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The Dirac Operator of a Commuting d-Tuple

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Given a commuting d-tuple $\bar{T} = (T_1, \ldots, T_d)$ of otherwise arbitrary operators on a Hilbert space, there is an associated Dirac operator $D_{\bar{T}}$. Significant attributes of the d-tuple are best expressed in terms of $D_{\bar{T}}$, including the Taylor spectrum and the notion of Fredholmness. In fact, *all* properties of $\bar{T}$ derive from its Dirac operator. We introduce a general notion of Dirac operator (in dimension $d = 1, 2, \ldots$) that is appropriate for multivariable operator theory. We show that every abstract Dirac operator is associated with a commuting d-tuple, and that two Dirac operators are isomorphic iff their associated operator d-tuples are unitarily equivalent. By relating the curvature invariant introduced in a previous paper to the index of a Dirac operator, we establish a stability result for the curvature invariant for pure d-contractions of finite rank. It is shown that for the subcategory of all such $\bar{T}$ that are (a) Fredholm and and (b) graded, the curvature invariant $K(\bar{T})$ is stable under compact perturbations. We do not know if this stability persists when $\bar{T}$ is Fredholm but ungraded, although there is concrete evidence that it does.

INTRODUCTION

We introduce an abstract notion of Dirac operator in complex dimension $d = 1, 2, \ldots$, and we show that this theory of Dirac operators actually coincides with the theory of commuting d-tuples of operators on a common Hilbert space H (see Theorem A of Section 3). The homology and cohomology of Dirac operators is discussed in general terms, and we relate the homological picture to classical spectral theory by describing its application to concrete problems involving the solution of linear equations of the form

$$T_1 x_1 + T_2 x_2 + \cdots + T_d x_d = y$$

given y and several commuting operators $T_1, T_2, \ldots, T_d$.

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These developments grew out of an attempt to understand the stability properties of a curvature invariant introduced in a previous paper (see [3], [4]), and to find an appropriate formula that expresses the curvature invariant as the index of some operator. The results are presented in Section 4 (see Theorem B and its corollary).

While there is a large literature concerning Taylor's cohomological notion of joint spectrum for commuting sets of operators on a Banach space, less attention has been devoted to the Dirac operator that emerges naturally in the context of Hilbert spaces (however, see Sections 4 through 6 of [6], where the operator $B+B^*$ is explicitly related to Taylor invertibility and the Fredholm property). We have made no attempt to compile a comprehensive list of references concerning the Taylor spectrum, but we do call the reader's attention to work of Albrecht [1], Curto [5, 6], Douglas and Voiculescu [7], McIntosh and Pryde [12], Putinar [14, 15], and Vasilescu [17, 18]. A more extensive list of references can be found in the survey [6]. Finally, I want to thank Stephen Parrott for useful remarks based on a draft of this paper, Ryszard Nest for helpful conversation, and Hendrik Lenstra for patiently enlightening me on homological issues.

1. PRELIMINARIES: CLIFFORD STRUCTURES AND THE CARS IN DIMENSION d

Since there is significant variation in the notation commonly used for Clifford algebras and CAR algebras, we begin with explicit statements of notation and terminology as it will be used below.

Let H be a complex Hilbert space and let d be a positive integer. By a Clifford structure on H (of real dimension $2d$) we mean a real-linear mapping $R\colon \mathbb{C}^d \to \mathscr{B}(H)$ of the $2d$-dimensional real vector space $\mathbb{C}^d$ into the space of self-adjoint operators on H that satisfies

$$R(z)^2 = \|z\|^2\,\mathbf{1}, \qquad z \in \mathbb{C}^d, \tag{1.1}$$

where for a d tuple $z=(z_1, ..., z_d)$ of complex numbers, $\|z\|$ denotes the Euclidean norm

$$\|z\|^2 = |z_1|^2 + \cdots + |z_d|^2.$$

Clifford structures can also be defined as real-linear maps R' of $\mathbb{C}^d$ into the space of skew-adjoint operators on H that satisfy $R'(z)^2 = -\|z\|^2\,\mathbf{1}$, and perhaps this is a more common formulation. Note however that such a structure corresponds to a Clifford structure R satisfying (1.1) by way of $R'(z) = iR(z)$.

Letting $e_1=(1, 0, ..., 0), ..., e_d=(0, ..., 0, 1)$ be the usual unit vectors in $\mathbb{C}^d$ we define operators $p_1, ..., p_d, q_1, ..., q_d \in \mathscr{B}(H)$ by $p_k=R(e_k)$, $q_k=R(ie_k)$, $k=1, ..., d$. The $2d$ operators $(r_1, ..., r_{2d})=(p_1, ..., p_d, q_1, ..., q_d)$ are self-adjoint, they satisfy

$$r_k r_j + r_j r_k = 2\delta_{jk}\mathbf{1}, \qquad 1 \leqslant j, k \leqslant 2d, \tag{1.2}$$

and the complex algebra they generate is a C^*-algebra isomorphic to $M_{2^d}(\mathbb{C})$.

While Clifford structures are real-linear maps of $\mathbb{C}^d$ there is an obvious way to complexify them, and once that is done one obtains a (complex-linear) representation of the canonical anticommutation relations. This sets up a bijective correspondence between Clifford structures and reprsentations of the anticommutation relations. The details are as follows. Since the $2d$-dimensional real vector space $\mathbb{C}^d$ comes with an *a priori* complex structure, any real-linear mapping R of $\mathbb{C}^d$ into the self adjoint operators of $\mathscr{B}(H)$ is the real part of a unique complex-linear mapping $C\colon \mathbb{C}^d \to \mathscr{B}(H)$ in the sense that

$$R(z) = C(z) + C(z)^*, \qquad z \in \mathbb{C}^d, \tag{1.3}$$

and C is given by $C(z)=\frac{1}{2}(R(z)-iR(iz))$, $z \in \mathbb{C}^d$. Corresponding to (1.2) one finds that the operators $c_k=C(e_k)=\frac{1}{2}(p_k-iq_k)$, $1 \leqslant k \leqslant d$ satisfy the canonical anticommutation relations

$$\begin{aligned} c_k c_j + c_j c_k &= 0 \\ c_k^* c_j + c_j c_k^* &= \delta_{jk}\mathbf{1}. \end{aligned} \tag{1.4}$$

Equivalently, the complex linear map $C\colon \mathbb{C}^d \to \mathscr{B}(H)$ satisfies

$$\begin{aligned} C(z)\, C(w) + C(w)\, C(z) &= 0, \\ C(w)^*\, C(z) + C(z)\, C(w)^* &= \langle z, w\rangle\, \mathbf{1} \end{aligned} \tag{1.5}$$

for $z, w \in \mathbb{C}^d$, $\langle z, w\rangle$ denoting the Hermitian inner product

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_d\bar{w}_d.$$

The $*$-algebra generated by the operators $C(z)$ contains the identity and is isomorphic to the matrix algebra $M_{2^d}(\mathbb{C})$.

Any two irreducible representations of the CAR algebra (in either of its presentations (1.4) or (1.5)) are unitarily equivalent. The standard irreducible representation of the CAR algebra is defined as follows. Let Z be a complex Hilbert space of finite dimension d, and let ΛZ be the exterior algebra over Z,

$$\Lambda Z = \Lambda^0 Z \oplus \Lambda^1 Z \oplus \Lambda^2 Z \oplus \cdots \oplus \Lambda^d Z$$

where $\Lambda^k Z$ denotes the kth exterior power of Z. By definition, $\Lambda^0 Z = \mathbb{C}$, and the last summand $\Lambda^d Z$ is also isomorphic to $\mathbb{C}$. $\Lambda^k Z$ is spanned by vectors of the form $z_1 \wedge z_2 \wedge \cdots \wedge z_k$, $z_k \in Z$, and the natural inner product on $\Lambda^k Z$ satisfies

$$\langle z_1 \wedge \cdots \wedge z_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle z_i, w_j \rangle),$$

the right side denoting the determinant of the $k \times k$ matrix of inner products $a_{ij} = \langle z_i, w_j \rangle$. ΛZ is a direct sum of the (complex) Hilbert spaces $\Lambda^k Z$, and it is a Hilbert space of complex dimension 2^d.

For $z \in Z$, the creation operator $C(z)$ maps $\Lambda^k Z$ to $\Lambda^{k+1} Z$, and acts on the generators as

$$C(z)\colon x_1 \wedge \cdots \wedge x_k \mapsto z \wedge x_1 \wedge \cdots \wedge x_k.$$

$C\colon Z \to \mathcal{B}(\Lambda Z)$ is an irreducible representation of the canonical anticommutation relations (1.5). One obtains the standard irreducible Clifford structure (1.1) by taking the real part of this representation $R(z) = C(z) + C(z)^*$.

Remarks. In the next section we will define Dirac operators in terms of Clifford structures. Because of the correspondence cited above, we could just as well have formulated this notion in terms of the anticommutation relations, avoiding Clifford structures entirely. We have chosen to use them because Clifford algebras are associated with the Dirac operators of Riemannian geometry, and perhaps also for reasons of taste, the single equation (1.1) being twice as elegant as the two equations of (1.5). On the other hand, we have found that proofs seem to go more smoothly with the anticommutation relations (1.5). The preceding observations show that nothing is lost in passing back and forth as needed.

We also want to emphasize that with any representation of either the Clifford relations (1.1) or the anticommutation relations (1.5) on a Hilbert space there are additional objects that are naturally associated with them, namely a gauge group, a number operator, and a $\mathbb{Z}_2$-grading of H. By a $\mathbb{Z}_2$-grading of a Hilbert space H we simply mean a decomposition of H into two mutually orthogonal subspaces

$$H = H_+ \oplus H_-.$$

Vectors in H_+ (resp. H_-) are called even (resp. odd). An operator $A \in \mathcal{B}(H)$ is said to be of odd degree if $AH_+ \subseteq H_-$ and $AH_- \subseteq H_+$, and the set of all such A is a self-adjoint linear subspace of $\mathcal{B}(H)$.

Proposition A. *Let* $R\colon \mathbb{C}^d \to \mathscr{B}(H)$ *be a Clifford structure and let* $\mathscr{A}$ *be the finite dimensional* C^**-algebra generated by the range of* R. *There is a unique strongly continuous unitary representation* Γ *of the circle group* $\mathbb{T}$ *on* H *satisfying*

$$\Gamma(\mathbb{T}) \subseteq \mathscr{A}$$
$$\Gamma(\lambda)\, R(z)\, \Gamma(\lambda)^* = R(\lambda z), \qquad \lambda \in \mathbb{T}, \quad z \in \mathbb{C}^d,$$

and such that the spectrum of Γ *starts at* 0 *in the sense that the spectral subspaces*

$$H_n = \{\xi \in H : \Gamma(\lambda)\, \xi = \lambda^n \xi \text{ for all } \lambda \in \mathbb{T}\}, \qquad n \in \mathbb{Z}$$

satisfy $H_n = \{0\}$ *for negative* n *and* $H_0 \neq \{0\}$.

The number operator N *is defined as the generator of the gauge group*

$$\Gamma(e^{it}) = e^{itN}, \qquad t \in \mathbb{R},$$

and is a self-adjoint element of $\mathscr{A}$ *having spectrum* $\{0, 1, 2, ..., d\}$. *The* $\mathbb{Z}_2$*-grading of* H *is defined by*

$$H_+ = \sum_{n \text{ even}} H_n, \qquad H_- = \sum_{n \text{ odd}} H_n.$$

Proof. One may check the validity of the proposition explicitly for the irreducible representation on $\Lambda\mathbb{C}^d$ described above. Since every Clifford structure is unitarily equivalent to a direct sum of copies of this irreducible one, Proposition A persists in the general case. ∎

Remark 1.6. One can single out these objects most explicitly in terms of the anticommutation relations $C\colon Z \to \mathscr{B}(H)$ (1.5) over any d-dimensional one-particle space Z. Here, $\mathscr{A}$ is the C^*-algebra generated by $C(Z)$ and Γ should satisfy $\Gamma(\lambda)\, C(z)\, \Gamma(\lambda)^* = \lambda C(z)$ for $z \in Z$, $\lambda \in \mathbb{T}$, along with the two requirements that (1) the spectrum of Γ should start at 0 and (2) the gauge automorphisms of $\mathscr{B}(H)$ should be inner in the sense that $\Gamma(\mathbb{T}) \subseteq \mathscr{A}$. The number operator and gauge group are given by

$$N = C(e_1)\, C(e_1)^* + \cdots + C(e_d)\, C(e_d)^*, \qquad \Gamma(e^{it}) = e^{itN}, \qquad t \in \mathbb{R}$$

$e_1, ..., e_d$ being any orthonormal basis for the complex Hilbert space Z. The $\mathbb{Z}_2$ grading is defined by the spectral subspaces of Γ (or equivalently, of N) as in Proposition A.

2. DIRAC OPERATORS AND TAYLOR INVERTIBILITY

A Dirac operator is a self-adjoint operator D acting on a Hilbert space H that has been endowed with a distinguished Clifford structure (1.1), satisfying three additional conditions. In order to keep the bookkeeping explicit, we include the Clifford structure as part of the definition.

DEFINITION. A Dirac operator of dimension d is a pair (D, R) consisting of a bounded self-adjoint operator D acting on a Hilbert space H and a Clifford structure $R: \mathbb{C}^d \to \mathscr{B}(H)$, satisfying

(D1) (symmetry about 0): $\Gamma(-1)\, D\Gamma(-1)^* = -D$,

(D2) (invariance of the Laplacian): $\Gamma(\lambda)\, D^2\Gamma(\lambda)^* = D^2$, $\lambda \in \mathbb{T}$,

(D3) $R(z)\, D + DR(z) \in \mathscr{A}'$, $z \in \mathbb{C}^d$,

where $\Gamma: \mathbb{T} \to \mathscr{B}(H)$ is the gauge group associated with R, and $\mathscr{A}$ is the C^*-algebra generated by the range of R.

Remarks. Let $H = H_+ \oplus H_-$ be the $\mathbb{Z}_2$-grading of H induced by the gauge group. (D1) is equivalent to requiring that $DH_+ \subseteq H_-$ and $DH_- \subseteq H_+$, i.e., that D should be an operator of odd degree. (D2) implies that the "Laplacian" D^2 associated with D should be invariant under the action of the gauge group as automorphisms of $\mathscr{B}(H)$. (D3) asserts that the "partial derivatives" of D must commute with the operators in $R(\mathbb{C}^d)$.

We have already pointed out that Clifford structures are interchangeable with representations C of the anticommutation relations (1.5). In terms of C, the definition of Dirac operator would be similar except that (D3) would be replaced with the following: $C(z)\, D + DC(z) \in \mathscr{A}'$, for every $z \in \mathbb{C}^d$.

There is a natural notion of isomorphism for Dirac operators, namely (D, R) (acting on H) is isomorphic to (D', R') (acting on H') if there is a unitary operator $U: H \to H'$ such that $UD = D'U$ and $UR(z) = R'(z)\, U$ for every $z \in \mathbb{C}^d$. Notice that the spectrum and multiplicity function of a Dirac operator are invariant under isomorphism, but of course the notion of isomorphism involves more than simple unitary equivalence of the operators D and D'.

We first show how to construct a Dirac operator, starting with a multioperator $(T_1, ..., T_d)$. Let $T_1, ..., T_d \in \mathscr{B}(H)$ be a commuting d-tuple of bounded operators, let Z be a d-dimensional Hilbert space (which may be thought of as $\mathbb{C}^d$), and let $C_0: Z \to \Lambda Z$ be the irreducible representation of the anticommutation relations (1.5) that was described in Section 1.

Consider the Hilbert space $\tilde{H} = H \otimes \Lambda Z$ and let $C(z) = \mathbf{1}_H \otimes C_0(z)$, $z \in Z$. C obviously satisfies (1.5). Fix any orthonormal basis $e_1, ..., e_d$ for Z and define an operator B on $\tilde{H}$ as

$$B = T_1 \otimes C_0(e_1) + \cdots + T_d \otimes C_0(e_d).$$

The pair (D, R) is defined as

$$D = B + B^*, \qquad R(z) = C(z) + C(z)^*, \qquad z \in Z. \tag{2.1}$$

If we use the orthonormal basis to identify Z with $\mathbb{C}^d$, the discussion of section 1 shows that R satisfies (1.1).

PROPOSITION. *(D, R) is a Dirac operator on $\tilde{H}$. For $\lambda = (\lambda_1, ..., \lambda_d) \in \mathbb{C}^d$, the Dirac operator of the translated d-tuple $(T_1 - \lambda_1 \mathbf{1}, ..., T_d - \lambda_d \mathbf{1})$ is (D_λ, R), where $D_\lambda = D - R(\lambda)$.*

Proof. Noting that the gauge group Γ is related to B by way of

$$\Gamma(\lambda)\, B\Gamma(\lambda)^* = \lambda B, \qquad \lambda \in \mathbb{T}, \tag{2.2}$$

we find that

$$\Gamma(\lambda)\, D\Gamma(\lambda)^* = \lambda B + \bar{\lambda} B^*$$

from which (D1) follows. (D3) follows after a straightforward computation using the anticommutation relations (1.4). In order to check (D2), notice first that $B^2 = 0$. Indeed, one has

$$B^2 = \sum_{i, j=1}^{d} T_i T_j \otimes C_0(e_i)\, C_0(e_j).$$

Since $T_i T_j = T_j T_i$ whereas $C_0(e_i)\, C_0(e_j) = -C_0(e_j)\, C_0(e_i)$, this sum must vanish.

It follows that $D^2 = B^*B + BB^*$. By (2.2), both BB^* and B^*B commute with the gauge group, hence so does D^2. The last sentence is immediate from (2.1). ∎

Remark. A routine verification shows that the isomorphism class of this Dirac operator (D, R) does not depend on the choice of orthonormal basis, and depends only on the commuting d-tuple $\bar{T} = (T_1, ..., T_d)$. For this reason we sometimes write $D_{\bar{T}}$ rather than (D, R), for the Dirac operator constructed from a multioperator $\bar{T}$.

Comments on homology, cohomology and the Taylor spectrum. Joseph Taylor [16] introduced a notion of invertibility (and therefore joint spectrum) for commuting d-tuples of operators $T_1, ..., T_d$ acting on a complex

Banach space. Taylor's notion of invertibility can be formulated as follows. Let

$$\tilde{H} = \tilde{H}_0 \oplus \tilde{H}_1 \oplus \cdots \oplus \tilde{H}_d$$

be the natural decomposition of $\tilde{H} = H \otimes \Lambda Z$ induced by the decomposition of the exterior algebra ΛZ into homogeneous forms of degree $k = 0, 1, \ldots, d$

$$\tilde{H}_k = H \otimes \Lambda^k Z.$$

The operator $B = T_1 \otimes c_1 + \cdots + T_d \otimes c_d$ of formula (1.6) satisfies

$$B\tilde{H}_k \subseteq \tilde{H}_{k+1}$$

and as we have already pointed out, $B^2 = 0$. Thus, the pair $\tilde{H}$, B defines a complex (the Koszul complex of the $\mathbb{C}[z_1, \ldots, z_d]$-module H), and when the range of B is closed and of finite codimension in $\ker B$, we can define the cohomology of this complex. Taylor defines the underlying d-tuple to be *invertible* if the cohomology is trivial: $B\tilde{H} = \ker B$. As we will see presently, for Hilbert spaces invertibility becomes a concrete property of the Dirac operator: *a d-tuple of commuting operators on H is Taylor-invertible if and only if its Dirac operator D is invertible in* $\mathscr{B}(H \otimes \Lambda\mathbb{C}^d)$.

The Taylor spectrum of a commuting d-tuple $\bar{T} = (T_1, \ldots, T_d)$ is defined as the set of all complex d-tuples $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ with the property that the translated d-tuple

$$(T_1 - \lambda_1 \mathbf{1}, \ldots, T_d - \lambda_d \mathbf{1})$$

is not invertible. In terms of the Dirac operator (D, R) of $\bar{T}$, this is the set of all $\lambda \in \mathbb{C}^d$ such that $D - R(\lambda)$ is not invertible. The relation between this "Clifford spectrum" and the ordinary spectrum of D is not very well understood.

The Taylor spectrum and Taylor's notion of invertibility are important not only because they lead to the "right" theorems about the spectrum in multivariable operator theory (see [6]), but also and perhaps more significantly, because they embody the correct multivariable generalization of classical spectral theory as it is defined in terms of solving linear equations.

In order to discuss the latter it is necessary to cast Taylor's cohomological picture of the joint spectrum into a homological picture; once that is done, a clear interpretation of the Taylor spectrum will emerge in terms of solving linear equations. In more detail, consider the canonical anticommutation relations in the form (1.4) and let $c_1, \ldots, c_d$ be the irreducible representation described in Section 1, where c_i acts as follows on the generators of $\Lambda^k\mathbb{C}^d$

$$c_i\colon z_1\wedge\cdots\wedge z_k\mapsto e_i\wedge z_1\wedge\cdots\wedge z_k,$$

$e_1, ..., e_d$ denoting an orthonormal basis for $\mathbb{C}^d$. Starting with a commuting d-tuple $T_1, ..., T_d\in\mathscr{B}(H)$, we have defined a cohomological boundary operator on $H\otimes\Lambda\mathbb{C}^d$ by

$$B=T_1\otimes c_1+\cdots+T_d\otimes c_d.$$

Instead, let us consider the homological boundary operator

$$\tilde{B}=T_1\otimes c_1^*+\cdots+T_d\otimes c_d^*. \tag{2.3}$$

Formula (2.1) defines a Dirac operator (D, R), and we now show that the operators

$$\tilde{D}=\tilde{B}+\tilde{B}^*,\qquad \tilde{R}(z)=R(\bar{z}),\qquad z\in\mathbb{C}^d$$

also define a Dirac operator $(\tilde{D}, \tilde{R})$, R being the Clifford structure of (2.1) and $\bar{z}$ denoting the natural conjugation in $\mathbb{C}^d$, for $z=(z_1, ..., z_d)$, $\bar{z}=(\bar{z}_1, ..., \bar{z}_d)$.

Proposition: Homology vs Cohomology. *The pair $(\tilde{D}, \tilde{R})$ is a Dirac operator on $H\otimes\Lambda\mathbb{C}^d$, and it is isomorphic to the Dirac operator (D, R) of* (2.1). *The gauge group $\tilde{\Gamma}$ of $(\tilde{D}, \tilde{R})$ is related to the gauge group Γ of (D, R) by $\tilde{\Gamma}(\lambda)=\lambda^d\Gamma(\lambda^{-1})$.*

Proof. Consider the annihilation opertors $a_k=c_k^*$, $1\leqslant k\leqslant d$. Obviously, the operators $a_1, ..., a_d$ and their adjoints form an irreducible set of operators satisfying (1.4), hence there is a unitary operator $U\in\mathscr{B}(\Lambda\mathbb{C}^d)$ such that $Uc_kU^*=c_k^*$, $k=1, ..., d$. Letting C_0 and $\tilde{C}_0$ be the corresponding anti-commutation relations in the form (1.5),

$$C_0(z)=z_1c_1+\cdots+z_dc_d,\qquad \tilde{C}_0(z)=z_1c_1^*+\cdots+z_dc_d^*,$$

we have $\tilde{C}_0(z)=C_0(\bar{z})^*$, and moreover $\tilde{C}_0(z)=UC_0(z)\,U^*$, $z\in\mathbb{C}^d$. It follows that the unitary operator $W=\mathbf{1}_H\otimes U\in\mathscr{B}(H\otimes\Lambda\mathbb{C}^d)$ satisfies $WC(z)\,W^*=C(\bar{z})^*$, $z\in\mathbb{C}^d$. Since

$$\tilde{R}(z)=R(\bar{z})=C(\bar{z})+C(\bar{z})^*=W(C(z)^*+C(z))\,W^*=WR(z)\,W^*$$

and since $\tilde{B}=WBW^*$, W implements an isomorphism of the pair (D, R) and the pair $(\tilde{D}, \tilde{R})$. Thus, $(\tilde{D}, \tilde{R})$ is a Dirac operator isomorphic to (D, R).

Letting $C_k = \mathbf{1} \otimes c_k$, $k = 1, \ldots, d$ the number operators $\tilde{N}$ and N of $(\tilde{D}, \tilde{R})$ and (D, R) are seen to be

$$\tilde{N} = C_1^* C_1 + \cdots + C_d^* C_d, \quad N = C_1 C_1^* + \cdots + C_d C_d^*,$$

so by the anticommutation relations (1.4) we have $\tilde{N} = d \cdot \mathbf{1} - N$, and the formula relating $\tilde{\Gamma}$ to Γ follows from Remark 1.6. ∎

In particular, the preceding proposition implies that the Taylor spectrum can be defined in either cohomological terms (using (D, R) and its associated coboundary operator B) or in homological terms (using $(\tilde{D}, \tilde{R})$ and its boundary operator $\tilde{B}$). It is the homological formulation that leads to the following interpretation.

Classical spectral theory starts with the problem of solving linear equations of the form $Tx = y$, where T is a given operator in $\mathscr{B}(H)$, y is a given vector in H, and x is to be found; T is said to be invertible when for every y there is a unique x. Taylor's notion of invertibility in its *homological* form provides the correct generalization to higher dimensions of this fundamental notion in dimension one. In dimension two for example, one has a pair T_1, T_2 of commuting operators acting on a Hilbert space H, and one is interested in solving equations of the form

$$T_1 x_1 + T_2 x_2 = y, \tag{2.4}$$

where y is a given vector in H. Of course the pair (x_1, x_2) is never uniquely determined by y, since if (x_1, x_2) solves this equation then so does (x_1', x_2') where $x_1' = x_1 + T_2 \zeta$ and $x_2' = x_2 - T_1 \zeta$ where $\zeta \in H$ is arbitrary. Equivalently,

$$x_1' = x_1 + T_1 \xi_{11} + T_2 \xi_{12}$$
$$x_2' = x_2 + T_1 \xi_{21} + T_2 \xi_{22},$$

where the vectors ξ_{ij}, $1 \leqslant i, j \leqslant 2$ satisfy $\xi_{ji} = -\xi_{ij}$ for all i, j but are otherwise arbitrary (note that $\xi_{11} = \xi_{22} = 0$ and $\xi_{12} = -\xi_{21} = \zeta$ above). Such perturbations (x_1', x_2') can be written down independently of any properties of the given operators T_1, T_2 (beyond commutativity, of course), and for that reason we will call them *tautological* perturbations of the given solution x_1, x_2. In order to understand how to solve such equations one needs to determine what happens *modulo* tautological perturbations, and for that one must look at the homology of (2.4).

Since we are in dimension two we can write

$$H \otimes \Lambda\mathbb{C}^2 = \Omega_0 \oplus \Omega_1 \oplus \Omega_2,$$

where $\Omega_0 = H$, $\Omega_1 = \{(x_1, x_2): x_k \in H\}$, and Ω_2 is parameterized as a space of "antisymmetric" sequences as

$$\Omega_2 = \{(\xi_{ij}): 1 \leqslant i, j \leqslant 2, \xi_{ij} = -\xi_{ji} \text{ for all } i, j\}.$$

Of course, Ω_2 is isomorphic to H by way of the map that associates to a vector $\zeta \in H$ the antisymmetric sequence $\xi_{11} = \xi_{22} = 0, \xi_{12} = \zeta, \xi_{21} = -\zeta$. The homological boundary operator $B = T_1 \otimes c_1^* + T_2 \otimes c_2^*$ of the complex

$$0 \leftarrow \Omega_0 \leftarrow \Omega_1 \leftarrow \Omega_2 \leftarrow 0 \tag{2.5}$$

acts as follows. On Ω_1, $B(x_1, x_2) = T_1 x_1 + T_2 x_2$, and on Ω_2

$$B(\xi_{ij}) = (T_1 \xi_{11} + T_2 \xi_{12}, T_1 \xi_{21} + T_2 \xi_{22}) = (T_2 \xi_{12}, -T_1 \xi_{12}).$$

Apparently, (2.4) has a solution iff y belongs to $B\Omega_1 = T_1 H + T_2 H$. Given a solution (x_1, x_2) of (2.4) and another pair of vectors (x_1', x_2'), (x_1', x_2') is also a solution iff the difference $(x_1 - x_1', x_2 - x_2')$ belongs to ker B. Given that (x_1', x_2') is a solution, then it is a tautological perturbation of (x_1, x_2) iff the difference $(x_1 - x_1', x_2 - x_2')$ belongs to $B\Omega_2$. Finally, the kernel of the boundary operator at Ω_2 is identified with $\ker T_1 \cap \ker T_2$. We conclude that the complex (2.5) is exact iff (a) $T_1 H + T_2 H = H$, (b) $\ker T_1 \cap \ker T_2 = \{0\}$, and (c) solutions of (2.4) are unique up to tautological perturbations. While the algebra is more subtle in higher dimensions the fundamental issues are the same, and that is why the Taylor spectrum is important in multivariable spectral theory.

We will not have to delve into homological issues here; but the above comments do show that the theory of abstract Dirac operators is rooted in concrete problems of linear algebra that are associated with solving linear equations involving commuting sets of operators.

Taylor's definition of invertibility can be reformulated in terms of the Dirac operator $D_{\bar{T}}$, and then extended to define Fredholm d-tuples and their index. In more detail, in the proof of the previous proposition we have already pointed out that $D^2 = B^*B + BB^*$; and since $B\tilde{H}$ and $B^*\tilde{H}$ are orthogonal, we conclude that $B\tilde{H} = \ker B$ iff D^2 is invertible.

Conclusion: A commuting d-tuple $(T_1, ..., T_d)$ is invertible if and only if its Dirac operator is invertible.

By a Fredholm d-tuple we mean one whose Dirac operator (D, R) is Fredholm in the sense that the self-adjoint operator D has closed range and finite dimensional kernel. The index of a Fredholm d-tuple is defined as follows. By property (D1) we have $D\tilde{H}_+ \subseteq \tilde{H}_-$ and $D\tilde{H}_- \subseteq \tilde{H}_+$. Thus we may consider the operator

$$D_+ = D\restriction_{H_+} \in \mathscr{B}(\tilde{H}_+, \tilde{H}_-),$$

whose adjoint is given by

$$D_+^* = D\restriction_{H_-} \in \mathscr{B}(\tilde{H}_-, \tilde{H}_+).$$

For a Fredholm d-tuple $\bar{T} = (T_1, ..., T_d)$, D_+ is a Fredholm operator from $\tilde{H}_+$ to $\tilde{H}_-$, and the index of $\bar{T}$ is defined by

$$\operatorname{ind}(\bar{T}) = \dim \ker(D_+) - \dim \ker(D_+^*).$$

One can define semi-Fredholm d-tuples similarly, but we do not require the generalization here.

3. DIRAC OPERATORS AND HILBERT MODULES OVER $\mathbb{C}[z_1, ..., z_d]$

In this section we prove the following result, which implies that Dirac operators (D, R) contain exactly the same geometric information as multioperators $\bar{T}$. In that sense, the Dirac operator of a d-contraction fills a position analogous to the Sz.-Nagy Foias characteristic operator function of a single contraction, for operator theory in higher dimensions.

THEOREM A. *For every d-dimensional Dirac operator* (D, R) *there is a commuting d-tuple* $\bar{T} = (T_1, ..., T_d)$ *acting on some other Hilbert space* H *such that* (D, R) *is isomorphic to* $D_{\bar{T}}$. *If* $\bar{T}' = (T'_1, ..., T'_d)$ *is another commuting d-tuple acting on* H', *then* $D_{\bar{T}}$ *and* $D_{\bar{T}'}$ *are isomorphic if and only if there is a unitary operator* $U: H \to H'$ *such that* $UT_k = T'_k U$ *for every* $k = 1, ..., d$.

Proof. Let K be the underlying Hilbert space of (D, R), so that $D = D^* \in \mathscr{B}(K)$ and $R: \mathbb{C}^d \to \mathscr{B}(K)$ is a Clifford structure (1.1) that satisfy (D1), (D2), (D3).

Consider the map $C: \mathbb{C}^d \to \mathscr{B}(K)$ defined by $C(z) = (1/2)(R(z) - iR(iz))$. The discussion of Section 1 implies that C satisfies the anticommutation relations (1.5), and $R(z) = C(z) + C(z)^*$. C is unitarily equivalent to a direct sum of copies of the standard irreducible representation C_0 of the anticommutation relations on $\Lambda\mathbb{C}^d$; thus by replacing (D, R) with an isomorphic copy we may assume that there is a Hilbert space H such that $K = H \otimes \Lambda\mathbb{C}^d$ and that $R(z) = C(z) + C(z)^*$ where $C(z)$ is defined on $H \otimes \Lambda\mathbb{C}^d$ by

$$C(z) = \mathbf{1}_H \otimes C_0(z), \qquad z \in \mathbb{C}^d.$$

We must exhibit a *commuting* set of operators $T_1, \ldots, T_d$ on H so that $D = B + B^*$ where

$$B = T_1 \otimes C_0(e_1) + \cdots + T_d \otimes C_0(e_d),$$

$e_1, \ldots, e_d$ being the usual orthonormal basis for $\mathbb{C}^d$.

To that end, let $\mathscr{A}$ be the finite dimensional C^*-algebra $\mathscr{A} = \mathbf{1}_H \otimes \mathscr{B}(\Lambda\mathbb{C}^d)$. The C^*-algebras generated by $R(\mathbb{C}^d)$ and $C(\mathbb{C}^d)$ are the same, and in fact

$$C^*(R(\mathbb{C}^d)) = C^*(C(\mathbb{C}^d)) = \mathscr{A}. \tag{3.1}$$

By (D1), $R(z)\, D + DR(z)$ must commute with $\mathscr{A}$ for every $z \in \mathbb{C}^d$ and, in view of the relation $C(e_k)^* = 2(R(e_k) + iR(ie_k))$ we have $C(e_k)^*\, D + DC(e_k)^* \in \mathscr{A}'$. Thus for every k there is a unique operator $T_k \in \mathscr{B}(H)$ such that

$$C(e_k)^*\, D + DC(e_k)^* = T_k \otimes \mathbf{1}_{\Lambda\mathbb{C}^d}. \tag{3.2}$$

For each $k = 1, \ldots, d$, let $c_k = C_0(e_k) \in \mathscr{B}(\Lambda\mathbb{C}^d)$, and consider the operator

$$B = T_1 \otimes c_1 + \cdots + T_d \otimes c_d \in \mathscr{B}(H \otimes \Lambda\mathbb{C}^d). \tag{3.3}$$

In order to show that $D = B + B^*$ we will make use of the following:

LEMMA. *Let $R\colon \mathbb{C}^d \to \mathscr{B}(K)$ be a Clifford structure on K and let $\Gamma\colon \mathbb{T} \to \mathscr{B}(K)$ be its gauge group. Every operator $A \in \mathscr{B}(K)$ satisfying $R(z)\, A + AR(z) = 0$ for every $z \in \mathbb{C}^d$ admits a decomposition $A = A_0 \Gamma(-1)$, where A_0 belongs to the commutant of $C^*(R(\mathbb{C}^d))$. In particular, such an operator must also be gauge invariant in the sense that $\Gamma(\lambda)\, A\Gamma(\lambda)^* = A$, $\lambda \in \mathbb{T}$.*

Proof. Since $\Gamma(-1)\, R(z)\, \Gamma(-1)^* = R(-z) = -R(z)$ it follows that $\Gamma(-1)$ anticommutes with $R(z)$ for every $z \in \mathbb{C}^d$. Since A also anticommutes with $R(z)$, the operator $A_0 = A\Gamma(-1)$ must commute with $R(z)$, and we have $A = A\Gamma(-1)^2 = A_0 \Gamma(-1)$ as required. The last assertion follows from this decomposition, because for every $\lambda \in \mathbb{T}$, $\Gamma(\lambda)$ belongs to the C^*-algebra generated by the range of R and hence commutes with both factors A_0 and $\Gamma(-1)$. ∎

We now show that for B as in (3.3) we have $D = B + B^*$. Indeed, since $C(e_k) = \mathbf{1}_H \otimes c_k$ and the c_k satisfy the anticommutation relations (1.4) we have

$$C(e_k)\, B + BC(e_k) = \sum_{j=1}^{d} T_j \otimes (c_k c_j + c_j c_k) = 0,$$

$$C(e_k)^*\, B + BC(e_k)^* = \sum_{j=1}^{d} T_j \otimes (c_k^* c_j + c_j c_k^*) = \sum_{j=1}^{d} T_j \otimes \delta_{jk} \mathbf{1} = T_k \otimes \mathbf{1}.$$

Using the definition of T_k (3.2) it follows from the preceding calculation that the difference $D - B - B^*$ must anticommute with all of the operators $C(e_j)$, $C(e_k)^*$, $1 \leqslant j, k \leqslant d$. Since $R(z) = C(z) + C(z)^*$ it follows that $D - B - B^*$ anticommutes with $R(z)$ for every $z \in \mathbb{C}^d$.

By the lemma, there is a (necessarily unique) operator $X \in \mathscr{B}(H)$ such that

$$D - B - B^* = X \otimes \Gamma_0(-1), \tag{3.4}$$

where $\Gamma_0 \colon \mathbb{T} \to \mathscr{B}(\Lambda \mathbb{C}^d)$ is the natural gauge action on $\Lambda \mathbb{C}^d$ and $\Gamma(\lambda) = \mathbf{1}_H \otimes \Gamma_0(\lambda)$. We want to show that $X = 0$. For that, recall that D is odd (property (D1)) and B is clearly odd by its definition (3.3). Hence $D - B - B^*$ is odd, so it must anticommute with the unitary operator $\Gamma(-1) = P_{H_+} - P_{H_-}$. On the other hand (3.4) implies that it commutes with $\Gamma(-1)$. Since $\Gamma(-1)$ is invertible, $D - B - B^* = 0$.

What remains to be proved is that the operators T_k of (3.2) commute with each other. Indeed, we claim first that $B^2 = 0$. Since we have established that $D = B + B^*$ we can write

$$D^2 = B^*B + BB^* + B^2 + B^{*2}. \tag{3.5}$$

From the definition of B (3.3) we have

$$\Gamma(\lambda)\, B \Gamma(\lambda)^* = \sum_{k=1}^{d} T_k \otimes \Gamma_0(\lambda)\, c_k \Gamma_0(\lambda)^* = \lambda \sum_{k=1}^{d} T_k \otimes c_k = \lambda B, \qquad \lambda \in \mathbb{T}.$$

It follows that B^*B and BB^* are invariant under the action of the gauge group, and that $\Gamma(\lambda)\, B^2 \Gamma(\lambda)^* = \lambda^2 B^2$. Thus

$$\Gamma(\lambda)\, D^2 \Gamma(\lambda)^* = B^*B + BB^* + \lambda^2 B^2 + \bar{\lambda}^2 B^{*2}. \tag{3.6}$$

Because of (D2), the left side of (3.6) does not depend on λ. Hence by equating Fourier coefficients on left and right we find that $B^2 = B^{*2} = 0$.

We can now show that the operators T_k defined by (3.2) mutually commute. Consider the operator C defined on $H \otimes \Lambda\mathbb{C}^d$ by

$$C = \sum_{1 \leqslant j < k \leqslant d} (T_j T_k - T_k T_j) \otimes c_j c_k. \tag{3.7}$$

Since the operators $\{c_j c_k : 1 \leqslant j < k \leqslant d\} \subseteq \mathscr{B}(\Lambda\mathbb{C}^d)$ are linearly independent, it is enough to show that $C = 0$. To see this, we use the anticommutation relations $c_k c_j + c_j c_k = \delta_{jk}\mathbf{1}$ to write

$$\begin{aligned} C &= \sum_{1 \leqslant j < k \leqslant d} T_j T_k \otimes c_j c_k - \sum_{1 \leqslant j < k \leqslant d} T_k T_j \otimes c_j c_k \\ &= \sum_{1 \leqslant j < k \leqslant d} T_j T_k \otimes c_j c_k + \sum_{1 \leqslant j < k \leqslant d} T_k T_j \otimes c_k c_j \\ &= \sum_{1 \leqslant p, q \leqslant d} T_p T_q \otimes c_p c_q = B^2 = 0. \end{aligned}$$

That completes the proof that every Dirac operator is associated with a commuting d-tuple.

Suppose now that we are given two commuting d-tuples $\bar{T}$ and $\bar{T}'$, acting on Hilbert spaces H and H'. It is obvious that if $U: H \to H'$ is a unitary operator satisfying $UT_k = T'_k U$ for every $k = 1, \ldots, d$, then $W = U \otimes \mathbf{1} : H \otimes \Lambda\mathbb{C}^d \to H' \otimes \Lambda\mathbb{C}^d$ is a unitary operator which implements an isomorphism of the respective Dirac operators.

Conversely, let $W: H \otimes \Lambda\mathbb{C}^d \to H' \otimes \Lambda\mathbb{C}^d$ be a unitary operator implementing an isomorphism of the respective Dirac operators (D, R) and (D', R') associated with $\bar{T}$ and $\bar{T}'$. Let $R_0: \mathbb{C}^d \to \Lambda\mathbb{C}^d$ be the irreducible Clifford structure defined in Section 1. Since $R(z) = \mathbf{1}_H \otimes R_0(z)$ and $R'(z) = \mathbf{1}_{H'} \otimes R_0(z)$, it follows that H and H' have the same dimension (namely the common multiplicity of the unitarily equivalent Clifford structures R and $R' = WRW^*$). Thus by replacing $\bar{T}'$ with a unitarily equivalent d-tuple, we can assume that $H = H'$, i.e., that both d-tuples act on the same Hilbert space H.

In these "coordinates," the relation

$$W(\mathbf{1}_H \otimes R_0(z))\, W^* = \mathbf{1}_H \otimes R_0(z), \qquad z \in \mathbb{C}^d$$

implies that W commutes with $\mathbf{1}_H \otimes \mathscr{B}(\Lambda\mathbb{C}^d)$, the C^*-algebra generated by $R(\mathbb{C}^d)$. Thus W decomposes $W = U \otimes \mathbf{1}_{\Lambda\mathbb{C}^d}$ where U is a uniquely determined unitary operator on H. Now according to the definition of Dirac operators (2.1), we have $D = B + B^*$, $D' = B' + B'^*$, where

$$B = T_1 \otimes c_1 + \cdots + T_d \otimes c_d, \qquad B' = T'_1 \otimes c_1 + \cdots + T'_d \otimes c_d,$$

$c_1, ..., c_d$ being the irrecucible representation of the canonical anticommutation relations (1.4) associated with R_0. Letting $C_k = \mathbf{1}_H \otimes c_k$ and using (1.4), a routine calculation gives

$$C_k^* D + DC_k^* = T_k \otimes \mathbf{1}, \quad C_k^* D' + D'C_k^* = T'_k \otimes \mathbf{1}, \qquad k = 1, ..., d.$$

Since $U \otimes \mathbf{1} = W$ commutes with all C_k^* and satisfies $WDW^* = D'$, it follows that for every $k = 1, ..., d$ we have

$$UT_kU^* \otimes \mathbf{1} = W(C_k^* D + DC_k^*) W^* = C_k D' + D'C_k^* = T'_k \otimes \mathbf{1},$$

and hence U implements a unitary equivalence of $\bar{T}$ and $\bar{T}'$. That completes the proof of Theorem A. ∎

Remark 3.8. It is worth pointing out that the proof of Theorem A shows how one may go directly from a Dirac operator (D, R) (acting on H) to the Koszul complex of its underlying d-tuple $\bar{T}$ (the operators $T_1, ..., T_d$ acting on some other Hilbert space) without making explicit reference to $\bar{T}$. Indeed, considering the spectral representation of the gauge group of R

$$\Gamma(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n E_n = \sum_{n=0}^{d} \lambda^n E_n, \qquad \lambda \in \mathbb{T},$$

and the operator $B = \sum_n E_{n+1} D E_n$, from the proof of Theorem A one finds that

$$B^2 = 0, \qquad D = B + B^*. \tag{3.9}$$

Moreover, the spectral subspaces $H_n = E_n H$ satisfy $BH_n \subseteq H_{n+1}$, $B^* H_n \subseteq H_{n-1}$, and the Koszul complex is given by

$$0 \to H_0 \to H_1 \to \cdots \to H_d \to 0$$

with cohomology defined by B.

4. STABILITY OF THE CURVATURE INVARIANT: GRADED CASE

Recall from [3] that a commuting d-tuple of operators $(T_1, ..., T_d)$ on a Hilbert space H is said to be graded if it is circularly symmetric in the sense that there is a strongly continuous unitary representation $\Gamma: \mathbb{T} \to \mathscr{B}(H)$ such that

$$\Gamma(\lambda)\, T_k \Gamma(\lambda)^* = \lambda T_k, \qquad k = 1, ..., d, \quad \lambda \in \mathbb{T}.$$

Many examples of graded d-contractions were described in [3]; in particular, all examples of d-contractions that were associated with projective algebraic varieties (and their finitely generated modules) are graded.

It was shown in ([3], see Theorem B) that the curvature invariant of a pure graded finite rank d-contraction is an integer, namely the Euler characteristic of a certain finitely generated algebraic module over $\mathbb{C}[z_1, ..., z_d]$ that is associated naturally with $\tilde{T}$. However, in the ungraded case this formula fails: both sides of this formula still make sense in the ungraded case, but examples are given in [3] for which they are unequal. This led us to ask in [3], [4] if $K(\bar{T})$ is an integer even when $\bar{T}$ is ungraded. That has been recently proved by Greene, Richter and Sundberg [10], and in fact the results of [10] show that the integer $K(\bar{T})$ can be expressed in terms of the (almost everywhere constant) rank of the boundary values of a certain operator-valued "inner" function that is naturally associated with $\bar{T}$ via dilation theory. A fuller discussion of this inner operator and its relation to $\bar{T}$ can be found in [2]. It is fair to say that the rank of this inner function is not easily computed in terms of the operator theory of $\bar{T}$.

It is also noteworthy that the asymptotic formula for the curvature (Theorem C of [3]) implies that it has certain stability properties; for example, the curvature is stable under the operation of restricting to an invariant subspace of finite codimension. But nothing was known about stability of the curvature invariant under more general compact perturbations.

These considerations led us to search for another formula for the curvature invariant that looks more like an index theorem in the sense that it equates the curvature invariant to the index of some operator. Such a formula would presumably lead to stability under compact perturbations, it would imply that the curvature invariant is in all cases an integer, and it would more closely resemble the Gauss-Bonnet-Chern formula in its modern incarnation as an index theorem (for example, see p. 311 of [9]). As a first step in this direction, we offer the following.

THEOREM B. *Let $\bar{T} = (T_1, ..., T_d)$ be a pure d-contraction of finite rank acting on a Hilbert space H. Assume that $\bar{T}$ is graded and let (D, R) be its Dirac operator. Then both* $\ker D_+$ *and* $\ker D_+^*$ *are finite dimensional and*

$$(-1)^d K(\bar{T}) = \dim \ker D_+ - \dim \ker D_+^*.$$

Remark. Note that we have *not* assumed that D is a Fredholm operator. However, when it is Fredholm we have the following stability.

COROLLARY. *Let $\bar{T} = (T_1, ..., T_d)$ and $\bar{T}' = (T'_1, ..., T'_d)$ be two pure d-contractions of finite rank acting on respective Hilbert spaces H, H'.*

Assume that both $\bar{T}$ and $\bar{T}'$ are graded, that $\bar{T}$ is Fredholm, and that they are unitarily equivalent modulo compacts in the sense that there is a unitary operator $U: H \to H'$ such that

$$UT_k - T'_k U \text{ is compact}, \qquad k = 1, ..., d.$$

Then $K(\bar{T}) = K(\bar{T}')$.

Proof of Corollary. Let (D, R) and (D', R') be the Dirac operators of $\bar{T}$ and $\bar{T}'$, acting on respective Hilbert spaces $\tilde{H} = H \otimes \Lambda\mathbb{C}^d$ and $\tilde{H}' = H' \otimes \Lambda\mathbb{C}^d$. The hypothesis implies that the unitary operator $W: U \otimes \mathbf{1}: \tilde{H} \to \tilde{H}'$ satisfies $WR(z) = \tilde{R}(z) W$ for all $z \in \mathbb{C}^d$, and $WD - D'W$ is compact. The first of these two relations implies that W implements an equivalence of the respective gauge groups $W\Gamma(\lambda) = \Gamma'(\lambda) W$, and hence W carries the $\mathbb{Z}_2$-grading of $\tilde{H}$ to that of $\tilde{H}'$. It follows that the restrictions of W to the even and odd subspaces of $\tilde{H}$ implement a unitary equivalence modulo compact operators of the two operators D_+ and D'_+. Since D_+ is Fredholm by hypothesis, D'_+ must be Fredholm as well, and moreover they must have the same index. From Theorem B we conclude that $K(\bar{T}) = K(\bar{T}')$. ∎

Before giving the proof of Theorem B, we recall some algebraic preliminaries. Let $\mathscr{A}$ be the complex polynomial algebra $\mathbb{C}[z_1, ..., z_d]$. By an $\mathscr{A}$-module we mean a complex vector space M that is endowed with a commuting d-tuple of linear operators $T_1, ..., T_d$, the module structure being defined by $f \cdot \xi = f(T_1, ..., T_d)\, \xi$, $f \in \mathscr{A}$, $\xi \in M$. M is said to be finitely generated if there is a finite set $\xi_1, ..., \xi_s$ of vectors in M such that

$$M = \{f_1 \cdot \xi_1 + \cdots + f_s \cdot \xi_s : f_1, ..., f_s \in \mathscr{A}\}.$$

The free module of rank 1 is defined to be $\mathscr{A}$ itself, with the module action associated with multiplication of polynomials. The free module of rank $r = 1, 2, ...$ is the direct sum of r copies of the free module of rank 1, with the obvious module action on r-tuples of polynomials.

Hilbert's Syzygy theorem implies that every finitely generated $\mathscr{A}$-module has a finite free resolution [8] in the sense that there is an exact sequence of $\mathscr{A}$-modules

$$0 \to F_n \to \cdots \to F_1 \to M \to 0 \tag{4.1}$$

where each F_k is a free module of finite rank. In [3], we defined the Euler characteristic of M in terms of finite free resolutions (4.1) as follows

$$\chi(M) = \sum_{k=1}^{n} (-1)^{k+1} \operatorname{rank}(F_k). \tag{4.2}$$

This integer does not depend on the particular resolution of M chosen to define it.

We must relate $\chi(M)$ to the alternating sum of the Betti numbers of the Koszul complex of M; since the latter is also called the Euler characteristic, we distinguish it from $\chi(M)$ by calling it the Euler number of M and by writing it as $e(M)$. The Euler number is defined as follows.

The Koszul complex of an $\mathscr{A}$-module M is defined as the $\mathscr{A}$-module

$$M \otimes \Lambda\mathbb{C}^d = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^d,$$

where $\Omega^k = M \otimes \Lambda^k\mathbb{C}^d$ is the submodule of k-forms, with coboundary operator

$$B = T_1 \otimes c_1 + \cdots + T_d \otimes c_d$$

exactly as we have done above in the case where M is a Hilbert space and the T_k are bounded linear operators. Letting B_k be the restriction of B to Ω^k we have a corresponding cohomology space $H^k(M) = \ker B_k / \operatorname{ran} B_{k-1}$ for $1 \leqslant k \leqslant d$, $H^0(M) = \ker B_0$, which may or may not be finite dimensional. M is said to be of *finite type* if $H^k(M)$ is finite dimensional for every $0 \leqslant k \leqslant d$, and in that case the Euler number is defined by

$$e(M) = \sum_{k=0}^{d} (-1)^k \dim H^k(M). \tag{4.3}$$

Taking M to be the free module $\mathscr{A}$ of rank one, it is well-known that $H^k(\mathscr{A}) = 0$ for $0 \leqslant k \leqslant d-1$ and that $H^d(\mathscr{A}) = \mathscr{A}/(z_1\mathscr{A} + \cdots + z_d\mathscr{A}) \cong \mathbb{C}$ is one-dimensional. It follows that for a free module F of arbitrary finite rank, we have

$$e(F) = (-1)^d \cdot \operatorname{rank} F. \tag{4.4}$$

The following result is part of the lore of commutative algebra; we sketch a proof for the reader's convenience.

Lemma 1. *Let $0 \to K \to L \to M \to 0$ be a short exact sequence of $\mathscr{A}$ modules, some two of which are of finite type. Then all are of finite type and we have*

$$e(L) = e(K) + e(M).$$

Proof. Letting $\kappa(N)$ denote the Koszul complex of an $\mathscr{A}$-module N, one sees that $\kappa(N)$ has $d+1$ nonzero terms, and the corresponding sequence of complexes

$$0 \to \kappa(K) \to \kappa(L) \to \kappa(M) \to 0$$

is exact. Thus by fundamental principles we obtain a long exact sequence of cohomology spaces which contains at most $3d+3$ nonzero terms. Two of any three consecutive terms in the latter sequence are finite dimensional because two of the three modules K, L, M are assumed to have finite dimensional cohomology. By exactness all cohomology spaces are finite dimensional and the alternating sum of their dimensions must be zero. The asserted formula follows. ∎

LEMMA 2. *Every finitely generated $\mathscr{A}$-module M is of finite type, and*

$$e(M) = (-1)^d \chi(M).$$

Proof. Choose a finite free resolution of M in the form (4.1)

$$0 \to F_n \to \cdots \to F_1 \to M \to 0.$$

Let $R_k \subseteq F_{k-1}$ be the image of F_k in F_{k-1}, $2 \leqslant k \leqslant n$, and let $R_1 \subseteq M$ be the image of F_1. Starting at the left of (4.1) we have a short exact sequence of modules

$$0 \to F_n \to F_{n-1} \to R_{n-1} \to 0,$$

the first two of which are of finite type. By Lemma 1, R_{n-1} is of finite type and

$$e(R_{n-1}) = e(F_{n-1}) - e(F_n).$$

Moving one step to the right, the same argument applied to

$$0 \to R_{n-1} \to F_{n-2} \to R_{n-2} \to 0$$

shows that R_{n-2} is of finite type and

$$e(R_{n-2}) = e(F_{n-2}) - e(R_{n-1}) = e(F_{n-2}) - e(F_{n-1}) + e(F_n).$$

Continuing in this way to the end of the sequence, we arrive at the conclusion that M is of finite type and

$$e(M) = \sum_{k=1}^{n} (-1)^{k+1} e(F_k) = (-1)^d \sum_{k=1}^{n} (-1)^{k+1} \operatorname{rank}(F_k) = (-1)^d \chi(M),$$

where in the second equality we have made use of (4.4). ∎

Proof of Theorem B. We are assuming that $\bar{T}$ is graded; this means that there is a continuous unitary representation of the circle group $U: \mathbb{T} \to \mathscr{B}(H)$ such that

$$U_\lambda T_k U_\lambda^* = \lambda T_k, \qquad 1 \leqslant k \leqslant d. \tag{4.5}$$

Let $\Delta = (\mathbf{1} - T_1 T_1^* - \cdots - T_d T_d^*)^{1/2}$ be the defect operator of $\bar{T}$. By hypothesis, Δ is of finite rank, and the canonical algebraic module M_H associated with $\bar{T}$

$$M_H = \operatorname{span}\{f(T_1, ..., T_d)\,\zeta : f \in \mathscr{A}, \zeta \in \Delta H\}$$

is a finitely generated $\mathscr{A}$ module. Because $\bar{T}$ is pure, M_H is dense in H (see [3], Proposition 5.4).

It follows that $M_H \otimes \Lambda\mathbb{C}^d$ is dense in $\tilde{H} = H \otimes \Lambda\mathbb{C}^d$. Let $D \in \mathscr{B}(\tilde{H})$ be the Dirac operator of $\bar{T}$. We will show that both $\ker D_+$ and $\ker D_+^*$ are finite dimensional subspaces of $M_H \otimes \Lambda\mathbb{C}^d$, and that in fact we have

$$\dim \ker(D_+) = \sum_{k \text{ even}} \dim H^k(M_H), \quad \dim \ker(D_+^*) = \sum_{k \text{ odd}} \dim H^k(M_H), \tag{4.6}$$

where $M_H \otimes \Lambda\mathbb{C}^d$ is viewed as the Koszul complex of M_H. Assuming for the moment that (4.6) has been established we find that

$$\dim \ker D_+ - \dim \ker D_+^* = \sum_{k=0}^{d} (-1)^k \dim H^k(M_H) = e(M_H),$$

and by Lemma 2 the right side is $(-1)^d\,\chi(M_H)$. By Theorem B of [3], the latter is $(-1)^d\,K(H)$, and the proof of Theorem B above will be complete.

In order to establish (4.6) we make use of the grading as follows. Let $c_1, ..., c_d$ be operators on $\mathbb{C}^d$ satisfying the anticommutation relations (1.4) and let

$$B = T_1 \otimes c_1 + \cdots + T_d \otimes c_d$$

be the coboundary operator on $\tilde{H}$. Since $D^2 = B^*B + BB^*$, the kernel of D is given by $\ker D = \ker B \cap \ker B^*$. Let $V: \mathbb{T} \to \mathscr{B}(\tilde{H})$ be the unitary representation corresponding to U, $V_\lambda = U_\lambda \otimes \mathbf{1}_{\Lambda\mathbb{C}^d}$, $\lambda \in \mathbb{T}$. By (4.5) we have

$$V_\lambda B V_\lambda^* = \lambda B,$$

and it follows that both $\ker B$ and $\ker B^*$ are invariant under the action of V. Since the spectral subspaces of U and V

$$H_k = \{\xi \in H : U_\lambda \xi = \lambda^k \xi, \lambda \in \mathbb{T}\}, \quad \tilde{H}_k = \{\zeta \in \tilde{H} : V_\lambda \zeta = \lambda^k \zeta, \lambda \in \mathbb{T}\}$$

are related by $\tilde{H}_k = H_k \otimes \Lambda\mathbb{C}^d$, it follows that both $\ker B$ and $\ker B^*$ decompose into orthogonal sums

$$\ker B = \sum_k \ker B \cap \tilde{H}_k, \quad \ker B^* = \sum_k \ker B^* \cap \tilde{H}_k.$$

We conclude that

$$\ker D = \sum_k \ker D \cap \tilde{H}_k = \sum_k \ker B \cap \ker B^* \cap \tilde{H}_k.$$

It was shown in [3], Proposition 5.4, that each H_k is a finite dimensional subspace of M_H, hence $\tilde{H}_k$ is a finite dimensional subspace of $M_H \otimes \Lambda\mathbb{C}^d$. Since the restriction B_{M_H} of B to $M_H \otimes \Lambda\mathbb{C}^d$ is the boundary operator of the Koszul complex of M_H it follows that for the restriction B_k of B to $M_H \cap \tilde{H}_k$ we have

$$\dim(\ker D \cap \tilde{H}_k) = \dim(\ker B \cap \ker B^* \cap \tilde{H}_k) = \dim(\ker B_k / \operatorname{ran} B_{k-1}).$$

Summing on k we find that

$$\dim \ker D = \dim(\ker B_{M_H} / \operatorname{ran} B_{M_H}).$$

The right side of the preceding formula is finite, because the Koszul complex of M_H has finite dimensional cohomology by Lemma 2.

By restricting this argument respectively to the even and odd subspaces of $\tilde{H}$, one finds in the same way that $\dim \ker D_+$ and $\dim \ker D_+^*$ are, respectively, the total dimensions of the even and odd cohomology of the Koszul complex of M_H, and that gives the two formulas of (4.6). ∎

Concluding remarks, examples, problems. It is natural to ask if Theorem B remains valid when one drops the hypothesis that $\bar{T}$ is graded. On the surface, this may appear a foolish question since it is not known if the Dirac operator associated with a finite rank pure d-contraction is Fredholm; and if it is not Fredholm then what does the index of D_+ mean? The Dirac operator is known to be Fredholm for classes of concrete examples (see the following proposition for some, and the discussion of Problems 1 and 2 for others), but the issue of Fredholmness for general pure finite rank d-contractions remains somewhat mysterious.

Nevertheless, Stephen Parrott has proved a result [13] for single operators that implies that Theorem B is true *verbatim* for the one-dimensional case $d=1$ and an arbitrary pure contraction T of finite rank, graded or not. His result implies that T is necessarily a Fredholm operator. Subsequently, R. N. Levy [11] gave a simpler alternate proof of Parrott's result.

To illustrate higher dimensional phenomena, we describe a class of examples of finite rank pure d-contractions $\bar{T}=(T_1, ..., T_d)$ in arbitrary dimension $d=1, 2, ...$. Most are ungraded. We show that these examples are Fredholm and we compute all three integer invariants (the index of the Dirac operator, the curvature invariant $K(\bar{T})$, and the Euler characteristic $\chi(\bar{T})$ of [3]). For some of these examples the formula $K(\bar{T})=\chi(\bar{T})$ of ([3], Theorem B) holds, but for most of them it fails. On the other hand, in all cases the formula of Theorem B above

$$(-1)^d K(\bar{T}) = \dim \ker D_+ - \dim \ker D_+^* \tag{4.7}$$

is satisfied. Indeed, we know of no examples for which (4.7) fails.

Fix $d=1, 2, ...$ and let r be a positive integer. Following the notation and terminology of [2, 3] we will consider the d-shift $\bar{S}=(S_1, ..., S_d)$ of multiplicity $r+1$. $\bar{S}$ acts on the Hilbert space $(r+1)\cdot H^2$, a direct sum of $r+1$ copies of the basic free Hilbert module $H^2=H^2(\mathbb{C}^d)$. We consider certain invariant subspaces $M\subseteq(r+1)\cdot H^2$ and their quotient Hilbert modules $H=(r+1)\cdot H^2/M$. The d-shift compresses to a pure d-contraction $\bar{T}=(T_1, ..., T_d)$ acting on H, and the rank of $\bar{T}$ is at most $r+1$. For the examples below, the rank is $r+1$ and $\bar{T}$ will have the properties asserted above. The subspaces M are defined as follows. Let $\phi_1, \phi_2, ..., \phi_r$ be a set of multipliers of H^2 and set

$$M=\{(f, \phi_1 f, \phi_2 f, ..., \phi_r f): f\in H^2\}\subseteq(r+1)\cdot H^2.$$

PROPOSITION. *Assume that the set of* $r+1$ *functions* $\{1, \phi_1, \phi_2, ..., \phi_r\}$ *is linearly independent, and let* $\bar{T}=(T_1, ..., T_d)$ *be the d-tuple of operators associated with the quotient Hilbert module* $H=(r+1)\cdot H^2/M$. $\bar{T}$ *is a pure d-contraction of rank* $r+1$, *it is Fredholm, and its index and curvature invariant are given by*

$$\dim\ker(D_+)-\dim\ker D_+^* = (-1)^d\cdot r, \qquad K(\bar{T})=r.$$

If each ϕ_k *is a homogeneous polynomial of some degree* n_k *then the Euler characteristic is also given by* $\chi(\bar{T})=r$. *If, on the other hand,* M *contains no nonzero element* $(p_0, p_1, ..., p_r)$ *with polynomial components* p_k, *then* $\chi(\bar{T})=r+1$.

Remark. For example, if each ϕ_k is the exponential of some nontrivial polynomial (or more generally, if no ϕ_k is a rational function), then the only $r+1$-tuple of polynomials $(p_0, p_1, ..., p_r)$ that belongs to M is the zero $r+1$-tuple.

Proof. We sketch the key elements of the argument.

Let us first deal first with the Euler characteristic. This invariant is associated with the finitely generalted algebraic $\mathbb{C}[z_1, ..., z_d]$-module

$$M_H = \text{span}\{f(T_1, ..., T_d)\,\zeta : f \in \mathbb{C}[z_1, ..., z_d], \zeta \in \Delta H\},$$

Δ being the finite rank defect operator $\Delta = (\mathbf{1} - T_1T_1^* - \cdots - T_dT_d^*)^{1/2}$. Realizing the quotient $H = (r+1)\cdot H^2/M$ as the orthogonal complement $M^\perp \subseteq (r+1)\cdot H^2$, let $E_0 \in \mathscr{B}((r+1)\cdot H^2)$ be the projection onto the $r+1$-dimensional space of constant vector functions. The operators $T_1, ..., T_d$ are obtained by compressing $S_1, ..., S_d$ to $M^\perp$, and a straghtforward computation shows that Δ is identified with the square root of the compression of E_0 to $M^\perp$. This operator is of rank $r+1$ because of the linear independence hypothesis on $\{1, \phi_1, ..., \phi_r\}$ (for example, see 8.4.3 of [2]). It follows that M_H is identified with the projection onto $M^\perp$ of the space of all vector polynomials

$$S = \{(p_0, p_1, ..., p_r) : p_k \in \mathbb{C}[z_1, ..., z_d]\}.$$

The $\mathbb{C}[z_1, ..., z_d]$-module action of a polynomial $f \in \mathbb{C}[z_1, ..., z_d]$ on M_H is given by

$$f \cdot P_M^\perp(p_0, p_1, ..., p_r) = P_M^\perp(fp_0, fp_1, ..., fp_r).$$

Now assume that M contains no nonzero element having polynomial components, and let $F = (r+1)\cdot \mathbb{C}[z_1, ..., z_d]$ denote the free $\mathbb{C}[z_1, ..., z_d]$-module of rank $r+1$. Consider the linear map

$$L: (f_0, f_1, ..., f_r) \in F \mapsto P_{M^\perp}(f_0, f_1, ..., f_r) \in M_H.$$

L is injective by hypothesis, its range is all of M_H, and it is obviously a homomorphism of $\mathbb{C}[z_1, ..., z_d]$-modules. Hence M_H is a free module of rank $r+1$ and its Euler characteristic is $r+1$. This shows that $\chi(\bar{T}) = r+1$ in this case.

On the other hand, if each ϕ_k is a homogeneous polynomial then one may extend the argument of the proof of ([2], Proposition 7.4, which addresses the case $r=1$ explicitly) in straightforward way to show that $\bar{T}$ is a graded d-contraction. It follows from Theorem B of [3] that $\chi(\bar{T}) = K(\bar{T})$. We will show momentarily that in all cases we have $K(\bar{T}) = r$, and this calculates the Euler characteristic for the asserted cases.

We show next that $\bar{T}$ is Fredholm of index $(-1)^d r$. For that, it is enough to show that $\bar{T}$ is similar to a Fredholm d-tuple whose index is known to be $(-1)^d r$. The latter d-tuple is the d-shift of multiplicity r. In more detail, consider the linear mapping A: $(r+1) \cdot H^2 \to r \cdot H^2$ defined by

$$A(f_0, f_1, f_2, ..., f_r) = (f_1 - \phi_1 f_0, f_2 - \phi_2 f_0, ..., f_r - \phi_r f_0).$$

It is clear that A is bounded, surjective, has kernel M, and intertwines the action of $\bar{S}$ (acting on $(r+1) \cdot H^2$) and the multiplicity r d-shift acting on $r \cdot H^2$. Thus A promotes to an invertible map of Hilbert spaces $\tilde{A}$: $H \to r \cdot H^2$ which implements a similarity of $\bar{T}$ and the d-shift of multiplicity r. The latter is a graded pure d-contraction of finite rank which is essentially normal by Proposition 5.3 of [2], and therefore Fredholm. Since the curvature invariant of the d-shift of multiplicity r is known to be r, Theorem B implies that its index is $(-1)^d r$.

We may not infer from this argument that $K(\bar{T}) = K(\bar{S}) = r$ since unlike the Fredholm index, the curvature invariant is not known to be invariant under similarity. In order to calculate $K(\bar{T})$ we appeal to a result of Greene, Richter and Sundberg [10] as follows. Identifying H with $M^\perp \subseteq (r+1) \cdot H^2$, we have already seen that the natural projection $L = P_{M^\perp}: (r+1) \cdot H^2 \to H$ is the minimal dilation of H in the sense of [2], and obviously L is a co-isometry with $L^*L = \mathbf{1} - P_M$. Now if one evaluates all of the functions in M at a point z in the open unit ball of $\mathbb{C}^d$, one obtains the following linear subspace of $\mathbb{C}^{r+1}$

$$M(z) = \{(\lambda, \lambda\phi_1(z), \lambda\phi_2(z), ..., \lambda\phi_r(z)): \lambda \in \mathbb{C}\}.$$

This is a one-dimensional space having codimension $(r+1) - 1 = r$, and the same assertion is valid for almost every point z on the boundary of the unit ball. By the results of [10], the codimension of $M(z) \subseteq \mathbb{C}^{r+1}$ is equal to $K(\bar{T})$ for almost every z in the boundary of the unit ball. Thus, $K(\bar{T}) = r$. ∎

A number of fundamental issues require clarification. We conclude by describing two related problems concerning the index theory of Dirac operators.

Problem 1. Is the Dirac operator of every pure finite rank d-contraction a Fredholm operator?

If the answer is yes then the index of D, $\dim \ker D_+ - \dim \ker D_+^*$, is well-defined. It would then be natural to ask if the curvature invariant of the d-tuple is related to the index of D by the formula (4.7) of Theorem B in general. If so, it would follow that the curvature invariant is stable under compact perturbations, similarity, and homotopy within the category of pure finite rank d-contractions.

Now if a pure finite rank d-contraction $\bar{T} = (T_1, ..., T_d)$ is *essentially normal* in the sense that its self-commutators $T_k T_j^* - T_j^* T_k$ are all compact, then it is not hard to see that its Dirac operator $D_{\bar{T}}$ is Fredholm. Thus an affirmative answer to the following would lead to significant progress on Problem 1 for the d-contractions that arise from modules associated with projective algebraic varieties.

Problem 2. Let $\bar{T}$ be a pure finite rank d-contraction which is graded. Do the self-commutators $T_k T_j^* - T_j^* T_k$ belong to the Schatten–von Neumann class $\mathscr{L}^p$ for every $p > d$?

We have shown in unpublished work that Problem 2 has an affirmative answer in the simplest cases (for example, when $H = H^2/M$ is the quotient of H^2 by a closed invariant subspace M that is generated by a set of *monomials* of the form $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}$, $\alpha_k \geqslant 0$). Note that since these examples are graded, Theorem B implies that the index formula (4.7) holds. However, such quotients are associated with somewhat trivial algebraic varieties: the essence of Problem 2 involves quotients of the form H^2/M where M is generated by a set of more general *homogeneous* polynomials. For example, we do not even know the answer to Problem 2 (or Problem 1 for that matter) for the particular case $H^2(\mathbb{C}^2)/M$ where M is the invariant subspace of $H^2(\mathbb{C}^2)$ generated by a single homogeneous polynomial of the form $p(z_1, z_2) = z_1^3 + \lambda z_2^3$, with λ a positive real constant.

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