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# ON THE EXISTENCE OF $E_0$-SEMIGROUPS

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Product systems are the classifying structures for semigroups of endomorphisms of $\mathcal{B}(H)$, in that two $E_0$-semigroups are cocycle conjugate iff their product systems are isomorphic. Thus it is important to know that every abstract product system is associated with an $E_0$-semigroup. This was first proved more than 15 years ago by rather indirect methods. Recently, Skeide has given a more direct proof. In this note we give yet another proof by an elementary construction.

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## 1. Formulation of the Result

There were two proofs of the above fact[2,4] (see also Ref. 3), both of which involved substantial analysis. In a recent paper, Michael Skeide[6] gave a more direct proof. In this note we present an elementary method for constructing an essential representation of any product system. Given the basic correspondence between $E_0$-semigroups and essential representations, the existence of an appropriate $E_0$-semigroup follows.

Our terminology follows the monograph.[3] Let $E = \{E(t) : t > 0\}$ be a product system and choose a unit vector $e \in E(1)$. *$e$ will be fixed throughout*. We consider the Fréchet space of all Borel-measurable sections $t \in (0, \infty) \mapsto f(t) \in E(t)$ that are locally square integrable

$$\int_0^T \|f(\lambda)\|^2 d\lambda < \infty \,, \qquad T > 0 \,. \tag{1.1}$$

**Definition 1.1.** A locally $L^2$ section $f$ is said to be *stable* if there is a $\lambda_0 > 0$ such that for almost every $\lambda \geq \lambda_0$, one has

$$f(\lambda + 1) = f(\lambda) \cdot e \,.$$

Note that a stable section $f$ satisfies $f(\lambda + n) = f(\lambda) \cdot e^n$ a.e. for all $n \geq 1$ whenever $\lambda$ is sufficiently large. The set of all stable sections is a vector space $\mathcal{S}$, and for any two sections $f$, $g \in \mathcal{S}$, $\langle f(\lambda + n), g(\lambda + n)\rangle$ becomes independent of $n \in \mathbb{N}$ (a.e.) when $\lambda$ is sufficiently large. Thus we can define a positive semidefinite inner product on $\mathcal{S}$ as follows:

$$\langle f, g\rangle = \lim_{n\to\infty} \int_n^{n+1} \langle f(\lambda), g(\lambda)\rangle d\lambda = \lim_{n\to\infty} \int_0^1 \langle f(\lambda + n), g(\lambda + n)\rangle d\lambda \,. \tag{1.2}$$

Let $\mathcal{N}$ be the subspace of $\mathcal{S}$ consisting of all sections $f$ that vanish eventually, in that for some $\lambda_0 > 0$ one has $f(\lambda) = 0$ for almost all $\lambda \geq \lambda_0$. One finds that $\langle f, f\rangle = 0$ iff $f \in \mathcal{N}$. Hence $\langle \cdot, \cdot\rangle$ defines an inner product on the quotient $\mathcal{S}/\mathcal{N}$, and its completion becomes a Hilbert space $H$ with respect to the inner product (1.2). Obviously, $H$ is separable.

There is a natural representation of $E$ on $H$. Fix $v \in E(t)$, $t > 0$. For every stable section $f \in \mathcal{S}$, let $\phi_0(v)f$ be the section

$$(\phi_0(v)f)(\lambda) = \begin{cases} v \cdot f(\lambda - t)\,, & \lambda > t\,, \\ 0\,, & 0 < \lambda \leq t\,. \end{cases}$$

Clearly $\phi_0(v)\mathcal{S} \subseteq \mathcal{S}$. Moreover, $\phi_0(v)$ maps null sections into null sections, hence it induces a linear operator $\phi(v)$ on $\mathcal{S}/\mathcal{N}$. The mapping $(t, v), \xi \in E \times \mathcal{S}/\mathcal{N} \mapsto \phi(v)\xi \in H$ is obviously Borel-measurable, and it is easy to check that $\|\phi(v)\xi\|^2 = \|v\|^2 \cdot \|\xi\|^2$ (see Sec. 2 for details). Thus we obtain a representation $\phi$ of $E$ on the completion $H$ of $\mathcal{S}/\mathcal{N}$ by closing the densely defined operators $\phi(v)(f + \mathcal{N}) = \phi_0(v)f + \mathcal{N}$, $v \in E(t)$, $t > 0$, $f \in \mathcal{S}$.

**Theorem 1.2.** *$\phi$ is an essential representation of $E$ on $H$.*

By Proposition 2.4.9 of Ref. 3, there is an $E$-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ that acts on $\mathcal{B}(H)$ and is associated with $\phi$ by way of

$$\alpha_t(X) = \sum_{n=1}^{\infty} \phi(e_n(t))X\phi(e_n(t))^*\,, \qquad X \in \mathcal{B}(H)\,, \qquad t > 0\,, \tag{1.3}$$

$e_1(t), e_2(t), \ldots$ denoting an arbitrary orthonormal basis for $E(t)$. Since $\phi$ is essential, $\alpha_t(\mathbf{1}) = \sum_n \phi(e_n(t))\phi(e_n(t))^* = \mathbf{1}$, $t > 0$. Thus we may conclude that the given product system $E$ can be associated with an $E_0$-semigroup.

## 2. Proof of Theorem 1.2

The following observation implies that we could just as well have defined the inner product of (1.2) by

$$\langle f, g\rangle = \lim_{T\to\infty} \int_T^{T+1} \langle f(\lambda), g(\lambda)\rangle d\lambda\,.$$

**Lemma 2.1.** *For any two stable sections $f$, $g$, there is a $\lambda_0 > 0$ such that*

$$\langle f, g\rangle = \int_T^{T+1} \langle f(\lambda), g(\lambda)\rangle d\lambda$$

*for all real numbers $T \geq \lambda_0$.*

**Proof.** Let $u : (0,\infty) \to \mathbb{C}$ be a Borel function satisfying $\int_0^T |u(\lambda)|d\lambda < \infty$ for every $T > 0$, together with $u(\lambda + 1) = u(\lambda)$ a.e. for sufficiently large $\lambda$. Then for $k \in \mathbb{N}$, the integral $\int_k^{k+1} u(\lambda)d\lambda$ becomes independent of $k$ when $k$ is large. We claim that for sufficiently large $T$ and the integer $n = n_T$ satisfying $T < n \leq T+1$, one has

$$\int_T^{T+1} u(\lambda)d\lambda = \int_n^{n+1} u(\lambda)d\lambda\,. \tag{2.1}$$

Note that Lemma 2.1 follows from (2.1) after taking $u(\lambda) = \langle f(\lambda), g(\lambda)\rangle$.

Of course, the formula (2.1) is completely elementary. The integral on the left decomposes into a sum $\int_T^n + \int_n^{T+1}$, and for large $T$ we can write

$$\int_T^n u(\lambda)d\lambda = \int_T^n u(\lambda+1)d\lambda = \int_{T+1}^{n+1} u(\lambda)d\lambda\,.$$

It follows that

$$\int_T^{T+1} u(\lambda)d\lambda = \left(\int_{T+1}^{n+1} + \int_n^{T+1}\right) u(\lambda)d\lambda = \int_n^{n+1} u(\lambda)d\lambda\,,$$

which proves (2.1). □

To show that $\phi$ is a representation, we must show that for every $t > 0$, every $v$, $w \in E(t)$, and every $f$, $g \in \mathcal{S}$ one has $\langle \phi_0(v)f, \phi_0(w)g\rangle = \langle v, w\rangle\langle f, g\rangle$. Indeed, for sufficiently large $n \in \mathbb{N}$ we can write

$$\begin{aligned}\langle \phi_0(v)f, \phi_0(w)g\rangle &= \int_n^{n+1} \langle \phi_0(v)f(\lambda), \phi_0(w)g(\lambda)\rangle d\lambda \\ &= \int_n^{n+1} \langle v\cdot f(\lambda - t), w \cdot g(\lambda - t)\rangle d\lambda \\ &= \langle v, w\rangle \int_n^{n+1} \langle f(\lambda - t), g(\lambda - t)\rangle d\lambda \\ &= \langle v, w\rangle \int_{n-t}^{n-t+1} \langle f(\lambda), g(\lambda)\rangle d\lambda = \langle v, w\rangle\langle f, g\rangle\,,\end{aligned}$$

where the final equality uses Lemma 2.1.

It remains to show that $\phi$ is an essential representation, and for that, we must calculate the adjoints of operators in $\phi(E)$. The following notation from Ref. 3 will be convenient.

**Remark 2.2.** Fix $s > 0$ and an element $v \in E(s)$; for every $t > 0$ we consider the left multiplication operator $\ell_v : x \in E(t) \mapsto v \cdot x \in E(s+t)$. This operator has an adjoint $\ell_v^* : E(s+t) \to E(s)$, which we write more simply as $v^*\eta = \ell_v^*\eta$, $\eta \in E(s+t)$. Equivalently, for $s < t$, $v \in E(s)$, $y \in E(t)$, we write $v^*y$ for $\ell_v^*y \in E(t-s)$. Note that $v^*y$ is undefined for $v \in E(s)$ and $y \in E(t)$ when $t \leq s$.

Given elements $u \in E(r)$, $v \in E(s)$, $w \in E(t)$, the "associative law"

$$u^*(v \cdot w) = (u^*v) \cdot w \tag{2.2}$$

makes sense when $r \leq s$ ($t > 0$ can be arbitrary), provided that it is suitably interpreted when $r = s$. Indeed, it is true *verbatim* when $r < s$ and $t > 0$, while if $s = r$ and $t > 0$, then it takes the form

$$u^*(v \cdot w) = \langle v, u\rangle_{E(s)} \cdot w\,, \qquad u, v \in E(s)\,, \qquad w \in E(t)\,. \tag{2.3}$$

**Lemma 2.3.** *Choose $v \in E(t)$. For every stable section $f \in \mathcal{S}$, there is a null section $g \in \mathcal{N}$ such that*

$$(\phi_0(v)^* f)(\lambda) = v^* f(\lambda + t) + g(\lambda)\,, \qquad \lambda > 0\,.$$

**Proof.** A straightforward calculation of the adjoint of $\phi_0(v) : \mathcal{S} \to \mathcal{S}$ with respect to the semidefinite inner product (1.2). □

Lemma 2.4 follows from the identification $E(t) \cong E(s) \otimes E(t-s)$ when $s < t$. We include a proof for completeness.

**Lemma 2.4.** *Let $0 < s < t$, let $v_1, v_2, \ldots$ be an orthornormal basis for $E(s)$ and let $\xi \in E(t)$. Then*

$$\sum_{n=1}^{\infty} \|v_n^*\xi\|^2 = \|\xi\|^2\,. \tag{2.4}$$

**Proof.** For $n \geq 1$, $\xi \in E(t) \mapsto v_n(v_n^*\xi) \in E(t)$ defines a sequence of mutually orthogonal projections in $\mathcal{B}(E(t))$. We claim that these projections sum to the identity. Indeed, since $E(t)$ is the closed linear span of the set of products $E(s)E(t-s)$, it suffices to show that for every vector in $E(t)$ of the form $\xi = \eta\cdot\zeta$ with $\eta \in E(s)$, $\zeta \in E(t-s)$, we have $\sum_n v_n(v_n^*\xi) = \xi$. For that, we can use (2.2) and (2.3) to write

$$v_n(v_n^*\xi) = v_n(v_n^*(\eta \cdot \zeta)) = v_n((v_n^*\eta) \cdot \zeta) = \langle \eta, v_n\rangle v_n \cdot \zeta\,,$$

hence

$$\sum_{n=1}^{\infty} v_n(v_n^*\xi) = \left(\sum_{n=1}^{\infty} \langle \eta, v_n\rangle v_n\right) \cdot \zeta = \eta \cdot \zeta = \xi\,,$$

as asserted. (2.4) follows after taking the inner product with $\xi$. □

**Proof of Theorem 1.2.** Since the subspaces $H_t = [\phi(E(t))H]$ satisfy $H_{s+t} = [\phi(E(s))H_t] \subseteq H_t$, it suffices to show that $H_1 = H$. For that, it is enough to show that for $\xi \in H$ of the form $\xi = f + \mathcal{N}$ where $f$ is a stable section

$$\left\langle \sum_{n=1}^{\infty} \phi(v_n)\phi(v_n)^*\xi, \xi \right\rangle = \sum_{n=1}^{\infty} \|\phi_0(v_n)^* f\|^2 = \|f\|^2 = \|\xi\|^2 , \tag{2.5}$$

$v_1, v_2, \ldots$ denoting an orthonormal basis for $E(1)$. Fix such a basis $(v_n)$ for $E(1)$ and a stable section $f$. Choose $\lambda_0 > 1$ so that $f(\lambda+1) = f(\lambda) \cdot e$ (a.e.) for $\lambda > \lambda_0$. For $\lambda > \lambda_0$ we have $\lambda + 1 > 1$, so Lemma 2.4 implies

$$\sum_{n=1}^{\infty} \|v_n^* f(\lambda+1)\|^2 = \|f(\lambda+1)\|^2 = \|f(\lambda)\cdot e\|^2 = \|f(\lambda)\|^2 , \qquad \text{(a.e.)} .$$

It follows that for every integer $N > \lambda_0$,

$$\begin{aligned} \sum_{n=1}^{\infty} \int_N^{N+1} \|v_n^* f(\lambda+1)\|^2 d\lambda &= \int_N^{N+1} \sum_{n=1}^{\infty} \|v_n^* f(\lambda+1)\|^2 d\lambda \\ &= \int_N^{N+1} \|f(\lambda)\|^2 d\lambda = \|f + \mathcal{N}\|_H^2 . \end{aligned}$$

Lemma 2.3 implies that when $N$ is sufficiently large, the left side is

$$\sum_{n=1}^{\infty} \int_N^{N+1} \|(\phi_0(v_n)^* f)(\lambda)\|^2 d\lambda = \sum_{n=1}^{\infty} \|\phi_0(v_n) f\|^2 ,$$

and (2.5) follows. □

**Remark 2.5.** (Nontriviality of $H$) Let $L^2((0,1];E)$ be the subspace of $L^2(E)$ consisting of all sections that vanish almost everywhere outside the unit interval. Every $f \in L^2((0,1];E)$ corresponds to a stable section $\tilde{f} \in \mathcal{S}$ by extending it from (0, 1] to $(0,\infty)$ by periodicity

$$\tilde{f}(\lambda) = f(\lambda - n)\cdot e^n , \qquad n < \lambda \le n+1 , \qquad n = 1, 2, \ldots ,$$

and for every $n = 1, 2, \ldots$ we have

$$\int_n^{n+1} \|\tilde{f}(\lambda)\|^2 d\lambda = \int_n^{n+1} \|f(\lambda - n)\cdot e^n\|^2 d\lambda = \int_0^1 f(\lambda)\|^2 d\lambda .$$

Hence the map $f \mapsto \tilde{f} + \mathcal{N}$ embeds $L^2((0,1];E)$ isometrically as a subspace of $H$; in particular, $H$ is not the trivial Hilbert space $\{0\}$.

**Remark 2.6.** (Purity) An $E_0$-semigroup $\alpha = \{\alpha_t : t \ge 0\}$ is said to be *pure* if the decreasing von Neumann algebras $\alpha_t(\mathcal{B}(H))$ have trivial intersection $\mathbb{C}\cdot\mathbf{1}$. The question of whether every $E_0$-semigroup is a cocycle perturbation of a pure one has been resistant.[3] Equivalently, is every product system associated with a *pure* $E_0$-semigroup? While the answer is yes for product systems of type $I$ and $II$, and it is yes for the type *III* examples constructed by Powers (see Ref. 5 or Chap. 13 of Ref. 3), it is unknown in general.

It is perhaps worth pointing out that we have shown that the examples of Theorem 1.2 are not pure; hence the above construction appears to be inadequate for approaching that issue. Since the proof establishes a negative result that is peripheral to the direction of this note, we have omitted it.

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