

Diagonals of self-adjoint operators

William Arveson and Richard V. Kadison

ABSTRACT. The eigenvalues of a self-adjoint $n \times n$ matrix A can be put into a decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_n)$, with repetitions according to multiplicity, and the diagonal of A is a point of \mathbb{R}^n that bears some relation to λ . The Schur-Horn theorem characterizes that relation in terms of a system of linear inequalities.

We prove an extension of the latter result for positive trace-class operators on infinite dimensional Hilbert spaces, generalizing results of one of us on the diagonals of projections. We also establish an appropriate counterpart of the Schur inequalities that relate spectral properties of self-adjoint operators in II_1 factors to their images under a conditional expectation onto a maximal abelian subalgebra.

1. Introduction

This paper presents some of the results of a project begun by the authors that is directed toward finding an appropriate common generalization of the Schur-Horn theorem (for matrices) to operators on an infinite-dimensional Hilbert space, and to operators in finite factors, in a form that would generalize work of one of us on projections in II_1 factors [Kad02a], [Kad02b].

That project continues, and remains unfinished. The results below are satisfactory in the case of type I factors, but are incomplete for finite factors. We are making these partial results public since there is renewed interest in these directions [She05], and it seems desirable to avoid duplication of effort. Other aspects of this work were presented in Section 5 of [Kad04].

We point out that while the results of Section 4 may appear to overlap with work of A. Neumann [Neu99], that is actually not the case. Neumann's results characterize the *closure* (in the ℓ^∞ norm) of the set of diagonals of self-adjoint operators with prescribed spectral properties. Here, on the other hand, we are concerned with the diagonals themselves, and not with their limits relative to any topology. For example, one should compare Theorem 15 of [Kad02b] – which characterizes the diagonals of projections – with the corresponding result of [Neu99] (Theorem 3.6 and Corollary 2.14) to understand the extent to which subtlety is lost when one

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takes the closure relative to the ℓ^∞ -norm. Along with the nature of the characterizations below, our methods also differ significantly from those of [Neu99]. We thank Daniel Markiewicz for calling our attention to the paper [Neu99] (also see [Neu02]).

2. The Results of Schur and Horn

We begin by stating the key assertion of Theorem 5 from Alfred Horn's 1954 paper [Hor54], which can be formulated as follows.

THEOREM 2.1 (Horn). *Let $\lambda_1 \geq \cdots \geq \lambda_n$ and $p_1 \geq \cdots \geq p_n$ be two decreasing sequences of real numbers satisfying $p_1 + \cdots + p_k \leq \lambda_1 + \cdots + \lambda_k$, $1 \leq k \leq n-1$, and $p_1 + \cdots + p_n = \lambda_1 + \cdots + \lambda_n$. Then there is a self-adjoint $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ whose diagonal entries are p_1, \dots, p_n .*

Another proof is offered at the end of this section. In a more coordinate-free formulation, Horn's theorem makes the following assertion. Let A be a self-adjoint operator on an n -dimensional Hilbert space H with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, and let $p_1 \geq \cdots \geq p_n$ be a decreasing sequence that relates to λ as in the hypothesis of Theorem 2.1. Then there is an orthonormal basis e_1, \dots, e_n for H such that

$$\langle Ae_k, e_k \rangle = p_k, \quad k = 1, \dots, n.$$

The converse of Theorem 2.1 is also true, and this is the part of the composite Schur-Horn theorem that is attributed to Schur [Sch23]: If there is a self-adjoint $n \times n$ matrix A with eigenvalue sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ with diagonal $p = (p_1, \dots, p_n)$, both written in decreasing order, then the inequalities

$$(2.1) \quad p_1 + \cdots + p_k \leq \lambda_1 + \cdots + \lambda_k, \quad 1 \leq k \leq n$$

of the hypothesis of Theorem 2.1 are satisfied, with equality holding for $k = n$. That implication follows from classical estimates going back to Weyl [Wey12] (see the proof of Theorem 4.1 below).

There are other formulations of the Schur-Horn theorem that borrow from classical inequalities [Har34], the most notable one being the following. Given a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of real numbers, let \mathcal{O}_λ be the set of all $n \times n$ self-adjoint matrices having eigenvalue sequence λ . Then the set $E(\mathcal{O}_\lambda)$ of all diagonals of matrices in \mathcal{O}_λ is the convex hull Λ of the set of points $\lambda \circ \pi \in \mathbb{R}^n$, $\pi \in S_n$, obtained by permuting the components of λ . Schur's part of the Schur-Horn theorem becomes the assertion $E(\mathcal{O}_\lambda) \subseteq \Lambda$ while Theorem 2.1 implies $E(\mathcal{O}_\lambda) \supseteq \Lambda$. These formulations are discussed in [Hor54].

The Schur-Horn theorem has led to generalizations in several directions. In 1973, Kostant [Kos73] put it into the context of actions of compact Lie groups (which generalize the unitary group $U(n)$). Later Atiyah [Ati82], and independently Guillemin and Sternberg [GS82], reformulated Kostant's result in the broader context of symplectic manifolds M acted on by a torus T , and showed that for every moment map Φ for the T -action, the range of Φ is the convex hull of the images of the T -fixed points of M . See [Knu00] for more detail.

There is some connection between the finite dimensional Schur-Horn theorem [Hor54] and Horn's subsequent work on the eigenvalues of sums of matrices that culminated in the inequalities conjectured in [Hor62], as described in [Knu00]. The Horn conjecture was recently proved, following work of Klyachko [Kly98] and the proof of the saturation conjecture by Knutson and Tao [KT99]. Thus, it may be

appropriate to point out that Friedland has obtained a generalization of Klyachko’s results to finite sums of positive trace-class operators acting on infinite dimensional Hilbert spaces [Fri00].

The purpose of this paper is to discuss two infinite dimensional formulations of the Schur-Horn Theorem. In Sections 3–4 we present a generalization of the Schur-Horn theorem to positive trace class operators on infinite dimensional Hilbert spaces. In Sections 5–7, we reformulate these issues in the context of finite factors, and we establish appropriate versions of the Schur inequalities. The II_1 version of Horn’s result (Theorem 2.1) is left as an open problem.

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PROOF OF THEOREM 2.1. We show how one can deduce Theorem 2.1 from two results of [Kad02a]. Let $p = (p_1, \dots, p_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ be two decreasing sequences satisfying the hypotheses of Theorem 2.1. By lemma 5 of [Kad02a], there is a sequence of points x_1, \dots, x_n in \mathbb{R}^n such that $x_1 = \lambda$, $x_n = p$, and for $k = 1, \dots, n - 1$, x_{k+1} can be expressed in terms of x_k as follows

$$(2.2) \quad x_{k+1} = t_k \cdot x_k + (1 - t_k) \cdot x_k \circ \tau_k$$

where t_k is a number in the unit interval, τ_k is a transposition in S_n , and where $x \circ \tau$ denotes $(x_{\tau(1)}, \dots, x_{\tau(n)}) \in \mathbb{R}^n$.

Given $x_1 = \lambda, \dots, x_n = p \in \mathbb{R}^n$, $t_1, \dots, t_{n-1} \in [0, 1]$ and $\tau_1, \dots, \tau_{n-1} \in S_n$ such that the relations (2.2) are satisfied, we exhibit a sequence of self-adjoint matrices A_1, \dots, A_n such that A_k has eigenvalue list λ and diagonal sequence x_k as follows. Theorem 6 of [Kad02a] asserts the following: Given a self-adjoint $n \times n$ matrix $A = (a_{ij})$ with diagonal sequence x , and given a transposition τ in S_n and a number $t \in [0, 1]$, there is a unitary matrix U such that the diagonal of UAU^* is $t \cdot x + (1-t) \cdot x \circ \tau$. The proof exhibits $U = (u_{ij})$ explicitly; if τ is the transposition (ij) then U coincides with the identity matrix except for the four terms $u_{ii}, u_{ij}, u_{ji}, u_{jj}$ specified by

$$\begin{pmatrix} u_{ii} & u_{ij} \\ u_{ji} & u_{jj} \end{pmatrix} = \begin{pmatrix} z \cos \theta & \sin \theta \\ -z \sin \theta & \cos \theta \end{pmatrix},$$

where z is a complex number of absolute value 1 such that $z a_{ij}$ is pure imaginary, and where θ satisfies $\cos^2 \theta = t$. Let A_1 be the diagonal matrix with diagonal $\lambda = (\lambda_1, \dots, \lambda_n) = x_1$. Given that A_1, \dots, A_k have been defined and satisfy the asserted conditions for $1 \leq k < n$, the above result implies that there is a unitary matrix U_k such that

$$\text{diag}(U_k A_k U_k^*) = t_k \cdot x_k + (1 - t_k) \cdot x_k \circ \tau_k.$$

Setting $A_{k+1} = U_k A_k U_k^*$ and continuing inductively, we obtain a sequence of matrices A_1, \dots, A_n whose last term $A_n = U_{n-1} \cdots U_1 A_1 U_1^* \cdots U_{n-1}^*$ is a self-adjoint matrix having eigenvalue list λ and diagonal p . \square

Part 1. Type I_∞ factors

We first give a generalization of the Schur-Horn theorem to the case of positive trace-class operators acting on a separable infinite-dimensional Hilbert space.

3. \mathcal{L}^1 -closed unitary orbits

Let H be a separable Hilbert space and let A be a positive compact operator on H . The sequence of eigenvalues of A can be put into decreasing order, with repetitions according to the multiplicity of positive terms in the sequence, to obtain a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$, and we have

$$\lambda_1 + \lambda_2 + \dots = \text{trace } A \in [0, +\infty].$$

Such a decreasing sequence λ will be called an *eigenvalue list*. The preceding formula shows that A is trace-class iff its eigenvalue list belongs to ℓ^1 , and the set of all eigenvalue lists in ℓ^1 is a weak*-closed cone, the weak*-topology on ℓ^1 arising from the identification of ℓ^1 with the dual of c_0 .

The eigenvalue list of A fails to be a complete invariant for unitary equivalence because it fails to detect zero eigenvalues except when A is of finite rank. For example, if A has infinitely many positive terms λ_k in its spectrum and has trivial kernel, then A and $A \oplus \mathbf{0}$ ($\mathbf{0}$ being an the zero operator on some space of positive dimension) cannot be unitarily equivalent despite the fact that both have the same eigenvalue list.

The state of affairs for trace-class operators is described as follows. We write $\mathcal{L}^1 = \mathcal{L}^1(H)$ for the Banach space of all trace-class operators on a Hilbert space H with respect to the trace norm

$$\|A\|_1 = \text{trace } |A|,$$

$|A|$ denoting the positive square root of A^*A . Given an eigenvalue list $\lambda \in \ell^1$, \mathcal{O}_λ will denote the set of all positive trace-class operators on H having λ as their eigenvalue list. Given a positive trace-class operator $A \in \mathcal{B}(H)$, $\mathcal{O}(A)$ will denote the *trace-norm closure* of the unitary orbit of A

$$\mathcal{O}(A) = \{UAU^* : U \in \mathcal{U}(H)\}^{-\|\cdot\|_1}.$$

Two trace-class operators A, B are said to be \mathcal{L}^1 -equivalent if there is a sequence of unitary operators U_1, U_2, \dots such that that $\|A - U_n B U_n^*\|_1 \rightarrow 0$ as $n \rightarrow \infty$; equivalently, $\mathcal{O}(A) = \mathcal{O}(B)$.

PROPOSITION 3.1. *Let A be a positive trace-class operator in $\mathcal{B}(H)$ and let λ be the eigenvalue list of A .*

- (i) $\mathcal{O}(A)$ is a Polish topological space on which the unitary group of H acts minimally.
- (ii) $\mathcal{O}(A) = \mathcal{O}_\lambda$; in particular, the eigenvalue list is a complete invariant for \mathcal{L}^1 -equivalence.
- (iii) Two positive trace-class operators A, B are \mathcal{L}^1 -equivalent iff $A \oplus \mathbf{0}$ and $B \oplus \mathbf{0}$ are unitarily equivalent, where $\mathbf{0}$ denotes the zero operator on an infinite dimensional Hilbert space.
- (iv) If λ has only finitely many nonzero terms, then \mathcal{O}_λ consists of a single unitary orbit $\{UAU^* : U \in \mathcal{U}(H)\}$.

PROOF. (i): \mathcal{O}_λ is a closed subset of \mathcal{L}^1 and therefore a separable complete metric space. The fact that the orbit of every point of $\mathcal{O}(A)$ under the action of $\mathcal{U}(H)$ is dense in $\mathcal{O}(A)$ follows from the fact that \mathcal{L}^1 -equivalence is a transitive relation.

(ii): Let B be another positive trace-class operator with eigenvalue list μ . We have to show that A and B are \mathcal{L}^1 -equivalent $\iff \lambda = \mu$. For the implication \implies we make use of the semiclassical inequality

$$\sum_{k=1}^{\infty} |\lambda_k - \mu_k| \leq \|A - B\|_1,$$

a proof of which can be found in the appendix of [Pow67]. Since B can be closely approximated in the norm of \mathcal{L}^1 by operators unitarily equivalent to A , this inequality implies that B must have the same eigenvalue list as A , hence $\mu = \lambda$. Conversely, if A and B are two positive trace-class operators with the same eigenvalue list

$$\lambda = (\lambda_1, \lambda_2, \dots),$$

then by the spectral theorem we can decompose A and B as follows

$$A = A_n + R_n, \quad B = B_n + S_n$$

where A_n and B_n are finite rank positive operators with eigenvalue list

$$(\lambda_1, \dots, \lambda_n, 0, 0, \dots)$$

and where the remainders R_n and S_n satisfy

$$\|R_n\|_1 = \|S_n\|_1 = \sum_{k=n+1}^{\infty} |\lambda_k|.$$

Since A_n and B_n are obviously unitarily equivalent for every $n = 1, 2, \dots$ and since $\|R_n\|_1$ and $\|S_n\|_1$ tend to zero as $n \rightarrow \infty$, it follows that there is a sequence of unitary operators U_1, U_2, \dots such that $\|B - U_n A U_n^*\|_1 \rightarrow 0$.

(iii) is a consequence of (ii), which asserts that A and B are \mathcal{L}^1 -equivalent iff they have the same eigenvalue list. Indeed, it is obvious that if A and B have the same eigenvalue list λ then $A \oplus \mathbf{0}$ and $B \oplus \mathbf{0}$ are unitarily equivalent; conversely, if $A \oplus \mathbf{0}$ and $B \oplus \mathbf{0}$ are unitarily equivalent then $A \oplus \mathbf{0}$ and $B \oplus \mathbf{0}$ must have the same eigenvalue list, hence so do A and B .

Finally, note that (iii) \implies (iv), since if A is a finite rank positive operator with eigenvalue list λ , then all but a finite number of components of λ are zero, hence A is unitarily equivalent to $A \oplus \mathbf{0}$, so that all operators in \mathcal{O}_λ are unitarily equivalent. \square

4. Diagonals of Trace Class Operators

Let H be a separable Hilbert space and let e_1, e_2, \dots be an orthonormal basis for H . The sequence of rank-one projections $E_k = [e_k], k = 1, 2, \dots$ generates a discrete maximal abelian subalgebra $\mathcal{A} \subseteq \mathcal{B}(H)$, and the map that replaces an operator A with the diagonal part (a_{11}, a_{22}, \dots) of its matrix $a_{ij} = \langle A e_j, e_i \rangle, i, j = 1, 2, \dots$, relative to (e_n) can be viewed as the unique trace preserving conditional expectation $E : \mathcal{B}(H) \rightarrow \mathcal{A}$

$$E(A) = \sum_{n=1}^{\infty} E_n A E_n = \sum_{n=1}^{\infty} a_{nn} E_n.$$

The following result provides an infinite-dimensional generalization of the Schur-Horn theorem. For a related result that characterizes the *norm-closure* of $E(\mathcal{O}(A))$ for a broader class of operators A , see [Neu99].

THEOREM 4.1. *Let \mathcal{A} be a discrete maximal abelian von Neumann algebra in $\mathcal{B}(H)$, let $E : \mathcal{B}(H) \rightarrow \mathcal{A}$ be the trace-preserving conditional expectation on \mathcal{A} and let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ be a decreasing sequence in ℓ^1 with nonnegative terms. Then $E(\mathcal{O}_\lambda)$ consists of all positive trace-class operators $B \in \mathcal{A}$ whose eigenvalue list $p = (p_1 \geq p_2 \geq \dots)$ satisfies*

$$(4.1) \quad p_1 + \dots + p_n \leq \lambda_1 + \dots + \lambda_n, \quad n = 1, 2, \dots$$

together with

$$(4.2) \quad p_1 + p_2 + \dots = \lambda_1 + \lambda_2 + \dots.$$

We will deduce Theorem 4.1 from the following more general assertion about the diagonals of positive compact operators.

THEOREM 4.2. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a discrete maximal abelian algebra with natural conditional expectation $E : \mathcal{B}(H) \rightarrow \mathcal{A}$. Let $A \in \mathcal{B}(H)$ be a positive compact operator with eigenvalue list $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, and let B be a positive compact operator in \mathcal{A} . The following are equivalent.*

- (i) *There is contraction $L \in \mathcal{B}(H)$ such that $E(L^*AL) = B$.*
- (ii) *The eigenvalue list $p = (p_1 \geq p_2 \geq \dots)$ of B satisfies*

$$p_1 + p_2 + \dots + p_n \leq \lambda_1 + \lambda_2 + \dots + \lambda_n, \quad n = 1, 2, \dots$$

We require some eigenvalue estimates that go back to work of Weyl [**Wey12**], [**Wey49**]. Let A be a positive compact operator with eigenvalue list $\lambda_1 \geq \lambda_2 \geq \dots$ and let \mathcal{P}_n be the set of all n -dimensional projections in $\mathcal{B}(H)$. Then we have

$$(4.3) \quad \sup_{P \in \mathcal{P}_n} \text{trace } AP = \max_{P \in \mathcal{P}_n} \text{trace } AP = \lambda_1 + \dots + \lambda_n,$$

the maximum being achieved on any n -dimensional projection whose range contains eigenvectors for $\lambda_1, \dots, \lambda_n$. Ky Fan's version of this result can be found on p. 22 of [**Bha87**].

The proof of Theorem 4.2 also requires a geometric result, asserting that if $p = (p_1, \dots, p_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ are two finite eigenvalue lists that satisfy the first n inequalities (4.1), then the components of λ can be reduced so as to preserve the first $n - 1$ inequalities, with equality in the n th.

LEMMA 4.3. *Let $p = (p_1, \dots, p_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ be two decreasing sequences of nonnegative reals of length $n = 1, 2, \dots$ satisfying*

$$(4.4) \quad p_1 + \dots + p_k \leq \lambda_1 + \dots + \lambda_k, \quad k = 1, 2, \dots, n.$$

There is a decreasing sequence $\mu = (\mu_1, \dots, \mu_n)$ such that

$$(4.5) \quad 0 \leq \mu_k \leq \lambda_k, \quad p_1 + \dots + p_k \leq \mu_1 + \dots + \mu_k,$$

for $1 \leq k \leq n$, and $p_1 + \dots + p_n = \mu_1 + \dots + \mu_n$.

PROOF. We argue by induction, the case $n = 1$ being obvious. Fix $n \geq 2$ and suppose that Lemma 4.3 is true for sequences of length $n - 1$. Let D be the set of all points $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ satisfying $\mu_1 \geq \dots \geq \mu_n \geq 0$ and $\mu_k \leq \lambda_k$, $1 \leq k \leq n$, and consider the compact convex set $K \subseteq \mathbb{R}^n$

$$K = \{\mu \in D : \mu_1 + \dots + \mu_k \geq p_1 + \dots + p_k, \quad k = 1, \dots, n - 1\}.$$

Since $f(x) = x_1 + \dots + x_n$ is a linear functional on \mathbb{R}^n , $f(K)$ is a closed interval $I \subseteq \mathbb{R}$. We have to show that $p_1 + \dots + p_n \in I$. For that, it suffices to show that

there are points $x, y \in K$ such that $f(x) \leq p_1 + \dots + p_n \leq f(y)$. Setting $y = \lambda \in K$, we have $p_1 + \dots + p_n \leq \lambda_1 + \dots + \lambda_n = f(y)$ by (4.4). For x , use the induction hypothesis to obtain numbers $\mu_1 \geq \dots \geq \mu_{n-1} \geq 0$ satisfying $0 \leq \mu_k \leq \lambda_k$, $\mu_1 + \dots + \mu_k \geq p_1 + \dots + p_k$, $1 \leq k \leq n-1$, and $\mu_1 + \dots + \mu_{n-1} = p_1 + \dots + p_{n-1}$. The point $x = (\mu_1, \dots, \mu_{n-1}, 0)$ belongs to K and satisfies $f(x) = p_1 + \dots + p_{n-1} \leq p_1 + \dots + p_n$. \square

PROOF OF THEOREM 4.2. (i) \implies (ii): Let e_1, e_2, \dots be an orthonormal basis for H with the property that $\langle Be_j, e_j \rangle = p_k$, $j = 1, 2, \dots$. Fixing k and letting E be the projection onto the span of e_1, \dots, e_k , we have

$$p_1 + \dots + p_k = \text{trace}(BE) = \text{trace}(L^*ALE) = \text{trace}(ALEL^*) \leq \text{trace}(AF)$$

where F is the projection onto the range of the positive contraction LEL^* . Since F is a projection of rank at most k , the estimate (4.3) implies

$$\text{trace}(AF) \leq \sup_{\dim F=k} \text{trace}(AF) = \lambda_1 + \dots + \lambda_k,$$

and (ii) follows.

(ii) \implies (i): Let B be a positive compact operator in \mathcal{A} whose eigenvalue list $p = (p_1 \geq p_2 \geq \dots)$ satisfies the inequalities (ii) and let e_1, e_2, \dots be an orthonormal basis for H such that $[e_1], [e_2], \dots$ are the minimal projections of \mathcal{A} . Since every permutation of the basis $\{e_k\}$ is implemented by a unitary operator $W \in \mathcal{B}(H)$ satisfying $WAW^* = A$, we may assume without essential loss that $Be_k = p_k e_k$, $k = 1, 2, \dots$.

We construct a sequence of operators $L_n \in \mathcal{B}(H)$, $n = 1, 2, \dots$, as follows. Consider the spectral representation of A

$$A = \sum_{k=1}^{\infty} \lambda_k \xi_k \otimes \bar{\xi}_k$$

where ξ_1, ξ_2, \dots is an orthonormal sequence in H consisting of eigenvectors of A . Fix n , let H_n be the linear span of ξ_1, \dots, ξ_n , and let A_n be the restriction of A to H_n . The eigenvalue list of A_n is $(\lambda_1, \dots, \lambda_n)$; so by Lemma 4.3, there is a decreasing sequence $\mu = (\mu_1, \dots, \mu_n)$ satisfying $0 \leq \mu_k \leq \lambda_k$ for $1 \leq k \leq n$, and

$$p_1 + \dots + p_k \leq \mu_1 + \dots + \mu_k, \quad k = 1, \dots, n,$$

with equality holding for $k = n$. The sequence μ of course depends on n but we suppress that in the notation since μ will soon disappear. Consider the operator B_n defined on H_n by requiring $B_n \xi_k = \mu_k \xi_k$, $1 \leq k \leq n$. The eigenvalue list of B_n dominates (p_1, \dots, p_n) as in the hypothesis of Horn's result Theorem 2.1. Thus there is an orthonormal basis $e_1^{(n)}, \dots, e_n^{(n)}$ for H_n such that

$$\langle B_n e_k^{(n)}, e_k^{(n)} \rangle = p_k, \quad k = 1, \dots, n.$$

Since $0 \leq B_n \leq A_n$ it follows that

$$p_k \leq \langle A_n e_k^{(n)}, e_k^{(n)} \rangle = \langle A e_k^{(n)}, e_k^{(n)} \rangle, \quad k = 1, \dots, n.$$

Let $L_n \in \mathcal{B}(H)$ be the operator defined by $L_n e_k = e_k^{(n)}$ for $k = 1, \dots, n$, and $L_n = 0$ on the orthocomplement of $[e_1^{(n)}, \dots, e_n^{(n)}]$.

We have constructed a sequence L_1, L_2, \dots of finite rank partial isometries in $\mathcal{B}(H)$ that satisfies the system of inequalities

$$(4.6) \quad p_k \leq \langle AL_n e_k, L_n e_k \rangle, \quad n \geq k \geq 1.$$

Since the unit ball of $\mathcal{B}(H)$ is sequentially compact in its weak operator topology, there is a subsequence $n_1 < n_2 < \dots$ and a contraction $L_\infty \in \mathcal{B}(H)$ such that $\langle L_{n_j}\eta, \zeta \rangle \rightarrow \langle L_\infty\eta, \zeta \rangle$ as $j \rightarrow \infty$, for every $\eta, \zeta \in H$. We claim that L_∞ satisfies

$$(4.7) \quad p_k \leq \langle AL_\infty e_k, L_\infty e_k \rangle, \quad k = 1, 2, \dots$$

To see that, fix k and note that for sufficiently large j , (4.6) implies

$$p_k \leq \langle AL_{n_j} e_k, L_{n_j} e_k \rangle.$$

As $j \rightarrow \infty$, $L_{n_j} e_k$ tends to $L_\infty e_k$ in the weak topology of H . Since A is a compact operator, $\|AL_{n_j} e_k - AL_\infty e_k\| \rightarrow 0$ as $j \rightarrow \infty$; hence the inner products $\langle AL_{n_j} e_k, L_{n_j} e_k \rangle$ converge to $\langle AL_\infty e_k, L_\infty e_k \rangle$, and (4.7) follows.

Finally, choose $t_1, t_2, \dots \in [0, 1]$ such that $p_k = t_k \langle AL_\infty u_k, L_\infty u_k \rangle$ for every k . Letting $D \in \mathcal{B}(H)$ be the contraction defined by $De_k = \sqrt{t_k} e_k$, $k \geq 1$, one finds that the operator $L = L_\infty D$ satisfies

$$\langle L^* A L e_k, e_k \rangle = p_k, \quad k = 1, 2, \dots,$$

and the required formula

$$E(L^* A L) = \sum_{k=1}^{\infty} E_k L^* A L E_k = \sum_{k=1}^{\infty} p_k E_k = B$$

follows. □

PROOF OF THEOREM 4.1. Let E_1, E_2, \dots be the minimal projections of \mathcal{A} and let e_1, e_2, \dots be an orthonormal basis for H such that E_k is the projection $[e_k]$, $k = 1, 2, \dots$

We show first that for every positive trace class operator $A \in \mathcal{B}(H)$ with eigenvalue list λ , the eigenvalue list $p = (p_1, p_2, \dots)$ of $B = E(A)$ must satisfy (4.1) and (4.2). By permuting the elements of the basis $\{e_k\}$ appropriately and changing notation, we may assume that $Be_k = p_k e_k$, $k = 1, 2, \dots$. Let P_n be the projection on $[e_1, \dots, e_n]$. Since A is a positive compact operator with eigenvalue list λ , we can make use of (4.3) to write

$$p_1 + \dots + p_n = \sum_{k=1}^n \langle Be_k, e_k \rangle = \text{trace } AP_n \leq \lambda_1 + \dots + \lambda_n.$$

Moreover, $p_1 + p_2 + \dots = \langle Au_1, u_1 \rangle + \langle Au_2, u_2 \rangle + \dots = \text{trace } A = \lambda_1 + \lambda_2 + \dots$.

Conversely, let p and λ be two summable eigenvalue lists that satisfy (4.1) – (4.2), and let B be a positive trace-class operator in \mathcal{A} with list p . Again, by relabeling the orthonormal basis $\{e_k\}$, we may assume that $Be_k = p_k e_k$, $k = 1, 2, \dots$. Choose any positive trace-class operator $A \in \mathcal{B}(H)$ having eigenvalue list λ , and let P be the projection onto the closure of AH . Theorem 4.2 implies that there is a contraction $L \in \mathcal{B}(H)$ satisfying $p_k = \langle AL e_k, L e_k \rangle$ for $k \geq 1$. By replacing L with PL if necessary, we may also assume that LH is contained in PH , and in that case we claim:

$$(4.8) \quad LL^* = P.$$

Indeed, $P - LL^* \geq 0$ because L is a contraction whose range is contained in PH , and it suffices to show that the positive operator $A^{1/2}(P - LL^*)A^{1/2} = A - A^{1/2}LL^*A^{1/2}$

has trace zero; equivalently, $\text{trace } A^{1/2}LL^*A^{1/2} = \text{trace } A$. Using $\text{trace } XX^* = \text{trace } X^*X$ for $X = A^{1/2}L$, we have

$$\begin{aligned} \text{trace } A^{1/2}LL^*A^{1/2} &= \text{trace } L^*AL = \sum_{n=1}^{\infty} \langle ALe_n, Le_n \rangle \\ &= p_1 + p_2 + \dots = \lambda_1 + \lambda_2 + \dots = \text{trace } A. \end{aligned}$$

L is a co-isometry by (4.8); hence it can be changed into a unitary operator $U : H \rightarrow H \oplus \ker L$ by making use of the projection $Q : H \rightarrow \ker L$ as follows: $U\xi = L\xi \oplus Q\xi$, $\xi \in H$. Now consider the operator

$$A_0 \oplus \mathbf{0} \in \mathcal{B}(PH \oplus \ker L),$$

A_0 denoting the restriction of A to $PH = \overline{AH}$. Since $Ue_k = Le_k \oplus Qe_k$, $k = 1, 2, \dots$, we have

$$\langle (A_0 \oplus \mathbf{0})Ue_k, Ue_k \rangle = \langle ALe_k, Le_k \rangle = p_k, \quad k = 1, 2, \dots$$

Therefore $U^*(A_0 \oplus \mathbf{0})U$ is a positive trace class operator in $\mathcal{B}(H)$ satisfying

$$E(U^*(A_0 \oplus \mathbf{0})U) = \sum_{k=1}^{\infty} p_k E_k = B.$$

Since $U^*(A_0 \oplus \mathbf{0})U$ has the same eigenvalue list as A , Proposition 3.1 implies that it must belong to $\mathcal{O}(A) = \mathcal{O}_\lambda$, and the proof is complete. \square

In the series [Kad02a], [Kad02b], one of us carried out a study of the possible diagonals of projections acting on a separable Hilbert space. The results of this paper do not address the most difficult case where the projection has infinite rank and infinite co-rank; but Theorem 4.1 does give the result of Theorem 13 of [Kad02b], as follows. We formulate that in terms of the Hilbert space ℓ^2 and its standard orthonormal basis $(u_k)_{k \geq 1}$, with the associated realization of operators as matrices relative to this basis.

COROLLARY 4.4. *Let $p = (p_1, p_2, \dots)$ be a sequence of numbers in the unit interval $0 \leq p_k \leq 1$, and let m be a positive integer. The following are equivalent*

- (i) *There is a rank m projection $P \in \mathcal{B}(\ell^2)$ whose matrix has p as its diagonal.*
- (ii) $p_1 + p_2 + \dots = m$.

PROOF. We prove the nontrivial implication (ii) \implies (i). Since (ii) implies that the sequence p_n converges to zero and since permutations of \mathbb{N} are implemented by unitary operators on $\ell^2(\mathbb{N})$ in the obvious way, it suffices to address the case where the sequence is decreasing $p_1 \geq p_2 \geq \dots$. The eigenvalue list of a projection of rank m is

$$\lambda = (\underbrace{1, \dots, 1}_{m \text{ times}}, 0, 0, \dots),$$

and \mathcal{O}_λ consists of all rank m projections in $\mathcal{B}(\ell^2)$. The hypothesis (ii), together with $0 \leq p_k \leq 1$, implies that $p_1 + \dots + p_n \leq \lambda_1 + \dots + \lambda_n$ holds for every $n \geq 1$. Hence Theorem 4.1 implies that there is an operator in \mathcal{O}_λ with diagonal sequence p . \square

Part 2. Type II_1 Factors

We turn now to the case of self-adjoint operators A in a finite factor. In this context, the appropriate counterpart of the eigenvalue list is a finite positive measure on the real line, called the spectral distribution of A . After working out the basic properties of the spectral distribution and relating it to approximate unitary equivalence, we establish a generalization of the Schur inequalities for II_1 factors.

5. Spectral distribution of a self-adjoint operator

In the remainder of this paper we work within the context of a *finite* factor \mathcal{M} with normalized trace τ . For every self-adjoint operator $A \in \mathcal{M}$ there is a unique probability measure m_A on the Borel subsets of the real line whose moments are given by

$$(5.1) \quad \int_{-\infty}^{\infty} \lambda^n dm_A(\lambda) = \tau(A^n), \quad n = 0, 1, 2, \dots$$

DEFINITION 5.1. The measure m_A is called the spectral distribution of A .

The purpose of these section is to discuss the basic properties of this invariant. The spectral distribution is the appropriate generalization to II_1 factors of the eigenvalue list invariant of self-adjoint $n \times n$ matrices. Indeed, if A is a self-adjoint $n \times n$ matrix with eigenvalue list $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$, then m_A is the discrete measure

$$m_A = \frac{1}{n}(\delta_{\lambda_1} + \dots + \delta_{\lambda_n}),$$

δ_λ denoting the unit point mass concentrated at $\lambda \in \mathbb{R}$. Equivalently, m_A assigns mass to singletons $\{\lambda\}$ of \mathbb{R} as follows

$$m_A(\lambda) = \begin{cases} \frac{1}{n}(\text{multiplicity of } \lambda), & \text{if } \lambda \in \sigma(A) \\ 0, & \text{otherwise.} \end{cases}$$

We require the following observation, which asserts that the spectral distribution of an operator in a II_1 factor can be arbitrary.

PROPOSITION 5.2. *Let $\mathcal{A} \subseteq \mathcal{M}$ be a MASA in a II_1 factor and let m be a compactly supported probability measure on the real line. Then there is a self-adjoint operator $A \in \mathcal{A}$ such that $m = m_A$.*

PROOF. \mathcal{A} contains a countably-generated nonatomic subalgebra, which must be isomorphic to $L^\infty[0, 1]$ in such a way that the restriction of the trace τ corresponds to the state of $L^\infty[0, 1]$ given by

$$\tau(f) = \int_0^1 f(x) dx, \quad f \in L^\infty[0, 1].$$

Thus it suffices to show that there is a real-valued function $f \in L^\infty[0, 1]$ such that $\int_0^1 f(x)^n dx = \int \lambda^n dm(\lambda)$ for $n = 0, 1, \dots$ or equivalently, for every Borel set $S \subseteq \mathbb{R}$,

$$(5.2) \quad \tau\{x \in [0, 1] : f(x) \in S\} = m(S),$$

where we abuse notation slightly by also writing τ for Lebesgue measure on the unit interval. Let K be the closed support of m . The pair (K, m) defines a separable measure algebra which may have a finite or countable number of atoms. On the

other hand, $([0, 1], \tau)$ gives rise to a nonatomic separable measure algebra. Letting x_1, x_2, \dots be the points of K having positive m -measure, we find a sequence of disjoint Borel sets $E_1, E_2, \dots \subseteq [0, 1]$ such that $\tau(E_k) = m(\{x_k\})$ for all k . Define f on $\cup_k E_k$ so that it takes on the constant value x_k throughout E_k . f is a measure-preserving map of $\cup_k E_k$ onto the atomic part of (K, τ) . Since $\tau([0, 1] \setminus \cup_k E_k) = m(K \setminus \{x_1, x_2, \dots\})$ and since the remaining parts of both measure spaces are nonatomic and separable, there is a surjective Borel map of $[0, 1] \setminus \cup_n E_n$ onto $K \setminus \{x_1, x_2, \dots\}$ that pushes Lebesgue measure forward to m , and we can use this map to extend the definition of f to all of $[0, 1]$ in the obvious way. \square

The eigenvalue list is a complete invariant for \mathcal{L}^1 -equivalence of positive trace-class operators in type I factors. We now show that the spectral distribution invariant occupies a similar position.

DEFINITION 5.3. Operators $A, B \in \mathcal{M}$ are said to be *approximately equivalent* if there is a sequence of unitary operators U_1, U_2, \dots in \mathcal{M} such that

$$(5.3) \quad \lim_{n \rightarrow \infty} \|U_n A U_n^* - B\| = 0.$$

The set of all operators in the norm-closed unitary orbit of A is written \mathcal{O}_A .

THEOREM 5.4. *Let A be a self-adjoint operator in \mathcal{M} , let m_A be the spectral distribution of A , and let \mathcal{O}_A be the norm-closed unitary orbit of A . Then \mathcal{O}_A is closed in the strong operator topology, and consists of all self-adjoint operators $B \in \mathcal{M}$ satisfying $m_B = m_A$.*

Before giving the proof we collect an elementary observation.

LEMMA 5.5. *Let $E_1 \leq E_2 \leq \dots \leq E_n$ and $F_1 \leq F_2 \leq \dots \leq F_n$ be two linearly ordered sets of projections in \mathcal{M} such that $\tau(E_k) = \tau(F_k)$ for $k = 1, \dots, n$. Then there is a unitary operator U in \mathcal{M} such that $U E_k U^* = F_k$, $k = 1, \dots, n$.*

PROOF. By adjoining the identity to the end of each list if necessary, we can assume that $E_n = F_n = \mathbf{1}$. Setting $E_0 = F_0 = 0$, the hypothesis implies that $\tau(E_k - E_{k-1}) = \tau(F_k - F_{k-1})$ for each $k = 1, \dots, n$. Since \mathcal{M} is a finite factor, projections with the same trace must be Murray-von Neumann equivalent. Thus there are partial isometries $U_1, \dots, U_n \in \mathcal{M}$ with $U_k^* U_k = E_k - E_{k-1}$ and $U_k U_k^* = F_k - F_{k-1}$ for all k . The projections $U_1^* U_1, \dots, U_n^* U_n$ add up to $E_n = \mathbf{1}$, and similarly $U_1 U_1^* + \dots + U_n U_n^* = \mathbf{1}$. It follows that $W = U_1 + \dots + U_n$ is a unitary operator in \mathcal{M} with the property $W(E_k - E_{k-1})W^* = F_k - F_{k-1}$ for every $k = 1, \dots, n$, hence $W E_k W^* = F_k$ for $k = 1, \dots, n$. \square

PROOF OF THEOREM 5.4. We will show that a self-adjoint operator B belongs to \mathcal{O}_A iff $m_B = m_A$. Once that is established, it will follow that \mathcal{O}_A is strongly closed, because the relation $m_B = m_A$ is characterized by the sequence of equations

$$(5.4) \quad \tau(B^n) = \tau(A^n), \quad n = 0, 1, 2, \dots$$

and each of the monomials $B \mapsto \tau(B^n)$ is strongly continuous on bounded subsets of \mathcal{M} .

Every operator $B \in \mathcal{M}$ that is unitarily equivalent to A must satisfy the formulas (5.4), and hence a norm-limit of such operators will satisfy the same formulas. It follows that $m_B = m_A$ for every B in the norm-closed unitary orbit of A .

Conversely, let A, B be self-adjoint operators of \mathcal{M} satisfying $m_A = m_B$, and let $[a, b]$ be an interval with the property that $m_A = m_B$ is supported in the interior

(a, b) . Then the spectra of A and B are both contained in (a, b) . For every $a \leq t \leq b$ let E_t (resp. F_t) be the spectral projection of A (resp. B) corresponding to the subinterval $[a, t] \subseteq \mathbb{R}$. Then by hypothesis we have

$$(5.5) \quad \tau(E_t) = m_A([a, t]) = m_B([a, t]) = \tau(F_t), \quad a \leq t \leq b.$$

Given $\epsilon > 0$ we can find a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$ fine enough that

$$\left| t - \sum_{k=1}^n t_k \chi_{(t_{k-1}, t_k]}(t) \right| \leq \epsilon, \quad a \leq t \leq b.$$

Letting A_0, B_0 be the operators

$$A_0 = \sum_{k=1}^n t_k (E_{t_k} - E_{t_{k-1}}), \quad B_0 = \sum_{k=1}^n t_k (F_{t_k} - F_{t_{k-1}}),$$

we find that $\|A - A_0\| \leq \epsilon$ and $\|B - B_0\| \leq \epsilon$. (5.5) implies $\tau(E_{t_k}) = \tau(F_{t_k})$ for every k , so by the Lemma there is a unitary operator $W \in \mathcal{M}$ such that $WE_{t_k}W^* = F_{t_k}$ for all k , hence $WA_0W^* = B_0$. An obvious estimate now implies $\|WAW^* - B\| \leq 2\epsilon$, and since ϵ is arbitrary it follows that A and B are approximately equivalent. \square

Theorem 5.4 implies that the spectral distribution is a complete invariant for approximate unitary equivalence, and it is natural to ask if two self-adjoint operators that are approximately equivalent must be unitarily equivalent, or at least conjugate by way of a $*$ -automorphism. The following class of examples shows that the answer is no.

Example. Let \mathcal{R} be a II_1 factor and let \mathcal{A} and \mathcal{B} be two MASAs in \mathcal{R} that are *not* conjugate by way of an automorphism of \mathcal{R} . For example, \mathcal{A} can be taken to be a regular MASA and \mathcal{B} a singular one. Since both \mathcal{A} and \mathcal{B} are isomorphic to $L^\infty[0, 1]$ by way of an isomorphism that carries the trace to Lebesgue measure, it follows that a) there is a $*$ -isomorphism α of \mathcal{A} onto \mathcal{B} satisfying $\tau(\alpha(X)) = \tau(X)$ for all $X \in \mathcal{A}$, and b) \mathcal{A} is the von Neumann algebra $W^*(A)$ generated by a single self-adjoint operator A . Let $B = \alpha(A)$. Then $\tau(A^n) = \tau(B^n)$ for every $n = 0, 1, 2, \dots$ and hence $m_A = m_B$. It follows from Theorem 5.4 that A and B are approximately equivalent. On the other hand, there is no unitary operator $U \in \mathcal{R}$ satisfying $UAU^* = B$, since that would imply that $\theta(X) = UXU^*$ is an automorphism of \mathcal{R} that carries $\mathcal{A} = W^*(A)$ onto $\mathcal{B} = W^*(B)$.

6. Schur-type inequalities for II_1 factors

The purpose of this section is to formulate an appropriate counterpart of the Schur inequalities for self-adjoint operators in II_1 factors. This is not the only formulation possible, and we refer the reader to Section 5 of [Kad04] for an alternate approach. Here, we seek to formulate the Schur inequalities in terms of spectral distributions. That formulation is based on the following observations.

PROPOSITION 6.1. *For any two compactly supported probability measures m, n on the real line, the following are equivalent:*

- (i) m and n have the same first moment

$$\int_{\mathbb{R}} \lambda dm(\lambda) = \int_{\mathbb{R}} \lambda dn(\lambda),$$

and for every $t \in \mathbb{R}$ we have

$$\int_t^\infty m([s, \infty)) ds \leq \int_t^\infty n([s, \infty)) ds.$$

(ii) m and n have the same first moment, and for every $t \in \mathbb{R}$ we have

$$\int_{[t, \infty)} (\lambda - t) dm(\lambda) \leq \int_{[t, \infty)} (\lambda - t) dn(\lambda).$$

(iii) For every continuous convex function defined on a closed interval $I = [c, d]$ that supports both measures m and n , we have

$$\int_I f(\lambda) dm(\lambda) \leq \int_I f(\lambda) dn(\lambda).$$

PROOF. The equivalence (i) \iff (ii) is a consequence of the classic integration by parts formula of Riemann-Stieltjes integration, which can be applied as follows. Fix $t \in \mathbb{R}$, let m be a compactly supported measure defined on \mathbb{R} , and choose $a, b \in \mathbb{R}$ so that $a < t < b$ and such that (a, b) contains the closed support of both m and n . Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be the decreasing function $\alpha(s) = m([s, \infty))$ and let f be the continuous increasing function $f(s) = \max(s - t, 0)$. An application of Theorem 9–6 of [Apo57] gives

$$\int_a^b f(x) d\alpha(x) + \int_a^b \alpha(x) df(x) = f(b)\alpha(b) - f(a)\alpha(a).$$

In this case, $\alpha(b) = f(a) = 0$, and straightforward computations show that

$$\begin{aligned} \int_a^b f(x) d\alpha(x) &= - \int_{[t, \infty)} (x - t) dm(x), \\ \int_a^b \alpha(x) df(x) &= \int_t^\infty m([x, \infty)) dx. \end{aligned}$$

It follows that

$$\int_{[t, \infty)} (\lambda - t) dm(\lambda) = \int_t^\infty m([s, \infty)) ds,$$

and the equivalence of (i) and (ii) follows.

(ii) \implies (iii): This will follow if we show that every continuous real-valued convex function f defined on a compact interval $I \subseteq \mathbb{R}$ can be uniformly approximated on I by functions

$$(6.1) \quad f(\lambda) = a + b\lambda + g(\lambda)$$

where a and b are real constants and g belongs to the cone generated by the “angular” functions

$$g_t(\lambda) = \max(\lambda - t, 0) = (\lambda - t)\chi_{[t, \infty)}(\lambda), \quad t \in \mathbb{R}.$$

To see how the approximation (6.1) is achieved, one first approximates f uniformly on I with a twice continuously differentiable convex function g . Since g' is an increasing function, it can be uniformly approximated by an increasing step function having the form $a + h(\lambda)$ where a is constant and h belongs to the cone generated by the step functions $\chi_{[t, \infty)}$, $t \in \mathbb{R}$. After one integrates this approximation of g' one obtains an approximation to $g(\lambda)$ of the form $a\lambda + b + \int h(\lambda) d\lambda$. Moreover, since the indefinite integral of a step function $\chi_{[t, \infty)}(\lambda)$ has the form $c + g_t(\lambda)$ where c is a constant, this approximation of g has the form (6.1).

The implication (iii) \implies (ii) is an immediate consequence of the fact that functions of the form (6.1) are continuous and convex. \square

DEFINITION 6.2. Let m and n be two compactly supported probability measures on the real line. We write $m \preceq n$ if m is dominated by n in the equivalent senses of Proposition 6.1, i.e., if $\int_{\mathbb{R}} \lambda \, dm(\lambda) = \int_{\mathbb{R}} \lambda \, dn(\lambda)$, and

$$(6.2) \quad \int_t^\infty m([s, \infty)) \, ds \leq \int_t^\infty n([s, \infty)) \, ds, \quad t \in \mathbb{R}.$$

The relation \preceq is obviously a partial ordering on the set of all compactly supported probability measures on the real line. Given two self-adjoint operators A, B in a II_1 factor \mathcal{M} , we interpret the relation $m_A \preceq m_B$ as the appropriate counterpart of the Schur inequalities (2.1) that relate the eigenvalue lists of A and B . This interpretation is justified by Proposition 6.1 and the following remarks.

REMARK 6.3 (Relation to the classical inequalities of Schur). Let τ be the normalized trace on the matrix algebra $M_n(\mathbb{C})$, and let A and B be self-adjoint $n \times n$ matrices. We have discussed the relation between the eigenvalue list of A and the spectral distribution m_A in Section 5. We now examine the relation between the integrals

$$\int_t^\infty m_A([s, \infty)) \, ds$$

appearing in (6.2) and the eigenvalue list $\lambda_1 \geq \dots \geq \lambda_n$ of A . For simplicity, we consider the case where the eigenvalues are simple ones. For every t in the interval $\lambda_{k+1} < t < \lambda_k$ one has $m([t, \infty)) = k/n$, and after a straightforward integration and cancellation one finds that for $\lambda_{k-1} < t < \lambda_k$,

$$\int_t^\infty m_A([s, \infty)) \, ds = \frac{\lambda_1 + \lambda_2 + \dots + \lambda_k - kt}{n}.$$

Let B be another self-adjoint matrix with eigenvalue list $\mu_1 \geq \dots \geq \mu_n$. The preceding formula shows that the system of inequalities

$$(6.3) \quad \int_t^\infty m_A([s, \infty)) \, ds \leq \int_t^\infty m_B([s, \infty)) \, ds, \quad t \in \mathbb{R},$$

differs somewhat from the system of classical Schur inequalities, which in terms of the normalized trace would assert

$$(6.4) \quad \frac{\lambda_1 + \dots + \lambda_k}{n} \leq \frac{\mu_1 + \dots + \mu_k}{n}, \quad k = 1, 2, \dots, n.$$

However, if $\tau(A) = \tau(B)$ then $\lambda_1 + \dots + \lambda_n = \mu_1 + \dots + \mu_n$; and in that event the inequalities (6.3) are *equivalent* to the Schur inequalities (6.4) because they are equivalent to the inequalities of assertion (iii) of Proposition 6.1. That is a consequence of classical results of Hardy, Littlewood and Polya [Har34] which are summarized in Theorem 1 of [Hor54]. The relevant result asserts that for two finite eigenvalue lists

$$\{\lambda_1 \geq \dots \geq \lambda_n\}, \quad \{\mu_1 \geq \dots \geq \mu_n\}$$

which satisfy $\lambda_1 + \dots + \lambda_n = \mu_1 + \dots + \mu_n$, the following are equivalent:

- (1) $\lambda_1 + \dots + \lambda_k \leq \mu_1 + \dots + \mu_k$, for every $k = 1, \dots, n$.

(2) For every convex function f defined on an interval containing all λ_i and μ_j , one has

$$\sum_{k=1}^n f(\lambda_k) \leq \sum_{k=1}^n f(\mu_k).$$

Thus, when taken together with the equivalence of (1) and (2), Proposition 6.1 implies that the system of inequalities (6.2) is an appropriate generalization of the Schur inequalities (6.4) to II_1 factors.

7. Proof of the Schur inequalities

We require a convexity inequality for operators in a II_1 factor. While related results can be found in the literature, we have been unable to find references appropriate for this particular result, and we include a proof for completeness. Let \mathcal{A} be a maximal abelian self-adjoint subalgebra of a II_1 factor \mathcal{M} with normalized trace τ , and let $E : \mathcal{M} \rightarrow \mathcal{A}$ be the τ -preserving conditional expectation.

PROPOSITION 7.1. *Let f be a real-valued continuous convex function defined on a compact interval $I = [a, b] \subseteq \mathbb{R}$. Then for every self-adjoint operator $A \in \mathcal{M}$ with spectrum contained in I , the spectrum of $E(A)$ is also contained in I and we have*

$$(7.1) \quad f(E(A)) \leq E(f(A)).$$

PROOF. For every self-adjoint operator $A \in \mathcal{M}$, we write A_+ for the positive part of A , defined by $A_+ = AP_+ = P_+A$ where P_+ is the spectral projection of A associated with the nonnegative real axis $[0, \infty)$. We claim first that

$$(7.2) \quad E(A)_+ \leq E(A_+).$$

Indeed, in the natural ordering of self-adjoint operators in \mathcal{M} we have $A \leq A_+$ and hence $E(A) \leq E(A_+)$. Thus $E(A_+)$ is a positive operator that dominates $E(A)$. Since \mathcal{A} is abelian, $E(A)_+$ is the smallest positive operator in \mathcal{A} that dominates $E(A)$, and (7.2) follows.

In order to prove (7.1), choose $r \in \mathbb{R}$ and let $g_r(\lambda) = \max(\lambda - r, 0)$. We may apply (7.2) to the operator $A - r\mathbf{1}$ to obtain

$$g_r(E(A)) = (E(A) - r\mathbf{1})_+ = E(A - r\mathbf{1})_+ \leq E((A - r\mathbf{1})_+) = E(g_r(A)).$$

It follows that for every convex function $f_0 : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$(7.3) \quad f_0(\lambda) = a + b\lambda + \sum_{k=1}^n c_k g_{r_k}(\lambda)$$

where $a, b, r_1, \dots, r_n \in \mathbb{R}$ and $c_1, \dots, c_n \geq 0$, one has

$$f_0(E(A)) \leq E(f_0(A)).$$

Since every continuous convex function $f : [a, b] \rightarrow \mathbb{R}$ can be uniformly approximated by functions f_0 of the form (7.3), one deduces (7.1) for continuous convex functions from these inequalities. □

The following result establishes the Schur inequalities for operators in a II_1 factor.

THEOREM 7.2. *Let \mathcal{A} be a MASA in \mathcal{M} and let $E : \mathcal{M} \rightarrow \mathcal{A}$ be the trace-preserving conditional expectation. For every self-adjoint operator A in \mathcal{M} , the spectral distribution of $B = E(A)$ is related to that of A by $m_B \preceq m_A$.*

PROOF. Let $[a, b]$ be the smallest closed interval containing $\sigma(A) \cup \sigma(B)$. Since both m_B and m_A are probability measures, Proposition 3.1 implies that $m_B \preceq m_A$ iff for every continuous convex function $f \in C[a, b]$,

$$\int_I f(\lambda) dm_B(\lambda) \leq \int_I f(\lambda) dm_A(\lambda).$$

Since the left side is $\tau(f(B)) = \tau(f(E(A)))$ and the right side is $\tau(f(A))$, the preceding inequality follows from formula (7.1). \square

Theorem 7.2 makes the following assertion about the norm-closed unitary orbit \mathcal{O}_A of a self-adjoint operator: $E(\mathcal{O}_A)$ is contained in the set of all self-adjoint operators $B \in \mathcal{A}$ satisfying $m_B \preceq m_A$. Thus, an affirmative reply to the following question would appear to be a natural generalization of Horn's Theorem for $n \times n$ matrices.

Problem. Let \mathcal{A} be a MASA in a II_1 factor \mathcal{M} , let $E : \mathcal{M} \rightarrow \mathcal{A}$ be the trace-preserving conditional expectation, and let A be a self-adjoint operator in \mathcal{M} . Does $E(\mathcal{O}_A)$ contain the set of all self-adjoint operators $B \in \mathcal{A}$ satisfying $m_B \preceq m_A$?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720
E-mail address: `arveson@mail.math.berkeley.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104
E-mail address: `kadison@math.upenn.edu`