{mn}(\omega)$  is a relatively open subset that is neither empty nor dense in  $E^{mn}(\omega)$ , and its probability  $p$  satisfies  $0 < p < 1$ .*

*Proof.* Note first that the set of faithful separable states must linearly span the self adjoint part  $S$  of the dual of  $\mathcal{B}(K \otimes H)$ ; equivalently, for every nonzero self adjoint operator  $x$ , there is a faithful separable state  $\omega$  such that  $\omega(x) \neq 0$ . Indeed, fixing  $x$ , we use the fact that the separable states obviously span  $S$  to find a separable state  $\omega$  for which  $\omega(x) \neq 0$ , and then we can make small changes in the decomposable vector states that sum to  $\omega$  so as to find a faithful separable state  $\omega'$  close enough to  $\omega$  that  $\omega'(x) \neq 0$ . Since the separable states of rank  $mn$  span  $S$ , we can find a basis for  $S$  consisting of separable states of rank  $mn$ .

Finally, since the convex hull of a basis for  $S$  consisting of states must contain a nontrivial open subset of the state space of  $\mathcal{B}(K \otimes H)$ , it follows that  $\text{Sep}(X^{mn})$  has nonempty interior and therefore has positive  $P^{mn}$ -measure. Theorem 7.9 implies  $0 < P^{mn}(\text{Sep}(X^{mn})) < 1$ , and the remaining assertions of Theorem 10.1 follow.  $\square$

## 11. Constructibility, Entanglement, and Zero-One Laws

In this section we digress in order to make some observations about set-theoretic issues that seem to add perspective to the results of Sects. 9 and 10, and which address the broader question of whether entanglement can be detected by way of a more detailed analysis of real-analytic varieties.

Let  $H, K$  be Hilbert spaces with  $n = \dim H \leq m = \dim K < \infty$  and fix  $r = 1, 2, \dots, mn$ . The subvarieties of  $V^r(H, K)$  (see Definition 2.5) generate a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of  $V^r(H, K)$ . This  $\sigma$ -algebra consists of Borel sets and it separates points of  $V^r(H, K)$ . In the context of descriptive set theory,  $\mathcal{A}$  consists of all Borel sets that can be constructed by way of a transfinite hierarchy of operations consisting of countable unions and complementations, starting with subvarieties. Let  $\mathcal{B}$  be the somewhat larger  $\sigma$ -algebra consisting of all Borel sets  $E \subseteq V^r(H, K)$  which agree almost surely with sets of  $\mathcal{A}$  in that there are sets  $A_1, A_2 \in \mathcal{A}$  such that  $A_1 \subseteq E \subseteq A_2$  and  $\mu(A_2 \setminus A_1) = 0$ ,  $\mu$  being the natural probability measure on  $V^r(H, K)$ .

Significantly, the “constructible” sets in  $\mathcal{A}$  and  $\mathcal{B}$  satisfy a zero-one law.

**Proposition 11.1.** *For every  $E \in \mathcal{B}$ ,  $\mu(E) = 0$  or 1.*

*Proof.* It clearly suffices to show that  $\mu \upharpoonright_{\mathcal{A}}$  is  $\{0, 1\}$ -valued. To prove that, let  $\mathcal{Z}$  be the family of all proper subvarieties  $Z \neq V^r(H, K)$ . By Proposition 2.6, every set in  $\mathcal{Z}$  has measure zero. Consider the family  $\mathcal{C}$  of all Borel subsets  $E \subseteq V^r(H, K)$  with the property that either  $E$  or its complement is contained in some countable union  $Z_1 \cup Z_2 \cup \dots$  of sets  $Z_k \in \mathcal{Z}$ . One checks easily that  $\mathcal{C}$  is closed under countable unions,

complementation, and it contains  $\mathcal{Z}$ . Hence  $\mathcal{C}$  is a  $\sigma$ -algebra containing  $\mathcal{A}$ . But for every set  $E \in \mathcal{C}$  we have  $\mu(E) = 0$  if  $E$  is contained in a countable union of sets from  $\mathcal{Z}$ , or  $\mu(E) = 1$  if the complement of  $E$  is contained in a countable union of sets from  $\mathcal{Z}$ . Hence  $\mu(E) = 0$  or  $1$ . In particular,  $\mu \upharpoonright_{\mathcal{A}}$  is  $\{0, 1\}$ -valued.  $\square$

Now fix a faithful state  $\omega$  of  $\mathcal{B}(H)$ , fix  $r = 1, 2, \dots, mn$ , and consider the space of all separable states in  $E^r(\omega)$ . The inverse image of this space under the parameterizing map  $v \in V^r(H, K) \mapsto \rho_v \in E^r(\omega)$ , namely

$$\text{Sep}(V^r(H, K)) = \{v \in V^r(H, K) : \rho_v \text{ is separable}\},$$

is a compact subspace of  $V^r(H, K)$ . Proposition 7.8 shows that its structure determines the properties of separable states in  $E^r(\omega)$ , and its complement determines the properties of entangled states in  $E^r(\omega)$ .

*Remark 11.2.* (Structure of  $\text{Sep}(V^r(H, K))$  for small  $r$ ). The key fact in the proof of Theorem 9.1 is that for relatively small values of  $r$ ,  $\text{Sep}(V^r(H, K))$  is contained in a proper subvariety  $A^*$ . It follows that  $\text{Sep}(V^r(H, K))$  belongs to the  $\sigma$ -algebra  $\mathcal{B}$  when  $r$  satisfies  $1 \leq r \leq n/2$ .

*Remark 11.3.* (Structure of  $\text{Sep}(V^r(H, K))$  for large  $r$ ). On the other hand, for large values of  $r$  the set  $\text{Sep}(V^r(H, K))$  has different properties. Indeed, Theorem 10.1 asserts that the probability of  $\text{Sep}(V^{mn}(H, K))$  is neither 0 nor 1, so that Proposition 11.1 implies that  $\text{Sep}(V^{mn}(H, K))$  cannot belong to the  $\sigma$ -algebra  $\mathcal{A}$  of “real-analytically constructible” sets, nor even to its somewhat larger relative  $\mathcal{B}$ . Perhaps this set-theoretic phenomenon helps to explain the computational difficulties that arise from attempts to decide whether a concretely presented state of a tensor product of matrix algebras is entangled.

Finally, note that for any  $r$ , (8.1) implies that  $\text{Sep}(V^r(H, K))$  can be expressed as an uncountable union of proper subvarieties  $\cup \{Z_\lambda : \lambda \in U(q)\}$  parametrized by the group  $U(q)$ ,  $q = m^2 n^2$ . But since the union is uncountable, that fact provides no information about whether  $\text{Sep}(V^r(H, K))$  belongs to the constructible  $\sigma$ -algebra  $\mathcal{A}$ .

## 12. Concluding Remarks

*Remark 12.1.* (States versus completely positive maps). While we have focused on states of matrix algebras and their extensions in this paper, all of the above results have equivalent formulations as statements about completely positive maps. In more concrete terms, note that with every  $r$ -tuple  $v = (v_1, \dots, v_r) \in V^r(H, K)$  one can associate a unit-preserving completely positive (UCP) map  $\phi_v : \mathcal{B}(K) \rightarrow \mathcal{B}(H)$  by way of

$$\phi_v(a) = \sum_{k=1}^r v_k^* a v_k, \quad a \in \mathcal{B}(K),$$

and there is a simple notion of *rank* in the category of completely positive maps in which  $\phi_v$  has rank  $\leq r$  (see [Arv03], Remark 9.1.3). Indeed, this map promotes to a homeomorphism  $\dot{v} \in X^r \mapsto \phi_v$  of  $X^r$  onto the space of UCP maps of rank  $\leq r$ . This parameterization  $v \mapsto \phi_v$  of UCP maps of rank  $\leq r$  corresponds to the parameterization  $v \mapsto \rho_v \in E^r(\omega)$  of (6.3) via

$$\rho_v(a \otimes b) = \langle (\phi_v(a) \otimes b) \xi_\omega, \xi_\omega \rangle, \quad a \in \mathcal{B}(K), \quad b \in \mathcal{B}(H). \quad (12.1)$$

Indeed, the bijective correspondence (12.1) between states and UCP maps exists independently of the issues taken up in this paper, and it is useful. For example, the connection between states of  $A \otimes M_n$  (where  $A$  is a unital  $C^*$ -algebra) and completely positive maps of  $A$  into  $M_n$  was first exploited in the proof of the extension theorem for completely positive maps (see Lemma 1.2.6 of [Arv69]).

*Remark 12.2.* (Quantum channels). A *quantum channel* is a completely positive map  $\psi : M' \rightarrow N'$  between the *duals* of matrix algebras  $M$  and  $N$  that carries states to states. Quantum channels are the adjoints of UCP maps. Indeed, the most general quantum channel  $\psi$  as above has the form  $\psi(\rho) = \rho \circ \phi$ ,  $\rho \in M'$ , where  $\phi : N \rightarrow M$  is a UCP map. In particular, *quantum channels of rank  $\leq r$  are parameterized by the same real-analytic noncommutative sphere that serves to parameterize UCP maps of rank  $\leq r$ .*

*Remark 12.3.* (Better estimates of the critical rank). Fix Hilbert spaces  $H, K$  of dimensions  $n \leq m$  respectively, and let  $v(n, m)$  be the largest integer such that the probability of entanglement in  $(X^r, P^r)$  is 1 for every  $r = 1, 2, \dots, v(n, m)$ . Together, Theorems 9.1 and 10.1 make the assertion

$$n/2 \leq v(n, m) < nm.$$

Our feeling is that each of these two bounds is far from best possible, and the problem of improving these bounds deserves further study.

*Remark 12.4.* (Bitraces). By a *bitrace* we mean a state  $\rho$  of  $\mathcal{B}(H \otimes H)$  such that  $\rho(a \otimes \mathbf{1}) = \rho(\mathbf{1} \otimes a) = \tau(a)$ ,  $a \in \mathcal{B}(H)$ ,  $\tau$  being the tracial state of  $\mathcal{B}(H)$ . There has been recent work on identifying the extremal bitraces, of which we mention only [Par05, PS07] and, in the equivalent context of UCP maps, [LS93]. After associating bitraces with UCP maps as in (12.1), one finds that bitraces are in one-to-one correspondence with the set of all UCP maps  $\phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$  that preserve the trace. In turn, the space of all trace-preserving UCP maps of rank  $\leq r$  corresponds to the subspace of  $V^r(H, H)$  consisting of all  $r$ -tuples  $v = (v_1, \dots, v_r)$  that satisfy

$$v_1^* v_1 + \dots + v_r^* v_r = v_1 v_1^* + \dots + v_r v_r^* = \mathbf{1}_H.$$

The latter equations define a proper subvariety of  $V^r(H, H)$  (Definition 2.5) that is neither homogeneous nor connected, and whose structure is considerably more complicated than that of  $V^r(H, H)$  itself. It is unclear to what extent the results of this paper have counterparts for bitraces.

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## Appendix

### Appendix A. Existence of Real-Analytic Structures

Theorem A.2 below is essentially known; but since it is basic to our main result, we include a proof. The argument we give makes use of the following result, which paraphrases a special case of Theorem 10.3.1 of [Die69]. It asserts that a real analytic map

of  $\mathbb{R}^n$  to  $\mathbb{R}^m$  whose derivative has constant rank can be realized locally as a linear map  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  after a real-analytic distortion of both coordinate systems. Let  $U, V$  be open subsets of  $\mathbb{R}^n$ . A real-analytic isomorphism of  $U$  on  $V$  is a bijection  $u : U \rightarrow V$  such that both  $u$  and  $u^{-1}$  are real-analytic mappings.

**Theorem A.1.** *Let  $D \subseteq \mathbb{R}^n$  be an open set and let  $f : D \rightarrow \mathbb{R}^m$  be a real-analytic mapping such that  $\text{rank } f'(x) = r$  is constant for  $x \in D$ . Then for every  $a \in D$ , there exist*

- (i) *a real-analytic isomorphism  $u$  of the open unit ball of  $\mathbb{R}^n$  onto an open set  $U \subseteq \mathbb{R}^n$  satisfying  $a \in U \subseteq D$ ,*
- (ii) *a real-analytic isomorphism  $v$  of the open unit ball of  $\mathbb{R}^m$  onto an open set  $V \subseteq \mathbb{R}^m$  satisfying  $f(U) \subseteq V$ ,*

*such that  $f|_U$  admits a factorization  $f = v \circ L \circ u^{-1}$ , where  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is the linear map  $L(x_1, \dots, x_n) = (x_1, \dots, x_r, 0, \dots, 0)$ .*

**Theorem A.2.** *Let  $H, K$  be finite-dimensional Hilbert spaces with  $\dim H \leq \dim K$ . Then the space  $\mathcal{S}$  of all isometries in  $\mathcal{B}(H, K)$  is a connected real-analytic manifold, and a homogeneous space relative to a smooth transitive action of the unitary group  $\mathcal{U}(K)$ . In particular, there is a unique probability measure on  $\mathcal{S}$  that is invariant under the  $\mathcal{U}(K)$ -action.*

*Proof.* To introduce a real-analytic structure on  $\mathcal{S}$ , consider the mapping  $f : \mathcal{B}(H, K) \rightarrow \mathcal{B}(H)$  given by  $f(v) = v^*v$ . If we view  $f$  as a real-analytic map of finite-dimensional real vector spaces, then the derivative of  $f$  at  $v \in \mathcal{B}(H, K)$  is the real-linear map  $f'(v) : h \in \mathcal{B}(H, K) \mapsto v^*h + h^*v \in \mathcal{B}(H)$ . The range of  $f'(v)$  is contained in the real vector space  $\mathcal{B}(H)^{\text{sa}}$  of self-adjoint operators on  $H$ .

Let  $D$  be the set of all  $v \in \mathcal{B}(H, K)$  such that  $v^*v$  is invertible. Then  $D$  is an open set containing  $\mathcal{S}$ , and we claim that  $f'(v)$  has range  $\mathcal{B}(H)^{\text{sa}}$  for every  $v \in D$ . Indeed, the most general real linear functional on  $\mathcal{B}(H)^{\text{sa}}$  has the form  $\omega(y) = \text{trace}(\Omega y)$  for some  $\Omega = \Omega^* \in \mathcal{B}(H)$ , and we have to show that if  $\omega$  annihilates the range of  $f'(v)$  for some  $v \in D$  then  $\omega = 0$ . Since  $\Omega = \Omega^*$ , we can replace  $h$  with  $\sqrt{-1}h$  in the formula

$$\text{trace}(\Omega(v^*h + h^*v)) = \omega(f'(v)(h)) = 0$$

to obtain  $\text{trace}(\Omega v^*h - h^*v)) = 0$ . After adding these two expressions we obtain  $\text{trace}(\Omega v^*h) = 0$  for all  $h \in \mathcal{B}(H, K)$ , hence  $\Omega v^* = 0$  for all  $v \in D$ . It follows that  $\Omega v^*v = 0$  and finally  $\Omega = 0$  since  $v^*v$  is invertible for every  $v \in D$ .

Hence the rank of  $f'(v)$  is constant throughout  $D$ . Theorem A.1 now implies that the subspace  $\mathcal{S} = \{v \in D : f(v) = \mathbf{1}_H\}$  of  $D$  can be endowed locally with a real-analytic structure, and moreover, that these local structures are mutually compatible with each other. Hence  $\mathcal{S}$  is a real-analytic submanifold of  $\mathcal{B}(H, K)$ .

For the remaining statements, fix  $u, v \in \mathcal{S}$ . We claim that there is a unitary operator  $w \in \mathcal{B}(K)$  such that  $wu = v$ . Indeed, since  $\|u\xi\| = \|v\xi\| = \|\xi\|$  for every  $\xi \in H$ , we can define an isometry  $w_0$  from the range of  $u$  to the range of  $v$  by setting  $w_0(u\xi) = v\xi$  for all  $\xi \in H$ . Since  $K$  is finite-dimensional,  $w_0$  can be extended to a unitary operator  $w \in \mathcal{U}(K)$ , and  $w$  satisfies  $wu = v$ . It follows that the natural action of  $\mathcal{U}(K)$  on  $\mathcal{S}$  is smooth and transitive.

The preceding observation implies that  $\mathcal{S}$  is arcwise connected. Indeed, for any two isometries  $u, v \in \mathcal{S}$ , there is a unitary operator  $w \in \mathcal{U}(K)$  such that  $wu = v$ ; and since the unitary group of  $K$  is arcwise connected, it follows that  $u$  can be connected to  $v$  by an arc of isometries.  $\square$

**Remark A.3.** (Identification of the invariant measure on  $\mathcal{S}$ ). The  $\mathcal{U}(K)$ -invariant probability measure  $\mu$  on  $\mathcal{S}$  can be described more concretely as follows. The space  $\mathcal{S}$  is embedded in the space of all operators  $\mathcal{B}(H, K)$ , and we can view the latter as a real Hilbert space with inner product

$$\langle a, b \rangle = \Re \operatorname{trace}(b^* a), \quad a, b \in \mathcal{B}(H, K).$$

The unitary group  $\mathcal{U}(K)$  acts as isometries of this real Hilbert space by left multiplication  $(u, a) \in \mathcal{U}(K) \times \mathcal{B}(H, K) \mapsto ua \in \mathcal{B}(H, K)$ . In turn, since the tangent spaces of  $\mathcal{S}$  are naturally embedded in  $\mathcal{B}(H, K)$ , this inner product gives rise to a Riemannian metric on  $\mathcal{S}$ , which in turn gives rise to a natural probability measure  $\tilde{\mu}$  after renormalization. Since the group  $\mathcal{U}(K)$  acts as isometries relative to the Riemannian structure of  $\mathcal{S}$ , the measure  $\tilde{\mu}$  must be invariant under the action of  $\mathcal{U}(K)$ , and hence  $\mu = \tilde{\mu}$ . In particular,  $\mu$  is mutually absolutely continuous with Lebesgue measure in smooth local coordinate systems for  $\mathcal{S}$ .

## Appendix B. Zeros of Real-Analytic Functions

While the result of this Appendix is well known, we lack a convenient reference and include a simple proof, the idea of which was shown to me by Michael Christ.

**Proposition B.1.** *Let  $D \subseteq \mathbb{R}^n$  be a connected open set and let  $f : D \rightarrow \mathbb{R}$  be a real-analytic function that does not vanish identically. Then the set of zeros of  $f$  has Lebesgue measure zero.*

*Proof.* Let  $Z = \{x \in D : f(x) = 0\}$ . It suffices to show that for every point  $a \in D$  there is an open set  $U$  containing  $a$  such that  $Z \cap U$  has measure zero. Choose a point  $a \in D$ . The power series expansion of  $f$  about  $a$  cannot have all zero coefficients, since that would imply that  $f$  vanishes on an open set, hence identically. Therefore some mixed partial of  $f$  of order  $N$  must be nonzero at  $a$ . This implies that the  $N^{\text{th}}$  derivative of  $f$  in some direction must be nonzero at  $a$ . By rotating the coordinate system of  $\mathbb{R}^n$  about  $a$ , we can assume that  $\partial^N f / \partial x_1^N$  is nonzero at  $a$ , and therefore on some open rectangle  $U$  centered at  $a$ . Let  $L$  be any line of the form  $x_2 = c_2, \dots, x_d = c_d$ , where  $c_2, \dots, c_d$  are constants. If  $L \cap U \neq \emptyset$ , then the restriction of  $f$  to  $L \cap U$  is a nonzero real-analytic function of the single variable  $x_1$ —which has isolated zeros. Hence the intersection of  $Z$  with  $L \cap U$  has linear Lebesgue measure zero. By Fubini's theorem,  $Z \cap U$  has measure zero.  $\square$

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