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The asymptotic lift of a completely positive map

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Abstract

Starting with a unit-preserving normal completely positive map $L : M \to M$ acting on a von Neumann algebra—or more generally a dual operator system—we show that there is a unique reversible system $\alpha : N \to N$ (i.e., a complete order automorphism α of a dual operator system N) that captures all of the asymptotic behavior of L, called the *asymptotic lift* of L. This provides a noncommutative generalization of the Frobenius theorems that describe the asymptotic behavior of the sequence of powers of a stochastic $n \times n$ matrix. In cases where M is a von Neumann algebra, the asymptotic lift is shown to be a W^*-dynamical system $(N, \mathbb{Z})$, and we identify $(N, \mathbb{Z})$ as the tail flow of the minimal dilation of L. We are also able to identify the Poisson boundary of L as the fixed algebra N^α. In general, we show the action of the asymptotic lift is trivial iff L is *slowly oscillating* in the sense that

$$\lim_{n \to \infty} \| \rho \circ L^{n+1} - \rho \circ L^n \| = 0, \quad \rho \in M_*.$$

Hence α is often a nontrivial automorphism of N. The asymptotic lift of a variety of examples is calculated.
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1. Introduction

Throughout this paper we use the term UCP map to denote a normal unit-preserving completely positive map $L : M_1 \to M_2$ of one dual operator system into another. While we are

primarily concerned with the dynamical properties of UCP maps $L : M \to M$ that act on a von Neumann algebra M, it is appropriate to broaden that category to include UCP self-maps of more general dual operator systems.

Stochastic $n \times n$ matrices $P = (p_{ij})$ describe the transition probabilities of n-state Markov chains. The asymptotic properties of the sequence of powers of the transition matrix govern the long-term statistical behavior of the process after initial transient fluctuations have died out [6, pp. 170–185]. A stochastic $n \times n$ matrix $P = (p_{ij})$ gives rise to a UCP map of the commutative von Neumann algebra $\mathbb{C}^n$ by way of

$$(Px)_i = \sum_{j=1}^{n} p_{ij} x_j, \quad 1 \leqslant i \leqslant n, \ x = (x_1, \ldots, x_n) \in \mathbb{C}^n,$$

and the classical Perron–Frobenius theory provides an effective description of the asymptotic behavior of the sequence $P, P^2, P^3, \ldots$ (see Section 9).

Recently, several emerging areas of mathematics have opened a vista of potential applications for noncommutative generalizations of the Frobenius theorems. The most obvious examples are quantum probability, noncommutative dynamics [2], and quantum computing [15]. Such considerations led us to initiate a study of "almost periodic" UCP maps on von Neumann algebras in [3]. Broadly speaking, those results allowed us to relate the asymptotic behavior of the powers of *certain* UCP maps on von Neumann algebras to the asymptotic behavior of ∗-automorphisms of other naturally associated von Neumann algebras.

However, the hypothesis of almost periodicity is very restrictive. For example, it forces the spectrum of the UCP map to have a discrete component consisting of eigenvalues on the unit circle. While this discrete spectrum appeared to be essential for the results of [3], it is missing in many important examples. Nevertheless, in *all* of the examples that we were able to penetrate, there was a "hidden" W^*-dynamical system that shared the same asymptotic behavior. That led us to suspect that a different formulation might be possible, in which the almost periodic hypothesis is eliminated entirely. This turned out to be true, but the new results require a complete reformulation in terms of UCP maps on dual operator systems.

The main results of this paper are Theorems 3.2, 5.1, 6.1, 7.1. While these general results are not as sharp as those of [3] when restricted to the case of almost periodic maps, we believe that is compensated for by the simplicity and full generality of the new setting. There are natural variations of all of the above results for one-parameter semigroups of normal completely positive maps that will be taken up elsewhere. We also note that a very recent paper of Størmer [21] complements the results of [3].

2. Reversible lifts of UCP maps

Recall that an *operator system* is a norm-closed self-adjoint linear subspace M of the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H, such that the identity operator $\mathbf{1}$ belongs to M. A *dual* operator system is an operator system that is closed in the weak∗-topology of $\mathcal{B}(H)$. We write M_* for the predual of such an operator system M, namely the norm-closed linear subspace of the dual of M consisting of all restrictions to M of normal linear functionals on $\mathcal{B}(H)$; and since M can be naturally identified with the dual of M_*, we refer to the M_*-topology as the weak∗-topology of M. A *normal* linear map of dual operator systems is a linear map $L : M \to N$ that is continuous relative to the respective weak∗-topologies. Of course, such a map is the adjoint of a unique bounded linear map of preduals, namely $\rho \in N_* \mapsto \rho \circ L \in M_*$.

While there is an abstract characterization of operator systems [5, Theorem 4.4], we shall not require the details of it here. Only rarely do we require a realization $M \subseteq \mathcal{B}(H)$ of M as a concrete dual operator system; but when we do, we require that the realization have the above properties—namely that M is a weak*-closed concrete operator system in $\mathcal{B}(H)$ and M_* consists of all restrictions of normal linear functionals of $\mathcal{B}(H)$. For the most part, our conventions and basic terminology follow those of the monographs [7] and [17].

We fix attention on the category whose objects are UCP self-maps

$$L : M \to M$$

acting on dual operator systems M, and whose morphisms are equivariant UCP maps. Thus, a homomorphism from $L_1 : M_1 \to M_1$ to $L_2 : M_2 \to M_2$ is a UCP map

$$E : M_1 \to M_2$$

satisfying $E \circ L_1 = L_2 \circ E$. In this case we say that L_1 is a *lifting* of L_2 through E, or simply a *lift* of L_2. By an *automorphism* of a dual operator system N we mean a UCP map $\alpha : N \to N$ having a UCP inverse $\alpha^{-1} : N \to N$.

Definition 2.1. A *reversible lift* of a UCP map $L : M \to M$ is a triple (N, α, E) consisting of an automorphism $\alpha : N \to N$ of another dual operator system N and a UCP map $E : N \to M$ satisfying $E \circ \alpha = L \circ E$.

A UCP map $L : M \to M$ has many reversible lifts, the simplest being the trivial lift $(\mathbb{C}, \mathrm{id}, \iota)$, where $\iota : \mathbb{C} \to M$ is the inclusion $\iota(\lambda) = \lambda \cdot \mathbf{1}_M$. We begin by pointing out that all reversible lifts must satisfy a system of asymptotic inequalities. In the usual way, $L : M \to M$ gives rise to a hierarchy of UCP maps $L_n : M^{(n)} \to M^{(n)}$, $n = 1, 2, \ldots$, in which $M^{(n)} = M_n(\mathbb{C}) \otimes M$ is the $n \times n$ matrix system over M and $L_n = \mathrm{id} \otimes L : M^{(n)} \to M^{(n)}$ is the naturally induced UCP map. Similarly, a reversible lifting (N, α, E) of $L : M \to M$ gives rise to a hierarchy of reversible liftings $(N^{(n)}, \alpha_n, E_n)$ of $L_n : M^{(n)} \to M^{(n)}$, one for every $n = 1, 2, \ldots$.

Proposition 2.2. *Let* (N, α, E) *be a reversible lift of a given UCP map* $L : M \to M$. *For every bounded linear functional* ρ *on* M, *the sequence of norms* $\|\rho \circ L^k\|$ *decreases with increasing* $k = 1, 2, \ldots,$ *and we have*

$$\|\rho \circ E\| \leqslant \lim_{k \to \infty} \|\rho \circ L^k\|. \tag{2.1}$$

Moreover, the inequalities (2.1) *persist throughout the hierarchy of liftings of* $L_n : M^{(n)} \to M^{(n)}$, *as* ρ *ranges over the dual of* $M^{(n)}$, $n \geqslant 1$.

Proof. Every UCP map is a contraction, hence $\|\rho \circ L^{k+1}\| \leqslant \|\rho \circ L^k\|$ for every $k \geqslant 0$. Moreover, for fixed $x \in N$ and $k \geqslant 0$ we can write

$$E(x) = E\big(\alpha^k\big(\alpha^{-k}(x)\big)\big) = L^k\big(E\big(\alpha^{-k}(x)\big)\big).$$

Since $\|E \circ \alpha^{-k}\| \leqslant 1$, we conclude that for every $\rho \in M'$,

$$\|\rho \circ E\| = \sup_{\|x\|=1} |\rho(E(x))| = \sup_{\|x\|=1} |\rho \circ L^k(E \circ \alpha^{-k}(x))| \leqslant \|\rho \circ L^k\|,$$

and (2.1) follows after passing to the limit as $k \to \infty$. The same argument applies throughout the hierarchy of liftings $(N^{(n)}, \mathrm{id}_n \otimes \alpha, \mathrm{id}_n \otimes E)$, after one replaces ρ with a bounded linear functional on $M^{(n)}$, $n = 1, 2, \ldots$. $\square$

Remark 2.3 *(Nondegeneracy).* Let (N, α, E) be a reversible lift of a UCP map $L : M \to M$. Since $E : N \to M$ is normal, it has a pre-adjoint $E_* : M_* \to N_*$, defined by $E_*(\rho) = \rho \circ E$, $\rho \in M_*$. Consider the range

$$E_*(M_*) = \{\rho \circ E : \ \rho \in M_*\}$$

of this map. $E_*(M_*)$ is a linear subspace of N_*, and note that it is invariant under the invertible isometry $\alpha_* \in \mathcal{B}(N_*)$:

$$E_*(M_*) \circ \alpha \subseteq E_*(M_*).$$

Indeed, that is immediate from equivariance and normality of L, since for $\rho \in M_*$ we have $\rho \circ E \circ \alpha = (\rho \circ L) \circ E \in M_* \circ E$. It follows that the sequence $E_*(M_*) \circ \alpha^n$, $n \in \mathbb{Z}$, defines a doubly infinite tower of subspaces of N_*

$$\cdots \subseteq E_*(M_*) \circ \alpha \subseteq E_*(M_*) \subseteq E_*(M_*) \circ \alpha^{-1} \subseteq E_*(M_*) \circ \alpha^{-2} \subseteq \cdots.$$

A straightforward application of the Hahn–Banach theorem shows that this tower is norm-dense in N_* if and only if for every $y \in N$ one has

$$E(\alpha^{-n}(y)) = 0, \quad n = 0, 1, 2, \ldots \quad \Rightarrow \quad y = 0. \tag{2.2}$$

A reversible lifting (N, α, E) of L is said to be *nondegenerate* if condition (2.2) is satisfied.

It is significant that when (2.2) fails, one can always replace (N, α, E) with a *nondegenerate* reversible lifting $(\tilde{N}, \tilde{\alpha}, \tilde{E})$ using the following device. We may assume that $M \subseteq \mathcal{B}(H)$ is realized as a concrete dual operator system. Let $\tilde{H} = \ell^2(\mathbb{Z}) \otimes H$ be the Hilbert space of all square-summable bilateral sequences from H and define a map $\theta : N \to \mathcal{B}(\tilde{H})$ by

$$(\theta(y)\xi)(n) = E(\alpha^{-n}(y))\xi(n), \quad \xi \in \tilde{H}, \ n \in \mathbb{Z}. \tag{2.3}$$

$\tilde{N}$ is defined as the weak*-closure of $\theta(N)$. The unitary shift defined on $\tilde{H}$ by $U\xi(n) = \xi(n-1)$, $n \in \mathbb{Z}$, implements a *-automorphism $\tilde{\alpha}(X) = UXU^*$ of $\mathcal{B}(\tilde{H})$ such that $\tilde{\alpha}(\tilde{N}) = \tilde{N}$; indeed, $\theta(N)$ is stable under shifts to the left or right because α is an automorphism of N. Note too that:

$$(U\theta(y)U^{-1}\xi)(n) = E(\alpha^{-n+1}(y))\xi(n) = L(E(\alpha^{-n}(y)))\xi(n), \quad \xi \in \tilde{H}, \ n \in \mathbb{Z}.$$

It follows that the map $\tilde{E} : \mathcal{B}(\tilde{H}) \to \mathcal{B}(H)$ that compresses an operator matrix in $\mathcal{B}(\tilde{H})$ to its 00th component restricts to a UCP map $\tilde{E} : \tilde{N} \to M$ satisfying $\tilde{E} \circ \tilde{\alpha} = L \circ \tilde{E}$. Thus, $(\tilde{N}, \tilde{\alpha}, \tilde{E})$ is a reversible lifting of L. It is a homomorphic image of (N, α, E) in the sense that the UCP map

$\theta : N \to \tilde{N}$ satisfies $\tilde{\alpha} \circ \theta = \theta \circ \alpha$ and $\tilde{E} \circ \theta = E$ (see Section 4 for a discussion of the category of reversible liftings of L). Finally, by examining components in the obvious way, one verifies directly that $(\tilde{N}, \tilde{\alpha}, \tilde{E})$ is nondegenerate.

3. Asymptotic lifts of UCP maps

In this section we show by a direct construction that there is a reversible lifting with favorable asymptotic properties and that, after degeneracies have been eliminated, it is *unique* up to natural isomorphism.

Definition 3.1. Let $L : M \to M$ be a UCP map on a dual operator system. An *asymptotic lift* of L is a reversible lifting (N, α, E) of L that satisfies nondegeneracy (2.2), such that the inequalities (2.1) become equalities for *normal* linear functionals throughout the entire matrix hierarchy

$$\| \rho \circ (\mathrm{id}_n \otimes E) \| = \lim_{k \to \infty} \| \rho \circ (\mathrm{id}_n \otimes L)^k \|, \quad \rho \in M_*^{(n)}, \ n = 1, 2, \ldots . \tag{3.1}$$

We come now to a basic result.

Theorem 3.2. *Every UCP map $L : M \to M$ of a dual operator system has an asymptotic lifting. If (N_1, α_1, E_1) and (N_2, α_2, E_2) are two asymptotic liftings for L, then there is a unique isomorphism of dual operator systems $\theta : N_1 \to N_2$ such that $\theta \circ \alpha_1 = \alpha_2 \circ \theta$ and $E_2 \circ \theta = E_1$.*

The existence assertion of Theorem 3.2 is proved by a direct construction involving inverse sequences, which are defined as follows:

Definition 3.3. Let $L : M \to M$ be a UCP map on a dual operator system. By an *inverse sequence* for L we mean a bilateral sequence $(x_n)_{n \in \mathbb{Z}}$ of elements of M satisfying $\sup_n \| x_n \| < \infty$, and

$$x_n = L(x_{n+1}), \quad n \in \mathbb{Z}. \tag{3.2}$$

The set of all inverse sequences for L is denoted $\mathcal{S}_L$, or more simply $\mathcal{S}$ when there is no cause for confusion.

Remark 3.4 (*Properties of inverse sequences*). The set $\mathcal{S}$ of all inverse sequences for $L : M \to M$ is a vector space that is closed under pointwise involution $(x_n) \mapsto (x_n^*)$, and it contains all "constant" scalar sequences of the form $(\ldots, \lambda \cdot \mathbf{1}, \lambda \cdot \mathbf{1}, \lambda \cdot \mathbf{1}, \ldots), \lambda \in \mathbb{C}$. More generally, the constant sequences $(\ldots, a, a, a, \ldots) \in \mathcal{S}$ correspond bijectively with the space of fixed elements $\{a \in M: L(a) = a\}$. Notice too that $\mathcal{S}$ is stable under shifting to the right or left; if $(x_n)_{n \in \mathbb{Z}}$ belongs to $\mathcal{S}$ then so does $(x_{n+k})_{n \in \mathbb{Z}}$ for every $k = 0, \pm 1, \pm 2, \ldots$.

Every element x_k of an inverse sequence (x_n) determines all of its predecessors uniquely, since $x_{k-1} = L(x_k)$ and, more generally, $x_r = L^{k-r}(x_k)$ for all $r \leqslant k$. On the other hand, x_k does *not* determine x_{k+1} uniquely, since there can be many solutions z of the equation $L(z) = x_k$. If we fix a particular solution z and replace x_{k+1} with z in the $(k + 1)$st spot, then it may not be possible to solve the equation $z = L(w)$ for $w \in M$; and *in that case there is no inverse sequence whose $(k + 1)$st term is the replaced element z and whose kth term is x_k.*

More generally, given an element $a \in M$, the question of whether or not there is an inverse sequence (x_n) satisfying $x_0 = a$ can be subtle.

Proof of Theorem 3.2. *Existence.* In order to construct an asymptotic lifting for L, we realize $M \subseteq \mathcal{B}(H)$ as a concrete weak*-closed operator system. Let $\ell^2 \otimes H$ be the Hilbert space of all sequences $n \in \mathbb{Z} \mapsto \xi_n \in H$ satisfying $\sum_n \|\xi_n\|^2 < \infty$, with its usual inner product.

We can realize the space $\mathcal{S}$ of all inverse sequences for L as an operator subspace $N \subseteq \mathcal{B}(\ell^2 \otimes H)$ by identifying an inverse sequence $(x_n) \in \mathcal{S}$ with the diagonal operator $D = \operatorname{diag}(x_n)$ defined by

$$(D\xi)_n = x_n \xi_n, \quad n \in \mathbb{Z}.$$

The operator norm of $\operatorname{diag}(x_n)$ is given by $\|\operatorname{diag}(x_n)\| = \sup_{n \geqslant 0} \|x_n\|$. Since L is a normal map, the relations defined by (3.2) are weak*-closed, and therefore the space $N = \{\operatorname{diag}(x_n): (x_n) \in \mathcal{S}\}$ is closed in the weak*-topology of $\mathcal{B}(\ell^2 \otimes H)$. Remark 3.4 implies that N is self-adjoint and contains the identity operator of $\mathcal{B}(\ell^2 \otimes H)$, hence N acquires the structure of a dual operator system.

Consider the right-shift $\alpha: N \to N$ of diagonal operators

$$\alpha\big(\operatorname{diag}(x_n)\big) = \operatorname{diag}(x_{n-1}).$$

Letting U be the unitary bilateral shift acting on $\ell^2 \otimes H$ by

$$(U\xi)_n = \xi_{n-1}, \quad n \in \mathbb{Z}, \ \xi \in \ell^2 \otimes H,$$

one finds that the associated *-automorphism $X \mapsto UXU^*$ of $B(\ell^2 \otimes H)$ implements the action of α on N. Hence α is a complete automorphism of the concrete operator system N. The natural inclusion of H in $\ell^2(\mathbb{Z}) \otimes H$, in which a vector $\xi \in H$ is identified with $\delta_0 \otimes \xi \in \ell^2(\mathbb{Z}) \otimes H$, gives rise to a normal map $X \mapsto P_H X{\upharpoonright}_H$ that restricts to a map $E: N \to M$ satisfying

$$E\big(\operatorname{diag}(x_n)\big) = x_0, \quad (x_n) \in \mathcal{S}.$$

One has

$$E \circ \alpha\big(\operatorname{diag}(x_n)\big) = E\big(\operatorname{diag}(x_{n-1})\big) = x_{-1} = L(x_0) = L \circ E\big(\operatorname{diag}(x_n)\big),$$

so that (N, α, E) becomes a reversible lift of L.

It remains to verify (2.2) and (3.1). For (2.2), suppose that $X = \operatorname{diag}(x_k)$ satisfies $E(\alpha^{-n}(X)) = x_n = 0$ for $n \geqslant 0$. Since the sequence (x_k) belongs to S, this implies that $x_j = L^{|j|}(x_0) = 0$ for negative j as well, hence (x_k) is the zero sequence.

For (3.1), it is enough to verify the system of nontrivial inequalities

$$\big\|\rho \circ (\operatorname{id}_n \otimes E)\big\| \geqslant \lim_{k \to \infty} \big\|\rho \circ (\operatorname{id}_n \otimes L)^k\big\|, \quad \rho \in M_*^{(n)}, \ n = 1, 2, \ldots. \tag{3.3}$$

Consider first the case $n = 1$, and choose $\rho \in M_*$. For each $k = 1, 2, \ldots$ we can find an element $u_k \in M$ satisfying $\|u_k\| = 1$ and $|\rho(L^k(u_k))| = \|\rho \circ L^k\|$. Consider the triangular array $s^k = (s_0^k, \ldots, s_k^k)$, $k = 1, 2, \ldots$, of elements of M defined by

$$\big(s_0^k, \ldots, s_k^k\big) = \big(L^k(u_k), L^{k-1}(u_k), \ldots, L(u_k), u_k\big), \quad k = 1, 2, \ldots.$$

Each component of every one of these sequences belongs to ball M, and $s_0^k = x_0$ for every $k \geqslant 1$. Moreover, the jth component s_j^k of any one of them is obtained from the $(j+1)$st by applying L,

$$s_j^k = L(s_{j+1}^k), \quad j = 0, 1, 2, \ldots, k-1. \tag{3.4}$$

Since the infinite Cartesian product ball $M \times$ ball $M \times \cdots$ is compact in its weak* product topology, there is a subnet $s^{k'}$ of the sequence s^k with the property that each of its components (note that each component is well defined for sufficiently large k') converges weak* to an element of ball M. Hence we can define a single infinite sequence $x_0, x_1, x_2, \ldots \in$ ball M by

$$x_j = \lim_{k'} s_j^{k'}, \quad j = 0, 1, 2, \ldots.$$

Since L is a normal map, the relations (3.4) imply that $x_{j+1} = L(x_j)$, $j = 0, 1, 2, \ldots$. If we continue the sequence into negative integers by setting $x_k = L^{|k|}(x_0)$ for $k \leqslant -1$, the result is a sequence $(x_n) \in \mathcal{S}$ satisfying $\sup_n \|x_n\| \leqslant 1$ and $E(\mathrm{diag}(x_n)) = x_0$. Thus we conclude that

$$\|\rho \circ E\| \geqslant |\rho(x_0)| = \lim_{k'} |\rho(L^{k'}(u_{k'}))| = \lim_{k'} \|\rho \circ L^{k'}\| = \lim_{k \to \infty} \|\rho \circ L^k\|,$$

and (3.3) follows.

Notice that this argument can be repeated *verbatim* to establish (3.3) throughout the matrix hierarchy for $n \geqslant 2$, since the inverse sequences for $\mathrm{id}_n \otimes L$ are bilateral sequences $(\tilde{x}_k)$ whose components $\tilde{x}_k$ are $n \times n$ matrices in $M^{(n)}$ that satisfy $(\mathrm{id}_n \otimes L)(\tilde{x}_{k+1}) = \tilde{x}_k$, $k \in \mathbb{Z}$. We conclude that (N, α, E) is an asymptotic lift of L. $\quad\square$

Turning now to the uniqueness issue for asymptotic lifts, we require the dual formulation of (3.1)—Lemma 3.6—the proof of which makes use of the following elementary result. Since we lack an appropriate reference, we sketch a proof of the latter for completeness.

Lemma 3.5. *Let X be a Banach space, let $K_1 \supseteq K_2 \supseteq \cdots$ be a decreasing sequence of nonempty weak*-compact convex subsets of the dual X' with intersection K_∞. Then for every weak*-continuous linear functional $\rho \in X''$,*

$$\sup\{|\rho(x)|: \ x \in K_n\} \downarrow \sup\{|\rho(x)|: \ x \in K_\infty\}, \quad as \ n \to \infty. \tag{3.5}$$

Proof. Fix ρ. The sequence of nonnegative numbers $\sup\{|\rho(x)|: \ x \in K_n\}$ obviously decreases with n, and its limit ℓ satisfies

$$\ell \geqslant \sup\{|\rho(x)|: \ x \in K_\infty\}.$$

To prove the opposite inequality choose, for every $n = 1, 2, \ldots$, an element $x_n \in K_n$ such that $|\rho(x_n)| = \sup\{|\rho(x)|: \ x \in K_n\}$. By compactness of the unit ball of X', there is a subnet $\{x_{n'}\}$ of $\{x_n\}$ that converges weak* to x_∞. The limit point x_∞ must belong to $K_\infty = \bigcap_n K_n$ because the K_n decrease with n, and since the numbers $|\rho(x_n)| = \sup\{|\rho(x)|: \ x \in K_n\}$ converge to ℓ, it follows from weak*-continuity of ρ that

$$\left|\rho(x_\infty)\right| = \lim_{n'}\left|\rho(x_{n'})\right| = \lim_{n\to\infty}\left|\rho(x_n)\right| = \ell.$$

Since $\ell = |\rho(x_\infty)| \leqslant \sup\{|\rho(x)|\colon x \in K_\infty\}$, the proof is complete. □

Lemma 3.6. *Let $L\colon M \to M$ be a UCP map on a dual operator system and let (N, α, E) be a reversible lift of L. The following are equivalent*:

(i) *Every $\rho \in M_*$ satisfies* (3.1)

$$\|\rho \circ E\| = \lim_{n\to\infty}\|\rho \circ L^n\|.$$

(ii) *Writing* ball X *for the closed unit ball of a normed space X, we have*

$$E(\text{ball }N) = \bigcap_{n=0}^{\infty} L^n(\text{ball }M). \tag{3.6}$$

Proof. (i) $\Rightarrow$ (ii). Choose an element $y \in$ ball N. Then for every $n = 0, 1, 2, \ldots$ we can write

$$E(y) = E\bigl(\alpha^n \circ \alpha^{-n}(y)\bigr) = L^n E\bigl(\alpha^{-n}(y)\bigr) \in L^n(\text{ball }M)$$

since $E(\alpha^{-n}(y)) \in$ ball M. Hence $E(\text{ball }N) \subseteq \bigcap_n L^n(\text{ball }M)$. For the opposite inclusion, note that both sides of $E(\text{ball }N) \subseteq \bigcap_n L^n(\text{ball }M)$ are circled weak*-compact convex subsets of M, so by a standard separation theorem it suffices to show that for every $\rho \in M_*$ one has

$$\sup\bigl\{\left|\rho(x)\right|\colon x \in E(\text{ball }N)\bigr\} = \sup\biggl\{\left|\rho(x)\right|\colon x \in \bigcap_n L^n(\text{ball }M)\biggr\}. \tag{3.7}$$

The left-hand side of (3.7) is $\|\rho \circ E\|$, while by Lemma 3.5, the right-hand side is

$$\lim_{n\to\infty}\sup\bigl\{\left|\rho(x)\right|\colon x \in L^n(\text{ball }M)\bigr\} = \lim_{n\to\infty}\|\rho \circ L^n\|.$$

An application of the hypothesis (i) now gives (3.7).

(ii) $\Rightarrow$ (i). Choose $\rho \in N_*$. For $n = 1, 2, \ldots$ let $K_n = L^n(\text{ball }M)$, and set $K_\infty = \bigcap_n K_n$. Lemma 3.5 implies that $\|\rho \circ L^n\| = \sup\{|\rho(y)|\colon y \in K_n\}$ decreases to $\sup\{|\rho(y)|\colon y \in K_\infty\}$ as $n \uparrow \infty$, while by (3.6),

$$\|\rho \circ E\| = \sup\bigl\{\left|\rho(y)\right|\colon y \in E(\text{ball }N)\bigr\} = \sup\bigl\{\left|\rho(y)\right|\colon y \in K_\infty\bigr\}.$$

Thus $\lim_n \|\rho \circ L^n\|$ is identified with $\|\rho \circ E\|$, and (i) follows. □

Lemma 3.7. *Let $L\colon M \to M$ be a UCP map of a dual operator system and let (N, α, E) be a reversible lift of L that satisfies*

$$\|\rho \circ E\| = \lim_{n\to\infty}\|\rho \circ L^n\|, \quad \rho \in M_*. \tag{3.8}$$

Let $K = \{z \in N: E(\alpha^k(z)) = 0, \ k \in \mathbb{Z}\}$. Then for every $y \in N$ we have

$$\sup_{k \in \mathbb{Z}} \left\| E\left(\alpha^k(y)\right) \right\| = \inf_{z \in K} \|y + z\|. \tag{3.9}$$

Proof. The inequality $\leqslant$ is apparent, since for $z \in K$ and $k \in \mathbb{Z}$ we have

$$\left\| E\left(\alpha^k(y)\right) \right\| = \left\| E\left(\alpha^k(y + z)\right) \right\| \leqslant \|y + z\|.$$

In order to prove $\geqslant$, we may assume that $\sup_k \|E(\alpha^k(y))\| = 1$. For fixed $n = 1, 2, \ldots$, note first that $E(\alpha^{-n}(y)) \in L^p(\text{ball } M)$ for every $p = 1, 2, \ldots$. Indeed, we have $E(\alpha^{-n}(y)) = L^p(E(\alpha^{-n-p}(y))$, and $\|E(\alpha^{-n-p}(y))\| \leqslant 1$. Lemma 3.6 implies that $\bigcap_p L^p(\text{ball } M) = E(\text{ball } N)$; and since α^n is an invertible isometry of N, we can find an element $y_n \in N$ satisfying $\|y_n\| \leqslant 1$ and $E(\alpha^{-n}(y_n)) = E(\alpha^{-n}(y))$.

Note that y and y_n satisfy the following relations:

$$E\left(\alpha^{-k}(y_n)\right) = E\left(\alpha^{-k}(y)\right), \quad k \in \mathbb{Z}, \ k \leqslant n. \tag{3.10}$$

Indeed, an application of L^{n-k} to both sides of $E(\alpha^{-n}(y_n)) = E(\alpha^{-n}(y))$ leads to

$$E\left(\alpha^{-k}(y_n)\right) = L^{n-k} E\left(\alpha^{-n}(y_n)\right) = L^{n-k} E\left(\alpha^{-n}(y)\right) = E\left(\alpha^{-k}(y)\right).$$

By weak*-compactness of the unit ball of N, there is a subnet y_{n_α} of the sequence y_n that converges weak* to an element $y_\infty \in N$ such that $\|y_\infty\| \leqslant 1$. Since each map $z \mapsto E(\alpha^{-k}(z))$ is weak*-continuous on the unit ball of N, the equations (3.10) imply that $E(\alpha^{-k}(y_\infty)) = E(\alpha^{-k}(y))$, $k \in \mathbb{Z}$. Hence $y_\infty - y$ belongs to K. It follows that

$$\sup_{k \in \mathbb{Z}} \left\| E\left(\alpha^k(y)\right) \right\| = 1 \geqslant \|y_\infty\| = \left\| y + (y_\infty - y) \right\| \geqslant \inf_{z \in K} \|y + z\|$$

proving the inequality $\geqslant$ of (3.9). $\quad\square$

Proof of Theorem 3.2. *Uniqueness.* Let (N, α, E) be the asymptotic lift of L constructed in the above existence proof and let $(\tilde{N}, \tilde{\alpha}, \tilde{E})$ be another one. Define a map $\theta: y \in \tilde{N} \to \theta(y) \in N$ as follows:

$$\theta(y) = \text{diag}(x_n), \quad \text{where } x_n = \tilde{E}\left(\tilde{\alpha}^{-n}(y)\right), \ n \in \mathbb{Z}.$$

Obviously, θ is a UCP map of $\tilde{N}$ into N satisfying $\theta \circ \tilde{\alpha} = \alpha \circ \theta$. Lemma 3.7 implies that θ is an injective isometry. Indeed, applying Lemma 3.7 repeatedly to the sequence of maps $\text{id}_n \otimes \theta : \tilde{N}^{(n)} \to N^{(n)}$, $n = 1, 2, \ldots$, we find that θ is a complete isometry. We claim that $\theta(\tilde{N}) = N$. Since θ is a weak*-continuous isometry, its range is a weak*-closed subspace of N, so to prove $\theta(\tilde{N}) = N$ it suffices to show that $\theta(\tilde{N})$ is weak*-dense in N. For that, let (x_k) be an inverse sequence satisfying $\sup_k \|x_k\| = 1$. Then x_n belongs to $E(\text{ball } N)$ for every $n \geqslant 1$; and by Lemma 3.6, $E(\text{ball } N) = \tilde{E}(\text{ball } \tilde{N})$. Hence there is an element $y_n \in \text{ball } \tilde{N}$ such that $x_n = \tilde{E}(\tilde{\alpha}^{-n}(y_n))$. As in the proof of (3.10), this implies $x_k = \tilde{E}(\tilde{\alpha}^{-k}(y_n))$ for all $k \leqslant n$. These equations imply that $\theta(y_1), \theta(y_2), \ldots$ converges component-by-component to $\text{diag}(x_k)$ in the weak*-topology. Therefore $\theta(y_n) \to \text{diag}(x_k)$ (weak*) as $n \to \infty$.

Hence θ is an equivariant isomorphism of $\tilde{N}$ on N. Since the zeroth component of $\theta(y)$ is $\tilde{E}(y)$, we also have $E \circ \theta = \tilde{E}$.

Finally, we claim that the two requirements $\theta \circ \tilde{\alpha} = \alpha \circ \theta$ and $E \circ \theta = \tilde{E}$ serve to determine such a UCP map θ uniquely. Indeed, if $\theta_1, \theta_2 : \tilde{N} \to N$ are two equivariant UCP maps satisfying $E \circ \theta_1 = E \circ \theta_2 = \tilde{E}$, then for every $y \in \tilde{N}$, $n \in \mathbb{Z}$ and $k = 1, 2$, we have

$$E\big(\alpha^n\big(\theta_k(y)\big)\big) = E\big(\theta_k\big(\tilde{\alpha}^n(y)\big)\big) = \tilde{E}\big(\tilde{\alpha}^n(y)\big)$$

so that $E(\alpha^n(\theta_1(y) - \theta_2(y))) = 0$, $n \in \mathbb{Z}$. Since (N, α, E) is nondegenerate, this implies $\theta_1(y) - \theta_2(y) = 0$. $\square$

4. The hierarchy of reversible lifts of L

We have emphasized that the asymptotic lift of a UCP map $L : M \to M$ is characterized by a family of asymptotic formulas (3.1). We now show that it is possible to characterize the asymptotic lift of L in a way that makes no explicit reference to the asymptotic behavior of the sequence $L, L^2, L^3, \dots$, but rather makes use of a natural ordering of the set of all (equivalence classes of) nondegenerate reversible liftings of L.

Throughout this brief section, $L : M \to M$ will be a fixed UCP map acting on a dual operator system M. We consider the category Rev_L whose objects are *nondegenerate* reversible liftings (N, α, E) of L; a homomorphism from (N_1, α_1, E_1) to (N_2, α_2, E_2) is by definition a UCP map $\theta : N_1 \to N_2$ that satisfies $\theta \circ \alpha_1 = \alpha_2 \circ \theta$ and $E_2 \circ \theta = E_1$. Significantly, a homomorphism in this category gives rise to an embedding of operator systems:

Proposition 4.1. *A homomorphism* $\theta : (N_1, \alpha_1, E_1) \to (N_2, \alpha_2, E_2)$ *of nondegenerate reversible lifts of L defines an injective map of N_1 to N_2.*

Proof. If $y \in N_1$ satisfies $\theta(y) = 0$, then for every $n \in \mathbb{Z}$ we have

$$E_1\big(\alpha_1^n(y)\big) = E_2\big(\theta\big(\alpha_1^n(y)\big)\big) = E_2\big(\alpha_2^n\big(\theta(y)\big)\big) = 0,$$

hence $y = 0$ by nondegeneracy (2.2) of (N_1, α_1, E_1). $\square$

When a homomorphism $\theta : (N_1, \alpha_1, E_1) \to (N_2, \alpha_2, E_2)$ exists, we write

$$(N_1, \alpha_1, E_1) \leqslant (N_2, \alpha_2, E_2).$$

There is an obvious notion of isomorphism in this category, namely that there should exist a map θ as above which has a UCP inverse $\theta^{-1} : N_2 \to N_1$. Obviously, both relations $\geqslant$ and $\leqslant$ hold between isomorphic elements of Rev_L. Conversely,

Proposition 4.2. *Any two nondegenerate reversible liftings (N_k, α_k, E_k) that satisfy $(N_1, \alpha_1, E_1) \leqslant (N_2, \alpha_2, E_2) \leqslant (N_1, \alpha_1, E_1)$ are isomorphic.*

Proof. By hypothesis, there are equivariant UCP maps $\theta : N_1 \to N_2$ and $\phi : N_2 \to N_1$ such that $E_2 \circ \theta = E_1$ and $E_1 \circ \phi = E_2$. Consider the composition $\phi \circ \theta : N_1 \to N_1$. We claim that $\phi \circ \theta$ is the identity map of N_1. Indeed, for every $y \in N_1$ and $n \in \mathbb{Z}$, one can write

$$E_1\big(\alpha_1^n\big((\phi \circ \theta)(y)\big)\big) = E_1\big(\alpha_1^n \circ \phi \circ \theta(y)\big) = E_1\big(\phi \circ \alpha_2^n \circ \theta(y)\big)$$
$$= E_2\big(\alpha_2^n \circ \theta(y)\big) = E_2\big(\theta \circ \alpha_1^n(y)\big) = E_1\big(\alpha_1^n(y)\big).$$

It follows that $E_1(\alpha_1^n(\phi \circ \theta(y) - y)) = 0$ for all $n \in \mathbb{Z}$. By the nondegeneracy hypothesis (2.2) we conclude that $\phi \circ \theta(y) = y$. By symmetry, $\theta \circ \phi$ is the identity map of N_2. Hence θ is an isomorphism. $\quad\square$

Proposition 4.2 implies that the isomorphism classes of Rev_L form a *bona fide* partially ordered set. There is a smallest element—the class of the trivial lift $(\mathbb{C}, \mathrm{id}, \iota)$, $\iota : \mathbb{C} \to M$ denoting the inclusion $\iota(\lambda) = \lambda \cdot \mathbf{1}_M$. The following result characterizes the class of an asymptotic lift as the largest element.

Proposition 4.3. *Let* $(N_\infty, \alpha_\infty, E_\infty)$ *be an asymptotic lift of L. Then every* $(N, \alpha, E) \in \mathrm{Rev}_L$ *satisfies* $(N, \alpha, E) \leqslant (N_\infty, \alpha_\infty, E_\infty)$.

Proof. As in the proof of Theorem 3.2, we can realize N_∞ as the space of all diagonal operators $N_\infty = \{\mathrm{diag}(x_n) : (x_n) \in \mathcal{S}\}$, α_∞ as the shift automorphism, and E_∞ as the 0th component map. Let $\theta : N \to N_\infty$ be the UCP map defined in (2.3). We have already pointed out in Remark 2.3 that θ is a homomorphism from (N, α, E) to $(N_\infty, \alpha_\infty, E_\infty)$. $\quad\square$

5. UCP maps on von Neumann algebras

In this section we prove that the asymptotic lift of a UCP map acting on a von Neumann algebra is actually a W^*-dynamical system.

Theorem 5.1. *Let* (N, α, E) *be the asymptotic lift of a UCP map* $L : M \to M$ *that acts on a von Neumann algebra M. Then N is a von Neumann algebra and α is a $*$-automorphism of N.*

We will deduce Theorem 5.1 from the following proposition, which is normally used to establish the existence and uniqueness of the Poisson boundary of a noncommutative space of "harmonic functions." The noncommutative Poisson boundary is a far-reaching generalization of the fact that the space of bounded harmonic functions in the open unit disk D is isometrically isomorphic to the abelian von Neumann algebra $L^\infty(\partial D, \frac{d\theta}{2\pi})$ of bounded measurable functions on the boundary ∂D of D. We sketch a proof for the reader's convenience; more detail can be found in [4] and [12]. The result itself appears to have been first discovered in [8, Corollary 1.6].

Proposition 5.2. *Let* $\Lambda : M \to M$ *be a UCP map on a von Neumann algebra and let H_Λ be the operator system of all harmonic elements of M*

$$H_\Lambda = \big\{x \in M : \Lambda(x) = x\big\}.$$

There is a unique associative multiplication $x, y \in H_\Lambda \mapsto x \circ y \in H_\Lambda$ that turns H_Λ into a von Neumann algebra with predual $(H_\Lambda)_*$, on which the group $\mathrm{aut}\, H_\Lambda$ of automorphisms of the operator system structure of H_Λ acts naturally as the group of all $*$-automorphisms.

Sketch of proof. *Uniqueness*. Given two such multiplications, the identity map defines a complete order isomorphism of one C^*-algebra structure to the other. Hence it is a $*$-isomorphism, and the two multiplications agree.

Existence. We claim that there is a completely positive idempotent linear map $E: M \to M$ that has range H_Λ. Indeed, if one topologizes the set of bounded linear maps of M to itself with the topology of point-weak*-convergence, then the set of completely positive unital maps on M becomes a compact space. Let E be any limit point of the sequence of averages

$$A_n = \frac{1}{n}\big(\Lambda + \Lambda^2 + \cdots + \Lambda^n\big), \quad n = 1, 2, \dots.$$

Using $\Lambda A_n = A_n \Lambda = \frac{n+1}{n} A_{n+1} - \frac{1}{n}\Lambda$, together with the straightforward estimate $\|A_n - A_{n+1}\| \leqslant \frac{2}{n+1}$, one finds that $\Lambda E = E\Lambda = E$, and from that follows $E^2 = E$ as well as $E(M) = H_\Lambda$.

Such an idempotent E allows one to introduce a Choi–Effros multiplication on H_Λ, $x \circ y = E(xy)$, $x, y \in H_\Lambda$, which makes H_Λ into a C^*-algebra [5, Theorem 3.1, Corollary 3.2]. Since H_Λ is weak*-closed, it is the dual of the Banach space $(H_\Lambda)_*$, and a theorem of Sakai [20, Theorem 1.16.7] implies that H_Λ is a von Neumann algebra with predual $(H_\Lambda)_*$. $\quad\square$

Proof of Theorem 5.1. Let (N, α, E) be the asymptotic lift constructed in the proof of Theorem 3.2, in which $N = \{\mathrm{diag}(x_n): (x_n) \in \mathcal{S}\}$, α is the bilateral shift automorphism $\alpha(X) = UXU^{-1}$, and $E(X) = X_{00}$, for $X \in \mathcal{B}(\tilde{H})$, $\tilde{H} = \ell^2(\mathbb{Z}) \otimes H$ being the Hilbert space of sequences introduced there. In order to prove the existence of a von Neumann algebra structure on N, we appeal to Proposition 5.2 as follows.

Consider the larger von Neumann algebra $\tilde{M} \supseteq N$ of all bounded diagonal operators $Y = \mathrm{diag}(y_n)$ with components $y_n \in M$, and let Λ be the map defined on $\tilde{M}$ by

$$\Lambda\big(\mathrm{diag}(y_n)\big) = \mathrm{diag}(z_n), \quad \text{where } z_n = L(y_{n+1}), \ n \in \mathbb{Z}.$$

Obviously, Λ is a UCP map of $\tilde{M}$ with the property $\Lambda(Y) = Y$ if and only if Y has the form $Y = \mathrm{diag}(x_n)$ with (x_n) an inverse sequence for L. Thus, N is the space of all Λ-fixed elements of $\tilde{M}$. An application of Proposition 5.2 completes the proof. $\quad\square$

6. Nontriviality of the asymptotic dynamics

A W^*-dynamical system $(A, \mathbb{Z})$ is considered trivial if the automorphism of A that implements the $\mathbb{Z}$-action is the identity automorphism. The purpose of this section is to show that the asymptotic lift of a UCP map is frequently a nontrivial dynamical system.

Theorem 6.1. *For every UCP map $L: M \to M$ of a dual operator system, the following are equivalent.*

(i) *The asymptotic lift (N, α, E) of L satisfies $\alpha(y) = y$, $y \in N$.*
(ii) *Every operator x in $\bigcap_n L^n(\mathrm{ball}\, M)$ satisfies $L(x) = x$.*
(iii) *The semigroup $\{L^n: n \geqslant 0\}$ oscillates slowly in the sense that*

$$\lim_{n \to \infty} \big\|\rho \circ L^n - \rho \circ L^{n+1}\big\| = 0, \quad \rho \in M_*.$$

Proof. (i) $\Rightarrow$ (ii). We have $E(\text{ball }N) = \bigcap_n L^n(\text{ball }M)$ by Lemma 3.6, so every $x \in \bigcap_n L^n(\text{ball }M)$ has the form $x = E(y)$ for some $y \in N$. Hence $L(x) = L(E(y)) = E(\alpha(y)) = E(y) = x$.

(ii) $\Rightarrow$ (iii). Fix $\rho \in M_*$. Another application of Lemma 3.6 implies that $E(\text{ball }N) = \bigcap_n L^n(\text{ball }M)$ is pointwise fixed by L, hence $\|(\rho - \rho \circ L) \circ E\| = 0$. Using (3.1), we obtain

$$\lim_{n \to \infty} \|\rho \circ L^n - \rho \circ L^{n+1}\| = \lim_{n \to \infty} \|(\rho - \rho \circ L) \circ L^n\| = \|(\rho - \rho \circ L) \circ E\|,$$

and the right-hand side is zero.

(iii) $\Rightarrow$ (i). The preceding formula implies that $\|(\rho - \rho \circ L) \circ E\| = 0$ for every $\rho \in M_*$, hence $E = L \circ E$. So for every $n \geqslant 1$ and every $y \in N$,

$$E\big(\alpha^{-n}\big(y - \alpha(y)\big)\big) = E\big(\alpha^{-n}(y)\big) - E\big(\alpha^{-n+1}(y)\big) = E\big(\alpha^{-n}(y)\big) - L\big(E\big(\alpha^{-n}(y)\big)\big) = 0$$

and $\alpha(y) = y$ follows from nondegeneracy (2.2). $\quad\square$

Remark 6.2 *(Matrix algebras).* Many UCP maps acting on matrix algebras are associated with nontrivial W^*-dynamical systems, because of the following observation: a UCP map L of $M_n = M_n(\mathbb{C})$ satisfies property (iii) of Theorem 6.1 $\Leftrightarrow$ $\sigma(L) \cap \mathbb{T} = \{1\}$. Indeed, to prove $\Rightarrow$ contrapositively, suppose there is a point λ in the spectrum of L that satisfies $\lambda \neq 1 = |\lambda|$. Choose an operator $x \in M_n$ satisfying $\|x\| = 1$ and $L(x) = \lambda x$, and choose $\rho \in M_*$ such that $\rho(x) = 1$. Then for all $n \geqslant 0$ we have

$$\|\rho \circ L^{n+1} - \rho \circ L^n\| \geqslant \big|\rho\big(L^{n+1}(x)\big) - \rho\big(L^n(x)\big)\big| = \big|\lambda^{n+1}\rho(x) - \lambda^n\rho(x)\big|$$
$$= |\lambda - 1|.$$

Hence $\{L^n\}$ does not oscillate slowly.

Conversely, if $\sigma(L) \cap \mathbb{T} = \{1\}$, then since points of $\sigma(L) \cap \mathbb{T}$ are associated with *simple* eigenvectors of L, the spectrum of the restriction L_0 of L to the range of id $- L$ is contained in the open unit disk $\{z: |z| < 1\}$, hence $\|L_0^n\| \to 0$ as $n \to \infty$. It follows that $L^{n+1} - L^n$ tends to zero (in norm, say) as $n \to \infty$; and that obviously implies condition (iii) of Theorem 6.1. We remark that the asymptotic behavior of $\|T^{n+1} - T^n\|$ for contractions T on Banach spaces has been much-studied; see [14] and references therein.

Elementary examples show that any finite subset of the unit circle that contains 1 and is stable under complex conjugation can occur as $\sigma(L) \cap \mathbb{T}$ for a UCP map L of a matrix algebra M_n (for appropriately large n). Hence there are many examples of UCP maps on finite-dimensional non-commutative von Neumann algebras whose asymptotic liftings are nontrivial finite-dimensional W^*-dynamical systems.

The asymptotic behavior of UCP maps on finite-dimensional algebras is discussed more completely in Section 9.

Remark 6.3 *(Automorphisms and endomorphisms).* Automorphisms are at the opposite extreme from slowly oscillating UCP maps. Indeed, if α is any automorphism of a dual operator system M, then α induces an isometry of the predual M_* via $\omega \mapsto \omega \circ \alpha$, and for every $\rho \in M_*$ and $n \in \mathbb{Z}$ we have

$$\|\rho \circ \alpha^{n+1} - \rho \circ \alpha^n\| = \|(\rho \circ \alpha - \rho) \circ \alpha^n\| = \|\rho \circ \alpha - \rho\|.$$

This formula obviously implies that the only slowly oscillating automorphism is the identity automorphism.

If α is merely an isometric UCP map on M and $M_\infty = \bigcap_n \alpha^n(M)$ is the "tail" operator system, then for every $\rho \in M_*$ we have

$$\left\| \rho \circ \alpha^{n+1} - \rho \circ \alpha^n \right\| = \left\| (\rho \circ \alpha - \rho) \circ \alpha^n \right\| = \left\| (\rho \circ \alpha - \rho) \restriction_{\alpha^n(M)} \right\|,$$

and the right-hand side decreases to $\left\| (\rho \circ \alpha - \rho) \restriction_{M_\infty} \right\|$ as $n \to \infty$ by Lemma 3.5. Hence α oscillates slowly iff it restricts to the identity map on M_∞.

7. Identification of the asymptotic dynamics

Let $L : M \to M$ be a UCP map acting on a von Neumann algebra M and let (N, α, E) be its asymptotic lift. We have seen that $(N, \mathbb{Z})$ is a W^*-dynamical system; indeed, the proof of Theorem 5.1 shows that the algebraic structure of N is that of the noncommutative Poisson boundary of an associated UCP map $\Lambda : \tilde{M} \to \tilde{M}$ on another von Neumann algebra.

In general, it can be a significant problem to find a concrete realization of the Poisson boundary, even when $M = L^\infty(X, \mu)$ is commutative (see [16]—this is called the *identification problem* in [13]). In the noncommutative case, there are only a few examples for which this problem has been effectively solved. Three of them are discussed in [12].

In this section we contribute to this circle of ideas by showing that the asymptotic lift of a UCP map L on a von Neumann algebra is isomorphic to the tail flow of the minimal dilation of L to a $*$-endomorphism whenever the minimal dilation has trivial kernel—which is automatic whenever L acts on a factor. We will clarify the precise relation between the asymptotic lift of L and the Poisson boundary of L in Section 8.

It will not be necessary to reiterate explicit details of the dilation theory of UCP maps acting on von Neumann algebras (see [2, Chapter 8]). Instead, we simply recall that every UCP map $L : M \to M$ acting on a von Neumann algebra can be dilated *minimally* to a normal unit-preserving $*$-endomorphism of a larger von Neumann algebra $\alpha : N \to N$ that contains $M = pNp$ as a corner. Any two minimal dilations of L are naturally isomorphic.

More generally, any unit-preserving normal isometric $*$-endomorphism $\alpha : N \to N$ of a von Neumann algebra N gives rise to a decreasing sequence of von Neumann subalgebras $N \supseteq \alpha(N) \supseteq \alpha^2(N) \supseteq \cdots$ whose intersection

$$A = \bigcap_{n \geqslant 0} \alpha^n(N)$$

is a von Neumann algebra with the property that the restriction of α to A is a $*$-*automorphism* of A. That W^*-dynamical system $(A, \mathbb{Z})$ (also written $(A, \alpha\restriction_A)$) is called the *tail flow* of the endomorphism $\alpha : N \to N$. The tail flow is clearly an interesting conjugacy invariant of the endomorphism α, but it has received little attention in the past.

The following result identifies the asymptotic lift of L as the tail flow of the minimal dilation of L whenever the minimal dilation has trivial kernel. Actually, we prove somewhat more, since the setting of Theorem 7.1 includes dilations that are not necessarily minimal.

Theorem 7.1. *Let $\alpha : N \to N$ be a unital normal isometric endomorphism of a von Neumann algebra N, let $p \in N$ be a projection in N satisfying $p \leqslant \alpha(p)$ and $\alpha^n(p) \uparrow \mathbf{1}_N$ as $n \uparrow \infty$. Let $M = pNp$ be the corresponding corner of N and let $L : M \to M$ be the UCP map defined by*

$$L(x) = p\alpha(x)p, \quad x \in M.$$

Then the asymptotic lift of $L : M \to M$ is isomorphic to $(A, \alpha\restriction_A, E)$, where $(A, \alpha\restriction_A)$ is the tail flow of α and $E : A \to M$ is the map $E(a) = pap$.

Proof. Obviously, $\alpha\restriction_A$ is an automorphism of the operator system structure of A, and E is a UCP map of A to $M = pNp$. We claim that the nondegeneracy requirement $E(\alpha^{-n}(a)) = 0$, $n \geqslant 1 \Rightarrow a = 0$ is satisfied. Indeed, fixing such an $a \in A$ and $n \geqslant 1$, we have

$$\alpha^n(p)a\alpha^n(p) = \alpha^n\big(p\alpha^{-n}(a)p\big) = \alpha^n\big(E\big(\alpha^{-n}(a)\big)\big) = 0, \quad n = 1, 2, \ldots.$$

Since $\alpha^n(p)$ converges strongly to $\mathbf{1}$ as $n \to \infty$, $\alpha^n(p)a\alpha^n(p)$ converges strongly to a, hence $a = 0$.

It remains to establish the formulas (3.1):

$$\lim_{k \to \infty} \big\| \rho \circ \big(\mathrm{id}_n \otimes L^k\big) \big\| = \big\| \rho \circ (\mathrm{id}_n \otimes E) \big\|, \tag{7.1}$$

for every $\rho \in (M_n \otimes M)_*$ and every $n = 1, 2, \ldots$. We first consider the case $n = 1$. Choose $\rho \in M_*$ and define $\bar{\rho} \in N_*$ by $\bar{\rho}(y) = \rho(pyp)$, $y \in N$. For every $k \geqslant 1$, we claim

$$\big\| \rho \circ L^k \big\| = \big\| \bar{\rho} \circ \alpha^k \big\| \quad \text{and} \quad \| \rho \circ E \| = \| \bar{\rho}\restriction_A \|. \tag{7.2}$$

Indeed, since $L^k(x) = p\alpha^k(x)p$ for all $x \in M$, it follows that for every $y \in N$ we have $\rho(L^k(pyp)) = \rho(p\alpha^k(pyp)p) = \rho(p\alpha^k(y)p) = \bar{\rho}(\alpha^k(y))$, and that identity clearly implies $\| \rho \circ L^k \| = \| \bar{\rho} \circ \alpha^k \|$. The second formula of (7.2) follows from the identity $\rho(E(a)) = \rho(pap) = \bar{\rho}(a)$ for $a \in A$.

If we now apply (3.5) to the decreasing sequence of weak*-compact sets $K_j = \alpha^j(\text{ball } N)$, $j = 1, 2, \ldots$, having intersection $K_\infty = \text{ball } A$, we find that $\| \bar{\rho} \circ \alpha^j \| = \sup\{|\rho(y)|\colon y \in K_j\}$ and $\| \bar{\rho}\restriction_A \| = \sup\{|\bar{\rho}(a)|\colon a \in \text{ball } A\}$, so by Lemma 3.5 we conclude

$$\lim_{k \to \infty} \big\| \rho \circ L^k \big\| = \lim_{k \to \infty} \big\| \bar{\rho} \circ \alpha^k \big\| = \| \bar{\rho}\restriction_A \| = \| \rho \circ E \|,$$

proving (7.1).

This argument applies *verbatim* to establish (7.1) throughout the matrix hierarchy. Indeed, for $n \geqslant 2$, the hypotheses of Theorem 7.1 carry over to the corner $M^{(n)} = p_n N^{(n)} p_n$, where $p_n = \mathbf{1}_{M_n} \otimes p$, with α replaced by $\mathrm{id}_n \otimes \alpha$. We have $p_n \leqslant (\mathrm{id}_n \otimes \alpha)(p_n) \leqslant (\mathrm{id}_n \otimes \alpha^2)(p_n) \leqslant \cdots \uparrow \mathbf{1}_{N^{(n)}}$, and for a fixed $\rho \in M_*^{(n)}$ there are appropriate versions of the formulas (7.2). $\quad\square$

8. Identification of the Poisson boundary

In this section we show how the Poisson boundary can be characterized in terms of the asymptotic lift. Since the asymptotic lift can often be calculated explicitly—either directly as in

Section 9 or using the tools of dilation theory and Theorem 7.1—this result contributes to the identification problem for noncommutative Poisson boundaries as discussed in Section 7.

Let $L : M \to M$ be a UCP map acting on a von Neumann algebra, and consider the dual operator system of all noncommutative harmonic elements

$$H_L = \{ x \in M \colon L(x) = x \}.$$

We have pointed out in Section 5 that H_L carries a unique von Neumann algebra structure, and that von Neumann algebra is called the Poisson boundary of the map $L : M \to M$. Given a $*$-automorphism of a von Neumann algebra N, we write $N^\alpha = \{ y \in N \colon \alpha(y) = y \}$ for its fixed subalgebra.

Proposition 8.1. *Let $L : M \to M$ be a UCP map on a von Neumann algebra, let H_L be its Poisson boundary, and let (N, α, E) be its asymptotic lift. Then the restriction of E to the fixed algebra N^α implements an isomorphism of von Neumann algebras $N^\alpha \cong H_L$.*

Proof. The equivariance property $E \circ \alpha = L \circ E$, implies $E(N^\alpha) \subseteq H_L$. For the opposite inclusion, every element $a \in H_L$ gives rise to an inverse sequence $\bar{a} = (\ldots, a, a, a, \ldots)$ and, after realizing (N, α, E) concretely as in Section 3, we conclude that $a = E(\operatorname{diag} \bar{a}) \in E(N^\alpha)$. Finally, a straightforward application of formula (3.9) of Lemma 3.7 throughout the matrix hierarchy shows that $E{\restriction}_{N^\alpha}$ is a complete isometry. Since both N^α and H_L are von Neumann algebras, $E{\restriction}_{H_L}$ is a $*$-isomorphism. $\quad\square$

9. Examples and concluding remarks

Theorem 7.1 identifies the asymptotic lift of a UCP map in terms of its minimal dilation. While one can often calculate properties of the minimal dilation in explicit terms (see [2, Chapter 8]), those computations can be cumbersome and sometimes difficult. On the other hand, given a specific UCP map, we have found that it is often easier to calculate its asymptotic lift directly in concrete terms. The purpose of this section is to illustrate that fact by carrying out calculations for some examples that require a variety of techniques.

9.1. Stochastic matrices

It is appropriate to begin with the classical commutative case having its origins in the theory of Markov chains. Let $P = (p_{ij})$ be an $n \times n$ matrix of nonnegative numbers satisfying $\sum_j p_{ij} = 1$ for every $i = 1, \ldots, n$. If we view the elements of the von Neumann algebra $M = \mathbb{C}^n$ as column vectors, then P gives rise to a UCP map on M by matrix multiplication. We now calculate the asymptotic lift of P, and we relate that to classical results of Frobenius [9,10], generalizing earlier results of Perron [18,19], on the asymptotic behavior of such matrices. To keep the discussion as simple as possible, we restrict attention to the case where P is *irreducible* in the sense that the only projections $e \in M$ satisfying $P(e) \leqslant e$ are $e = 0$ and $e = \mathbf{1}$ (but see Remark 9.4). We write $\sigma(P)$ for the spectrum of the linear operator $P : \mathbb{C}^n \to \mathbb{C}^n$; $\sigma(P)$ is a subset of the closed unit disk that contains the eigenvalue 1.

In this context, Theorem 2 of [11, see pp. 65–75] can be paraphrased as follows.

Theorem 9.1. *Let P be an irreducible stochastic $n \times n$ matrix. Then there is a k, $1 \leqslant k \leqslant n$, such that $\sigma(P) \cap \mathbb{T} = \{ 1, \zeta, \zeta^2, \ldots, \zeta^{k-1} \}$ is the set of all kth roots of unity, $\zeta = e^{2\pi i / k}$, each ζ^j being*

a simple eigenvalue. When $k > 1$, there is a permutation matrix U such that UPU^{-1} has cyclic form

$$UPU^{-1} = \begin{pmatrix} 0 & C_0 & 0 & \ldots & 0 \\ 0 & 0 & C_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & C_{k-2} \\ C_{k-1} & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{9.1}$$

in which the C_j are rectangular submatrices, and where the zero submatrices along the diagonal are all square.

Notice that we can eliminate the unitary permutation matrix U entirely by replacing P with a suitably relabeled stochastic matrix, and we do so. For each $j = 0, 1, \ldots, k - 1$, let e_j be the projection corresponding to the domain subspace of the block C_j. These projections are mutually orthogonal, have sum $\mathbf{1}$, and satisfy $P(e_j) \leqslant e_{j+1}$ (addition modulo k). Making use of $P(\mathbf{1}) = \mathbf{1}$, we find that in fact, $P(e_j) = e_{j+1}$ for every j. Let N be the linear span of $e_0, \ldots, e_{k-1}$ and let α be the restriction of P to N. Clearly N is a $*$-subalgebra of M and $\alpha : N \to N$ is the $*$-automorphism associated with this cyclic permutation of the minimal projections $e_0, e_1, \ldots, e_{k-1}$ of N.

Proposition 9.2. *The asymptotic lift of $P : M \to M$ is the triple (N, α, E), where $E : N \subseteq M$ is the inclusion map.*

Proof. For $k > 1$ as above, consider the UCP map $P^k : M \to M$. The spectrum of P^k consists of the eigenvalue 1, together with other spectral points in the open unit disk $\{|z| < 1\}$, the eigenvalue 1 having eigenspace N. Hence the sequence of powers $P^k, P^{2k}, P^{3k}, \ldots$ converges to an idempotent linear map Q on M having range N. Clearly Q is a UCP projection onto N. For every $\rho \in M_*$, the norms $\|\rho \circ P^r\|$ decrease as r increases, hence

$$\lim_{r \to \infty} \|\rho \circ P^r\| = \lim_{m \to \infty} \|\rho \circ (P^k)^m\| = \|\rho \circ Q\| = \|\rho{\upharpoonright}_N\| = \|\rho \circ E\|. \tag{9.2}$$

The same argument applies throughout the matrix hierarchy over M, and we conclude that the two criteria of Definition 3.1 are satisfied. Hence (N, α, E) is the asymptotic lift of P. $\square$

Remark 9.3 (*Asymptotics of the powers of P*). The idempotent UCP map

$$Q = \lim_{m \to \infty} P^{mk}$$

exhibited in the proof of Proposition 9.2 provides a precise sense in which the asymptotic lift (N, α, E) contains all of the asymptotic information about the sequence of powers $P, P^2, P^3, \ldots$. Indeed, we claim that there are positive constants c, r with $r < 1$ such that

$$\|P^n - \alpha^n Q\| \leqslant cr^n, \quad n = 1, 2, 3, \ldots. \tag{9.3}$$

This follows from the fact that Q is an idempotent that commutes with P with the property that $\alpha = P{\upharpoonright}_{N = \mathrm{ran}\, Q}$ f has spectrum in the unit circle and $P{\upharpoonright}_{\mathrm{ran}(\mathrm{id} - Q)}$ has spectrum in $\{|z| < 1\}$.

Choosing a number $r < 1$ so that $\{|z| < r\}$ contains the spectrum of $P\restriction_{\mathrm{ran(id}-Q)}$, the spectral radius formula of elementary Banach algebra theory implies that the sequence

$$r^{-n}\|P^n - \alpha^n Q\| = r^{-n}\|P^n - P^n Q\| = \left(r^{-1}\|P^n\restriction_{\mathrm{ran(id}-Q)}\|^{1/n}\right)^n$$

tends to zero as $n \to \infty$, hence there is a $c > 0$ so that (9.3) is satisfied.

Remark 9.4 *(Reducibility and noncommutativity).* In the following subsection, we generalize the above result to the case of UCP maps acting on noncommutative finite-dimensional von Neumann algebras. In particular, the discussion of Section 9.2 applies equally to the reducible cases not covered in the statement of Theorem 9.1.

9.2. Finite-dimensional von Neumann algebras

One can compute the asymptotic lift of a UCP map on a finite-dimensional von Neumann algebra explicitly, using nothing but elementary methods along with the Choi–Effros multiplication. Since these results have significance for quantum computing [15], we sketch that calculation in some detail.

Let $L : M \to M$ be a UCP map on a finite-dimensional von Neumann algebra. For every point $\lambda \in \sigma(L) \cap \mathbb{T}$ let

$$N_\lambda = \left\{x \in M \colon L(x) = \lambda x\right\}$$

and let

$$N = \sum_{\lambda \in \sigma(L) \cap \mathbb{T}} N_\lambda$$

be the sum of these maximal eigenspaces. The identity operator belongs to N, and from the property $L(x^*) = L(x)^*$, $x \in M$, one deduces $N^* = N$. Hence N is an operator system such that the restriction $\alpha = L\restriction_N$ of L to N is a *diagonalizable* UCP map with spectrum $\sigma(L) \cap \mathbb{T}$. Indeed, it is not hard to show that α is a UCP automorphism of N.

We digress momentarily to point out that in the classical setting of Frobenius' result for irreducible stochastic matrices P as formulated in Theorem 9.1, N coincides with the span of the projections $e_0, \ldots, e_{k-1}$ constructed above, the eigenvector associated with a kth root of unity $\lambda \in \sigma(P) \cap \mathbb{T}$ being given by

$$x_\lambda = e_0 + \bar{\lambda}e_1 + \bar{\lambda}^2 e_2 + \cdots + \bar{\lambda}^{k-1} e_{k-1}.$$

It follows that, in such commutative cases, N is closed under multiplication.

In the more general setting under discussion here, $\sigma(L) \cap \mathbb{T}$ need not consist of roots of unity and N need not be closed under multiplication. But in all cases N can be made into a von Neumann algebra. The most transparent proof of that fact uses the following observation of Kuperberg ([15], also see [3, Theorem 2.6]).

Lemma 9.5. *There is an increasing sequence of integers $n_1 < n_2 < \cdots$ such that $L^{n_1}, L^{n_2}, \ldots$ converges to an idempotent UCP map Q with the property $N = Q(M)$. In fact, Q is the unique idempotent limit point of the sequence of powers $L, L^2, L^3, \ldots$.*

One now uses the idempotent Q to introduce a Choi–Effros multiplication

$$x \circ y = Q(xy), \quad x, y \in N,$$

in N, thereby making it into a finite-dimensional von Neumann algebra. Hence (N, α) becomes a (typically noncommutative) W^*-dynamical system, and the inclusion $E : N \subseteq M$ becomes a UCP map satisfying $E \circ \alpha = L \circ E$.

Proposition 9.6. *The triple (N, α, E) is the asymptotic lift of $L : M \to M$.*

Sketch of proof. The inclusion of N in M is injective, hence the nondegeneracy condition (2.2) is satisfied. Thus we need only show that equality holds in the formulas (3.1), and that follows by a proof paralleling that of (9.2). Indeed, letting $n_1 < n_2 < \cdots$ be a sequence such that $L^{n_k} \to Q$ as in Lemma 9.5, one finds that for every bounded linear functional ρ on M,

$$\lim_{n \to \infty} \| \rho \circ L^n \| = \lim_{k \to \infty} \| \rho \circ L^{n_k} \| = \| \rho \circ Q \| = \| \rho {\restriction} N \| = \| \rho \circ E \|.$$

Obviously, the same argument can be promoted throughout the matrix hierarchy, as one allows ρ to range over the dual of $M^{(n)}$, $n = 1, 2, \ldots$. $\square$

Remark 9.7 (*Asymptotics of the powers of L*). Making use of the idempotent Q much as in Remark 9.3, one finds that (N, α, E) contains all asymptotic information about the sequence $L, L^2, L^3, \ldots$ because of the following precise estimate: There are positive constants c, r such that $r < 1$ and

$$\| L^n - \alpha^n Q \| \leqslant c r^n, \quad n = 1, 2, \ldots. \tag{9.4}$$

9.3. UCP maps on II_1 factors

We now calculate the asymptotic lifts of a family of nontrivial UCP maps acting on the hyperfinite II_1 factor R, the point being to show that the asymptotic lifts of these maps can be arbitrary $*$-automorphisms of R. There are many variations on these examples that exhibit a variety of phenomena in other von Neumann algebras. Here, we confine ourselves to the simplest nontrivial cases.

Let τ be the tracial state of R, and let P be any normal UCP map of R having τ as an absorbing state in the sense that for every normal state ρ of R, one has

$$\lim_{k \to \infty} \| \rho \circ P^k - \tau \| = 0. \tag{9.5}$$

Of course, the simplest such map is $P(x) = \tau(x)\mathbf{1}$; but we have less trivial examples in mind. For example, let $A_\theta = C^*(U, V)$ be the irrational rotation C^*-algebra, where U, V are unitaries satisfying $UV = e^{2\pi i \theta} VU$ and θ is an irrational number. Then for every number λ in the open unit interval $(0, 1)$ one can show that there is a completely positive unit-preserving map P_λ that is defined uniquely on A_θ by its action on monomials:

$$P_\lambda(U^p V^q) = \lambda^{|p|+|q|} U^p V^q, \quad p, q \in \mathbb{Z}.$$

Moreover, since in every finite linear combination $x = \sum_{p,q} a_{p,q} U^p V^q$, the coefficients satisfy $|a_{p,q}| \leqslant \|x\|$, $p, q \in \mathbb{Z}$, a straightforward estimate leads to

$$\sup_{\|x\| \leqslant 1} \left\| P_\lambda^k(x) - \tau(x)\mathbf{1} \right\| \leqslant \sum_{(p,q) \neq (0,0)} \lambda^{(|p|+|q|)^k} \leqslant \lambda^{k-1} \sum_{(p,q) \neq (0,0)} \lambda^{|p|+|q|} = c\lambda^k,$$

for some positive constant c. We conclude that

$$\lim_{k \to \infty} \left\| P_\lambda^k - \tau(\cdot)\mathbf{1} \right\| = 0,$$

and in particular, (9.5) holds for every state ρ of $C^*(U, V)$.

The GNS construction applied to the tracial state of A_θ now provides a representation π of A_θ with the property that the weak closure of $\pi(A_\theta)$ is R, and there is a unique UCP map P on R such that $P(\pi(x)) = \pi(P_\lambda(x))$ for all $x \in A_\theta$. The preceding paragraph implies that the extended map $P: R \to R$ has the asserted property (9.5).

Similar examples of UCP maps acting on other II_1 factors can be constructed by replacing R with the group von Neumann algebras of discrete groups with infinite conjugacy classes (see [3, Proposition 4.4]).

Choose a UCP map $P: R \to R$ having property (9.5), choose an arbitrary $*$-automorphism α of R, and consider the UCP map $L = P \otimes \alpha$ defined uniquely on the spatial tensor product $R \otimes R$ by

$$L(x \otimes y) = P(x) \otimes \alpha(y), \quad x, y \in R.$$

Since $R \otimes R$ is isomorphic to R, one can think of L as a UCP map on R.

Proposition 9.8. *Let $E: R \to R \otimes R$ be the map $E(x) = \mathbf{1} \otimes x$, $x \in R$. The asymptotic lift of $L = P \otimes \alpha$ is the triple (R, α, E).*

Sketch of proof. E is clearly an injective UCP map of von Neumann algebras satisfying the equivariance condition $E \circ \alpha = L \circ E$. Thus it remains only to verify the formulas (3.1). For that, we will prove a stronger asymptotic relation. Let $Q: R \otimes R \to \mathbf{1} \otimes R$ be the τ-preserving conditional expectation

$$Q(x \otimes y) = \tau(x)y, \quad x, y \in R.$$

We claim that for all $\rho \in (R \otimes R)_*$, one has

$$\lim_{k \to \infty} \left\| \rho \circ L^k - \rho \circ (\mathrm{id}_R \otimes \alpha)^k \circ Q \right\| = 0. \tag{9.6}$$

Indeed, since $(R \otimes R)_*$ is the norm-closed linear span of functionals of the form $\rho_1 \otimes \rho_2$ with $\rho_k \in R_*$, obvious estimates show that it suffices to prove (9.6) for decomposable functionals of the form $\rho_1 \otimes \rho_2$, where $\rho_1, \rho_2 \in R_*$. Fix $\rho = \rho_1 \otimes \rho_2$ of this form. Using the decomposition

$$\left(L^k - (\mathrm{id}_R \otimes \alpha)^k \circ Q \right)(x \otimes y) = \left(P^k(x) - \tau(x)\mathbf{1} \right) \otimes \alpha^k(y),$$

we find that $\|\rho_1 \otimes \rho_2(L^k - (\mathrm{id}_R \otimes \alpha)^k \circ Q)\|$ decomposes into a product

$$\begin{aligned}
\left\|(\rho_1 \otimes \rho_2)\big(\big(P^k - \tau(\cdot)\mathbf{1}\big) \otimes \alpha^k\big)\right\| &= \left\|\rho_1 \circ \big(P^k - \tau(\cdot)\mathbf{1}\big)\right\| \cdot \left\|\rho_2 \circ \alpha^k\right\| \\
&= \left\|\rho_1 \circ \big(P^k - \tau(\cdot)\mathbf{1}\big)\right\| \cdot \|\rho_2\| \\
&= \left\|\rho_1 \circ P^k - \tau\right\| \cdot \|\rho_2\|,
\end{aligned}$$

which by (9.5), tends to zero as $k \to \infty$.

Note that (9.6) leads immediately to the case $n = 1$ of (3.1), since

$$\lim_{k\to\infty} \left\|\rho \circ L^k\right\| = \lim_{k\to\infty}\left\|\rho \circ \big(\mathrm{id}_R \otimes \alpha^k\big) \circ Q\right\| = \lim_{k\to\infty}\left\|\rho \circ Q \circ \alpha^k\right\| = \|\rho \circ Q\| = \|\rho \circ E\|.$$

With trivial changes, these arguments can be repeated throughout the matrix hierarchy, after one identifies $M_n \otimes (R \otimes R)$ with $(M_n \otimes R) \otimes R$. We omit those mind-numbing details. $\square$

9.4. The CCR heat flow

The three asymptotic assertions (9.3), (9.4), (9.6) are much stronger than the requirements of (3.1), and one might ask if those stronger results can be established for the asymptotic lifts (N, α, E) of more general UCP maps on von Neumann algebras $L : M \to M$. In each of the preceding examples, it was possible to identify N with a dual operator subsystem $M_\infty \subseteq M$ (namely the range $E(N)$ of E), α with the restriction $\alpha_\infty = L{\upharpoonright}_{M_\infty}$ of L to N, and E with the inclusion map $\iota : N \subseteq M$. There was also an idempotent completely positive projection Q mapping M onto M_∞, and together, these objects gave rise to the asymptotic relations

$$\lim_{k\to\infty}\left\|\rho \circ L^k - \rho \circ \alpha_\infty^k \circ Q\right\| = 0, \quad \rho \in M_*. \tag{9.7}$$

The relative strength of (9.7) and (3.1) is clearly seen if one reformulates the case $n = 1$ of (3.1) in this context as the following assertion:

$$\lim_{k\to\infty}\left|\left\|\rho \circ L^k\right\| - \left\|\rho \circ \alpha_\infty^k \circ Q\right\|\right| = \lim_{k\to\infty}\left|\left\|\rho \circ L^k\right\| - \|\rho \circ E\|\right| = 0$$

(see the proof of Proposition 9.8).

So it is natural to ask if the stronger relations (9.7) can be established more generally, at least in cases where $N = M_\infty \subseteq M$ is a subspace of M and $\alpha = \alpha_\infty$ is obtained by restricting L to M_∞. That is true, for example, in the more restricted context of [3]. The purpose of these remarks is to show that the answer is no in general, by describing examples with the two properties $M_\infty \subseteq M$ and $\alpha = L{\upharpoonright}_{M_\infty}$, but for which there is no idempotent completely positive map Q satisfying (9.7).

The CCR heat flow is a semigroup of UCP maps $\{P_t : t \geqslant 0\}$ acting on the von Neumann algebra $M = \mathcal{B}(H)$ [1]. Consider the single UCP map $L = P_{t_0}$ for some fixed $t_0 > 0$. This map has the following two properties: (a) there is no normal state $\rho \in M_*$ satisfying $\rho \circ L = \rho$, and (b) for any two normal states $\rho_1, \rho_2 \in M_*$ one has

$$\lim_{k\to\infty}\left\|\rho_1 \circ L^k - \rho_2 \circ L^k\right\| = 0. \tag{9.8}$$

We claim first that the asymptotic lift of this L is the triple $(\mathbb{C}, \mathrm{id}, \iota)$, where ι is the inclusion of $\mathbb{C}$ in M, $\iota(\lambda) = \lambda \cdot \mathbf{1}_M$. Indeed, to sketch the proof of the key assertion—namely the case $n = 1$ of

formula (3.1)—choose $\rho \in M_*$. We have $\|\rho \circ \iota\| = |\rho(\mathbf{1})|$, so that (3.1) reduces in this case to the formula

$$\lim_{k \to \infty} \|\rho \circ L^k\| = |\rho(\mathbf{1})|, \quad \rho \in M_*. \tag{9.9}$$

It is a simple exercise to show that (9.9) and (9.8) are in fact equivalent assertions about L—note for example that (9.8) is the special case of (9.9) in which $\rho(\mathbf{1}) = 0$—and of course the restriction of L to $\mathbb{C} \cdot \mathbf{1}$ is the identity map $\alpha_\infty = \mathrm{id}$. This argument promotes naturally throughout the matrix hierarchy over M, hence $(\mathbb{C}, \mathrm{id}, \iota)$ is the asymptotic lift of L.

We now show that one cannot obtain formulas like (9.7) for this example.

Proposition 9.9. *There is no completely positive projection Q of M on $\mathbb{C} \cdot \mathbf{1}$ that satisfies*

$$\lim_{k \to \infty} \|\rho \circ L^k - \rho \circ L^k \circ Q\| = 0, \quad \rho \in M_*. \tag{9.10}$$

Proof. Indeed, a completely positive idempotent Q with range $\mathbb{C} \cdot \mathbf{1}$ would have the form $Q(x) = \omega(x)\mathbf{1}$, $x \in M$, where ω is a state of M. For any normal state ρ we have $\rho \circ L^k \circ Q(x) = \rho(L^k(\omega(x)\mathbf{1})) = \omega(x)$, $x \in M$, hence in this case (9.10) makes the assertion

$$\lim_{k \to \infty} \|\rho \circ L^k - \omega\| = 0;$$

i.e., ω is an *absorbing state* for L. But an absorbing state ω for L is a *normal* state that satisfies $\omega \circ L = \omega$, contradicting property (a) above. $\quad\square$

Remark 9.10 *(Further examples).* One can generalize this construction based on L. For example, let α be a $*$-automorphism of $\mathcal{B}(H)$ and consider the UCP map $L \otimes \alpha$ defined on $M = \mathcal{B}(H \otimes H)$. One can show that $L \otimes \alpha$ has asymptotic lift $(\mathcal{B}(H), \alpha, E)$ where $E : \mathcal{B}(H) \to M$ is the UCP map $E(x) = \mathbf{1} \otimes x$, much as in the proof of Proposition 9.8. With suitable choices of α (specifically, for any α that has a normal invariant state), one can also show that there is no completely positive projection $Q : M \to \mathbf{1} \otimes \mathcal{B}(H)$ that satisfies (9.10). Thus, even though a UCP map on a von Neumann algebra always has an asymptotic lift, *there are many examples for which one cannot expect precise asymptotic formulas such as* (9.7).

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