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THE STRUCTURE OF SPIN SYSTEMS

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A spin system is a sequence of self-adjoint unitary operators $U_1, U_2, \ldots$ acting on a Hilbert space H which either commute or anticommute, $U_iU_j = \pm U_jU_i$ for all i, j; it is called irreducible when $\{U_1, U_2, \ldots\}$ is an irreducible set of operators. There is a unique infinite matrix (c_{ij}) with $0, 1$ entries satisfying

$$U_iU_j = (-1)^{c_{ij}} U_jU_i\,, \qquad i, j = 1, 2, \ldots\,.$$

Every matrix (c_{ij}) with $0, 1$ entries satisfying $c_{ij} = c_{ji}$ and $c_{ii} = 0$ arises from a nontrivial irreducible spin system, and there are uncountably many such matrices.

In cases where the commutation matrix (c_{ij}) is of "infinite rank" (these are the ones for which infinite dimensional irreducible representations exist), we show that the C^*-algebra generated by an irreducible spin system is the CAR algebra, an infinite tensor product of copies of $M_2(\mathbb{C})$, and we classify the irreducible spin systems associated with a given matrix (c_{ij}) up to approximate unitary equivalence.

That follows from a structural result. The C^*-algebra generated by the universal spin system $u_1, u_2, \ldots$ of (c_{ij}) decomposes into a tensor product $C(X) \otimes \mathcal{A}$, where X is a Cantor set (possibly finite) and $\mathcal{A}$ is either the CAR algebra or a finite tensor product of copies of $M_2(\mathbb{C})$. We describe the nature of this decomposition in terms of the "symplectic" properties of the $\mathbb{Z}_2$-valued form

$$\omega(x, y) = \sum_{p,q=1}^{\infty} c_{pq}x_qy_p\,,$$

x, y ranging over the free infninite dimensional vector space over the Galois field $\mathbb{Z}_2$.

1. Introduction

A spin system is a sequence $u_1, u_2, \ldots$ of self-adjoint unitary elements of some unital C^*-algebra which commute up to phase in the sense that

$$u_iu_j = \lambda_{ij}u_ku_j\,, \qquad i, j = 1, 2, \ldots$$

where the λ_{ij} are complex numbers. Since $u_iu_ju_i^{-1} = \lambda_{ij}u_j$ and $u_j^2 = \mathbf{1}$, it follows that each λ_{ij} is -1 or $+1$. Thus there is a unique matrix of zeros and ones c_{ij} such that the commutation relations become

$$u_iu_j = (-1)^{c_{ij}}u_ju_i\,, \qquad i, j = 1, 2, \ldots\,. \tag{1.1}$$

The matrix (c_{ij}) is easily seen to be symmetric, and has zeros along the main diagonal. A concrete spin system $U_1, U_2, \ldots \subseteq \mathcal{B}(H)$ is said to be irreducible when $\{U_1, U_2, \ldots\}$ is an irreducible set of operators. The purpose of this paper is to determine the structure of the C^*-algebra generated by an irreducible spin system associated with a given 0-1 matrix (c_{ij}), and to classify such spin systems up to "approximate" unitary equivalence (Theorem 4.1).

1.1. *Quantum spin systems*

Spin systems arise naturally in several contexts, including the theory of quantum spin systems ([4, Sec. 6.2]), and in the theory of quantum computing (especially, systems involving a large or infinite number of qubits). For example, suppose we are given a mutually commuting sequence $\theta_1, \theta_2, \ldots$ of involutive $*$-automorphisms of $\mathcal{B}(H)$, i.e. $\theta_j^2 = \mathrm{id}$, $\theta_k\theta_j = \theta_j\theta_k$ for all $j, k = 1, 2, \ldots$ (one can imagine that θ_k represents reversing the state of a two-valued quantum observable located at the kth site). For each k one can find a unitary operator U_k such that $\theta_k(A) = U_k A U_k^{-1}$, $A \in \mathcal{B}(H)$, and by replacing U_k with λU_k for an appropriate $\lambda \in \mathbb{T}$ if necessary, we can arrange that $U_k^2 = \mathbf{1}$. Since $\theta_i\theta_j = \theta_j\theta_i$ it follows that U_i and U_j must commute up to phase, hence there is a unique number $c_{ij} \in \{0, 1\}$ such that (1.1) is satisfied. The matrix $C = (c_{ij})$ does not depend on the choices made and is therefore an invariant attached to the original sequence of automorphisms $\bar{\theta} = (\theta_1, \theta_2, \ldots)$. The sequence $\bar{\theta}$ is *ergodic* in the sense that its fixed algebra is $\mathbb{C} \cdot \mathbf{1}$ if and only if every spin system $\bar{U} = (U_1, U_2, \ldots)$ associated with it is irreducible.

1.2. *Remarks on rank*

Consider the commutation matrix (c_{ij}) associated with a spin system (1.1). If all coefficients c_{ij} vanish then $C^*(u_1, u_2, \ldots)$ is commutative. More generally, $C^*(u_1, u_2, \ldots)$ degenerates whenever (c_{ij}) is of finite rank, where the rank is defined as follows. Considering $\mathbb{Z}_2 = \{0, 1\}$ as the two-element Galois field we may consider vector spaces over $\mathbb{Z}_2$, and in particular we can form the free infinite dimensional vector space $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots$ over $\mathbb{Z}_2$. Elements of Γ are sequences $x = (x_1, x_2, \ldots)$, $x_k \in \mathbb{Z}_2$, which vanish eventually. The dual of Γ is identified with the vector space $\mathbb{Z}_2^\infty$ of all sequences $y = (y_1, y_2, \ldots)$, $y_k \in \mathbb{Z}_2$. The commutation matrix (c_{ij}) gives rise to a linear operator $C : \Gamma \to \mathbb{Z}_2^\infty$ by way of $(Cx)_k = \sum_{j=1}^\infty c_{kj}x_j$, $k = 1, 2, \ldots$. The *rank* of the matrix is defined by

$$\operatorname{rank}(c_{ij}) = \dim(C\Gamma)\,.$$

Actually, what we have defined is the *column rank* of the matrix (c_{ij}), but because (c_{ij}) is a symmetric matrix its column and row ranks are the same. We will see below that the rank is finite iff the center of $C^*(u_1, u_2, \ldots)$ is of finite codimension in $C^*(u_1, u_2, \ldots)$ iff every irreducible spin system satisying (1.1) acts on a finite dimensional Hilbert space. Thus we are primarily concerned with the nondegenerate cases in which the commutation matrix (c_{ij}) is of infinite rank.

1.3. *Remarks on existence and universality*

A sequence $u_1, u_2, \ldots$ of unitary operators satisfying a given set of noncommutative equations $f_k(u_1, u_2, \ldots) = 0$, $k = 1, 2, \ldots$ (we leave the precise nature of the noncommutative polynomials f_k unspecified) is said to be universal if every sequence $U_1, U_2, \ldots \in \mathcal{B}(H)$ of concrete unitary operators (acting on a separable Hilbert space) that satisfies the equations can be obtained from it via a representation $\pi : C^*(u_1, u_2, \ldots) \to \mathcal{B}(H)$ such that $\pi(u_k) = U_k$, $k = 1, 2, \ldots$.

Of course, for bad choices of f_k (e.g. $f_k(x_1, x_2, \ldots) = x_k$) there may be no unitary solutions to the set of equations except on the trivial Hilbert space $H = \{0\}$ (note that every meaningful operator equation is satisfied on the trivial Hilbert space). But in all cases there is a universal solution ... consider the direct sum of all concrete unitary solutions. Any two universal solutions $(u_1, u_2, \ldots)$ and $(v_1, v_2, \ldots)$ are equivalent in the sense that there is a unique $*$-isomorphism $\theta : C^*(u_1, u_2, \ldots) \to C^*(v_1, v_2, \ldots)$ satisfying $\theta(u_k) = v_k$ for every k. Thus the C^*-algebra generated by a universal sequence of solutions to the given set S of equations is uniquely determined by S.

Given an arbitrary matrix $C = (c_{ij})$ of zeros and ones satisfying the consistency requirements $c_{ij} = c_{ji}$ and $c_{jj} = 0$ for all $i, j = 1, 2, \ldots$, we consider the C^*-algebra $\mathcal{A}_C = C^*(u_1, u_2, \ldots)$ generated by a *universal* spin system satisfying (1.1). The set of distinct matrices (c_{ij}) satisfying these conditions is of cardinality $2^{\aleph_0}$, and each of them is associated with a nontrivial spin system (1.1) (see Proposition 1.1). We determine the structure of these C^*-algebras $\mathcal{A}_C$ in Theorem 3.1.

1.4. *Spin systems in characteristic p*

We have found it helpful, even simplifying, to consider the natural generalization of spin systems to characteristic p where p is an arbitrary prime. By a spin system in characteristic p we mean a sequence of unitary operators $u_1, u_2, \ldots$ which are pth roots of unity in the sense that $u_j^p = \mathbf{1}$ for every j, and which satisfy commutation relations of the form

$$u_i u_j = \zeta^{c_{ij}} u_j u_i \,, \qquad i, j = 1, 2, \ldots \,, \tag{1.2}$$

where $\zeta = e^{2\pi i/p}$, and where $c_{ij} \in \{0, 1, \ldots, p-1\} = \mathbb{Z}_p$. Notice that each c_{ij} is uniquely determined by (1.2). If we regard $\mathbb{Z}_p$ as a finite field in the usual way, then the matrix is *skew*-symmetric in that $c_{ij} = -c_{ji}$ for every $i, j = 1, 2, \ldots$.

The reason for considering the cases $p > 2$ can be clearly seen when one specializes the previous paragraph to $p = 2$. Indeed, in the two-element field we have $x = -x$ for every x, hence a skew-symmetric matrix over $\mathbb{Z}_2$ is the same as a symmetric matrix with zeros along the main diagonal. We found that viewing (c_{ij}) as a skew-symmetric matrix led in the right direction, whereas viewing it as a symmetric matrix with zeros along the diagonal led nowhere. Thus the case $p = 2$ can be misleading, and for that reason we consider the more general case of spin systems (1.2) in characteristic p.

Fixing a prime p, assume we are given a skew-symmetric matrix (c_{ij}) of elements of the Galois field $\{0, 1, \ldots, p-1\} = \mathbb{Z}_p$. Since $\mathbb{Z}_p$ is a field, we can form the free infinite dimensional vector space Γ over $\mathbb{Z}_p$; elements of Γ are sequences $x = (x_1, x_2, \ldots)$ of elements of $\mathbb{Z}_p$ satisfying $x_k = 0$ for all but a finite number of k. The coefficients c_{ij} give rise to a bilinear form $\omega : \Gamma \times \Gamma \to \mathbb{Z}_p$ by way of

$$\omega(x, y) = \sum_{i,j=1}^{\infty} c_{ij} x_j y_i \,, \qquad x, y \in \Gamma \,. \tag{1.3}$$

This bilinear form is **skew-symmetric** in that it satisfies $\omega(x, y) = -\omega(y, x)$ for all $x, y \in \Gamma$, and it will occupy a central position throughout the sequel. The structure of such forms is described in Theorem 2.1 and Corollary 2.1.

Consider now the C^*-algebra $\mathcal{A}$ generated by a sequence of unitary elements $u_1, u_2, \ldots$ satisfying $u_k^p = \mathbf{1}$ and the commutation relations (1.2). A **word** is a finite product of elements from $\{u_1, u_2, \ldots\}$, and it is convenient regard the identity $\mathbf{1}$ as the empty word. The set of linear combinations of words is a dense $*$-subalgebra of $\mathcal{A}$ which contains $\mathbf{1}$. Using the commutation relations (1.2), every word can be written in the form $\lambda u_1^{n_1} u_2^{n_2} \cdots u_r^{n_r}$ where λ is a complex scalar. Thus we may use the elements of Γ to parameterize a spanning set of words as follows,

$$w_x = u_1^{x_1} u_2^{x_2} \ldots \,, \qquad x = (x_1, x_2, \ldots) \in \Gamma \,,$$

and one finds that

$$w_x w_y = \zeta^{\omega(x,y)} w_y w_x \,, \qquad x, y \in \Gamma \,, \tag{1.4}$$

where $\zeta = e^{2\pi i/p}$ and $\omega : \Gamma \times \Gamma \to \mathbb{Z}_p$ is the bilinear form (1.3).

We will occasionally make use of a second bilinear form $Q : \Gamma \times \Gamma \to \mathbb{Z}_p$,

$$Q(x, y) = \sum_{1 \le i < j} c_{ij} x_j y_i \,, \qquad x, y \in \Gamma \,. \tag{1.5}$$

Q is related to ω by $\omega(x, y) = Q(x, y) - Q(y, x)$, and it is a straightforward computation to verify the "Weyl" relations

$$w_x w_y = \zeta^{Q(x,y)} w_{x+y} \,, \qquad x, y \in \Gamma \,. \tag{1.6}$$

We conclude the introduction with a remark about the existence of solutions of (1.2) for arbitrary coefficient matrices (c_{ij}). For $p = 2$, this generalizes the examples of finite dimensional spin systems described in [3].

Proposition 1.1. *Let $p = 2, 3, \ldots$ be a prime and let (c_{ij}) be an arbitrary skew-symmetric matrix over the Galois field $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$. Then there a Hilbert space $H \neq \{0\}$ and a sequence of unitary operators $U_1, U_2, \ldots \in \mathcal{B}(H)$ such that $U_k^p = \mathbf{1}$ and $U_j U_k = \zeta^{c_{jk}} U_k U_j$ for every $j, k = 1, 2, \ldots$, where $\zeta = e^{2\pi i/p}$.*

Proof. Regarding $\mathbb{Z}_p$ as an additive abelian group, consider the unitary operators S, V defined on the p-dimensional Hilbert space $\ell^2(\mathbb{Z}_p)$ by

$$Sf(k) = f(k+1)\,, \qquad Vf(k) = \zeta^k f(k)\,, \qquad f \in \ell^2(\mathbb{Z}_p)\,, \qquad k \in \mathbb{Z}_p\,.$$

We have $S^p = V^p = \mathbf{1}$, $SV = \zeta VS$, and in fact $SV^k = \zeta^k V^k S$ for all $k \in \mathbb{Z}$.

Consider the L^2-space of the compact abelian group $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots$. We can realize $L^2(G)$ as the infinite tensor product of copies of $\ell^2(\mathbb{Z}_p)$ along the stabilizing vector $u \in \ell^2(\mathbb{Z}_p)$ where u is the constant function $u(k) = 1$, $k \in \mathbb{Z}_p$. Thus for any finite sequence $A_1, \ldots, A_r$ of operators on $\ell^2(\mathbb{Z}_p)$ we can form the operator

$$A_1 \otimes \cdots \otimes A_r \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \in \mathcal{B}(L^2(G))\,.$$

The unitary operators $U_1, U_2, \ldots$ are defined on $L^2(G)$ in terms of the given coefficients c_{ij} as follows; $U_1 = S \otimes \mathbf{1} \otimes \mathbf{1} \otimes \ldots$ and for $k = 2, 3, \ldots$

$$U_k = V^{c_{1k}} \otimes \cdots \otimes V^{c_{k-1k}} \otimes S \otimes \mathbf{1} \otimes \mathbf{1} \otimes \ldots\,.$$

One can verify that $U_k^p = \mathbf{1}$, and $U_j U_k = \zeta^{c_{jk}} U_k U_j$ for $1 \leq k < j$. □

As the preceding remarks on Quantum spin systems show, commutation relations of the form $uv = \lambda vu$ where $\lambda \in \mathbb{T}$ arise naturally whenever one considers commuting $*$-automorphisms of $\mathcal{B}(H)$. Indeed, many natural contexts lead to projective representations of groups involving similar commutation relations. For example, they are associated with ergodic actions of compact groups on C^*-algebras (see [2, 5] and references therein). Such commutation relations are so ubiquitous that we have made no effort to compile references to the related literature, even for the case of spin systems. Finally, we point out that the results of this paper generalize certain results in [6–9] which concern spin systems for which the commutation matrix depends only on the separation $c_{ij} = f(i-j)$.

2. Symplectic Forms in Characteristic p

In this section we work out the symplectic linear algebra that underlies the results described above. Throughout, F denotes a field, the primary cases being the Galois field $F = \mathbb{Z}_p$ of characteristic p where p is any prime including 2. Γ denotes the free infinite dimensional vector space over F, consisting of all sequences $x = (x_1, x_2, \ldots)$ of elements $x_k \in F$ satisfying $x_k = 0$ for all but a finite number of k. The addition and scalar multiplication are defined pointwise,

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots)\,, \qquad \lambda \cdot x = (\lambda x_1, \lambda x_2, \ldots)$$

λ being an element of F. A vector space V over F is said to be *countably generated* if it contains a sequence $v_1, v_2, \ldots$ such that every element of v is a finite linear combination of elements of $\{v_1, v_2, \ldots\}$. Finite dimensional vector spaces are countably generated, and for every countably generated vector space V over F there is a linear map $L : \Gamma \to V$ such that $V = L\Gamma$.

We are concerned with skew-symmetric bilinear forms $B : \Gamma \times \Gamma \to F$. The **kernel** of such a bilinear form is the subspace $K = \{x \in \Gamma : B(x, \Gamma) = \{0\}\}$. B is called a **symplectic** form when it is skew-symmetric and has kernel $\{0\}$, and a **symplectic vector space** is a pair (V, B) consisting of a countably generated vector space V over F and a symplectic bilinear form $B : V \times V \to F$. Two symplectic vector spaces (V, B) and $(V', B\prime)$ are **congruent** if there is a linear isomorphism $L : V \to V'$ satisfying $B'(Lx, Ly) = B(x, y)$ for all $x, y \in V$.

Any skew-symmeteric bilinear form defined on a vector space $B : V \times V \to F$ gives rise to a symplectic vector space as follows. Letting K be the kernel of B, B promotes natrually to a bilinear form $\omega : V/K \times V/K \to F$,

$$\omega(x + K, y + K) = B(x, y)\,, \qquad x, y \in V\,. \tag{2.1}$$

$(V/K, \omega)$ is a symplectic vector space, and it is the trivial symplectic vector space only when $B = 0$.

Definition 2.1. The rank of B is the dimension of the vector space $V/\ker B$.

The rank of B is a nonnegative integer or ∞. We will see presently that when it is finite it must be an even integer $n = 2r$, $r = 1, 2, \ldots$.

Remark 2.1. Let $C = (c_{ij})$ be the commutation matrix associated with the relations (1.1). We have given a different definition of rank in the introduction, and we want to point out that the rank defined in the introduction is the same as the rank of the skew-symmetric form ω associated to it by (1.3). To see that consider the linear map $L : \Gamma \to \mathbb{Z}_2^\infty$ defined by

$$Lx = (\omega(u_1, x), \omega(u_2, x), \ldots)\,,$$

where $u_1, u_2, \ldots$ is the usual sequence of basis vectors in Γ, $u_k(j) = \delta_{kj}$. Noting that the kth component of Lx is $\omega(u_k, x) = \sum_j c_{kj} x_j$, one sees that the range of L is the linear span of the columns of (c_{ij}) and hence its dimension is the rank of the matrix (c_{ij}). On the other hand, the kernel of L is exactly $\ker \omega$, so that $\operatorname{rank} C = \dim L\Gamma = \dim(\Gamma/\ker \omega)$, as asserted.

Let (V, ω) be a symplectic vector space. By a **symplectic basis** for V we mean a pair of sequences $e_1, e_2, \ldots, f_1, f_2, \ldots \in V$ with the properties

$$\omega(e_i, e_j) = 0\,, \qquad \omega(f_i, f_j) = 0\,, \qquad \omega(e_i, f_j) = \delta_{ij}\,, \tag{2.2}$$

for all $i, j = 1, 2, \ldots$ and which span V in the sense that every element of V is a finite linear combination of the elements $\{e_i, f_j\}$. The sequences are allowed to be either finite or infinite, but if one of them is finite then the other is also finite of the same length. A simple argument shows that any finite set of $2r$ vectors $e_1, \ldots, e_r, f_1, \ldots, f_r$ which satisfy the relations (2.2) must be linearly independent. Thus a symplectic basis for V is a countable Hamel basis, and in particular V is countably generated.

2.1. *The standard examples*

We describe the standard models of symplectic vector spaces of dimension $n = 2, 4, 6, \ldots, \infty$ over an arbitrary field F. Consider first the case $n = \infty$. Let $F^\infty = F \oplus F \oplus \ldots$ be the vector space of all infinite sequences $x = (x_1, x_2, \ldots)$, where $x_k \in F$ and $x_k = 0$ for all but a finite number of k. The symplectic space $(V_\infty, \omega_\infty)$ is defined by $V_\infty = F^\infty \oplus F^\infty$ and

$$\omega_\infty((x, y), (x', y')) = \sum_{k=1}^{\infty} y_k x'_k - x_k y'_k \,.$$

$(V_\infty, \omega_\infty)$ is a countably generated infinite dimensional symplectic vector space, and it has a natural symplectic basis $\{e_j, f_k\}$, defined by

$$e_k = (u_k, 0)\,, \qquad f_k = (0, u_k)\,, \qquad k = 1, 2, \ldots$$

where u_k is the standard unit vector $u_k(j) = \delta_{kj}$.

For $n = 2r$ finite, we take V_n to be the $2r$ dimensional subspace $F^r \oplus F^r \subseteq V_\infty$ and define ω_n by restricting ω_∞ to V_n.

The following result implies that any two countably generated symplectic vector spaces of the same dimension are congruent.

Theorem 2.1. *Let F be a field of arbitrary characteristic.*

(1) *Every countably generated symplectic vector space (V, ω) over F has a symplectic basis. When the dimension of V is finite it must be even*, $\dim V = 2r$, $r = 1, 2, \ldots$.

(2) *Let ω be a skew-symmetric bilinear form on a countably generated vector space V, let K be the kernel of ω and let L be any vector space complement $V = K \oplus L$. Then the restriction ω_L of ω to L is a symplectic form. If L' is any other complement $V = K \oplus L'$, then the symplectic spaces (L, ω_L) and $(L', \omega_{L'})$ are congruent.*

Proof of (1). Assume first that V is finite dimensional and nonzero. Choose any vector $e_1 \neq 0$ in V. By nondegeneracy, there is a vector $f_1 \in V$ with $\omega(e_1, f_1) = 1$. The following result provides the inductive step.

Lemma 2.1. *Let (V, ω) be a finite dimensional symplectic vector space, let $S \subseteq V$ be a subspace such that the restriction of ω to $S \times S$ is nondegenerate, and let K be its symplectic complement*

$$K = \{x \in V : \omega(x, S) = \{0\}\}\,.$$

Then $V = S \oplus K$.

Proof. Obviously, $S \cap K = \{0\}$ because the restriction of ω to $S \times S$ is nondegenerate. We have to show that $V = S + K$, and since the intersection of these two spaces is trivial it suffices to show that $\dim S + \dim K = \dim V$.

Assuming $S \neq \{0\}$, let $v_1, \ldots, v_r$ be a basis for S, and consider the linear map $L : V \to F^r$ defined by

$$Lx = (\omega(x, v_1), \ldots, \omega(x, v_r))\,, \qquad x \in V\,.$$

The kernel of L is K, and we claim that $LV = F^r$. To prove that we show that the only linear functional $f : F^r \to F$ that vanishes on LV is $f = 0$. Indeed, writing

$$f(t_1, \ldots, t_r) = \sum_{k=1}^{r} \lambda_k t_k\,,$$

for certain $\lambda_j \in F$, the vector $v = \sum_k \lambda_k v_k \in S$ satisfies $\omega(x, v) = f(Lx) = 0$ for all $x \in V$. Since ω is nondegenerate we must have $v = 0$, hence $\lambda_1 = \cdots = \lambda_r = 0$, hence $f = 0$. We conclude that

$$\dim V = \dim \operatorname{ran} L + \dim \ker L = \dim F^r + \dim K = \dim S + \dim K\,,$$

since $\dim S = r = \dim F^r$. □

Inductively, suppose we have vectors $e_1, \ldots, e_r, f_1, \ldots, f_r \in V$ which satisfy the symplectic requirements (2.2) insofar as they make sense, and let S be the subspace of V spanned by $\{e_k, f_j : 1 \leq j, k \leq r\}$. Since $\{e_k, f_j\}$ is a symplectic basis for the restriction of ω to $S \times S$, the latter must be nondegenerate. By Lemma 2.1, we have $V = S + K$ where $K = \{x \in V : \omega(x, S) = \{0\}\}$. Thus we can choose a nonzero vector e_{r+1} in K. Since $\omega(e_{r+1}, S) = \{0\}$ and $V = S + K$, there must be a vector $f_{r+1} \in K$ for which $\omega(e_{r+1}, f_{r+1}) = 1$. An inductive argument completes the proof in the case where V is finite dimensional.

Remark 2.2. Notice that the preceding argument implies that in a finite dimensional symplectic vector space (V, ω), any set of vectors $e_1, \ldots, e_r, f_1, \ldots, f_r \in V$, which satisfy the relations (2.2), can be enlarged to a symplectic basis for V. It also shows that a finite dimensional symplectic vector space over an arbitrary field has even dimension $2 \cdot r$, $r = 1, 2, \ldots$.

Turning now to the infinite dimensional case, we claim that there is an increasing sequence of finite dimensional subspaces $E_1 \subseteq E_2 \subseteq \cdots \subseteq V$ with $\cup_n E_n = V$, such that the restriction of ω to $E_n \times E_n$ is nondegenerate for every n. Suppose for the moment that this has been established. The preceding paragraphs show that we can find a symplectic basis for E_1. Since the restriction of ω to $E_2 \times E_2$ is a symplectic form on E_2, the preceding remark implies that this symplectic set can be enlarged to a symplectic basis for E_2. Continuing inductively, we obtain an increasing sequence of symplectic sets, each one being a basis for its corresponding linear span E_n, $n = 1, 2, \ldots$, and their union is a symplectic basis for $\cup_n E_n = V$.

Thus we have reduced the proof of (B1) to showing how to construct such a sequence $E_1 \subseteq E_2 \subseteq \ldots$. In order to carry out the inductive step, we require

Lemma 2.2. *Let (V, ω) be a symplectic vector space and let E be a finite dimensional subspace of V. Then there is a subspace $E' \supseteq E$ of dimension at most $2 \cdot \dim E$ such that the restriction of ω to $E' \times E'$ is nondegenerate.*

Proof. Let $K = \{x \in E : \omega(x, E) = 0\}$ be the kernel of the restriction of ω to $E \times E$, and let $k_1, \ldots, k_r$ be a basis for K. We claim that there are vectors $\ell_1, \ldots, \ell_r \in V$ such that

$$\omega(k_i, \ell_j) = \delta_{ij}\,, \qquad 1 \le i, j \le r\,. \tag{2.3}$$

To see that, consider the r-dimensional vector space $F^r = \{(t_1, \ldots, t_r) : t_i \in F\}$, and consider the linear map $L : V \to F^r$ defined by

$$L(x) = (\omega(k_1, x), \omega(k_2, x), \ldots, \omega(k_r, x))\,, \qquad x \in V\,.$$

We have to show that L is onto: $L(V) = F^r$. To prove that, we show that the only linear functional $f : F^r \to F$ which vanishes on the range of L is the zero functional. Choosing such an f, we can write

$$f(t_1, \ldots, t_r) = \lambda_1 t_1 + \cdots + \lambda_r t_r$$

for a unique r-tuple of scalars $\lambda_k \in F$. Since $f(L(x)) = 0$ for all $x \in V$ we have

$$\omega\left(\sum_{j=1}^{r} \lambda_j k_j, x\right) = \sum_{j=1}^{r} \lambda_j \omega(k_j, x) = f(L(x)) = 0\,.$$

By nondegeneracy, we must have $\sum_j \lambda_j k_j = 0$, hence $\lambda_1 = \cdots = \lambda_r = 0$ because $k_1, \ldots, k_r$ are linearly independent, thus (2.3) is proved.

Setting $L = \operatorname{span}\{\ell_1, \ldots, \ell_r\}$, notice that (2.3) implies that the restriction of ω to $K \times L$ is nondegenerate in the sense that for every $k \in K$,

$$\omega(k, \ell) = 0\,, \qquad \text{for all } \ell \in L \Rightarrow k = 0\,, \tag{2.4}$$

while for every $\ell \in L$,

$$\omega(k, \ell) = 0\,, \qquad \text{for all } k \in K \Rightarrow \ell = 0\,. \tag{2.5}$$

Choose such a set of vectors $\ell_1, \ldots, \ell_r \in V$, let $L = \operatorname{span}\{\ell_1, \ldots, \ell_r\}$, and define $E' = E + L$. We show that the restriction of ω to $E' \times E'$ is nondegenerate. For that, suppose that $z \in E'$ has the property that $\omega(z, z') = 0$ for every $z' \in E'$. We can write $z = x + \ell$ where $x \in E$ and $\ell \in L$. Then

$$\omega(z, z') = \omega(x, z') + \omega(\ell, z') = 0$$

for all $z' \in E'$. Picking $z' \in K$ and noting that $\omega(x, K) = \{0\}$ (by definition of K), we conclude that $\omega(\ell, z') = 0$ for all $z' \in K$. Because of (2.5), we conclude that $\ell = 0$. Hence $\omega(x, E') = \{0\}$. Since $x \in E \subseteq E'$ this implies that x is an element of K for which $\omega(x, E') = 0$. By (2.4), this implies $x = 0$. $\square$

We now construct the sequence E_n. Since V is countably generated there is a spanning sequence of nonzero vectors $v_1, v_2, \ldots \in V$; we will construct an increasing sequence E_n of finite dimensional subspaces such that E_n contains $v_1, \ldots, v_n$ and the restriction of ω to E_n is nondegenerate. Since $v_1 \neq 0$ and ω is nondegenerate, choose any $w \in v$ such that $\omega(v_1, w) = 1$, and set $E_1 = \operatorname{span}\{v_1, w\}$. The restriction of ω to E_1 is nondegenerate because $\{v_1, w\}$ is a symplectic basis.

Suppose now that we have finite dimensional subspaces $E_1 \subseteq \cdots \subseteq E_n$ such that E_k contains $v_1, \ldots, v_k$ and the restriction of ω to each E_k is nondegenerate. Applying Lemma 2.2 to the space spanned by E_n and v_{n+1}, we find a finite dimensional space E_{n+1} containing both v_{n+1} and E_n such that the restriction of ω to $E_{n+1} \times E_{n+1}$ is nondegenerate. An induction completes the proof of Theorem 2.1(1).

In order to prove Theorem 2.1(2), consider the natural symplectic space $(V/K, \omega)$ described above. We claim that for every subspace L of V satisfying $L \cap K = \{0\}$ and $L + K = V$, the symplectic spaces (L, ω_L) and $(V/K, \omega)$ are congruent; i.e. there is a linear isomorphism $T : L \to V/K$ such that

$$\omega(Tx, Ty) = B(x, y) = \omega_L(x, y)\,, \qquad x, y \in L\,, \tag{2.6}$$

where ω_L is the restriction of B to $L \times L$. To see that, define $Tx = x + K$, $x \in L$. T is a linear isomorphism because L is a complement of K, and (2.6) follows because for any $x, y \in V$ we have $\omega(x + K, y + K) = B(x, y)$ by definition of ω, so when $x, y \in L$ we have (2.6).

For any other subspace L' with $V = K \oplus L'$, $(L', \omega_{L'})$ is also congruent to $(V/K, \omega)$, hence it is congruent to (L, ω_L). $\square$

Corollary 2.1. *Any two countably generated symplectic vector spaces of the same dimension $n = 2r$, $r = 1, 2, \ldots, \infty$ are congruent.*

Proof. Let (V, ω) be a symplectic vector space of dimension $n = 2r$, $r = 1, 2, \ldots, \infty$. By Theorem 2.1, we can find a (finite or infinite) symplectic basis $\{e_k, f_j\}$ for V, and once we have that there is an obvious way to transform (V, ω) congruently to the standard example (V_n, ω_n). $\square$

2.2. *Examples of commutation matrices*

The above results have concrete implications about how to exhibit sequences of unitary operators that generate the infinite dimensional CAR algebra; they also provide a systematic method for generating all possible skew-symmetric matrices $C = (c_{ij})$ with entries in $\mathbb{Z}_2$ which are *nondegenerate* in the sense that their associated bilinear forms

$$\omega_C(x, y) = \sum_{i,j=1}^{\infty} c_{ij} x_j y_i\,, \qquad x, y \in \Gamma \tag{2.7}$$

have trivial kernel. We abuse our own terminology somewhat in calling such a matrix C *symplectic*. Starting with any countably infinite symplectic vector space (V, ω)

over $\mathbb{Z}_2$, such as the standard example $(V_\infty, \omega_\infty)$ described above, let $v_1, v_2, \ldots$ be any Hamel basis for V and define

$$c_{ij} = \omega(v_i, v_j)\,, \qquad i, j = 1, 2, \ldots\,.$$

One verifies directly that $C = (c_{ij})$ is a symplectic matrix. Moreover, the Corollary of Theorem 2.1 implies that every symplectic matrix arises in this way from some basis $v_1, v_2, \ldots$ for V.

One can view this construction in more concrete operator-theoretic terms by making use of the standard self-adjoint generators of the CAR algebra as follows. Consider the Clifford algebra $\mathcal{C}$ generated by an infinite sequence $W_1, W_2, \ldots$ of unitary operators satisfying

$$W_i W_j + W_j W_i = 2\delta_{ij}\mathbf{1}\,, \qquad i, j = 1, 2, \ldots\,.$$

Since W_i and W_j anticommute when $i \neq j$, the commutation matrix $A = (a_{ij})$ associated with a Clifford sequence is

$$a_{ij} = \begin{cases} 1\,, & i \neq j\,, \\ 0\,, & i = j\,, \end{cases}$$

and its associated form is

$$\omega_A(x, y) = \sum_{p \neq q} x_q y_p = \left(\sum_k x_k\right)\left(\sum_k y_k\right) - \sum_k x_k y_k\,.$$

One verifies easily that ω_A is nondegenerate. Choosing an arbitrary Hamel basis $v_1, v_2, \ldots$ for Γ, we obtain the most general symplectic matrix $C = (c_{ij})$ as follows

$$c_{ij} = \omega_A(v_i, v_j) = \sum_{p \neq q} v_i(q) v_j(p)\,. \tag{2.8}$$

Each element v_k in this basis is associated with a word in the original sequence (W_n), namely $U_k = W_1^{v_k(1)} W_2^{v_k(2)} \ldots$. The unitary operators $U_1, U_2, \ldots$ satisfy

$$U_i U_j = (-1)^{c_{ij}} U_j U_i \qquad i, j = 1, 2, \ldots \tag{2.9}$$

and, after multiplication by suitable phase factors, $U_1, U_2, \ldots$ becomes a spin system which generates the Clifford algebra $\mathcal{C}$.

3. The Universal C^*-algebra

The purpose of this section is to prove

Theorem 3.1. *Let $p = 2, 3, \ldots$ be a prime and let $u_1, u_2, \ldots$ be a universal sequence of unitary operators satisfying $u_k^p = \mathbf{1}$ for all k and the commutation relations* (1.2). *Let $\omega : \Gamma \times \Gamma \to \mathbb{Z}_p$ be the skew-symmetric form* (1.3) *and let $n = 2r$ be its rank, $r = 1, 2, \ldots, \infty$.*

Then $C^*(u_1, u_2, \ldots)$ is isomorphic to $C(X) \otimes \mathcal{B}$, where X is a totally disconnected compact metrizable space, and where $\mathcal{B} = M_{p^r}(\mathbb{C})$ if r is finite and is a UHF algebra of type p^∞ if $r = \infty$.

The center $C(X)\otimes\mathbf{1}$ is the closed linear span of the set of words $\{w_x : x \in \ker\omega\}$. $C^*(u_1, u_2, \ldots)$ is simple iff its center is trivial if and only if ω is a symplectic form.

Remark 3.1. R 3.1 Since every quotient of $C(X)$ for X a compact totally disconnected metrizable space is of the form $C(Y)$ for Y of the same type, it follows that any sequence of unitary operators $U_1, U_2, \ldots$ that satisfies $U_k^p = \mathbf{1}$ and the relations (1.2), whether it is universal or not, must generate a C^*-algebra of the same general type $C(Y)\otimes\mathcal{B}$ as the universal one $C(X)\otimes\mathcal{B}$. If $\{U_1, U_2, \ldots\}$ is irreducible and ω is of infinite rank, then X reduces to a point and $C^*(U_1, U_2, \ldots)$ is a UHF algebra of type p^∞.

A version of Theorem 3.1 was proved in [6] in the case of a spin system of characteristic 2 with translation-invariant relations (1.2), i.e. $c_{ij} = c_{i+k,j+k}$ for any i and j and any non-negative integer k. In that setting it is shown that $C^*(u_1, u_2, \ldots)$ is isomorphic to the CAR-algebra if and only if the sequence $\ldots, c_{13}, c_{12}, c_{11}, c_{12}, c_{13}, \ldots$ is *not* periodic. Hence the aperiodicity of this sequence is equivalent to ω being a symplectic form in the statement of Theorem 3.1. The result in this special case was also obtained in [9] using a much different approach.

Before giving the proof of Theorem 3.1, we require two elementary results.

Lemma 3.1. *L 3.1 Let $\mathcal{A}$ be a unital C^*-algebra which is generated by two mutually commuting unital C^*-subalgebras $\mathcal{Z}$, $\mathcal{B}$ with the properties*

(i) *$\mathcal{Z}\cong C(X)$ is commutative, and*
(ii) *$\mathcal{B}$ is a UHF algebra.*

Then $\mathcal{Z}$ is the center of $\mathcal{A}$ and $\mathcal{A}\cong C(X)\otimes\mathcal{B}$.

Proof. The proof is straightforward and we merely sketch the argument. Suppose first that the subalgebra $\mathcal{B}$ is finite dimensional, hence isomorphic to the matrix algebra $M_n(\mathbb{C})$ for some $n = 1, 2, \ldots$. Pick a set of matrix units e_{ij}, $1 \le i, j \le n$ for $\mathcal{B}$. Thus $e_{ij}e_{kl} = \delta_{jk}e_{il}$, $e_{ij}^* = e_{ji}$, and $e_{11} + \cdots + e_{nn} = \mathbf{1}$. Using these relations and the fact that the elements of $\mathcal{Z}$ commute with the e_{ij} one finds that for arbitrary $z_{ij}\in\mathcal{Z}$, $1 \le i, j \le n$,

$$\sum_{i,j=1}^{n} z_{ij}e_{ij} = 0 \Rightarrow z_{ij} = 0\,, \qquad \text{for all } 1 \le i, j \le n.$$

Thus if we consider $\mathcal{Z}\otimes\mathcal{B}$ to be the C^*-algebra $M_n(\mathcal{Z})$ then the preceding observation shows that the natural $*$-homomorphism $\pi : M_n(\mathcal{Z}) \to \mathcal{A}$ defined by

$$\pi((z_{ij})) = \sum_{i.j=1}^{n} z_{ij}e_{ij}$$

is injective; it also has dense range, hence it is a $*$-isomorphism which carries the center of $M_n(\mathcal{Z})$ onto $\mathcal{Z}$.

In the general case, $\mathcal{B}$ is the norm closure of an increasing sequence of algebras $\mathcal{B}_n$ of the above type. The preceding argument shows that the natural surjective $*$-homomorphism $\pi : \mathcal{Z} \otimes \mathcal{B} \to \mathcal{A}$ restricts to an isometric $*$-homomorphism on each $\mathcal{Z} \otimes \mathcal{B}_n$, hence it is an isometric $*$-isomorphism. □

Lemma 3.2. *L 3.2 Let p be a positive integer and let V and W be unitary operators in some C^*-algebra satisfying $V^p = W^p = \mathbf{1}$ and $VW = \zeta WV$ where $\zeta = e^{2\pi i/p}$. Then $C^*(V, W) \cong M_p(\mathbb{C})$.*

Proof. Since $W^p = \mathbf{1}$, the spectrum $\sigma(W)$ of W is contained in the set of pth roots of unity, and because $VWV^{-1} = \zeta W$, $\sigma(W)$ is invariant under multiplication by ζ. Hence $\sigma(W) = \{1, \zeta, \zeta^2, \ldots, \zeta^{p-1}\}$. Letting P_k be the spectral projection of W corresponding to the eigenvalue ζ^k, $k = 0, 1, \ldots, p-1$, the commutation relation $V^i W V^{-i} = \zeta^i W^j$ implies that $V^i P_j = P_{i+j} V^i$, where the sum $i + j$ is interpreted modulo p. Together with $V^p = \mathbf{1}$, this implies that the operators $e_{ij} = V^{i-j} P_j$, $0 \leq i, j \leq p-1$, are a set of $p \times p$ matrix units which have $C^*(V, W)$ as their linear span. □

Proof of Theorem 3.1. Fix a universal sequence $u_1, u_2, \ldots$ as above and let $\mathcal{Z}$ be the closed linear span of the words of the form $w_x = u_1^{x_1} u_2^{x_2} \ldots$ where $x \in \ker \omega$. Notice that because $w_x w_y = (-1)^{\omega(x,y)} w_y w_x$, it follows that every word w_x with $x \in \ker \omega$ belongs to the center of $C^*(u_1, u_2, \ldots)$. Note that for each $x \in \Gamma$ we can choose a scalar $\lambda_x \in \mathbb{T}$ with the property that $(\lambda_x w_x)^p = \mathbf{1}$. It is possible to do this because the relation $w_s w_t = \zeta^{Q(s,t)} w_{s+t}$ implies that w_x^p is a scalar multiple of $w_{px} = w_0 = \mathbf{1}$. One can specify λ_x explicitly, but it is not necessary to do so. Thus $\mathcal{Z}$ is a commutative AF algebra isomorphic to $C(X)$ for X a compact metrizable totally disconnected space. Because of Lemma 3.1, it is enough to show that there is a UHF algebra $\mathcal{B} \subseteq C^*(u_1, u_2, \ldots)$ of the asserted type such that $C^*(u_1, u_2, \ldots)$ is generated by $\mathcal{Z} \cup \mathcal{B}$.

By Theorem 2.1, Γ decomposes into a direct sum of vector spaces $\Gamma = \ker \omega \oplus L$, where the restriction of ω to $L \times L$ is a symplectic form, and where $\dim L$ is the rank of ω. Since L is a vector space, the relation $w_x w_y = \zeta^{Q(x,y)} w_{x+y}$ implies that $\mathcal{B} = \overline{\mathrm{span}}\{w_x : x \in L\}$ is a C^*-subalgebra of $\mathcal{A}$. Moreover, since $\Gamma = \ker \omega + L$, the set of products of words of the form $w_x w_y = \zeta^{Q(x,y)} w_{x+y}$, $x \in \ker \omega$, $y \in L$ have $\mathcal{A}$ as their closed linear span. Thus $\mathcal{Z} \cup \mathcal{B}$ generates $\mathcal{A}$.

It remains to show that $\mathcal{B}$ is a UHF algebra of the asserted type. Suppose first that $\dim L = 2r$ is finite. By Theorem 2.1, we can find a symplectic basis $e_1, \ldots, e_r, f_1, \ldots, f_r$ for the symplectic vector space (L, ω_L) obtained by restricting ω to L. Consider the operators $V_1, \ldots, V_r, W_1, \ldots, W_r$ defined by

$$V_k = \lambda_{e_k} w_{e_k}\,, \qquad W_k = \lambda_{f_k} w_{f_k}\,, \tag{3.1}$$

where the scalars λ_x are as above. Every $x \in L$ is a linear combination of elements of $e_1, \ldots, e_r, f_1, \ldots, f_r$, hence the set of all products $V_1^{m_1} \ldots V_r^{m_r} W_1^{n_1} \ldots, W_r^{n_r}$,

$m_1, \ldots, m_r, n_1, \ldots, n_r = 0, 1, \ldots, p-1$, spans $\mathcal{B}$. We have already arranged that $V_k^p = W_k^p = \mathbf{1}$ for every k. Note that for all $i, j = 1, \ldots, r$

$$V_i V_j = V_j V_i \,, \qquad W_i W_j = W_j W_i \,, \qquad V_i W_j = \zeta^{\delta_{ij}} W_j V_i \,, \tag{3.2}$$

δ_{ij} denoting the Kronecker delta. Indeed, these relations are immediate consequences of the basic formula $w_x w_y = \zeta^{\omega(x,y)} w_y w_x$ and the fact that $\{e_i, f_j\}$ is a symplectic set for ω. It follows from (3.2) that the C^*-algebras $C^*(V_i, W_i)$ and $C^*(V_j, W_j)$ commute for $i \neq j$; and by Lemma 3.2 each $C^*(V_k, W_k)$ is isomorphic to $M_p(\mathbb{C})$. Thus $\mathcal{B}$ is isomorphic to a tensor product of r compies of $M_p(\mathbb{C})$.

If $\dim L$ is infinite, then another application of Theorem 2.1 provides an infinite symplectic basis $e_1, e_2, \ldots, f_1, f_2, \ldots$ for L. We define $V_1, V_2, \ldots, W_1, W_2, \ldots$ by (3.1) as before, and these operators satisfy (3.2). In this case, the C^*-algebra $\mathcal{B}$ generated by V_i, W_j commutes with $\mathcal{Z}$, and is generated by an increasing sequence of subalgebras $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots$

$$\mathcal{B}_n = C^*(V_1, \ldots, V_n, W_1, \ldots, W_n) \,, \qquad n = 1, 2, \ldots \,.$$

The preceding paragraph shows that $\mathcal{B}_n$ is isomorphic $M_{p^n}(\mathbb{C})$. Hence $\mathcal{B}$ is a UHF algebra of type p^∞.

The assertions of the third paragraph of Theorem 3.1 are obvious consequences of what has already been proved. □

4. Irreducible Spin Systems

Let $C = (c_{ij})$ be a matrix of zeros and ones, fixed throughout the remainder of this section; in order to rule out the degeneracies described in the introduction, we also assume that (c_{ij}) is of infinite rank. Thus, the $\mathbb{Z}_2$-valued bilinear form

$$\omega(x, y) = \sum_{p,q=1}^{\infty} c_{pq} x_q y_p \,, \qquad x, y \in \Gamma \tag{4.1}$$

associated with $C = (c_{ij})$ has the property that $\Gamma / \ker \omega$ is infinite dimensional, $\ker \omega$ being the linear subspace $\{x \in \Gamma : \omega(x, \Gamma) = \{0\}\} \subseteq \Gamma$.

The purpose of this section is to classify the irreducible spin systems associated with C. Thus we consider irreducible spin systems $\bar{U} = (U_1, U_2, \ldots)$ acting on an infinite dimensional Hilbert space H, satisfying

$$U_i U_j = (-1)^{c_{ij}} U_j U_i \,, \qquad i, j = 1, 2 \ldots \,. \tag{4.2}$$

Theorem 3.1 implies that $C^*(U_1, U_2, \ldots)$ is the CAR algebra, and since the CAR algebra is a simple C^*-algebra not of type I, there can be no meaningful classification of such sequences $\bar{U}$ up to unitary equivalence. The equivalence relation that is appropriate for irreducible spin systems is weaker than unitary equivalence, and is defined as follows. Two spin systems $\bar{U}$ and $\bar{V}$, acting on infinite dimensional Hilbert spaces H and K, respectively, are said to be equivalent (written $\bar{U} \sim \bar{V}$) if there is a sequence of unitary operators $W_1, W_2, \ldots : H \to K$ such that

$$\lim_{n \to \infty} \| W_n U_k W_n^{-1} - V_k \| = 0 \qquad k = 1, 2, \ldots \,.$$

We first introduce an invariant for irreducible spin systems $\bar{U}$. For every $x \in \Gamma$ there is a word

$$W_x = U_1^{x_1} U_2^{x_2} \dots , \qquad x \in \Gamma ,$$

and we have $W_x W_y = (-1)^{Q(x,y)} W_{x+y}$ for all $x, y \in \Gamma$, where $Q : \Gamma \times \Gamma \to \mathbb{Z}_2$ is the bilinear form (1.5). If $x \in \ker \omega$ then by (1.4), W_x commutes with all words, and by irreducibility it must be a scalar multiple of the identity

$$W_x = f(x)\mathbf{1} , \qquad x \in \ker \omega .$$

This defines a function $f : \ker \omega \to \mathbb{T}$ satisfying the functional equation

$$f(x)f(y) = (-1)^{Q(x,y)} f(x+y) , \qquad x, y \in \ker \omega . \tag{4.3}$$

f is called the **standard invariant** associated with the irreducible spin system $\bar{U}$. Notice that (4.3) implies that $f(0) = 1$. Since $f(x)^2 = (-1)^{Q(x,x)} f(2x) = (-1)^{Q(x,x)} f(0) = \pm 1$, it follows that f must take values in the multiplicative group of fourth roots of unity,

$$f(x)^4 = 1 , \qquad x \in \ker \omega .$$

Proposition 4.1. *Let $\bar{U} = (U_1, U_2, \dots)$ and $\bar{U}' = (U_1', U_2', \dots)$ be two irreducible spin systems on Hilbert spaces H, H' which satisfy the relations C, and let π, π' be the representations of the universal C^*-algebra $\mathcal{A}_C = C^*(u_1, u_2, \dots)$ defined by $\pi(u_k) = U_k$, $\pi'(u_k) = U_k'$, $k = 1, 2, \dots$. The following are equivalent.*

(i) $\bar{U} \sim \bar{U}'$.
(ii) $\ker \pi = \ker \pi'$.
(iii) *$\bar{U}$ and $\bar{U}'$ have the same standard invariant.*
(iv) *For every $n = 1, 2, \dots$, there is a unitary operator $W_n : H \to H'$ such that*

$$W_n U_k W_n^{-1} = U_k' , \qquad k = 1, 2, \dots, n .$$

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i) are straightforward. We prove (iii) $\Rightarrow$ (iv). For that we require another elementary result (see [1]).

Lemma 4.1. *Let $\mathcal{B}$ be a finite dimensional C^*-algebra and let π_1, π_2 be two faithful nondegenerate representations of $\mathcal{B}$ on Hilbert spaces H_1, H_2 such that $\pi_j(\mathcal{B}) \cap \mathcal{K}_j = \{0\}$ for $j = 1, 2$, $\mathcal{K}_j$ denoting the compact operators on H_j. Then π_1 and π_2 are unitarily equivalent.*

Proof. For every nonzero minimal central projection E of $\mathcal{B}$ there is a (finite dimensional) irreducible representation σ_E of $\mathcal{B}$ which carries E to $\mathbf{1}$. Let σ be the direct sum $\oplus_E \sigma_E$. σ is a finite dimensional representation having uniform multiplicity one; and it is enough to show that any faithful nondegenerate representation π of $\mathcal{B}$ on a Hilbert space H, for which $\pi(\mathcal{B})$ contains no nonzero compact operators, is unitarily equivalent to an infinite direct sum of copies of σ.

Fixing such a representation π, for every nonzero central projection $E \in \mathcal{B}$, $\pi(E)$ must be an infinte dimensional projection, hence it defines a subrepresentation on $\pi(E)H$ which is unitarily equivalent to an infinite direct sum of copies of σ_E. Since π is nondegenerate, these subrepresentations sum to π, hence π is equivalent to an infinite direct sum of copies of σ. □

Let f and f' be the respective standard invariants of $\bar{U}$ and $\bar{U}'$. Assuming that $f = f'$ as in (iii), we have to verify (iv), and by replacing the spin system $\bar{U}'$ with a unitarily equivalent one, we may assume that both $\bar{U}$ and $\bar{U}'$ act on the same Hilbert space. Consider the words $w_x = u_1^{x_1} u_2^{x_2} \ldots$ in $\mathcal{A}_C$ corresponding to elements $x \in \ker \omega$. Since $f = f'$ we have

$$\pi(w_x) = U_1^{x_1} U_2^{x_2} \cdots = f(x)\mathbf{1} = f'(x)\mathbf{1} = \pi'(w_x)\,, \qquad x \in \ker \omega\,.$$

Since by Theorem 3.1, the central words of this type have the center $\mathcal{Z}$ of $\mathcal{A}_C$ as their closed linear span, it follows that $\pi \restriction_{\mathcal{Z}} = \pi' \restriction_{\mathcal{Z}}$. Theorem 3.1 also implies that $\mathcal{A}_C$ is isomorphic to $C(X) \otimes \mathcal{C}$ where $\mathcal{C}$ is the CAR algebra, hence any two irreducible representations that agree on the center must have the same kernel (corresponding to some point $p \in X$). Hence $\ker \pi = \ker \pi'$.

It follows that for every operator $A \in \mathcal{A}_C$ we have $\|\pi(A)\| = \|\pi'(A)\|$, hence there is a unique $*$-isomorphism $\alpha : C^*(U_1, U_2, \ldots) \to C^*(U_1', U_2', \ldots)$ such that $\alpha \circ \pi = \pi'$. Both of these are simple unital C^*-algebras (they are isomorphic to the CAR algebra), and hence contain no nonzero compact operators. Noting that the restriction of α to $C^*(U_1, \ldots, U_n)$ is a $*$-isomorphism onto $C^*(U_1', \ldots, U_n')$ which carries the n-tuple of operators $(U_1, \ldots, U_n)$ to $(U_1', \ldots, U_n')$, an application of Lemma 4.1 implies that each of these restrictions is implemented by a unitary operator $W_n \in \mathcal{B}(H)$, and (iv) follows. □

We now discuss how the irreducible spin systems associated with a commutation matrix C can be described and classified in terms of any one of them. For any irreducible spin system $\bar{U} = (U_1, U_2, \ldots)$, we consider the spin systems that can be obtained from it by changing phases as follows. For every sequence of numbers $\gamma = (\gamma_1, \gamma_2, \ldots)$ in $\{0, 1\} = \mathbb{Z}_2$ consider the sequence of unitary operators

$$\bar{U}^\gamma = ((-1)^{\gamma_1} U_1, (-1)^{\gamma_2} U_2, \ldots)\,.$$

It is clear that $\bar{U}^\gamma$ is an irreducible spin system satisfying the same commutation relations as $\bar{U}$. We now show that these "phase shifted" versions of $\bar{U}$ provide all possible standard invariants.

Lemma 4.2. *Let $\bar{U}$ be an irreducible spin system, let $f : \ker \omega \to \mathbb{T}$ be its standard invariant, and let $g : \ker \omega \to \mathbb{T}$ be any function satifying the same functional Eq.* (4.3)

$$g(x)g(y) = (-1)^{Q(x,y)} g(x+y)\,, \qquad x, y \in \ker \omega\,.$$

Then there is a $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathbb{Z}_2^\infty$ such that g is the standard invariant of $\bar{U}^\gamma$.

Proof. For $\gamma \in \mathbb{Z}_2^\infty$, we can express the standard invariant f^γ of $\bar{U}^\gamma$ in terms of the standard invariant f of $\bar{U}$ as follows. For every $x \in \Gamma$ the word for $\bar{U}^\gamma$ is

$$(-1)^{\sum_k \gamma_k x_k} U_1^{x_1} U_2^{x_2} \dots ,$$

hence for $x \in \ker \omega$ we have

$$f^\gamma(x) = (-1)^{\sum_k \gamma_k x_k} f(x) . \tag{4.4}$$

Now both g and f satisfy (4.3), hence the function $h : \ker \omega \to \mathbb{T}$ defined by $h(x) = g(x)/f(x)$ satisfies

$$h(x+y) = h(x)h(y) , \qquad x, y \in \ker \omega .$$

Notice too that since $x + x = 0$ for all $x \in \ker \omega$ we have $h(x)^2 = h(x)h(x) = h(x+x) = h(0) = 1$. It follows that $h(x) = \pm 1$ for all $x \in \ker \omega$. Thus there is a unique function $\theta : \ker \omega \to \{0, 1\} = \mathbb{Z}_2$ satisfying

$$g(x)/f(x) = h(x) = (-1)^{\theta(x)} , \qquad x \in \ker \omega , \tag{4.5}$$

and we have $\theta(x+y) = \theta(x) + \theta(y)$ relative to the addition in the field $\mathbb{Z}_2$ because $h(x+y) = h(x)h(y)$ for $x, y \in \ker \omega$.

We may consider $\theta : \ker \omega \to \mathbb{Z}_2$ as a linear functional defined on the vector space $\ker \omega \subseteq \Gamma$. A familiar argument implies that a linear functional defined on a subspace of a vector space can be extended to a linear functional defined on the entire space. Thus we may find a function $\tilde{\theta} : \Gamma \to \mathbb{Z}_2$ such that $\tilde{\theta}(x+y) = \tilde{\theta}(x) + \tilde{\theta}(y)$ for all $x, y \in \Gamma$ and which restricts to θ on $\ker \omega$. Letting $u_1, u_2, \dots$ be the usual basis of unit vectors for Γ, $u_k(j) = \delta_{kj}$, we define $\gamma = (\gamma_1, \gamma_2, \dots) \in \mathbb{Z}_2^\infty$ by $\gamma_k = \tilde{\theta}(u_k)$, $k = 1, 2, \dots$. For every $x = (x_1, x_2, \dots) \in \Gamma$ we have $\tilde{\theta}(x) = \sum_{k=1}^\infty \tilde{\theta}(u_k) x_k = \sum_{k=1}^\infty \gamma_k x_k$. Substituting the latter into (4.5), we find that

$$g(x) = (-1)^{\theta(x)} f(x) = (-1)^{\sum_{k=1}^\infty \gamma_k x_k} f(x) , \qquad x \in \ker \omega .$$

By (4.4), this is the standard invariant f^γ of $\bar{U}^\gamma$. □

Theorem 4.1. *Let $\bar{U}$ be any irreducible spin system satisfying the commutation relations* (1.1) *and let ω be the skew-symmetric form* (1.3). *Every irreducible spin system satisying the same commutation relations is equivalent to $\bar{U}^\gamma$ for some $\gamma \in \mathbb{Z}_2^\infty$. Given two sequences γ, γ' in $\mathbb{Z}_2^\infty$, the spin systems $\bar{U}^\gamma$ and $\bar{U}^{\gamma'}$ are equivalent iff γ and γ' define the same linear functional on* $\ker \omega$ *in the sense that*

$$\sum_{k=1}^\infty \gamma_k x_k = \sum_{k=1}^\infty \gamma'_k x_k , \qquad x \in \ker \omega .$$

In particular, if $\ker \omega$ *is of finite dimension d as a vector space over $\mathbb{Z}_2$, then there are exactly 2^d equivalence classes of irreducible spin systems associated with C. If* $\ker \omega$ *is infinite dimensional then the set of distinct equivalence classes of irreducible spin systems has the cardinality of the continuum $2^{\aleph_0}$.*

Proof. Fix an irreducible spin system $\bar{U}$ as above, and let γ and γ' be two sequences in $\mathbb{Z}^\infty$. We show first that $\bar{U}^\gamma \sim \bar{U}^{\gamma'} \iff$

$$\sum_{k=1}^{\infty}(\gamma_k)x_k = \sum_{k=1}^{\infty}\gamma'_k x_k\,, \qquad x \in \ker\omega\,. \tag{4.6}$$

Indeed, lettting f^γ and $f^{\gamma'}$ be the standard invariants for $\bar{U}^\gamma$ and $\bar{U}^{\gamma'}$, we see from (4.4) that

$$f^\gamma(x) = (-1)^{\sum_k \gamma_k x_k} f(x)\,, \qquad f^{\gamma'}(x) = (-1)^{\sum_k \gamma'_k x_k} f(x)\,, \qquad x \in \ker\omega$$

and hence $f^\gamma = f^{\gamma'}$ if and only if (4.6) holds. By the characterization (iii) of Proposition 4.1, that is equivalent to $\bar{U}^\gamma \sim \bar{U}^{\gamma'}$.

Now let $V = (V_1, V_2, \ldots)$ be an arbitrary irreducible spin system associated with C, and let $g : \ker\omega \to \mathbb{T}$ be its standard invariant. Lemma 4.2 implies that there is a $\gamma \in \mathbb{Z}_2^\infty$ such that $g = f^\gamma$, and by Part (iii) of Proposition 4.1, we conclude that $\bar{V} \sim \bar{U}^\gamma$.

It remains only to establish the results on cardinality, and in view of what has been proved, we simply have to count the distinct functions $g : \ker\omega \to \mathbb{T}$ that satisfy the functional Eq. (4.3). Letting f be the standard invariant of $\bar{U}$, the proof of Lemma 4.2 shows that every such g is obtained from it by way of

$$g(x) = (-1)^{\theta(x)} f(x)\,, \qquad x \in \ker\omega\,,$$

where $\theta : \ker\omega \to \mathbb{Z}_2$ is a (necessarily unique) linear functional. Thus the set of all such g is in bijective correspondence with the set of linear functionals on $\ker\omega$. If $\ker\omega$ is of finite dimension d then by choosing a basis $e_1, \ldots, e_d$ for $\ker\omega$ we find that the set of all such θ is in bijective correspondence with the set of all functions from $\{e_1, \ldots, e_d\}$ to $\mathbb{Z}_2$, and the cardinality of that set is 2^d.

If $\ker\omega$ is infinite dimensional, then since it is a countably generated vector space it has a Hamel basis $\{e_1, e_2, \ldots\}$. As in the preceding paragraph, the set of all standard invariants is in bijective correspondence with the set of all linear functionals on $\ker\omega$, which in turn corresponds bijectively with the set of all functions from $\{e_1, e_2, \ldots\}$ to $\mathbb{Z}_2$, a set of cardinality $2^{\aleph_0}$. □

We have indicated in (2.8) how one can generate all possible symplectic matrices $C = (c_{ij})$ over $\mathbb{Z}_2$. When the commutation matrix is symplectic one has the following uniqueness result.

Corollary 4.1. *Let $C = (c_{ij})$ be an infinite matrix of zeros and ones which is skew-symmetric and nondegenerate. Then any two irreducible spin systems satisfying the commutation relations $U_iU_j = (-1)^{c_{ij}} U_jU_i$ are approximately unitarily equivalent.*

Proof. Somce ω_C is nondegenerate, Theorem 4.1 implies that there is just one equivalence class of irreducible spin systems associated with C. □

Remark 4.1. R 4.1 In such cases the C^*-algebra $\mathcal{A}_C$ associated with C is the CAR algebra, and is therefore simple not of type I. In view of Proposition 4.1, the corollary remains valid *verbatim* if one deletes the irreducibility hypothesis.

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References

1. W. Arveson, *An Invitation to C^*-algebras*, Graduate Texts in Mathematics 39, Springer-Verlag, reprinted 1999.
2. O. Bratteli, G. Eliott and P.E.T. Jorgensen, Decomposition of unbounded derivations into invariant and approximately inner parts, *J. Für Reine Angew. Math.* **346** (1984), 166–193.
3. P. Biane, Free hypercontractivity *Commun. Math. Phys.* **184** (1997), 457–474.
4. O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, 2nd edn., Vol. 2, Springer-Verlag 1996.
5. P. E. T. Jorgensen, A structure theorem for Lie algebras of unbounded derivations in C^*-algebras, *Compositio Math.* **52** (1984), 85–98.
6. R. Powers and G. Price, Cocycle conjugacy classes of shifts on the hyperfinite II_1 factor, *J. Funct. Anal.* **121** (1994), 275–295.
7. G. Price, Cocycle conjugacy classes of shifts on the hyperfinite II_1 factor, *J. Operator Theory* **156** (1998), 177–195.
8. G. L. Price, Shifts on the hyperfinite II_1 factor, *J. Funct. Anal.* **156** (1998), 121–169.
9. S. Vik, Fock representation of the binary shift algebra, *Math. Scand.* **88** (2001), 257–278.