International Journal of Mathematics
Vol. 15, No. 3 (2004) 289–312

ASYMPTOTIC STABILITY I: COMPLETELY POSITIVE MAPS

WILLIAM ARVESON
Department of Mathematics, University of California, Berkeley, CA 94720
arveson@math.berkeley.edu

Received 27 January 2004

We show that for every "locally finite" unit-preserving completely positive map P acting on a C^*-algebra, there is a corresponding $*$-automorphism α of another unital C^*-algebra such that the two sequences $P, P^2, P^3, \ldots$ and $\alpha, \alpha^2, \alpha^3, \ldots$ have the same *asymptotic* behavior. The automorphism α is uniquely determined by P up to conjugacy. Similar results hold for normal completely positive maps on von Neumann algebras, as well as for one-parameter semigroups.

These results are operator algebraic counterparts of the classical theory of Perron and Frobenius on the structure of square matrices with nonnegative entries.

Keywords: Completely positive maps; C^*-dynamical systems.

Mathematics Subject Classification 1991: 46L55, 46L09, 46L40

1. Introduction

The purpose of this paper is to show that many completely positive maps on C^*-algebras, and normal completely positive maps on von Neumann algebras, have significant asymptotic stability properties. This material arose in connection with our work on an asymptotic spectral invariant for single automorphisms of C^*-algebras, and for one-parameter semigroups of endomorphisms of von Neumann algebras. Those applications will be taken up in a subsequent paper. However, since they provide the motivation and conceptual foundation for the discussion below, we offer the following remarks concerning the problem that inspired this work, and describe the connections between the results of this paper and noncommutative dynamics.

Let $n \geq 2$ be a positive integer. The noncommutative Bernoulli shift of rank n is the automorphism of the hyperfinite II_1 factor R that is associated with the bilateral shift acting on the UHF algebra

$$\mathcal{A}_n = \bigotimes_{k=-\infty}^{+\infty} A_k ,$$

each factor A_k being the C^*-algebra of $n \times n$ matrices. The GNS construction applied to the tracial state gives rise to a representation of $\mathcal{A}_n$ whose weak closure is R; and since the shift on $\mathcal{A}_n$ preserves the trace it can be extended naturally to a $*$-automorphism of R, which we denote by σ_n. The problem of whether σ_m is conjugate to σ_n for $m \neq n$ was solved by Connes and Størmer [10] by introducing a noncommutative generalization of the Kolmogorov–Sinai entropy of ergodic theory. The entropy of the noncommutative shift of rank n was computed in [10], and was found to have the value $\log n$, thereby settling the issue of conjugacy of the various shifts σ_n.

We now consider another construction of automorphisms of von Neumann algebras that resembles the construction of finite state Markov processes from their transition probability matrices. This construction begins with a pair (A, P), consisting of a finite dimensional C^*-algebra A and a completely positive linear map $P : A \to A$ satisfying $P(\mathbf{1}) = \mathbf{1}$. For technical reasons we require that $P(e) \neq 0$ for every minimal central projection $e \in A$; there is no essential loss if one thinks of A as a full matrix algebra $M_n(\mathbb{C})$ — and in that case the technical hypothesis is automatically satisfied. There is a "noncommutative Markov process" that can be constructed from (A, P) as follows. Briefly, a dilation theorem of Bhat [6, 7], as formulated in [5, Chap. 8], gives rise to a pair (M_0, σ_0) consisting of a von Neumann algebra M_0 and a normal $*$-endomorphism $\sigma_0 : M_0 \to M_0$ that is appropriately related to the pair (A, P). The technical hypothesis implies that σ_0 is isometric, and one may then show that the endomorphism σ_0 can be extended appropriately to a $*$-automorphism of a larger von Neumann algebra $M \supseteq M_0$. Let us denote the latter automorphism by σ^P.

By analogy with the theory of noncommutative Bernoulli shifts, we were led to conjecture that σ^P and σ^Q were generically not conjugate. The proof of that called for a new invariant, since the Connes–Størmer entropy is inappropriate for two reasons. First, σ^P typically acts on a type I von Neumann algebra M, and second, the construction of M involves free products of copies of A and not tensor products [3], (see [19] for the significance of that fact). There is an asymptotic invariant for automorphisms α of von Neumann algebras (as well as for C^*-algebras) that we call the *asymptotic spectrum* $Sp_\infty(\alpha)$. We do not define $Sp_\infty(\alpha)$ here, but we do point out that it is a subset of the unit circle, perhaps finite, and is typically not closed. The fact is that the asymptotic spectrum is relatively easy to compute and serves to distinguish between the various σ^P. The key result on the computation of the asymptotic spectrum is the following.

Theorem 1.1. *Let (A, P) be a pair consisting of a von Neumann algebra A and a normal completely positive map $P : A \to A$ satisfying $P(\mathbf{1}) = \mathbf{1}$, and let (M, σ^P) be the associated W^*-dynamical system. Then under appropriate hypotheses that include all pairs (A, P) with finite-dimensional A, one has*

$$SP_\infty(\sigma^P) = \sigma_p(P) \cap \mathbb{T}\,,$$

where $\sigma_p(P)$ denotes the point spectrum of P.

Thus, σ^P and σ^Q are not conjugate whenever P and Q have a different set of eigenvalues on the unit circle, and hence there is a continuum of non-conjugate automorphisms σ^P. The proof of Theorem 1.1 has two components: (a) the development of properties of $Sp_\infty(\alpha)$, and (b) an analysis of the asymptotic stability properties of completely positive maps. In this paper we concentrate on (b), the discussion (a) will be taken up elsewhere.

We now describe the contents of this paper in somewhat more detail. We are concerned with the *asymptotic* behavior of the powers of a completely positive map $P : A \to A$, where A is either a C^*-algebra or a von Neumann algebra (in which case P is assumed to be normal). In order to illustrate the simplest case of that phenomenon, consider a normal completely positive map P of $\mathcal{B}(H)$ into itself satisfying $P(\mathbf{1}) = \mathbf{1}$. In this setting there is a natural generalization of the ergodic-theoretic notion of mixing: P is said to be *mixing* if there is a normal state ω on $\mathcal{B}(H)$ with the property

$$\lim_{n\to\infty} \omega(AP^n(B)) = \omega(A)\omega(B)\,, \quad A, B \in \mathcal{B}(H)\,. \tag{1.1}$$

If such an ω exists, then it must be invariant in the sense that $\omega \circ P = \omega$; indeed, ω is the *unique* normal P-invariant state. In case ω is faithful, it is not hard to see that (1.1) is equivalent to the following somewhat stronger absorption property: for every normal state ρ on $\mathcal{B}(H)$ one has

$$\lim_{n\to\infty} \|\rho \circ P^n - \omega\| = 0\,. \tag{1.2}$$

The formula (1.2) represents the simplest form of the asymptotic stability that such maps P can have. Our objective is to develop a generalization of this kind of stability that is flexible enough to apply to a broad class of completely positive maps on von Neumann algebras (Theorem 5.4) and C^*-algebras (Theorem 4.1).

In order to keep the discussion as simple and focused as possible, we shall fix attention on pairs (A, P) consisting of a C^*-algebra A (perhaps without unit) and a completely positive map $P : A \to A$ satisfying $\|P\| = 1$, making occasional comments about how formulations must be modified for normal maps on von Neumann algebras. In this case, the idea of asymptotic stability is formulated as follows.

The most rigid completely positive maps are called *quasiautomorphisms* below. Roughly speaking, a quasiautomorphism is a completely positive contraction $Q : A \to A$ whose behavior away from its null space $\ker Q = \{z \in A : Q(z) = 0\}$ is identical to that of a $*$-automorphism α of a secondary C^*-algebra B in the following sense: the powers of α and the powers of Q can be related to each other by a pair of completely positive contractions $\theta : A \to B$ and $\theta_* : B \to A$ satisfying $\theta \circ \theta_* = \mathrm{id}_B$ (see Definition 3.3). Significantly, when a C^*-dynamical system (B, α) is related in this way to Q then it is uniquely determined by Q up to conjugacy. We consider that a pair (A, P) is *asymptotically stable* if there is a (necessarily unique) quasiautomorphism $Q : A \to A$ with the following property:

$$\lim_{n\to\infty} \|P^n(a) - Q^n(a)\| = 0\,, \quad a \in A\,. \tag{1.3}$$

See Theorem 4.1. Given the relation between Q and (B, α) described above, one may conclude that when (A, P) is stable in the sense of (1.3), then the *asymptotic* properties of the sequence $P, P^2, P^3, \ldots$ are identical with the asymptotic properties of the sequence of automorphisms $\alpha, \alpha^2, \alpha^3, \ldots$. In particular, once one knows the C^*-dynamical system (B, α), one knows everything about the asymptotics of (A, P). This stability result for completely positive maps on C^*-algebras generalizes the classical Perron–Frobenius theorem on the structure of square matrices with nonnegative entries. The connection between the stability assertion (1.3) and the Perron–Frobenius theorem is discussed more fully in Remark 4.2.

The appropriate formulation of asymptotic stability for normal completely positive maps $P : M \to M$ on von Neumann algebras differs significantly from the formulation C^*-algebras, since it involves elements $\rho \in M_*$ of the predual rather than elements $a \in M$. The appropriate formulation is this: there should be a unique *normal* quasiautomorphism $Q : M \to M$ with the property that for every normal linear functional $\rho \in M_*$, one has

$$\lim_{n\to\infty} \|\rho \circ P^n - \rho \circ Q^n\| = 0\,. \tag{1.4}$$

In this case, there is a W^*-dynamical system (N, α) associated with Q as in the case of C^*-algebras, and which is unique up to conjugacy.

With this formulation of stability for von Neumann algebras, simple mixing of the type (1.2) becomes a special case of (1.4) as follows. Let $P : \mathcal{B}(H) \to \mathcal{B}(H)$ be a normal completely positive unit-preserving map for which there is a normal state ω on $\mathcal{B}(H)$ satisfying the strong mixing requirement (1.2). Let Q be the normal map of $\mathcal{B}(H)$ defined by $Q(x) = \omega(x)\mathbf{1}$, $x \in \mathcal{B}(H)$. Then Q is a quasiautomorphism with range $\mathbb{C} \cdot \mathbf{1}$, having the property that for every normal state ρ of $\mathcal{B}(H)$,

$$\rho \circ Q^n = \omega\,, \quad n = 1, 2, \ldots\,.$$

It follows that (1.2) and (1.4) make the same assertion in this case. The W^*-dynamical system associated with this Q is the trivial one $(\mathbb{C}, \mathrm{id})$, id denoting the identity automorphism of the one-dimensional von Neumann algebra $\mathbb{C}$.

These developments rest on some very general results for contractions acting on Banach spaces, and logic requires that we first work out this basic material in Sec. 2. We discuss the properties of quasiautomorphisms in Sec. 3, and then give applications to C^*-algebras and von Neumann algebras in Secs. 4–5, including examples. Similar results are valid for one-parameter semigroups, though beyond a few basic considerations in Sec. 6, applications to semigroups are not developed here.

The main hypothesis invoked in Theorems 4.1 and 5.4 is not necessary for the main conclusion concerning stability, and it is reasonable to ask if *every* normal unit-preserving completely positive map on $\mathcal{B}(H)$ is stable. The purpose of the last section is to show that there is a naturally-occurring class of such maps that are unstable.

2. Locally Finite Contractions

In this section we establish a general result in the category of Banach spaces, with contractions as maps. A *contraction* is an operator $T \in \mathcal{B}(X)$ acting on a complex Banach space X that satisfies $\|T\| \leq 1$. We are concerned with the structure of contractions and with the asymptotic properties of their associated semigroups $\mathbf{1}, T, T^2, \ldots$. By an *automorphism* we mean an invertible isometry $U \in \mathcal{B}(X)$.

Remark 2.1. Given a pair of Banach spaces X_1, X_2 and an automorphism $U \in \mathcal{B}(X_1)$, there are many ways to introduce a norm on the algebraic direct sum $X_1 \dot{+} X_2$ so as to obtain a Banach space $X_1 \oplus X_2$ with the property $\|x_1\| \leq \|x_1 + x_2\|$ for all $x_k \in X_k$. Settling on one of these norms, one can then define a contraction $T \in \mathcal{B}(X_1 \oplus X_2)$ as the direct sum of operators $T = U \oplus \mathbf{0}$, $\mathbf{0}$ denoting the zero operator on X_2. This is the most general example of a quasiautomorphism, a concept defined more concisely in operator-theoretic terms as follows.

Definition 2.2. A contraction $T \in \mathcal{B}(X)$ is called a *quasiautomorphism* if the restriction of T to its range TX is an automorphism of TX.

The range of a quasiautomorphism is necessarily closed, and every quasiautomorphism T admits a unique "polar decomposition" $T = UE$, where E is an idempotent contraction with range TX, and U is an automorphism of TX. Indeed, U is the restriction of T to its range, and E is the composition $U^{-1}T$. The projection E commutes with T, and we have $T^n = U^n E$ for every $n = 1, 2, 3, \ldots$. Quasiautomorphisms are in an obvious sense the most rigid contractions.

We emphasize that the term quasiautomorphism will be used below in other categories with more structure, and the attributes of quasiautomorphisms vary from one category to another. For example, when we work with completely positive maps on C^*-algebras in Sec. 4, quasiautomorphisms inherit the properties of maps in that category.

The purpose of this section is to relate the *asymptotic* behavior of a broad class of contractions to that of quasiautomorphisms. These are the *locally finite* contractions, whose action on vectors is characterized as follows. The linear span of a finite set of vectors $x_1, \ldots, x_n \in X$ is denoted by $[x_1, \ldots, x_n]$.

Proposition 2.3. *Let T be a contraction on a Banach space X. For every vector $x \in X$, the following are equivalent*:

(i) *For every $\epsilon > 0$, there is a positive integer N such that*

$$\operatorname{dist}(T^n x, [x, Tx, T^2 x, \ldots, T^N x]) \leq \epsilon\,, \quad n = 0, 1, 2, \ldots .$$

(ii) *For every $\epsilon > 0$, there is a finite-dimensional subspace $F \subseteq X$ such that*

$$\operatorname{dist}(T^n x, F) \leq \epsilon\,, \quad n = 0, 1, 2, \ldots .$$

(iii) *The norm-closure of the orbit $\{x, Tx, T^2 x, \ldots\}$ is compact.*

Proof. The implication (i)⇒(ii) is trivial.

(ii)⇒(iii): It suffices to show that for every $\epsilon > 0$, the orbit of x $O_x = \{x, Tx, T^2x, \ldots\}$ can be covered by a finite union of balls of radius ϵ. To that end, fix $\epsilon > 0$ and let F be a finite dimensional subspace of X such that every point of O_x is within $\epsilon/2$ of $K = \{f \in F : \|f\| \leq \|x\| + 1\}$. Since K is compact it may be covered by a finite union of balls $B_1 \cup \cdots \cup B_r$ of radius at most $\epsilon/2$. After doubling the radius of each ball B_i, one obtains a finite union of balls of radius ϵ that covers O_x.

(iii)⇒(i): Fix $\epsilon > 0$. Since O_x is dense in its closure and its closure is compact, there is an $N \geq 1$ such that every point of O_x is within ϵ of $\{x, Tx, T^2x, \ldots, T^Nx\}$, and (i) follows. □

Definition 2.4. A contraction T acting on a Banach space X is called *locally finite* if every vector $x \in X$ satisfies the conditions of Proposition 2.3.

Remark 2.5. A vector $x \in X$ that is *algebraic* in the sense that $p(T)x = 0$ for some nonzero polynomial $p(z)$ obviously satisfies condition (i). A vector x will satisfy (ii) when it remains localized under the action of the nonnegative powers of T in the sense that no subsequence of $x, Tx, T^2x, \ldots$ can wander in an essential way through infinitely many dimensions.

A straightforward argument shows that in general, the set of vectors that satisfy condition (iii) of Proposition 2.3 is a closed linear subspace of X that is invariant under the set of all operators in $\mathcal{B}(X)$ that commute with T. It follows that T will be locally finite if, for example, the set of all algebraic vectors has X as its closed linear span.

We fix attention on eigenvectors x of T whose eigenvalues have maximum absolute value: $Tx = \lambda x$, where $|\lambda| = 1$. Such an x is called a *maximal eigenvector*. The *point spectrum* of an operator T is the set of all eigenvalues of T, written $\sigma_p(T)$, and of course the point spectrum can be empty. An invertible isometry $U \in \mathcal{B}(X)$ is said to be *diagonalizable* if X is spanned by the (necessarily maximal) eigenvectors of U, and for such operators $\sigma_p(U)$ is dense in the spectrum of U. Our use of the term *diagonalizable* is not universal; for example, diagonalizable unitary operators are often said to have *pure point spectrum*. Nevertheless, this terminology will be convenient. The asymptotic behavior of locally finite contractions is described as follows.

Theorem 2.6. *For every locally finite contraction T acting on a Banach space X there is a unique quasiautomorphism $S \in \mathcal{B}(X)$ such that*

$$\lim_{n\to\infty} \|T^n x - S^n x\| = 0\,, \quad x \in X\,. \tag{2.1}$$

The restriction U of $S = UE$ to its range is diagonalizable, and we have

$$\sigma_p(U) = \sigma_p(T) \cap \mathbb{T}\,, \tag{2.2}$$

where $\mathbb{T}$ is the unit circle. The projection E is characterized as the unique idempotent in the set $\mathcal{L}$ of strong limit points of the powers of T

$$\mathcal{L} = \bigcap_{n=1}^{\infty} \{T^n, T^{n+1}, T^{n+2}, \dots\}^{-\text{strong}}, \tag{2.3}$$

and U is the restriction of T to EX.

Remark 2.7. Equation (2.1) asserts that a locally finite contraction has the same asymptotic behavior as an automorphism. According to (2.2), the point spectrum of U consists of all eigenvalues of T that are associated with maximal eigenvectors. A consequence of the characterization (2.3) is that the projection E will share the salient features of $\{T^n : n \geq 1\}$. For example, when T is a completely positive contraction acting on a C^*-algebra whose powers do *not* tend to zero in the strong operator topology, then E will be a completely positive idempotent of norm 1.

The proof of Theorem 2.6 will make use of the following known result from the theory of almost periodic representations of groups — in our case the group is $\mathbb{Z}$. That material generalizes work of Harald Bohr (for the group $\mathbb{R}$ [8]) to arbitrary groups, the generalization being due to von Neumann and others (see [14, pp. 245–261], and [13, pp. 310–312]).

Lemma 2.8. *Let U be an invertible isometry acting on a Banach space X, and suppose that the $\mathbb{Z}$-orbit $\{U^n x : n \in \mathbb{Z}\}$ of every vector $x \in X$ is relatively norm-compact. Then U is diagonalizable.*

We also require the following observation.

Lemma 2.9. *Let T be a contraction on a Banach space X such that the identity operator belongs to the strong closure of $\{T, T^2, T^3, \dots\}$. Then T is an automorphism of X.*

Proof. For each $x \in X$ there is a sequence $n_1, n_2, \dots$ of positive integers such that $T^{n_k} x \to x$, and therefore $\|T^{n_k} x\| \to \|x\|$, as $k \to \infty$. Since $n_k \geq 1$, for each k it follows that $\|Tx\| \geq \|x\|$, hence $\|Tx\| = \|x\|$. Similarly, x belongs to the closure of TX. These observations show that T is an isometry with dense range, hence it is invertible. □

Proof of Theorem 2.6. Let $G \subseteq X$ be the (conceivably empty) set of all maximal eigenvectors of T and let $\tilde{T}$ be the restriction of T to $M = \overline{\operatorname{span}}\, G$.

We claim first that $\tilde{T}$ is an invertible isometry. According to Lemma 2.9, that will follow if we prove that the identity operator $\mathbf{1}_M$ of M belongs to the strong closure of $\{\tilde{T}, \tilde{T}^2, \tilde{T}^3, \dots\}$. For that, let $\{x_1, \dots, x_r\}$ be a finite subset of G. It suffices to show that there is an increasing sequence $n_1 < n_2 < \cdots$ of integers such that

$$\lim_{k\to\infty} \|T^{n_k} x_i - x_i\| = 0, \quad i = 1, \dots, r. \tag{2.4}$$

Noting that $Tx_i = \lambda_i x_i$ for $\lambda_1, \dots, \lambda_r \in \mathbb{T}$, we make use of a familiar result from Diophantine analysis which asserts that for every finite choice of elements $\lambda_1, \dots, \lambda_r$ in the multiplicative group $\mathbb{T}$, there is an increasing sequence $n_1 < n_2 < \dots \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \lambda_i^{n_k} = 1\,, \quad i = 1, \dots, r\,.$$

This sequence $n_1, n_2, \dots$ obviously satisfies (2.4). Thus Lemma 2.9 implies that $\tilde{T}$ is an invertible isometry.

Now let N be the asymptotic null space

$$N = \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}\,.$$

We will show that N and M are complementary subspaces in the sense that $N \cap M = \{0\}$ and $N + M = X$.

Indeed,

$$N \cap M = \{0\} \tag{2.5}$$

follows from the preceding paragraph. For if if z is a vector in $N \cap M$ then since T restricts to an isometry on M we have $\|z\| = \|T^n z\|$ for every $n = 1, 2, \dots$, while since $z \in N$ we have $\|T^n z\| \to 0$ as $n \to \infty$. Hence $z = 0$.

For every $x \in X$ consider the set of limit points

$$K_\infty(x) = \bigcap_{n=1}^{\infty} \overline{\{T^n x, T^{n+1}x, T^{n+2}x, \dots\}},$$

the bar denoting closure in the norm of X. We claim that $K_\infty(x) \subseteq M$ for every $x \in X$. To prove that, fix x and choose $z \in K_\infty(x)$. We claim first that there is a sequence $n_1 < n_2 < \dots$such that

$$\lim_{k\to\infty} T^{n_k} z = z, \tag{2.6}$$

the convergence being in norm. Indeed, by definition of $K_\infty(x)$ there is a sequence $m_1 < m_2 < \dots$ such that $T^{m_k}x$ converges to z. We may assume that the m_k increase as rapidly as desired by passing to a subsequence, and we choose m_k so that the sequence of differences $n_k = m_{k+1} - m_k$ increases to ∞. Writing $z = T^{m_k}x + (z - T^{m_k}x)$ and estimating in the obvious way, we obtain

$$\begin{aligned}\|T^{n_k}z - z\| &\le \|T^{n_k}T^{m_k}x - T^{m_k}x\| + 2\|z - T^{m_k}x\| \\ &= \|T^{m_{k+1}}x - T^{m_k}x\| + 2\|z - T^{m_k}x\|\,,\end{aligned}$$

hence $\|T^{n_k}z - z\| \to 0$ as $k \to \infty$. Keeping $z \in K_\infty(x)$ fixed and choosing a sequence $n_1 < n_2 < \dots$ satisfying (2.6), we consider the set of vectors

$$M_z = \{y \in X : \lim_{k\to\infty} T^{n_k} y = y\}\,.$$

M_z is a closed linear subspace of X that contains z, and it is invariant under all operators that commute with T. We claim that the restriction $\tilde{T}$ of T to M_z is an invertible isometry. Indeed, from the definition of M_z it follows that $\tilde{T}^{n_k}$ converges strongly to the identity operator of M as $k \to \infty$, hence the assertion follows after another application of Lemma 2.9. Noting that $\tilde{T}^{-k}$ belongs to the strong closure of $\{\tilde{T}, \tilde{T}^2, \tilde{T}^3, \ldots\}$ for every integer $k \geq 0$, it follows that $\tilde{T}$ satisfies the hypotheses of Lemma 2.8. We conclude from that result that M_z is spanned by maximal eigenvectors of T, and is therefore a subspace of M. In particular, $z \in M$.

We show now that $N + M = X$. Choose $x \in X$. We will exhibit a vector $e \in M$ such that $\|T^n x - T^n e\| \to 0$ as $n \to \infty$. Indeed, the distance from $T^n x$ to $K_\infty(x)$ must decrease to zero as $n \to \infty$ because $K_\infty(x)$ is the intersection of the decreasing sequence of *compact* sets $\{T^n x, T^{n+1}x, T^{n+2}x, \ldots\}^-$. Thus there is a sequence $k_n \in K_\infty(x)$ with the property $\|T^n x - k_n\| \to 0$ as $n \to \infty$. We have shown above that $K_\infty(x) \subseteq M$ and that the restriction of T to M is an isometry. It follows the restriction of T to the compact metric space $K_\infty(x)$ defines an isometry of metric spaces. Since by the definition of $K_\infty(x)$ it is clear that $TK_\infty(x)$ is dense in $K_\infty(x)$, it follows that $TK_\infty(x) = K_\infty(x)$. So for every n there is an element $\ell_n \in K_\infty(x)$ such that $k_n = T^n \ell_n$ and $\|k_n\| = \|\ell_n\|$, hence

$$\lim_{n\to\infty} \|T^n x - T^n \ell_n\| = 0\,. \tag{2.7}$$

Finally, by compactness of $K_\infty(x)$ there is a subsequence $m_1 < m_2 < \ldots$ such that $\ell_{m_k} \to e \in K_\infty(x)$ as $k \to \infty$. From (2.7), we deduce that

$$\lim_{k\to\infty} \|T^{m_k}x - T^{m_k}e\| = \lim_{k\to\infty} \|T^{m_k}x - T^{m_k}\ell_{m_k}\| = 0\,.$$

It follows that $\lim_{k\to\infty} \|T^{m_k}(x - e)\| = 0$. This implies that $x - e \in N$ because the sequence of norms $\|T^n(x - e)\|$ decreases with n. Thus $x = (x - e) + e$ is exhibited as an element of $N + M$.

Let E be the idempotent defined by $E \restriction_N = 0$ and $Ey = y$ for $y \in M$. We claim that E is a strong limit point of the sequence $\{T, T^2, T^3, \ldots\}$ of powers of T. For that, it suffices to show that for every integer $N \geq 1$, every $\epsilon > 0$, every finite set $x_1, \ldots, x_n$ of maximal eigenvectors of T, and every finite set $z_1, \ldots, z_m \in N$, there is an integer $p \geq N$ such that

$$\|T^p x_k - x_k\| \leq \epsilon \quad \text{and} \quad \|T^p z_j\| \leq \epsilon\,, \quad 1 \leq k \leq n\,, \quad 1 \leq j \leq m\,. \tag{2.8}$$

Writing $Tx_k = \lambda_k x_k$, $1 \leq k \leq n$, the Diophantine approximation employed above shows that there is an infinite set of positive integers p such that $|\lambda_1^p - 1| \leq \epsilon, \ldots, |\lambda_n^p - 1| \leq \epsilon$. Since $\|T^p z_1\|, \ldots, \|T^p z_m\|$ all tend to zero with large p, it is apparent that we can satisfy (2.8) with infinitely many values of p. This shows that E is a strong cluster point of $\{T^n : n \geq 1\}$, and in particular $\|E\| \leq 1$.

If we set $S = TE$, then we may conclude from the preceding discussion that S is a quasiautomorphism satisfying both (2.1) and (2.2).

We claim now that E is the only idempotent that can be a strong cluster point of $\{T^n : n \geq 1\}$. Let F be such another such limit point. Since both E and F are idempotents, to show that $F = E$ it suffices to show that $\ker F \subseteq \ker E = N$ and

$FX \subseteq EX = M$. If z is any vector in the kernel of F, then there is a sequence $n_k \to \infty$ such that $T^{n_k}z \to Fz = 0$, hence $\|T^{n_k}z\| \to 0$, as $k \to \infty$. The latter implies that $\lim_{n\to\infty} \|T^n z\| = 0$ since the norms $\|T^n z\|$ decrease with n, hence $z \in N$. If y is a vector in the range of F then there is a sequence $m_k \to \infty$ such that $T^{m_k}y \to Fy = y$, and in particular $y \in K_\infty(y)$. We have already proved that $K_\infty(y) \subseteq M$, and therefore $y \in M = EX$.

It remains to show that $S = TE$ is the only quasiautomorphism in $\mathcal{B}(X)$ that satisfies (2.1). Let $R = UF$ be the polar decomposition of another quasiautomorphism such that $\lim_{n\to\infty} \|S^n x - R^n x\| = 0$ for every $x \in X$. We claim that $F = E$ and $U = T\restriction_M$. Since both E and F are idempotents, the first assertion will follow if we show that $N = \ker E \subseteq \ker F$ and that $M = EX \subseteq FX$. Indeed, if $x \in \ker E$ then (2.1) implies that

$$\|Fx\| = \lim_{n\to\infty} \|U^n Fx\| = \lim_{n\to\infty} \|T^n Ex - U^n Fx\| = \lim_{n\to\infty} \|S^n x - R^n x\| = 0\,,$$

hence $x \in \ker F$. $M \subseteq FX$ will follow if we show that every maximal eigenvector x for T belongs to the range of F. Writing $Tx = Sx = \lambda x$ for some $\lambda \in \mathbb{T}$, we have $\|x - \bar{\lambda}^n R^n x\| = \|S^n x - R^n x\| \to 0$ as $n \to \infty$, which implies that the distance from x to $FX = RX$ is zero. Finally, to show that $U = T\restriction_M$, choose $x \in FX = EX$ and write

$$\begin{aligned}\|Tx - Ux\| &= \|U^n(Tx - Ux)\| = \|U^n Tx - S^{n+1}x + (S^{n+1}x - U^{n+1}x)\| \\ &\le \|R^n Tx - S^n Tx\| + \|S^{n+1}x - R^{n+1}x\|\,.\end{aligned}$$

As $n \to \infty$, both terms on the right tend to zero by hypothesis. It follows that $\|Tx - Ux\| = 0$, and therefore $U = T\restriction_{EX}$. □

Remark 2.10. Perhaps it is worth pointing out that the set $\mathcal{L}$ of strong cluster points (2.3) is compact in its relative strong operator topology. Indeed, $\mathcal{L}$ is a compact topological group with respect to operator multiplication, whose unit is E. Since we do not require this fact, we omit the proof.

3. Quasiautomorphisms of C^*-Algebras

The notion of quasiautomorphism must be interpreted appropriately when it is applied to completely positive maps on C^*-algebras. The purpose of this section is to make some observations that show how a quasiautomorphism of a C^*-algebra can be related to to an ordinary $*$-automorphism of a different C^*-algebra; and that in fact the C^*-dynamical system associated with the quasiautomorphism is unique up to conjugacy.

By a *CP contraction* we mean a completely positive linear map $P : A \to A$ defined on a C^*-algebra A such that $\|P\| \le 1$. If A has a unit $\mathbf{1}$, then a completely positive map $P : A \to A$ is a CP contraction if and only if $\|P(\mathbf{1})\| \le 1$; but in general, we may speak of CP contractions even when A fails to posses a unit. Throughout the section, A will denote a C^*-algebra. We first show how, starting with an automorphism of another C^*-algebra that is suitably related to A, one obtains a CP contraction on A with special features. We use the term *C^*-dynamical system* to denote a pair (B, β) consisting of a C^*-algebra B and a $*$-automorphism $\beta : B \to B$. Two C^*-dynamical systems (B_1, β_1) and (B_2, β_2) are said to be *conjugate* if there is a $*$-isomorphism $\theta : B_1 \to B_2$ satisfying $\theta \circ \beta_1 = \beta_2 \circ \theta$.

Proposition 3.1. *Let (B, β) be a C^*-dynamical system and let $\theta : A \to B$, $\theta_* : B \to A$ be a pair of completely positive contractions satisfying*

$$\theta \circ \theta_* = \mathrm{id}_B .$$

Let $P : A \to A$ be the CP contraction defined by

$$P = \theta_* \circ \beta \circ \theta \,. \tag{3.1}$$

Then $E = \theta_* \circ \theta$ is an idempotent CP contraction on A with range $P(A)$, $\ker E = \ker P$, $PE = EP = P$, and the restriction α of P to $E(A)$ is a surjective completely isometric map with the property $P^n = \alpha^n \circ E$, for every $n = 1, 2, \ldots$.

If $(\tilde{B}, \tilde{\beta})$ is another C^*-dynamical system that is similarly related to P, $P = \tilde{\theta} \circ \tilde{\beta} \circ \tilde{\theta}_*$, where $\tilde{\theta} : A \to \tilde{B}$ and $\tilde{\theta}_* : \tilde{B} \to A$ are completely positive contractions with $\tilde{\theta} \circ \tilde{\theta}_* = \mathrm{id}_{\tilde{B}}$, then the C^*-dynamical systems $(\tilde{B}, \tilde{\beta})$ and (B, β) are naturally conjugate.

Proof. Since $\theta \circ \theta_* = \mathrm{id}_B$, it follows that θ is surjective, θ_* is injective, and

$$E^2 = \theta_* \circ \theta \circ \theta_* \circ \theta = \theta_* \circ \theta = E \,.$$

Since θ and $\beta \circ \theta$ are both surjective,

$$\mathrm{ran} P = \theta_*(\beta(\theta(A))) = \theta_*(B) = \theta_*(\theta(A)) = \mathrm{ran} E \,; \tag{3.2}$$

and since θ_* and $\theta_* \circ \beta$ are both injective,

$$\ker P = \ker \theta_* \circ \beta \circ \theta = \ker \theta = \ker \theta_* \circ \theta = \ker E \,. \tag{3.3}$$

Similarly, one verifies directly that the restriction α of P to $E(A)$ satisfies $\alpha^n(E(a)) = \theta_* \circ \beta^n \circ \theta(E(a))$ for $a \in A$, $n = 1, 2, \ldots$. In particular, α is the restriction of $\theta_* \circ \beta \circ \theta$ to $E(A)$, a completely isometric surjective map of $E(A)$ onto itself.

Suppose that $(\tilde{B}, \tilde{\beta})$ is another C^*-dynamical system and $\tilde{\theta} : A \to B$ and $\tilde{\theta}_* : B \to A$ are CP contractions satisfying $\tilde{\theta} \circ \tilde{\theta}_* = \mathrm{id}_{\tilde{B}}$ and $P = \tilde{\theta}_* \circ \tilde{\beta} \circ \tilde{\theta}$. In this case we have a second CP idempotent $\tilde{E} : A \to A$ defined by $\tilde{E} = \tilde{\theta}_* \circ \tilde{\theta}$, and we claim that $\tilde{E} = E$. Indeed, since both $\tilde{E}$ and E are idempotents it suffices to show that they have the same kernel and the same range; and (3.2) and (3.3) imply that $\ker \tilde{E} = \ker P = \ker E$ and $\mathrm{ran}\tilde{E} = \mathrm{ran}P = \mathrm{ran}E$.

We define CP contractions $\phi : B \to \tilde{B}$ and $\tilde{\phi} : \tilde{B} \to B$ by

$$\phi = \tilde{\theta} \circ \theta_* \,, \quad \tilde{\phi} = \theta \circ \tilde{\theta}_* \,.$$

We have

$$\phi \circ \tilde{\phi} = \tilde{\theta} \circ \theta_* \circ \theta \circ \tilde{\theta}_* = \tilde{\theta} \circ E \circ \tilde{\theta}_* = \tilde{\theta} \circ \tilde{E} \circ \tilde{\theta}_* = \mathrm{id}_{\tilde{B}}^2 = \mathrm{id}_{\tilde{B}}\,.$$

Similarly, $\tilde{\phi} \circ \phi = \mathrm{id}_B$, so that the maps ϕ, $\tilde{\phi}$ are completely isometric completely positive maps that are inverse to each other. Since B and $\tilde{B}$ are both C^*-algebras, we may conclude that ϕ is a $*$-isomorphism of B onto $\tilde{B}$ with inverse $\tilde{\phi}$.

It remains to show that $\phi \circ \beta = \tilde{\beta} \circ \phi$. Composing the identity

$$\theta_* \circ \beta \circ \theta = \tilde{\theta}_* \circ \tilde{\beta} \circ \tilde{\theta} = P\,,$$

on the left with $\tilde{\theta}$ gives

$$\phi \circ \beta \circ \theta = \tilde{\theta} \circ \tilde{\theta}_* \circ \tilde{\beta} \circ \tilde{\theta} = \tilde{\beta} \circ \tilde{\theta}\,,$$

and after composing with θ_* on the right we obtain $\theta \circ \beta = \tilde{\beta} \circ \theta$. □

Proposition 3.2. *For every CP contraction $P : A \to A$ on a C^*-algebra A, the following are equivalent.*

(i) *P admits a factorization $P = \alpha \circ E$ where $E : A \to A$ is an idempotent completely positive contraction and α is a completely isometric linear map of $E(A)$ onto itself.*
(ii) *There is a C^*-dynamical system (B, β) that is related to P as in* (3.1).

If A has a unit $\mathbf{1}$ and $P(\mathbf{1}) = \mathbf{1}$, then (i) *can be replaced with*

(i)′ *The restriction of P to $P(A)$ is a surjective complete isometry.*

Proof. The implication (ii)⇒(i) follows from Proposition 3.1.

(i)⇒(ii): The hypothesis (i) obviously implies that $EP = P = PE$.

A result of Choi and Effros [9] implies that $E(A)$ is a C^*-algebra with respect to the multiplication defined on it by $x \bullet y = E(xy)$, for $x, y \in E(A)$ (one uses the norm of A and the vector space operations and $*$-operation inherited from A). Let B be this C^*-algebra.

We may consider E as a completely positive contraction of A onto B; let θ be that map, and let θ_* be the natural inclusion of $B = E(A) \subseteq A$. Obviously, $\theta \circ \theta_* = \mathrm{id}_B$. By hypothesis, the restriction of P to $E(A)$ is a surjective completely isometric map of the operator space $E(A)$ onto itself, and therefore it defines a completely isometric linear map B onto itself, which we denote by β. We have to show that β is a $*$-automorphism of B and the maps θ, θ_* relate β to P as in (3.1).

We claim first that β is also a positive linear map on B, i.e. $\beta(x^* \bullet x) \geq 0$ for every $x \in E(A)$. To see that, fix $x \in E(A)$, choose a positive linear functional ρ on B, and consider the linear functional defined on A by $\omega(a) = \rho(\theta(a)) = \rho(E(a))$, $a \in A$. ω is a positive linear functional on A, hence

$$\rho(\beta(x^* \bullet x) = \rho(P(E(x^*x))) = \rho(E(P(x^*x))) = \omega(P(x^*x)) \geq 0\,.$$

Since ρ is an arbitrary positive linear functional on B, $\beta(x^* \bullet x) \geq 0$ follows, hence β is a positive linear map. An obvious variation of this argument (that we omit) shows that β induces a positive linear map on every matrix algebra $M_n \otimes B$ over B, hence β is a completely positive linear map that is also completely isometric. It follows that β is a $*$-automorphism of B.

Finally, to check that (3.1) is satisfied, we have

$$\theta_* \circ \beta \circ \theta(a) = P(E(a)) = P(a)\,, \quad a \in A\,,$$

since $P \circ E = \alpha \circ E = P$.

Finally, assuming that A has a unit $\mathbf{1}$ and $P(\mathbf{1}) = \mathbf{1}$, then $E(A)$ is an operator system, and a unit-preserving linear map of one operator system to another is completely positive if and only if it is completely contractive [1]. So in this case (i)$'$ is equivalent to (i). □

Definition 3.3. A completely positive contraction $P : A \to A$ is called a *quasiautomorphism* if the conditions of Proposition 3.2 are satisfied.

Some concrete examples of quasiautomorphisms are given in Sec. 4. The preceding remarks support the following point of view: *the nontrivial behavior of the powers of a quasiautomorphism is identical with the behavior of the powers of a uniquely determined automorphism of a C^*-algebra.*

4. Applications to C^*-Algebras

In this section we describe an application of Theorem 2.6 to completely positive maps on C^*-algebras and describe how that result provides a noncommutative generalization of the Perron–Frobenius theorem. We also exhibit a variety of examples of locally finite completely positive maps on infinite-dimensional C^*-algebras.

The term *locally finite* applies to completely positive contractions exactly as stated in Definition 2.4, and of course in this context the strong operator topology is the topology of point-norm convergence: a net of linear maps $L_i : A \to A$ converges strongly to a linear map $L : A \to A$ if and only if one has

$$\lim_{i\to\infty} \|L_i(a) - L(a)\| = 0\,, \quad a \in A\,.$$

Theorem 4.1. *Let A be a C^*-algebra and let $P : A \to A$ be a locally finite completely positive contraction. Then there is a unique quasiautomorphism $Q = \alpha \circ E$ of A such that*

$$\lim_{n\to\infty} \|P^n(a) - Q^n(a)\| = 0\,, \quad a \in A\,. \tag{4.1}$$

The completely positive idempotent E is characterized as the unique idempotent in the set of strong cluster points of $\{P, P^2, P^3, \ldots\}$, α is the restriction of P to the operator space $E(A)$, and $E(A)$ is the norm-closed linear span of the set of all maximal eigenvectors of P.

Proof. By Theorem 2.6, there is a unique idempotent E in the set of strong limit points of $\{P, P^2, \ldots\}$. Being a limit in the strong operator topology of a net of powers of P, E is a completely positive contraction. Moreover, the general results of Theorem 2.6 imply that P restricts to an isometry α of $E(A)$ onto itself in such a way that the map $Q = \alpha \circ E$ satisfies

$$\lim_{n\to\infty} \|P^n(a) - Q^n(a)\| = 0\,, \quad a \in A\,.$$

We claim that α is a *completely* isometric linear map of $E(A)$ onto itself. Indeed, for fixed $n = 2, 3, \ldots$, consider the map of $M_n \otimes A$ defined on $n \times n$ matrices over A by

$$\mathrm{id}_n \otimes P : (a_{ij}) \mapsto (P(a_{ij}))\,.$$

$\mathrm{id}_n \otimes P$ satisfies the same hypotheses as P, and since $\mathrm{id}_n \otimes \alpha$ is the restriction of $\mathrm{id}_n \otimes P$ to $M_n \otimes E(A)$, we may argue exactly as above to conclude that $\mathrm{id}_n \otimes \alpha$ is isometric on $\mathrm{id}_n \otimes E(A)$. Hence α is completely isometric.

It follows that $Q = \alpha \circ E$ is a quasiautomorphism of A in the sense of Definition 3.3. The remaining assertions, including the uniqueness of Q, now follow from Theorem 2.6. □

Remark 4.2 (Relation to the Perron–Frobenius Theory). Frobenius' generalization [11] of Perron's theorem [16, 17] on square matrices with positive entries can be viewed as a result that provides information about the structure and properties of positive linear maps acting on finite-dimensional commutative C^*-algebras. Indeed, every $n \times n$ matrix with nonnegative entries acts naturally on complex column vectors as a positive linear map, and every positive linear map of $\mathbb{C}^n$ arises in that way. Recall too that a positive linear map on a commutative C^*-algebra is automatically completely positive.

In order to simplify the following remarks, we start with a positive linear map $P : A \to A$ on a finite-dimensional commutative C^*-algebra A satisfying $P(\mathbf{1}) = \mathbf{1}$, in which case both the norm and spectral radius of P are 1. Thus the first assertion of the Perron–Frobenius theorem, namely that there is a nonzero positive element of A that is fixed under P, is automatic. For purposes of this discussion, the principal assertions of Frobenius' result [12, p. 65] can be paraphrased for positive maps as follows.

Perron–Frobenius Theorem. Assume further that P is *irreducible* in the sense that the only projections $e \in \mathcal{A}$ satisfying $P(e) \leq e$ are $e = \mathbf{0}$ and $e = \mathbf{1}$, and let $\{\lambda_0, \ldots, \lambda_{k-1}\}$ be the distinct eigenvalues of P that lie on the unit circle, $1 \leq k \leq \dim A$.

Then each λ_j is a simple eigenvalue and $\lambda_0, \ldots, \lambda_{k-1}$ are the distinct kth roots of unity; hence we can arrange that $\lambda_j = \zeta^j$, where $\zeta = e^{2\pi i/k}$. Moreover, the matrix of P relative to a basis of minimal projections of A has the form UCU^{-1}, where U

is a permutation matrix and C is a "cyclic" matrix of rectangular blocks

$$C = \begin{pmatrix} 0 & C_0 & 0 & \dots & 0 \\ 0 & 0 & C_1 & \dots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & 0 & \dots & C_{k-2} \\ C_{k-1} & 0 & 0 & \dots & 0 \end{pmatrix},$$

in which the diagonal blocks are square.

We now describe how these combined assertions about the structure of P fit naturally into the context of Theorem 4.1. The displayed cyclic structure of C, together with the fact that $P = UCU^{-1}$, implies that there is a set of mutually orthogonal projections $e_0, \dots, e_{k-1} \in A$, with $e_0 + \cdots + e_{k-1} = \mathbf{1}$, which are permuted cyclically by P in the sense that

$$P(Ae_i) \subseteq Ae_{i \dot{+} 1}, \tag{4.2}$$

where $\dot{+}$ denotes addition modulo k. Let B be the C^*-subalgebra of A spanned by the projections $e_0, \dots, e_{k-1}$. Equation (4.2) implies that $P(e_i) \le e_{i \dot{+} 1}$, and after summing on i we find that equality must hold for each i because $e_0 + \cdots + e_{k-1} = \mathbf{1}$ and $P(\mathbf{1}) = \mathbf{1}$. Thus the restriction of P to B is the $*$-automorphism α of B determined by $\alpha(e_i) = e_{i \dot{+} 1}$, $0 \le i \le k-1$.

This automorphism α is the "isometric" part of the quasiautomorphism Q that is associated with P by Theorem 4.1. To see that, one first observes that B is spanned by the set of maximal eigenvectors of P. Indeed, an elementary argument shows that B is spanned by the set of elements

$$x_\ell = e_0 + \bar{\zeta}^\ell e_1 + \bar{\zeta}^{2\ell} e_2 + \cdots + \bar{\zeta}^{(k-1)\ell} e_{k-1}, \quad 0 \le \ell \le k-1,$$

and clearly $P(x_\ell) = \zeta^\ell x_\ell$ for all ℓ. Since the eigenvalues $\lambda_\ell = \zeta^\ell$ are all simple, it follows that $B = [e_0, \dots, e_{k-1}] = [x_0, \dots, x_{k-1}]$ is the space spanned by all maximal eigenvectors. Thus, Theorem 4.1 implies that the $*$-automorphism $\alpha = P \restriction_B$ is related to Q by $Q = \alpha \circ E$ where E is the unique idempotent limit point of $\{P^n\}$. We now identify E. Since B is spanned by the maximal eigenvectors of P, the proof of Theorem 2.6 implies that we have a direct sum decomposition of finite-dimensional vector spaces

$$A = B \oplus \{z \in A : \lim_{n\to\infty} \|P^n(z)\| = 0\}.$$

Thus the sequence $P^k, P^{2k}, P^{3k}, \dots$ converges to an idempotent with range B, and another application of Theorem 4.1 shows that $E = \lim_n P^{nk}$.

In fact, given the relation between quasiautomorphisms and $*$ automorphisms in Proposition 3.2, it is not hard to turn this argument around to deduce [12, Theorem 2] (including the permutation formula (4.2)) from Theorem 4.1 above; and in this sense *one can regard Theorem* 4.1 *as a generalization of the Perron–Frobenius theorem to C^*-algebras.*

Our search for a result like Theorem 4.1 was inspired in part by a recent observation of Greg Kuperberg on the existence of idempotent limits of powers of a completely positive map on a finite-dimensional C^*-algebra.

Theorem 4.3 (Kuperberg). *Let A be a finite-dimensional C^*-algebra and let $P : A \to A$ be a unital completely positive map. There is a sequence of integers $0 < n_1 < n_2 < \cdots$ such that P^{n_k} converges to a unique completely positive idempotent map $E : A \to A$.*

The uniqueness assertion means that E does not depend on the sequence n_k in the sense that if $m_1 < m_2 < \cdots$ is another increasing sequence for which the powers P^{m_k} converge to an *idempotent* F, then $F = E$. An elementary proof of Theorem 4.3 is sketched in [15].

Of course, any unital completely positive linear map that acts on a finite dimensional C^*-algebra must be locally finite, and therefore satisfies the hypotheses of Theorem 4.1. But there are many others as well, and we now briefly describe some examples that act on familiar C^*-algebras.

Example. Let G be a discrete group, let $U : G \to \mathcal{B}(\ell^2(G))$ be the regular representation of G on its natural Hilbert space, and consider the reduced group C^*-algebra $\mathcal{A} = C^*\{U_x : x \in G\}$ of G. For every state ρ of $\mathcal{A}$ there is a naturally associated positive definite function $\phi : G \to \mathbb{C}$, defined by $\phi(x) = \rho(U_x)$, and one has $|\phi(x)| \leq \phi(e) = \rho(\mathbf{1}) = \mathbf{1}$ for all $x \in G$. Notice first that the "kernel" of ϕ

$$K = \{x \in G : |\phi(x)| = 1\}$$

is a subgroup of G and the restriction of ϕ to K is a character of K. Indeed, the GNS construction provides us with a unitary representation $V : G \to H$ and a cyclic vector ξ for V so that

$$\phi(x) = \langle V_x\xi, \xi\rangle\,, \quad x \in G\,.$$

Noting that $\|V_x\xi - \phi(x)\xi\|^2 = 2 - 2|\phi(x)|^2$ for any $x \in G$, it follows that $|\phi(x)| = 1$ if and only if ξ is an eigenvector for V_x in the sense that $V_x\xi = \phi(x)\xi$. Thus K is a subgroup on which ϕ is multiplicative.

Proposition 4.4. *For every state ρ of $\mathcal{A}$, there is a unique completely positive linear map $P : \mathcal{A} \to \mathcal{A}$ satisfying*

$$P(U_x) = \rho(U_x)U_x\,, \quad x \in G\,. \tag{4.3}$$

P is a locally finite map with the following properties.

Let $\mathcal{B}$ be the C^*-subalgebra of $\mathcal{A}$ generated by $\{U_x : |\rho(U_x)| = 1\}$. Then

(i) *The restriction α of P to $\mathcal{B}$ is a $*$-automorphism of $\mathcal{B}$.*
(ii) *There is a unique completely positive map E defined on $\mathcal{A}$ by*

$$E(U_x) = \begin{cases} U_x\,, & |\rho(U_x)| = 1 \\ 0\,, & |\rho(U_x)| < 1\,, \end{cases}$$

and E is an idempotent with range $\mathcal{B}$.

(iii) *For every* $A \in \mathcal{A}$ *we have*

$$\lim_{n\to\infty} \|P^n(A) - \alpha^n \circ E(A)\| = 0.$$

Sketch of Proof. To see that there is a completely positive map $P : \mathcal{A} \to \mathcal{A}$ satisfying (4.3), consider the unitary representation

$$W_x = U_x \otimes U_x \in \mathcal{B}(\ell^2(G) \otimes \ell^2(G)), \quad x \in G.$$

By [20, Proposition 4.2], W is weakly contained in the regular representation, so there is a representation $\pi : \mathcal{A} \to \mathcal{B}(\ell^2(G) \otimes \ell^2(G))$ satisfying $\pi(U_x) = U_x \otimes U_x$, $x \in G$. Letting $Q : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ be the slice map $Q(A \otimes B) = \rho(A)B$, one finds that the composition $Q \circ \pi$ satisfies (4.3). The proofs of the remaining assertions are straightforward. □

I want to thank Marc Rieffel for [20]. There are many variations of this example, including some natural examples acting on irrational rotation C^*-algebras. These examples all have the feature that the range $E(\mathcal{A})$ of the completely positive idempotent

$$E \in \bigcap_{n\geq 0} \overline{\{P^n, P^{n+1}, \dots\}}^{\text{strong}}$$

is already C^*-subalgebra of $\mathcal{A}$. However, that is an artifact of this class of examples, and perhaps it is worth pointing out that in general, $E(\mathcal{A})$ need not be a subalgebra of $\mathcal{A}$. The following examples illustrate the point.

Example. Let $\mathcal{T}$ be the Toeplitz C^*-algebra, the C^*-algebra generated by the simple unilateral shift. The familiar exact sequence of C^*-algebras

$$0 \to \mathcal{K} \to \mathcal{T} \underset{\pi}{\to} C(\mathbb{T}) \to 0 \tag{4.4}$$

has a positive linear lifting $\pi_* : C(\mathbb{T}) \to \mathcal{T}$ which carries a symbol $f \in C(\mathbb{T})$ to its associated Toeplitz operator T_f. Every homeomorphism $h : \mathbb{T} \to \mathbb{T}$ gives rise to a $*$-automorphism β of $C(\mathbb{T})$ via $\beta(f) = f \circ h$, and after fixing h one obtains a quasiautomorphism $Q : \mathcal{T} \to \mathcal{T}$ as in Proposition 3.1,

$$Q(A) = \pi_*(\beta(\pi(A))), \quad A \in \mathcal{T}.$$

In more explicit terms, every operator in $\mathcal{T}$ admits a unique decomposition $A = T_f + K$, where $f \in C(\mathbb{T})$, $K \in \mathcal{K}$ [2], and we have

$$Q(T_f + K) = T_{f\circ h}, \quad f \in C(\mathbb{T}), \quad K \in \mathcal{K}.$$

This quasiautomorphism has "polar decomposition" $Q = \alpha \circ E$, where $E : \mathcal{T} \to \mathcal{T}$ is the completely positive idempotent

$$E(T_f + K) = T_f, \quad f \in C(\mathbb{T}), \quad K \in \mathcal{K},$$

and α is the completely isometric linear map defined on $E(\mathcal{T})$ by

$$\alpha(T_f) = T_{f\circ h}, \quad f \in C(\mathbb{T}).$$

Note that in these examples, $E(\mathcal{T})$ is the space of all Toeplitz operators with continuous symbol, an operator system that is not a C^*-subalgebra of $\mathcal{T}$.

As a variation on this example, let P_0 be a unital completely positive map that acts on a finite-dimensional C^*-algebra $\mathcal{A}$, and let $P = P_0 \otimes Q$ be the completely positive linear map on $\mathcal{A} \otimes \mathcal{T}$ that satisfies

$$P(X \otimes A) = P_0(X) \otimes Q(A)\,, \quad X \in \mathcal{A}\,, \quad A \in \mathcal{T}\,. \tag{4.5}$$

Since P_0 acts on a finite-dimensional C^*-algebra it is locally finite, hence there is a unique quasiautomorphism $Q_0 : \mathcal{A} \to \mathcal{A}$ satisfying

$$\lim_{n\to\infty} \|P_0^n(X) - Q_0^n(X)\| = 0\,, \quad X \in \mathcal{A}\,.$$

It follows that

$$\lim_{n\to\infty} \|P^n(X \otimes A) - Q_0^n(X) \otimes Q^n(A)\| = 0\,, \quad X \in \mathcal{A}\,, \quad A \in \mathcal{T}\,.$$

Thus, $Q_0 \otimes Q$ is the quasiautomorphism of $\mathcal{A} \otimes \mathcal{T}$ that is asymptotically associated with P. The range of $Q_0 \otimes Q$ is certainly an operator system, but it is never a C^*-subalgebra of $\mathcal{A} \otimes \mathcal{T}$.

It goes without saying that one can obtain a great variety of such examples by replacing the Toeplitz diagram (4.4) with other linearly split short exact sequences of C^*-algebras.

5. Applications to von Neumann Algebras

We now describe how Theorem 4.1 must be modified for normal completely positive maps acting on von Neumann algebras.

Definition 5.1. By a *quasiautomorphism* of a von Neumann algebra M we mean a *normal* unit-preserving completely positive linear map $P : M \to M$, such that $P(M)$ is norm-closed and P restricts to a completely isometric linear map of $P(M)$ onto itself.

Remark 5.2 (Structure of Quasiautomorphisms). Note first that the range $P(M)$ of a quasiautomorphism must be weak*-closed. Indeed, since P is normal we may consider the natural action P_* of P on the predual of M: $P_*(\rho) = \rho \circ P$, $\rho \in M_*$. P_* is a completely positive contraction on M_*, and its range is norm-closed because its adjoint P has norm-closed range. At this point we can appeal to the elementary result asserting that the adjoint of an operator with norm-closed range must have weak*-closed range.

Let $\alpha : P(M) \to P(M)$ be the restriction of P to its range. α is a unital surjective normal isometry that acts on a dual operator system. Its inverse α^{-1} is therefore normal as well, and $E = \alpha^{-1} \circ P \restriction_{P(M)}$ defines a normal completely positive idempotent with range $P(M)$ that fixes the unit of M and satisfies $EP = PE = P$. Thus we have exhibited a unique "polar decomposition" $P = \alpha \circ E$.

As in the more general case of C^*-algebras discussed in the proof of Proposition 3.2, $P(M)$ can be made into a C^*-algebra by introducing the multiplication $x \bullet y = E(xy)$, and with respect to this structure α becomes an automorphism of C^*-algebras. Moreover, since in this case $P(M)$ is weak*-closed, it is naturally identified with the dual of the Banach space $P(M)_* = M_*/P(M)_\perp$, where $P(M)_\perp$ is the pre-annihilator of $P(M)$. A familiar theorem of Sakai [18, Theorem 1.16.7] implies that the C^*-algebra $P(M)$ is a von Neumann algebra with respect to this multiplication.

We conclude: *every quasiautomorphism P of a von Neumann algebra M has a unique representation $P = \alpha \circ E$ where $E : M \to M$ is a normal completely positive idempotent with range $P(M)$ and α is a $*$-automorphism of the natural von Neumann algebra structure of $E(M)$ associated with the multiplication $x \bullet y = E(xy)$, $x, y \in M$.*

The appropriate notion of local finiteness for the category of von Neumann algebras involves the action of normal maps on the predual as follows.

Definition 5.3. Let P be a normal completely positive map on a von Neumann algebra M satisfying $P(\mathbf{1}) = \mathbf{1}$. P is said to be *locally finite* if for every normal state ω of M, the set of normal states $\{\omega \circ P^n : n = 0, 1, 2, \ldots\}$ is relatively compact in the norm topology of M_*.

Since every element of M_* is a linear combination of normal states, we see that a map $P : M \to M$ satisfying the conditions of Definition 5.3 has the property the norm-closure of $\{\rho \circ P^n : n \geq 0\}$ is compact for every $\rho \in M_*$; and therefore $P_*(\rho) = \rho \circ P$ defines a locally finite contraction in $\mathcal{B}(M_*)$.

Theorem 5.4. *Let M be a von Neumann algebra with separable predual and let $P : M \to M$ be a locally finite normal completely positive map satisfying $P(\mathbf{1}) = \mathbf{1}$. There is a unique quasiautomorphism $Q = \alpha \circ E$ of M such that for every normal state ρ of M, one has*

$$\lim_{n \to \infty} \|\rho \circ P^n - \rho \circ Q^n\| = 0\,.$$

The completely positive map E is characterized as the unique idempotent for which there is a sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\lim_{k \to \infty} \|\rho \circ P^{n_k} - \rho \circ E\| = 0\,, \quad \rho \in M_*\,, \tag{5.1}$$

α is the restriction of P to the dual operator system $E(M)$, and $E_*(M_*)$ is the norm-closed linear span of the set of all maximal eigenvectors of P_*.

Proof. Applying Theorem 2.6 to the locally finite contraction $P_* \in \mathcal{B}(M_*)$ defined by $P_*(\rho) = \rho \circ P$, one obtains a unique quasiautomorphism $Q_* : M_* \to M_*$ with the property

$$\lim_{n \to \infty} \|\rho \circ P^n - Q_*^n(\rho)\| = 0\,, \quad \rho \in M_*\,.$$

Letting $Q : M \to M$ be the adjoint of Q_*, one finds that Q is a quasiautomorphism of M, and the rest follows from Theorem 2.6. □

Of course, one can drop the separability hypothesis on the predual of M at the cost of replacing the sequential limit (5.1) with an appropriate more general assertion.

6. Semigroups

Let X be a Banach space. By a *contraction semigroup* we mean a semigroup $T = \{T_t : t \geq 0\}$ of operators on X satisfying $\|T_t\| \leq 1$ that is strongly continuous in the sense that for each $x \in X$, the function $t \in [0, \infty) \mapsto T_t x$ moves continuously in the norm of X. Notice that we have not specified that $T_0 = \mathbf{1}$, so that in general T_0 is simply an idempotent contraction.

Proposition 6.1. *For every contraction semigroup $T = \{T_t : t \geq 0\}$ acting on a Banach space X and every vector $x \in X$, the following are equivalent.*

(i) *The norm-closure of the orbit $\{T_t x : t \geq 0\}$ is compact.*
(ii) *For some $s > 0$, the norm closure of $\{x, T_s x, T_s^2 x, \ldots\}$ is compact.*

Sketch of Proof. The implication (i)⇒(ii) is trivial. We sketch the proof of (ii)⇒(i). Choose $s > 0$ such that the closure K_x of $\{x, T_s x, T_s^2 x, \ldots\}$ is compact. It suffices to show that the union

$$\bigcup_{0 \leq r \leq s} T_r K_x \tag{6.1}$$

is compact, since the set (6.1) obviously contains $\{T_t x : t \geq 0\}$. Consider the map of $[0, \infty) \times X$ to X defined by $(t, x) \mapsto T_t x$. This map is continuous (with respect to the product topology of $[0, \infty) \times X$ and the norm topology of X) because T is strongly continuous. Since the union (6.1) is the range of the restriction of this map to the compact subspace $[0, s] \times K_x \subseteq [0, \infty) \times X$, it follows that the set (6.1) is compact. □

Definition 6.2. A contraction semigroup $T = \{T_t : t \geq 0\}$ is said to be *locally finite* the conditions of Proposition 6.1 are satisfied for every $x \in X$.

Notice that Proposition 6.1 implies that the semigroup T will be locally finite whenever T_1 is a locally finite contraction.

Let $T = \{T_t : t \geq 0\}$ be a contraction semigroup with the property that T_t is a quasiautomorphism of X for every $t \geq 0$. Then $E = T_0$ is an idempotent contraction that commutes with $\{T_t : t \geq 0\}$, and a straightforward argument (that we omit) shows that $T_t X = EX$ for every $t \geq 0$, that $U_t = T_t \restriction_{EX}$ is an automorphism of EX, and that we have the "polar decomposition"

$$T_t = U_t E\,, \quad t \geq 0\,. \tag{6.2}$$

In particular, the most general semigroup of quasiautomorphisms T is obtained from a semigroup U of automorphisms by a direct sum procedure $T_t = U_t \oplus 0$, $t > 0$ analogous to the one spelled out in Remark 2.1.

Let $x \in X$ be a nonzero eigenvector for $T = \{T_t : t \geq 0\}$. Then there is a complex number λ in the upper half-plane $\{z = x + iy : y \geq 0\}$ such that

$$T_t x = e^{it\lambda} x\,, \quad t \geq 0\,. \tag{6.3}$$

Such a λ belongs to the point spectrum of the generator of T; we abuse notation slightly by writing $\sigma_p(T)$ for the set of all complex numbers λ satisfying (6.3). The eigenvector x is said to be *maximal* if $\|T_t x\| = \|x\|$ for $t \geq 0$; thus, x is maximal if and only if $\lambda \in \mathbb{R}$. Corresponding to Theorem 2.6, we have:

Theorem 6.3. *Let $T = \{T_t : t \geq 0\}$ be a locally finite contraction semigroup acting on a Banach space X, satisfying $T_0 = \mathbf{1}$. There is a unique semigroup $S = \{S_t : t \geq 0\}$ of quasiautomorphisms such that*

$$\lim_{t\to\infty} \|T_t x - S_t x\| = 0\,, \quad x \in X\,.$$

Let E be the idempotent $E = S_0$, so that $S_t = U_t E$ for $t \geq 0$ as in (6.2)*. Then the generator of U is diagonalized by the set of maximal eigenvectors, and its point spectrum is given by*

$$\sigma_p(U) = \sigma_p(T) \cap \mathbb{R}\,.$$

The projection $E = S_0$ is characterized as the unique idempotent in the set $\mathcal{L}$ of strong limit points of $\{T_t : t \geq 0\}$

$$\mathcal{L} = \bigcap_{\alpha>0} \{T_t : t \geq \alpha\}^{-\text{strong}}\,,$$

and U_t is the restriction of T_t to EX, $S_t = T_t E$, $t \geq 0$.

Sketch of Proof. The argument is merely a variation of the proof of Theorem 2.6, requiring little more than a change of notation. Let M be the closed linear span of the set of all maximal eigenvectors for T and let N be the space of all asymptotically null vectors

$$N = \{x \in X : \lim_{t\to\infty} \|T_t x\| = 0\}\,.$$

In order to show that $X = N \dotplus M$, one first shows that the restriction $\tilde{T}$ of the semigroup T to M has the property that the identity operator belongs to the strong closure of $\{\tilde{T}_t : t \geq \alpha\}$ for every $\alpha > 0$ by the same method used in the proof of Theorem 2.6. From that, along with an appropriate variation of Lemma 2.9 for semigroups, it follows that $\tilde{T}$ is a semigroup of invertible isometries with the property that for every $s \geq 0$, the inverse of $\tilde{T}_s$ belongs to the strong closure of $\{\tilde{T}_t : t \geq 0\}$. Thus the strong closure of $\{\tilde{T}_t : t \geq 0\}$ contains the one-parameter group generated by $\{\tilde{T}_t\colon t \geq 0\}$. For each vector $x \in X$ one introduces the set of limit points

$$K_\infty(x) = \bigcap_{\alpha>0} \overline{\{T_t x : t \geq \alpha\}},$$

and one shows that $K_\infty(x) \subseteq M$ by the method of Theorem 2.6, except that now one must replace references to almost periodic functions on the group $\mathbb{Z}$ with references to almost periodic functions on the group $\mathbb{R}$. Once one knows that $K_\infty(x) \subseteq M$ for every $x \in X$, the decomposition $X = N \dotplus M$ follows readily as in the case of single contractions.

The remaining assertions of Theorem 6.3 are straightforward. □

With these general results in hand, one can establish a natural counterpart of Theorem 5.4 for semigroups of normal completely positive maps acting on von Neumann algebras. We leave the explicit formulation of that result for the reader.

7. An Unstable Example

It is natural to ask whether *all* completely positive contractions are stable. More precisely, can the key hypothesis of local finiteness can be dropped entirely from Theorems 4.1 and 5.4, provided that one is willing to give up the secondary conclusions? For example, in the second group of examples described in Sec. 4, there are many homeomorphisms $h : \mathbb{T} \to \mathbb{T}$ for which the completely positive map P of (4.5) is not locally finite and has no maximal eigenvectors, even though in all such cases P is asymptotically related to a quasiautomorphism Q as in (4.1). Such examples show that local finiteness is not *necessary* for the stability assertion of Theorem 4.1.

We conclude by briefly describing an example of a normal unital completely positive map $P : \mathcal{B}(H) \to \mathcal{B}(H)$ for which the principal conclusion of Theorem 5.4 fails in the sense that there does not exist a quasiautomorphism $Q : \mathcal{B}(H) \to \mathcal{B}(H)$ with the property

$$\lim_{n\to\infty} \|\rho \circ P^n - \rho \circ Q^n\| = 0\,, \tag{7.1}$$

for every normal state ρ of $\mathcal{B}(H)$. A significant feature of this example is that it very nearly satisfies the hypothesis of Theorem 5.4 in the following sense: there is a subspace $\mathcal{S} \subseteq \mathcal{B}(H)_*$ of codimension one such that $\{\rho \circ P^n : n \geq 0\}$ is relatively norm-compact for every $\rho \in \mathcal{S}$.

The example is based on the heat flow of the canonical commutation relations $\{P_t : t \geq 0\}$ of [4], a semigroup of normal completely positive maps on $\mathcal{B}(H)$ that is *pure* in the sense that for any pair ρ_1, ρ_2 of normal states of $\mathcal{B}(H)$ one has

$$\lim_{t\to\infty} \|\rho_1 \circ P_t - \rho_2 \circ P_t\| = 0\,,$$

while on the other hand, there is no normal state ω of $\mathcal{B}(H)$ satisfying $\omega \circ P_t = \omega$, $t \geq 0$. If we fix $t_0 > 0$ and set $P = P_{t_0}$, then

$$\lim_{n\to\infty} \|\rho_1 \circ P^n - \rho_2 \circ P^n\| = 0 \tag{7.2}$$

for normal states ρ_1, ρ_2, and P cannot leave any normal state ω invariant.

The relation (7.2), together with a simple compactness argument, implies that for fixed $A \in \mathcal{B}(H)$ the sequence $P(A), P^2(A), P^3(A), \ldots$ is asymptotically a scalar sequence in the sense that there is a sequence of complex numbers $\lambda_1, \lambda_2, \ldots$ (which depends on A) such that

$$\lim_{n\to\infty} (P^n(A) - \lambda_n \mathbf{1}) = 0\,,$$

in the weak operator topology. So if there were a quasiautomorphism Q satisfying (7.1), then the range of Q would be $\mathbb{C}\cdot\mathbf{1}$. Since the von Neumann algebra associated with the range of Q in Remark 5.2 is in this case $\mathbb{C}$ and since a $*$-automorphism of $\mathbb{C}$ is the identity map, such a quasiautomorphism Q would simply be a normal idempotent with range $\mathbb{C}\cdot\mathbf{1}$. Thus Q would have the form $Q(A) = \omega(A)\mathbf{1}$, where ω is a normal state of $\mathcal{B}(H)$. Since $\omega \circ Q = \omega$, we have $\|\omega \circ P^n - \omega\| = \|\omega \circ P^n - \omega \circ Q^n\|$ for every $n \geq 1$; and (7.1) implies that $\|\omega \circ P^n - \omega \circ Q^n\| \to 0$ as $n \to \infty$. We conclude that $\|\omega \circ P^n - \omega\| \to 0$ as $n \to \infty$. Therefore $\omega \circ P = \omega$, contradicting the second property of P cited above.

Finally, let $\mathcal{S}$ be the codimension one subspace of $\mathcal{B}(H)_*$ consisting of all normal linear functionals ρ on $\mathcal{B}(H)$ satisfying $\rho(\mathbf{1}) = 0$. Every $\rho \in \mathcal{S}$ can be decomposed into a sum

$$\rho = \lambda(\rho_1 - \rho_2) + i\mu(\rho_3 - \rho_4)\,,$$

where λ and μ are real scalars and each ρ_k is a normal state. By (7.2), we have $\|\rho \circ P^n\| \to 0$ as $n \to \infty$, and therefore $\{\rho \circ P^n : n \geq 0\}$ is relatively norm-compact for every ρ in the subspace $\mathcal{S}$.

Acknowledgment

The author is supported by NSF grant DMS-0100487.

References

[1] W. Arveson, Subalgebras of C^*-algebras, *Acta Math.* **123** (1969) 141–224.
[2] W. Arveson, *A Short Course on Spectral Theory*, Graduate Texts in Mathematics, Vol. 209 (Springer-Verlag, New York, 2001).
[3] W. Arveson, Generators of noncommutative dynamics, *Ergodic Theory Dynam. Syst.* **22** (2002) 1017–1030, arXiv:math.OA/0201137.
[4] W. Arveson, The heat flow of the CCR algebra, *Bull. London Math. Soc.* **34** (2002) 73–83, arXiv:math.OA/0005250.
[5] W. Arveson, *Noncommutative Dynamics and E-Semigroups*, Monographs in Mathematics (Springer-Verlag, New York, 2003).

[6] B. V. R. Bhat, An index theory for quantum dynamical semigroups, *Trans. Amer. Math. Soc.* **348** (1996) 561–583.

[7] B. V. R. Bhat, Minimal dilations of quantum dynamical semigroups to semigroups of endomorphisms of C^*-algebras, *J. Ramanujan Math. Soc.* **14**(2) (1999) 109–124.

[8] H. Bohr, *Almost Periodic Functions* (Chelsea, New York, 1947).

[9] M.-D. Choi and E. Effros, Injectivity and operator spaces, *J. Funct. Anal.* **24**(2) (1977) 156–209.

[10] A. Connes and E. Størmer, Entropy of automorphisms of II_1 von Neumann algebras, *Acta Math.* **34** (1975) 289–306.

[11] G. Frobenius, Über matrizen aus nicht negativen elementen, *Sitzber. Akad. Wiss. Berlin, Phys. Math. Kl.* (1912), 456–477.

[12] F. R. Gantmacher, *Applications of the Theory of Matrices* (Interscience Publishers, New York, 1959).

[13] E. Hewitt and K. Ross, *Abstract Harmonic Analysis II*, Grundlehren Vol. 152 (Springer-Verlag, New York, 1970), second printing (1994).

[14] E. Hewitt and K. Ross, *Abstract Harmonic Analysis I*, Grundleheren Vol. 115, 2nd edn. (Springer-Verlag, New York, 1979).

[15] G. Kuperberg, The capacity of hybrid quantum memory, *IEEE Trans. Inf. Theory*, 2003 (to appear), arXiv:quant-ph/0203105 v2.

[16] O. Perron, Grundlagen für eine theorie des jacobischen kettenbruchalgorithmus, *Math. Ann.* **64** (1907) 1–76.

[17] O. Perron, Zur theorie der matrices, *Math. Ann.* **64** (1907) 248–263.

[18] S. Sakai, C^*-algebras and W^*-algebras, Classics in Mathematics (Springer-Verlag, New York, 1998), reprinted from the 1971 edition.

[19] E. Størmer, A survey of noncommutative dynamical entropy, in *Encyclopaedia of Mathematical Sciences*, eds. M. Rørdam and E. Størmer, Vol. 126 (Springer-Verlag, Heidelberg, 2002), pp. 147–198, arXiv:math.OA/0007010.

[20] J.-M. Vallin, C^*-algèbres de Hopf et C^*-algèbres de Kac, *Proc. London Math Soc.* **50**(3) (1985) 131–174.