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THE HEAT FLOW OF THE CCR ALGEBRA

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Abstract

Let $Pf(x) = -if'(x)$ and $Qf(x) = xf(x)$ be the canonical operators acting on an appropriate common dense domain in $L^2(\mathbb{R})$. The derivations $D_P(A) = i(PA - AP)$ and $D_Q(A) = i(QA - AQ)$ act on the $*$-algebra $\mathscr{A}$ of all integral operators having smooth kernels of compact support, for example, and one may consider the noncommutative 'Laplacian', $L = D_P^2 + D_Q^2$, as a linear mapping of $\mathscr{A}$ into itself.

L generates a semigroup of normal completely positive linear maps on $\mathscr{B}(L^2(\mathbb{R}))$, and this paper establishes some basic properties of this semigroup and its minimal dilation to an E_0-semigroup. In particular, the author shows that its minimal dilation is pure and has no normal invariant states, and he discusses the significance of those facts for the interaction theory introduced in a previous paper.

There are similar results for the canonical commutation relations with n degrees of freedom, where $1 \leqslant n < \infty$.

1. *Discussion and basic results*

Consider the canonical operators P and Q, acting on an appropriate common dense domain in $L^2(\mathbb{R})$:

$$P = \frac{1}{i} \cdot \frac{d}{dx}, \qquad Q = \text{multiplication by } x.$$

These operators can be used to define unbounded derivations (say on the dense $*$-algebra $\mathscr{A}$ of all integral operators having kernels which are smooth and of compact support) by

$$D_P(X) = i(PX - XP), \qquad D_Q(X) = i(QX - XQ), \quad \text{where } X \in \mathscr{A}.$$

Thinking of these derivations as noncommutative counterparts of $\partial/\partial x$ and $\partial/\partial y$, we define a 'Laplacian' $L : \mathscr{A} \to \mathscr{A}$ by

$$L = D_P^2 + D_Q^2. \tag{1.1}$$

Throughout this paper, we shall use the term *CP semigroup* to denote a semigroup $\phi = \{\phi_t : t \geqslant 0\}$ of normal completely positive linear maps on the algebra $\mathscr{B}(H)$ of all bounded operators on a separable Hilbert space H, which preserves the unit $\phi_t(\mathbf{1}) = \mathbf{1}$, and which is continuous in the natural sense (namely $\langle \phi_t(A)\xi, \eta \rangle$ should be continuous in t for fixed $\xi, \eta \in H$ and $A \in \mathscr{B}(H)$). The purpose of this section is to exhibit concretely a CP semigroup whose generator can be identified with the operator mapping L of equation (1.1); see Theorem 1.10.

Let $U_t = e^{itQ}$ and $V_t = e^{itP}$ be the two unitary groups associated with Q, P:

$$U_t f(x) = e^{itx} f(x), \qquad V_t f(x) = f(x + t), \quad \text{where } f \in L^2(\mathbb{R}).$$

These two groups satisfy the canonical commutation relations $V_t U_s = e^{ist} U_s V_t$ for

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$s, t \in \mathbb{R}$. It is more convenient to make use of the CCRs in Weyl's form. For every $z = (x, y) \in \mathbb{R}^2$, the Weyl operator

$$W_z = e^{ixy/2} U_x V_y \tag{1.2}$$

is unitary, is strongly continuous in z, and satisfies the Weyl relations

$$W_{z_1} W_{z_2} = e^{i\omega(z_1,z_2)} W_{z_1+z_2}, \tag{1.3}$$

where ω is the symplectic form on $\mathbb{R}^2$ given by

$$\omega((x, y), (x', y')) = \tfrac{1}{2}(x'y - xy'). \tag{1.4}$$

A strongly continuous mapping $z \mapsto W_z \in \mathscr{B}(H)$ into the unitary operators on some Hilbert space H which satisfies equation (1.3) is called a *Weyl system*. It is well known that the Weyl system (1.2) is irreducible, and hence the space of all finite linear combinations of the W_z is a unital strongly dense $*$-subalgebra of $\mathscr{B}(L^2(\mathbb{R}))$. The Stone–von Neumann theorem implies that every Weyl system is unitarily equivalent to a direct sum of copies of the concrete Weyl system (1.2).

Proceeding heuristically for a moment, we let D_P and D_Q be the derivations above. After formally differentiating the relation $V_t U_s = e^{ist} U_s V_t$, we find that

$$\begin{array}{rclcrcl} D_P(U_x) & = & ixU_x, & \quad & D_P(V_y) & = & 0, \\ D_Q(U_x) & = & 0, & \quad & D_Q(V_y) & = & iyU_y, \end{array}$$

and hence the action of $L = D_P^2 + D_Q^2$ on the Weyl system (1.2) is given by

$$L(W_z) = -(x^2 + y^2)W_z = -|z|^2 W_z, \quad \text{where } z = (x, y) \in \mathbb{R}^2.$$

After formally exponentiating, we find that for $t \geqslant 0$, the operator mapping $\phi_t = \exp(tL)$ for $t \geqslant 0$ can be expected to satisfy

$$\phi_t(W_z) = e^{-t|z|^2} W_z, \quad \text{where } z \in \mathbb{R}^2, t \geqslant 0. \tag{1.5}$$

REMARKS. A number of authors have considered completely positive semigroups defined on a Weyl system by formulas such as equation (1.5), using techniques similar to those of Proposition 1.7 below (see [**3**, pp. 128–129] or [**4**], for two notable examples). We include a full discussion of these basic issues, since in Section 3 we require details of the construction that are not easily found in the literature.

We also remark that virtually all of the results below have straightforward generalizations to the case in which P and Q are replaced with the canonical operators $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$, associated with n degrees of freedom. Indeed, the generalization amounts to little more than a reinterpretation of notation. On the other hand, while Proposition 1.7 remains valid (with the same proof) for infinitely many degrees of freedom (see [**3**], loc. cit.), we do not know if that is the case for the more precise results of Section 3.

In order to define the CCR heat flow rigorously, we take equation (1.5) as our starting point, and deduce the existence of the semigroup and its basic properties from the following general result. Consider the Banach space $M(\mathbb{R}^2)$ of all complex-valued measures μ on $\mathbb{R}^2$ having finite total variation $\|\mu\|$. We regard $M(\mathbb{R}^2)$ as a commutative Banach algebra with unit relative to the usual convolution of measures

$$\mu * \nu(S) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi_S(z + w)\, d\mu(z)\, d\nu(w).$$

It will be convenient to define the Fourier transform of a measure $\mu \in M(\mathbb{R}^2)$ in terms of the symplectic form ω of equation (1.4):

$$\hat{\mu}(\zeta) = \int_{\mathbb{R}^2} e^{i\omega(\zeta,z)}\, d\mu(z). \tag{1.6}$$

Remark. While this definition of the Fourier transform differs from the usual one, which involves the Euclidean inner product of $\mathbb{R}^2$,

$$\langle (x,y),(x',y')\rangle = xx' + yy',$$

rather than the symplectic form ω, it is equivalent to it in a natural way. Indeed, since ω is nondegenerate, there is a unique invertible skew symmetric linear operator Ω on the two-dimensional real vector space $\mathbb{R}^2$ satisfying $\omega(z,z') = \langle \Omega z, z'\rangle$ for all $z, z' \in \mathbb{R}^2$. Hence one can pass back and forth from the usual Fourier transform of a measure to the one above by the invertible linear change of variables given by composing the transformed measure with either Ω or $\Omega^{-1} = -4\Omega$.

Proposition 1.7. *Let $\{W_z : z \in \mathbb{R}^2\}$ be an irreducible Weyl system acting on a Hilbert space H. For every complex measure $\mu \in M(\mathbb{R}^2)$ there is a unique normal completely bounded linear map $\phi_\mu : \mathscr{B}(H) \to \mathscr{B}(H)$ satisfying*

$$\phi_\mu(W_z) = \hat{\mu}(z) W_z, \quad \text{where } z \in \mathbb{R}^2.$$

One has $\phi_\mu \circ \phi_\nu = \phi_{\mu * \nu}$, and $\|\phi_\mu\|_{cb} \leqslant \|\mu\|$, where $\|\psi\|_{cb}$ denotes the completely bounded norm of an operator mapping ψ. When μ is a positive measure, ϕ_μ is a completely positive map.

Proof. Fix $\mu \in M(\mathbb{R}^2)$. The uniqueness of the mapping ϕ_μ is apparent from the irreducibility hypothesis on the Weyl system, since the set of all linear combinations of the W_z, for $z \in \mathbb{R}^2$, is a unital $*$-algebra which is weak*-dense in $\mathscr{B}(H)$.

For existence, we display $\phi_\mu(A)$ for $A \in \mathscr{B}(H)$ as a weak integral

$$\phi_\mu(A) = \int_{\mathbb{R}^2} W_{2^{-1/2}\zeta} A W^*_{2^{-1/2}\zeta}\, d\mu(\zeta), \tag{1.8}$$

namely the operator defined by the bounded sesquilinear form on the right of equation (1.9):

$$\langle \phi_\mu(A)\xi, \eta\rangle = \int_{\mathbb{R}^2} \left\langle W_{2^{-1/2}\zeta} A W^*_{2^{-1/2}\zeta}\xi, \eta \right\rangle d\mu(\zeta), \quad \text{where } \xi, \eta \in H. \tag{1.9}$$

A straightforward estimate shows that $\|\phi_\mu(A)\| \leqslant \|A\| \cdot \|\mu\|$; after promoting ϕ_μ to $n \times n$ matrices over $\mathscr{B}(H)$, a similar estimate shows that $\|\phi_\mu\|_{cb} \leqslant \|\mu\|$. Obviously, ϕ_μ is completely positive when μ is a positive measure.

Formula (1.9), together with a straightforward application of the bounded convergence theorem, implies that when $A_1, A_2, \ldots$ is a (necessarily bounded) sequence in $\mathscr{B}(H)$ which converges weakly to A, one has

$$\lim_{n\to\infty} \langle \phi_\mu(A_n)\xi, \eta\rangle = \langle \phi_\mu(A)\xi, \eta\rangle, \quad \text{where } \xi, \eta \in H.$$

It follows that ϕ_μ is a normal linear map.

Finally, from the commutation relation (1.3) we find that

$$W_{2^{-1/2}\zeta} W_z W^*_{2^{-1/2}\zeta} = W_{2^{-1/2}\zeta} W_z W_{-2^{-1/2}\zeta} = e^{-\omega(\zeta,z)} W_z,$$

and hence equation (1.8) implies that $\phi_\mu(W_z) = \hat{\mu}(z) W_z$. □

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Theorem 1.10. *Let $W = \{W_z : z \in \mathbb{R}^2\}$ be an irreducible Weyl system. Then there is a unique CP semigroup $\phi = \{\phi_t : t \geqslant 0\}$ satisfying*

$$\phi_t(W_z) = e^{-t|z|^2} W_z, \quad \text{where } z \in \mathbb{R}^2. \tag{1.11}$$

The only bounded normal linear functional ρ for which $\rho \circ \phi_t = \rho$ for all $t \geqslant 0$ is $\rho = 0$. In particular, there is no normal state of $\mathscr{B}(H)$ which is invariant under ϕ.

Proof. For each $t \geqslant 0$, $u_t(z) = e^{-t|z|^2}$ is a continuous function of positive type, which takes the value 1 at $z = 0$. Thus it is the Fourier transform of a unique probability measure $\mu_t \in M(\mathbb{R}^2)$. We shall require an explicit formula for the Gaussian measure μ_t later on; but for purposes of this section we require nothing more than its existence and uniqueness.

Since $u_s(z)u_t(z) = u_{s+t}(z)$ for all $z \in \mathbb{R}^2$, it follows that $\mu_s * \mu_t = \mu_{s+t}$. Hence Proposition 1.7 implies that there is a semigroup $\phi = \{\phi_t : t \geqslant 0\}$ of normal completely positive maps on $\mathscr{B}(H)$ which satisfies equation (1.11). It is a simple matter to check that the required continuity of ϕ_t in t follows from the continuity of the right-hand side of equation (1.11) in t for fixed z.

Suppose now that ρ is a normal linear functional which is invariant under ϕ. Then for every $z \in \mathbb{R}^2$ and every $t \geqslant 0$, the definition of ϕ_t implies that

$$\rho(W_z) = \rho(\phi_t(W_z)) = e^{-t|z|^2}\rho(W_z),$$

and for fixed $z \neq 0$, the right-hand side tends to 0 as $t \to \infty$. Hence $\rho(W_z) = 0$ for every $z \neq 0$; by strong continuity on the unit ball it follows that $\rho(\mathbf{1}) = \omega(W_0) = 0$. Hence ρ vanishes on the irreducible $*$-algebra spanned by W_z, for $z \in \mathbb{R}^2$, and by normality it follows that $\rho = 0$. □

Remarks. We point out that while ϕ has no normal invariant states, it does have a normal invariant weight, namely the trace, in that

$$\operatorname{trace}(\phi_t(A)) = \operatorname{trace}(A)$$

for every positive operator $A \in \mathscr{B}(L^2(\mathbb{R}))$ and every $t \geqslant 0$. One sees this immediately from equation (1.8). It follows that ϕ_t leaves the C^*-algebra $\mathscr{K}$ of all compact operators invariant, where $\phi_t(\mathscr{K}) \subseteq \mathscr{K}$. Since K is the C^*-algebra associated with the canonical commutation relations (more precisely, $\mathscr{K}$ is the enveloping C^*-algebra of the Banach $*$-algebra of all Weyl integral operators associated with the canonical commutation relations with a finite number of degrees of freedom), this justifies viewing the semigroup of restrictions $\{\phi_t \restriction_{\mathscr{K}} : t \geqslant 0\}$ as the heat flow of the canonical commutation relations.

We also remark that one can deduce the existence of other CP semigroups along similar lines. For example, the proof of Theorem 1.10 implies that there is a 'Cauchy' semigroup $\psi = \{\psi_t : t \geqslant 0\}$ which is defined uniquely by the requirement

$$\psi_t(W_z) = e^{-t(|x|+|y|)} W_z, \quad \text{where } z = (x, y) \in \mathbb{R}^2,$$

and which has properties similar to those discussed above for $\phi = \{\phi_t : t \geqslant 0\}$.

2. *Harmonic analysis of the commutation relations*

A classical theorem of Beurling asserts that singletons obey spectral synthesis. More precisely, if G is a locally compact abelian group and f is an integrable

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function on G whose Fourier transform vanishes at a point p in the dual of G, then there is a sequence of functions $f_n \in L^1(G)$ such that $\|f - f_n\| \to 0$ as $n \to \infty$, and such that the Fourier transform of each f_n vanishes identically on some open neighborhood U_n of p. The purpose of this section is to present a noncommutative version of that result, which will be required in Section 3.

Let $\{W_z : z \in \mathbb{R}^2\}$ be an irreducible Weyl system acting on a Hilbert space H. (For example, one may take the Weyl system (1.2) acting on $L^2(\mathbb{R}^2)$.) For every trace-class operator $A \in \mathscr{L}^1(H)$, we consider the following analogue of the Fourier transform $\hat{A} : \hat{\mathbb{R}}^2 \to \mathbb{C}$:

$$\hat{A}(z) = \operatorname{trace}(AW_z), \quad \text{where } z \in \mathbb{R}^2.$$

This transform $A \in \mathscr{L}^1(H) \mapsto \hat{A}$ shares many features in common with the commutative Fourier transform. For example, using the concrete realization (1.2), it is quite easy to establish a version of the Riemann–Lebesgue lemma

$$\lim_{|z|\to\infty} \hat{A}(z) = 0,$$

for every $A \in \mathscr{L}^1(H)$. What we actually require is the following analogue of Beurling's theorem, which lies somewhat deeper.

THEOREM 2.1. *Let $A \in \mathscr{L}^1(H)$, and let $\zeta \in \mathbb{R}^2$ be such that* $\operatorname{trace}(AW_\zeta) = 0$. *There is a sequence $A_n \in \mathscr{L}^1(H)$ and a sequence of open neighborhoods U_n of ζ such that*

$$\operatorname{trace}(A_n W_z) = 0, \quad \text{where } z \in U_n,$$

and such that $\operatorname{trace}|A - A_n| \to 0$ *as $n \to \infty$.*

Proof. By replacing A with AW_ζ and making obvious use of the canonical commutation relations (1.3), we may immediately reduce to the case $\zeta = 0$. We find it more convenient to establish the dual assertion of Theorem 2.1. For that, consider the following linear subspaces of $\mathscr{B}(H)$

$$\mathscr{S}_\epsilon = \overline{\operatorname{span}}\{W_z : |z| \leqslant \epsilon\}, \quad \text{where } \epsilon > 0,$$

the closure being taken relative to the weak* topology on $\mathscr{B}(H)$. Obviously, the spaces $\mathscr{S}_\epsilon$ decrease as ϵ decreases, and the identity operator belongs to $\mathscr{S}_\epsilon$ for every $\epsilon > 0$. The pre-annihilator of $\mathscr{S}_\epsilon$ is identified with the space of all trace-class operators A satisfying

$$\hat{A}(z) = \operatorname{trace}(AW_z) = 0, \quad \text{where } |z| \leqslant \epsilon. \tag{2.2}$$

LEMMA 2.3. *Let $\{W_z : z \in \mathbb{R}^2\}$ be an arbitrary Weyl system acting on a separable Hilbert space H. Then $\cap\{\mathscr{S}_\epsilon : \epsilon > 0\} = \mathbb{C} \cdot \mathbf{1}$.*

Proof of Lemma 2.3. Let $\mathscr{S}_0$ denote the intersection $\cap\{\mathscr{S}_\epsilon : \epsilon > 0\}$. We have already remarked that the inclusion $\supseteq$ is obvious. For the opposite one, consider the von Neumann algebra $\mathscr{M}$ generated by $\{W_z : z \in \mathbb{R}^2\}$. The Stone–von Neumann theorem implies that M is a factor (of type I_∞). We will show that $\mathscr{S}_0$ is contained in the center of $\mathscr{M}$.

For that, choose $T \in \mathscr{S}_0$, and consider the operator-valued function $z \mapsto W_z T W_z^*$. We have to show that this function is constant; equivalently, we shall show that for fixed ξ and η in H, the function

$$z \in \mathbb{R}^2 \mapsto \langle W_z T W_z^* \xi, \eta\rangle \tag{2.4}$$

is constant. Since the function of (2.4) is bounded and continuous, it suffices to show that its spectrum (in the sense of spectral synthesis for functions in $L^\infty(\mathbb{R}^2)$) is the singleton $\{0\}$: this is the dual formulation of Beurling's theorem cited above. Thus we have to show that for every function $f \in L^1(\mathbb{R}^2)$ whose Fourier transform

$$\hat{f}(\zeta) = \int_{\mathbb{R}^2} e^{i\omega(z,\zeta)} f(z)\, dz$$

vanishes throughout a neighborhood of the origin $\zeta = 0$, we have

$$\int_{\mathbb{R}^2} f(z)\,\langle W_z T W_z \xi, \eta\rangle\; dz = 0. \tag{2.5}$$

Fix such an $f \in L^1(\mathbb{R}^2)$ and choose $\epsilon > 0$ small enough so that $\hat{f}(\zeta) = 0$ for all ζ satisfying $|\zeta| \leqslant \epsilon$. Since the linear functional

$$X \in \mathscr{B}(H) \mapsto \int_{\mathbb{R}^2} f(z)\,\langle W_z X W_z^* \xi, \eta\rangle\; dz$$

is weak*-continuous and T belongs to the weak*-closed linear span of operators of the form W_ζ with $|\zeta| \leqslant \epsilon$, to prove equation (2.5) it suffices to show that for every ζ with $|\zeta| \leqslant \epsilon$, we have

$$\int_{\mathbb{R}^2} f(z)\,\langle W_z W_\zeta W_z^* \xi, \eta\rangle\; dz = 0. \tag{2.6}$$

Using the canonical commutation relations, we can write

$$W_z W_\zeta W_z^* = e^{i\omega(z,\zeta)} W_{z+\zeta} W_{-z} = e^{\omega(\zeta,-z)} W_\zeta = e^{i\omega(z,\zeta)} W_\zeta.$$

Hence the left-hand side of equation (2.6) becomes

$$\int_{\mathbb{R}^2} f(z) e^{i\omega(z,\zeta)}\,\langle W_\zeta \xi, \eta\rangle\; dz = \hat{f}(\zeta)\,\langle W_\zeta \xi, \eta\rangle\,,$$

and the latter term vanishes because $\hat{f}(\zeta) = 0$ for $|\zeta| \leqslant \epsilon$. □

To complete the proof of Theorem 2.1, choose an operator $A \in \mathscr{L}^1(H)$ satisfying

$$\hat{A}(0) = \operatorname{trace}(A) = 0,$$

and consider the linear functional ρ defined on $\mathscr{B}(H)$ by $\rho(T) = \operatorname{trace}(AT)$. It is clear that ρ vanishes on $\mathbb{C} \cdot \mathbf{1}$. The linear spaces $\mathscr{S}_\epsilon$ are weak*-closed, and they decrease to $\mathscr{S}_0 = \mathbb{C} \cdot \mathbf{1}$ as ϵ decreases to 0, by Lemma 2.3. Since ρ is weak*-continuous we must have

$$\lim_{\epsilon \to 0} \|\rho \restriction_{\mathscr{S}_\epsilon}\| = \|\rho \restriction_{\mathbb{C}\cdot\mathbf{1}}\| = 0.$$

Thus we can choose a sequence $\epsilon_n \downarrow 0$ so that $\|\rho \restriction_{\mathscr{S}_{\epsilon_n}}\| \leqslant 1/n$ for every $n = 1, 2, \ldots.$ We have already pointed out that the pre-annihilator of $\mathscr{S}_{\epsilon_n}$ is identified with all trace class operators B satisfying

$$\hat{B}(z) = \operatorname{trace}(B W_z) = 0, \quad \text{where } |z| \leqslant \epsilon_n. \tag{2.7}$$

Since $\|\rho \restriction_{\mathscr{S}_{\epsilon_n}}\|$ is the trace-norm distance from A to the pre-annihilator of $\mathscr{S}_{\epsilon_n}$, we conclude that there is a sequence of operators $B_n \in \mathscr{L}^1(H)$ that satisfy $\operatorname{trace}(B_n W_z) = 0$ for $|z| \leqslant \epsilon_n$, such that $\operatorname{trace}|A - B_n| \leqslant 2/n$, as asserted. □

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3. *Purity and dilation theory*

An E_0-semigroup is a CP semigroup $\alpha = \{\alpha_t : t \geqslant 0\}$, acting on $\mathscr{B}(H)$, such that the individual maps are endomorphisms, $\alpha_t(AB) = \alpha_t(A)\alpha_t(B)$, for $A, B \in \mathscr{B}(H)$. An E_0-semigroup α is called *pure* if its 'tail' von Neumann algebra is trivial:

$$\bigcap_{t \geqslant 0} \alpha_t(\mathscr{B}(H)) = \mathbb{C} \cdot \mathbf{1}. \tag{3.1}$$

It is known that an E_0-semigroup is pure if and only if for any pair of normal states ρ_1 and ρ_2 of $\mathscr{B}(H)$, we have

$$\lim_{t \to \infty} \|\rho_1 \circ \alpha_t - \rho_2 \circ \alpha_t\| = 0; \tag{3.2}$$

see [**1**].

If a pure E_0-semigroup α has a normal invariant state ω, then the characterization (3.2) implies that ω must be an *absorbing* state in the sense that, for every normal state ρ of $\mathscr{B}(H)$, one has

$$\lim_{t \to \infty} \|\rho \circ \alpha_t - \omega\| = 0. \tag{3.3}$$

Conversely, if for an arbitrary E_0-semigroup α there is a state ω of $\mathscr{B}(H)$ which is absorbing in the sense that equation (3.3) is satisfied for every normal state ρ of $\mathscr{B}(H)$, then ω must be a normal invariant state, and thus by equation (3.2) α must be a pure E_0-semigroup.

In the theory of interactions worked out in [**2**], pure E_0-semigroups occupy a central position, especially those for which there is a normal invariant (and therefore absorbing) state. A natural question that emerges from the theory of interactions is whether or not every pure E_0-semigroup must have a normal invariant state. Now, since the state space of $\mathscr{B}(H)$ is weak*-compact, a routine application of the Markov–Kakutani fixed-point theorem shows shows that every E_0-semigroup must have invariant states; but invariant states obtained by such methods need not be normal. In this section we exhibit a concrete E_0-semigroup which is pure, but which has no *normal* invariant states. This is a result which was asserted (without proof) in [**2**]. This E_0-semigroup is obtained from the CP semigroup of Theorem 1.10 by a dilation procedure.

In order that the minimal dilation of a CP semigroup to an E_0-semigroup should satisfy equation (3.1), it is necessary and sufficient that the CP semigroup should satisfy property (3.2) (see Proposition 3.5). Thus we generalize the definition of a pure E_0-semigroup as follows.

DEFINITION 3.4. A CP semigroup ϕ acting on $\mathscr{B}(H)$ is called *pure* if for every pair of normal states ρ_1, ρ_2 of $\mathscr{B}(H)$ we have

$$\lim_{t \to \infty} \|\rho_1 \circ \phi_t - \rho_2 \circ \phi_t\| = 0.$$

PROPOSITION 3.5. *Let $\phi = \{\phi_t : t \geqslant 0\}$ be a pure CP semigroup which has no normal invariant state, and let α be its minimal dilation to an E_0-semigroup. Then α satisfies equation* (3.1)*, and has no normal invariant state.*

Proof. The proof is straightforward, but we require results from [**1**]. We find that [**1**, Proposition 2.4] implies that α satisfies equation (3.1).

To see that α has no normal invariant state, we can assume that α acts on $\mathscr{B}(H)$

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for some Hilbert space H, and that there is a closed subspace $K \subseteq H$ such that ϕ is the compression of α onto $\mathscr{B}(K) = P\mathscr{B}(H)P$, where P denotes the projection of H onto K. We have $\alpha_t(P) \uparrow \mathbf{1}$ because α is minimal over P. So if ω is any normal state of $\mathscr{B}(H)$ which is invariant under α, then we have

$$\omega(P) = \lim_{t\to\infty} \omega(\alpha_t(P)) = \omega(\mathbf{1}) = 1.$$

Thus the restriction of ω to $\mathscr{B}(K) = P\mathscr{B}(H)P$ defines a normal ϕ-invariant state on $\mathscr{B}(K)$, contradicting the hypothesis on ϕ. □

In the remainder of this section, we show that the CP semigroup defined in Theorem 1.10 is pure. Once that has been established, Proposition 3.5 implies that its minimal dilation is an E_0-semigroup with properties asserted in the discussion above.

THEOREM 3.6. *The CP semigroup ϕ defined in equation* (1.11) *is pure.*

Before giving the proof, we require the following lemma.

LEMMA 3.7. *For each $t > 0$, let μ_t be the Gaussian measure on $\mathbb{R}^2$ whose Fourier transform* (1.6) *is given by*

$$\hat{\mu}_t(z) = e^{-t|z|^2}, \quad \text{where } z \in \mathbb{R}^2,$$

and choose $\delta > 0$. There is a family ν_t, where $t > 0$, of probability measures on $\mathbb{R}^2$ such that

(i) *$\hat{\nu}_t(z) = 0$ for all $|z| \geqslant \delta$ and every t, and*

(ii) *$\lim_{t\to\infty} \|\mu_t - \nu_t\| = 0$, where $\|\cdot\|$ denotes the norm of the measure algebra $M(\mathbb{R}^2)$.*

Proof. For $t > 0$, measure μ_t is given by $d\mu_t = u_t(x, y)\,dx\,dy$, where u_t is the density

$$u_t(x, y) = \frac{1}{\pi t} e^{-(x^2+y^2)/4t}.$$

The function $f_t = \sqrt{u_t}$ belongs to $L^2(\mathbb{R}^2)$, and in fact $\|f_t\|_2 = 1$.

Choose a function $g \in L^1(\mathbb{R}^2)$ whose Fourier transform

$$\hat{g}(\zeta) = \int_{\mathbb{R}^2} e^{i\omega(\zeta,z)} g(z)\,dz$$

satisfies $0 \leqslant \hat{g}(\zeta) \leqslant 1$ for all ζ, and

$$\hat{g}(\zeta) = \begin{cases} 1, & \text{for } 0 \leqslant |\zeta| \leqslant \delta/4, \\ 0, & \text{for } |\zeta| \geqslant \delta/2. \end{cases}$$

Consider the convolution $g * f_t \in L^2(\mathbb{R}^2)$ and the positive finite measure

$$d\nu_t = |g * f_t|^2\,dx\,dy.$$

We claim first that the Fourier transform of ν_t lives in the disk $|\zeta| \leqslant \delta$. Indeed, letting U_ζ be the unitary operator on $L^2(\mathbb{R}^2)$, and letting T_ζ be the unitary operator on $L^2(\hat{\mathbb{R}}^2)$, given by

$$U_\zeta F(z) = e^{i\omega(\zeta,z)} F(z) \quad \text{and} \quad T_\zeta G(w) = G(w + \zeta),$$

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respectively, we have by the Plancherel theorem

$$\hat{\nu}_t(\zeta) = \langle U_\zeta(g * f_t), g * f_t \rangle_{L^2(\mathbb{R}^2)} = \left\langle T_\zeta(\hat{g}\hat{f}_t), \hat{g}\hat{f}_t \right\rangle_{L^2(\hat{\mathbb{R}}^2)}$$

$$= \int_{\hat{\mathbb{R}}^2} (\hat{g}\hat{f}_t)(w+\zeta)\overline{(\hat{g}\hat{f}_t)}(w)\, dw.$$

When $|\zeta| \geqslant \delta$, the integrand on the right vanishes identically in w because $\hat{g}\hat{f}_t$ is supported in the disk of radius $\delta/2$. Hence $\hat{\nu}_t(\zeta) = 0$ for $|\zeta| \geqslant \delta$.

To establish property (ii), it is enough to show that

$$\lim_{t\to\infty} \|f_t - g * f_t\|_2 = 0, \tag{3.8}$$

since by the Schwarz inequality

$$\begin{aligned} \|\mu_t - \nu_t\| &= \int_{\mathbb{R}^2} |f_t^2 - |g * f_t|^2|\, dz \\ &\leqslant \int_{\mathbb{R}^2} |f_t - g * f_t| \cdot |f_t + |g * f_t||\, dz \\ &\leqslant \|f_t - g * f_t\|_2 \cdot \|f_t + |g * f_t|\|_2 \\ &\leqslant \|f_t - g * f_t\|_2(\|f_t\|_2 + \|g * f_t\|_2). \end{aligned}$$

To establish limit (3.8), we use the Plancherel theorem again to write

$$\int_{\mathbb{R}^2} |f_t(z) - g * f_t(z)|^2\, dz = \int_{\hat{\mathbb{R}}^2} |\hat{f}(\zeta) - \hat{g}(\zeta)\hat{f}_t(\zeta)|^2\, d\zeta = \int_{\mathbb{R}^2} |1 - \hat{g}(\zeta)|^2 \cdot |\hat{f}_t|^2\, d\zeta.$$

The function $|1 - \hat{g}(\zeta)|$ is bounded above by 1, and it vanishes throughout the disk $0 \leqslant |\zeta| \leqslant \delta/4$. Hence the term on the right is dominated by

$$\int_{\{|\zeta|\geqslant\delta/4\}} |\hat{f}_t(\zeta)|^2\, d\zeta. \tag{3.9}$$

In order to estimate the integral (3.9), we require the explicit formula

$$f_t(x,y) = \sqrt{u_t(x,y)} = \frac{1}{\sqrt{\pi t}} e^{-(x^2+y^2)/8t}.$$

The Fourier transform of f_t has the form

$$\hat{f}_t(\zeta) = K\sqrt{t}e^{-2t|\zeta|^2}$$

where K is a positive constant; hence equation (3.9) evaluates to

$$K^2 t \int_{\{|\zeta|\geqslant\delta/4\}} e^{-4t|\zeta|^2}\, d\zeta = K^2 \int_{S_t} e^{-4(u^2+v^2)}\, du\, dv,$$

where $S_t = \{(u,v) : \sqrt{u^2+v^2} \geqslant (\delta/4)\sqrt{t}\}$. As $t \to \infty$, the sets S_t decrease to $\emptyset$, and hence the right-hand side of the previous expression tends to 0, and equation (3.8) is proved.

The positive measures ν_t are not necessarily probability measures, but in view of the established property (ii), $\nu_t(\mathbb{R}^2)$ must be arbitrarily close to $\mu_t(\mathbb{R}^2) = 1$ when t is large. Hence we can rescale ν_t in an obvious way to achieve $\nu_t(\mathbb{R}^2) = 1$ for all $t > 0$, as well as the properties (i) and (ii) of Lemma 3.7. □

Proof of Theorem 3.6. Let W_z, for $z \in \mathbb{R}^2$, be an irreducible Weyl system acting on a Hilbert space H, and let $\phi = \{\phi_t : t \geqslant 0\}$ be the CP semigroup defined by the

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condition

$$\phi_t(W_z) = e^{-t|z|^2} W_z, \quad \text{where } z \in \mathbb{R}^2.$$

Choose a pair of normal states, ρ_1 and ρ_2 on $\mathscr{B}(H)$, and consider their difference, $\omega = \rho_1 - \rho_2$. We have to show that

$$\lim_{t\to\infty} \|\omega \circ \phi_t\| = 0. \tag{3.10}$$

For that, let A be the self-adjoint trace-class operator defined by trace$(AT) = \omega(T)$, for $T \in \mathscr{B}(H)$, and choose $\epsilon > 0$. Since trace$A = 0$, Theorem 2.1 implies that we can find a self-adjoint trace-class operator A_0 such that trace$(A_0 W_z) = 0$ for every z in some neighborhood U of $z = 0$, and trace$|A - A_0| \leqslant \epsilon$. It follows that the normal linear functional $\omega_0(T) = \text{trace}(A_0 T)$ satisfies $\|\omega - \omega_0\| \leqslant \epsilon$ and $\omega_0(W_z) = 0$ for $z \in U$.

By Lemma 3.5, we can find probability measures v_t, for $t > 0$, such that $\hat{v}_t(z)$ vanishes for $z \notin U$ and $\|\mu_t - v_t\|$ tends to 0 as $t \to \infty$. For each $t > 0$, let ψ_t be the completely positive map defined by Proposition 1.7:

$$\psi_t(W_z) = \hat{v}_t(z) W_z, \quad \text{where } z \in \mathbb{R}^2.$$

In order to prove equation (3.10), we decompose the linear functional $\omega \circ \phi_t$ into a sum of three terms as follows:

$$\omega \circ \phi_t = (\omega - \omega_0) \circ \phi_t + \omega_0 \circ (\phi_t - \psi_t) + \omega_0 \circ \psi_t. \tag{3.11}$$

The third term on the right of equation (3.11) is zero, because for every $z \in \mathbb{R}^2$, we have

$$\omega_0(\psi_t(W_z)) = \omega_0(\hat{v}_t(z) W_z) = \hat{v}_t(z)\omega_0(W_z) = 0,$$

since $\hat{v}_t(z)$ vanishes when $z \notin U$ and $\omega_0(W_z)$ vanishes when $z \in U$. (Recall that the linear span of the W_z for $z \in \mathbb{R}^2$ is a strongly dense $*$-subalgebra of $\mathscr{B}(H)$.) The first term on the right of equation (3.11) is estimated for arbitrary t by

$$\|(\omega - \omega_0) \circ \phi_t\| \leqslant \|\omega - \omega_0\| \leqslant \epsilon.$$

In order to estimate the second term, note that

$$\|\phi_t - \psi_t\| \leqslant \|\mu_t - v_t\| \tag{3.12}$$

for every $t > 0$. Indeed, considering the measure $\sigma_t = \mu_t - v_t \in M(\mathbb{R}^2)$, we can write

$$\phi_t(W_z) - \psi_t(W_z) = \hat{\mu}_t(z) W_z - \hat{v}_t(z) W_z = \hat{\sigma}_t(z) W_z.$$

It follows from Proposition 1.7 that the completely bounded norm of the operator mapping $\phi_t - \psi_t$ is at most $\|\sigma_t\| = \|\mu_t - v_t\|$; hence (3.12).

From equation (3.11) and these estimates, we may conclude that

$$\limsup_{t\to\infty} \|\omega \circ \phi_t\| \leqslant \epsilon + \lim_{t\to\infty} \|\mu_t - v_t\| + 0 = \epsilon.$$

Since ϵ is arbitrary, the limit (3.10) is proved. □

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