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Interactions in Noncommutative Dynamics

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To the memory of Irving Segal

Abstract: A mathematical notion of interaction is introduced for noncommutative dynamical systems, i.e., for one parameter groups of $*$-automorphisms of $\mathcal{B}(H)$ endowed with a certain causal structure. With any interaction there is a well-defined "state of the past" and a well-defined "state of the future". We describe the construction of many interactions involving cocycle perturbations of the CAR/CCR flows and show that they are nontrivial. The proof of nontriviality is based on a new inequality, relating the eigenvalue lists of the "past" and "future" states to the norm of a linear functional on a certain C^*-algebra.

Introduction, Summary of Results

In this paper we are concerned with one-parameter groups of $*$-automorphisms, of the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H, which carry a particular kind of causal structure. More precisely, A **history** is a pair (U, M) consisting of a one-parameter group $U = \{U_t : t \in \mathbb{R}\}$ of unitary operators acting on a separable infinite-dimensional Hilbert space H and a type I subfactor $M \subseteq \mathcal{B}(H)$ which is invariant under the automorphisms $\gamma_t(X) = U_t X U_t^*$ for negative t, and which has the following two properties:

(irreducibility)

$$(\bigcup_{t\in\mathbb{R}} \gamma_t(M))'' = \mathcal{B}(H), \tag{0.1}$$

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(trivial infinitely remote past)

$$\bigcap_{t\in\mathbb{R}} \gamma_t(M) = \mathbb{C}\cdot\mathbf{1}. \tag{0.2}$$

We find it useful to think of the group $\{\gamma_t : t \in \mathbb{R}\}$ as representing the flow of time in the Heisenberg picture, and the von Neumann algebra M as representing bounded observables that are associated with the "past". However, this paper is concerned with purely mathematical issues concerning the dynamical properties of histories, with problems concerning their existence and construction, and especially with the issue of nontriviality (to be defined momentarily).

An E_0**-semigroup** is a one-parameter semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of unit-preserving $*$-endomorphisms of a type I_∞ factor M, which is continuous in the natural sense [2–8,10,11,29–33]. The subfactors $\alpha_t(M)$ decrease as t increases, and α is called **pure** if $\cap_t \alpha_t(M) = \mathbb{C}\mathbf{1}$. There are two E_0-semigroups α^-, α^+ associated with any history, α^- being the one associated with the "past" by restricting γ_{-t} to M for $t \geq 0$ and α^+ being the one associated with the "future" by restricting γ_t to the commutant M' for $t \geq 0$. By an **interaction** we mean a history with the additional property that there are normal states ω_-, ω_+ of M, M' respectively such that ω_- is invariant under the action of α^- and ω_+ is invariant under the action of α^+. Both α^- and α^+ are pure E_0-semigroups, and when a pure E_0-semigroup has a normal invariant state then that state is uniquely determined, see (4.1) below. Thus ω_- (resp. ω_+) is the unique normal invariant state of α^- (resp. α^+).

Remarks. Since the state space of any unital C^*-algebra is weak*-compact, the Markov-Kakutani fixed point theorem implies that every E_0-semigroup has invariant states. But there is no reason to expect that there is a *normal* invariant state. Indeed, we have examples (unpublished) of pure E_0-semigroups which have no normal invariant states. Notice too that ω_-, for example, is defined *only* on the algebra M of the past. Of course, ω^- has many extensions to normal states of $\mathcal{B}(H)$, but none of these normal extensions need be invariant under the action of the group γ. In fact, we will see that if there is a *normal* γ-invariant state defined on all of $\mathcal{B}(H)$ then the interaction must be trivial.

In order to define a trivial interaction we must introduce a C^*-algebra of "local observables". For every compact interval $[s, t] \subseteq \mathbb{R}$ there is an associated von Neumann algebra

$$\mathcal{A}_{[s,t]} = \gamma_t(M) \cap \gamma_s(M)'. \tag{0.3}$$

Notice that since $\gamma_s(M) \subseteq \gamma_t(M)$ are both type I factors, so is the relative commutant $\mathcal{A}_{[s,t]}$. Clearly $\mathcal{A}_I \subseteq \mathcal{A}_J$ if $I \subseteq J$, and for adjacent intervals $[r, s]$, $[s, t]$, $r \leq s \leq t$ we have

$$\mathcal{A}_{[r,t]} = \mathcal{A}_{[r,s]} \otimes \mathcal{A}_{[s,t]}, \tag{0.4}$$

in the sense that the two factors $\mathcal{A}_{[r,s]}$ and $\mathcal{A}_{[s,t]}$ mutually commute and generate $\mathcal{A}_{[r,t]}$ as a von Neumann algebra. The automorphism group γ permutes the algebras $\mathcal{A}_I$ covariantly,

$$\gamma_t(\mathcal{A}_I) = \mathcal{A}_{I+t}, \quad t \in \mathbb{R}. \tag{0.5}$$

Finally, we define the local C^*-algebra $\mathcal{A}$ to be the *norm* closure of the union of all the $\mathcal{A}_I$, $I \subseteq \mathbb{R}$. $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(H)$ which is strongly dense and invariant under the action of the automorphism group γ.

Remarks. It may be of interest to compare the local structure of the C^*-algebra $\mathcal{A}$ to its commutative counterpart, namely the local algebras associated with a stationary random distribution with independent values at every point [19]. More precisely, suppose that we are given a random distribution ϕ; i.e., a linear map from the space of real-valued test functions on $\mathbb{R}$ to the space of real-valued random variables on some probability space (Ω, P). With every compact interval $I = [s, t]$ with $s < t$ one may consider the weak*-closed subalgebra $\mathcal{A}_I$ of $L^\infty(\Omega, P)$ generated by random variables of the form $e^{i\phi(f)}$, f ranging over all test functions supported in I. When the random distribution ϕ is stationary and has independent values at every point, this family of subalgebras of $L^\infty(\Omega, P)$ has properties analogous to (0.4) and (0.5), in that there is a one-parameter group of measure preserving automorphisms $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ of $L^\infty(\Omega, P)$ which satisfies (0.5), and instead of (0.4) we have the assertion that the algebras $\mathcal{A}_{[r,s]}$ and $\mathcal{A}_{[s,t]}$ are *probabilistically independent* and generate $\mathcal{A}_{[r,t]}$ as a weak*-closed algebra.

One should keep in mind, however, that this commutative analogy has serious limitations. For example, we have already pointed out that in the case of interactions there is typically no normal γ-invariant state on $\mathcal{B}(H)$, and there is no reason to expect any normal state of $\mathcal{B}(H)$ to decompose as a product state relative to the decompositions of (0.4).

There is also some common ground with the Boolean algebras of type I factors of Araki and Woods [1], but here too there are significant differences. For example, the local algebras of (0.3) and (0.4) are associated with intervals (and more generally with finite unions of intervals), but not with more general Borel sets as in [1]. Moreover, here the translation group acts as automorphisms of the given structure whereas in [1] there is no assumption of "stationarity" with respect to translations.

For our purposes, the local C^*-algebra $\mathcal{A}$ has two important features. First, it gives us a way of comparing ω_- and ω_+. Indeed, both states ω_- and ω_+ extend *uniquely* to γ-invariant states $\bar{\omega}_-$ and $\bar{\omega}_+$ of $\mathcal{A}$. We sketch the proof for ω_-.

Proposition 0.1. *There is a unique γ-invariant state $\bar{\omega}_-$ of $\mathcal{A}$ such that*

$$\bar{\omega}_- \restriction_{\mathcal{A}_I} = \omega_- \restriction_{\mathcal{A}_I}$$

for every compact interval $I \subseteq (-\infty, 0]$.

Proof. For existence of the extension, choose any compact interval $I = [a, b]$ and any operator $X \in \mathcal{A}_I$. Then for sufficiently large $s > 0$ we have $I - s \subseteq (-\infty, 0]$ and for these values of s, $\omega_-(\gamma_{-s}(X))$ does not depend on s because ω_- is invariant under the action of $\{\gamma_t : t \leq 0\}$. Thus we can define $\bar{\omega}_-(X)$ unambiguously by

$$\bar{\omega}_-(X) = \lim_{t \to -\infty} \omega_-(\gamma_t(X)).$$

This defines a positive linear functional $\bar{\omega}_-$ on the unital $*$-algebra $\cup_I \mathcal{A}_I$, and now we extend $\bar{\omega}_-$ to all of $\mathcal{A}$ by norm-continuity. The extended state is clearly invariant under the action of γ_t, $t \in \mathbb{R}$.

The proof of uniqueness of the extension is straightforward, and we omit it. □

It is clear from the proof of Proposition 0.1 that these extensions of ω_- and ω_+ are *locally normal* in the sense that their restrictions to any localized subalgebra $\mathcal{A}_I$ define normal states on that type I factor.

Definition 0.2. *The interaction (U, M), with past and future states ω_- and ω_+, is said to be trivial if $\bar{\omega}_- = \bar{\omega}_+$.*

More generally, the norm $\|\bar{\omega}_- - \bar{\omega}_+\|$ gives some measure of the "strength" of the interaction, and of course we have $0 \leq \|\bar{\omega}_- - \bar{\omega}_+\| \leq 2$.

If there is a normal state ρ of $\mathcal{B}(H)$ which is invariant under the action of γ, then since ω_- (resp. ω_+) is the unique normal invariant state of α_- (resp. α_+) we must have $\rho \restriction_M = \omega_-$, $\rho \restriction_{M'} = \omega_+$, and hence $\bar{\omega}_- = \bar{\omega}_+ = \rho \restriction_{\mathcal{A}}$ by the uniqueness part of Proposition 0.1. In particular, *if the interaction is nontrivial then neither $\bar{\omega}_-$ nor $\bar{\omega}_+$ can be extended from $\mathcal{A}$ to a normal state of its strong closure $\mathcal{B}(H)$.*

The second important feature of $\mathcal{A}$ is that there is a definite "state of the past" and a definite "state of the future" in the following sense.

Proposition 0.3. *For every $X \in \mathcal{A}$ and every normal state ρ of $\mathcal{B}(H)$ we have*

$$\lim_{t \to -\infty} \rho(\gamma_t(X)) = \bar{\omega}_-(X), \quad \lim_{t \to +\infty} \rho(\gamma_t(X)) = \bar{\omega}_+(X).$$

Proof. Consider the first limit formula. The set of all $X \in \mathcal{A}$ for which this formula holds is clearly closed in the operator norm, hence it suffices to show that it contains $\mathcal{A}_I$ for every compact interval $I \subseteq \mathbb{R}$.

We will make use of the fact (discussed more fully at the beginning of Sect. 5) that if ρ is any normal state of M and A is an operator in M then

$$\lim_{t \to -\infty} \rho(\gamma_t(A)) = \omega_-(A),$$

see formula (4.1). Choosing a real number T sufficiently negative that $I + T \subseteq (-\infty, 0]$, the preceding remark shows that for the operator $A = \gamma_T(X) \in M$ we have $\lim_{t \to -\infty} \rho(\gamma_t(A)) = \omega_-(A)$, and hence

$$\lim_{t \to -\infty} \rho(\gamma_t(X)) = \lim_{t \to -\infty} \rho(\gamma_{t-T}(\gamma_T(X))) = \omega_-(\gamma_T(X)) = \bar{\omega}_-(X).$$

The proof of the second limit formula is similar. □

Thus, whatever (normal) state ρ one chooses to watch evolve over time on operators in $\mathcal{A}$, it settles down to become $\bar{\omega}_+$ in the distant future, it must have come from $\bar{\omega}_-$ in the remote past, and the limit states do not depend on the choice of ρ. For a trivial interaction, nothing happens over the long term: for fixed X and ρ the function $t \in \mathbb{R} \mapsto \rho(\gamma_t(X))$ starts out very near some value (namely $\bar{\omega}_-(X)$), exhibits transient fluctuations over some period of time, and then settles down near the same value again. For a nontrivial interaction, there will be a definite change from the limit at $-\infty$ to the limit at $+\infty$ (for some choices of $X \in \mathcal{A}$).

A number of questions arise naturally. 1) How does one construct examples of interactions? 2) How does one determine if a given interaction is nontrivial? 3) What C^*-dynamical systems can occur as the C^*-algebras of local observables associated with an interaction? The purpose of this paper is to provide an effective partial solution of problem 1) and a complete solution of problem 2). The latter involves an inequality which we feel is of some interest in its own right. These results are summarized as follows.

By an *eigenvalue list* we mean a decreasing sequence of nonnegative real numbers $\lambda_1 \geq \lambda_2 \geq \ldots$ with finite sum. Every normal state ω of a type I factor is associated with a positive operator of trace 1, whose eigenvalues counting multiplicity can be arranged

into an eigenvalue list which will be denoted $\Lambda(\omega)$. If the factor is finite dimensional, we still consider $\Lambda(\omega)$ to be an infinite list by adjoining zeros in the obvious way. Given two eigenvalue lists $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ and $\Lambda' = \{\lambda'_1 \geq \lambda'_2 \geq \dots\}$, we will write

$$\|\Lambda - \Lambda'\| = \sum_{k=1}^{\infty} |\lambda_k - \lambda'_k|$$

for the ℓ^1-distance from one list to the other. A classical result implies that if ρ and σ are normal states of a type I factor M, then we have

$$\|\Lambda(\rho) - \Lambda(\sigma)\| \leq \|\rho - \sigma\|$$

(see Sect. 3).

Combining the results of [7] with the results of Sect. 1 below, we obtain the following result on the existence of interactions having arbitrary *finite* eigenvalue lists.

Theorem A. *Let $n = 1, 2, \dots, \infty$ and let Λ_- and Λ_+ be two eigenvalue lists, each of which has only finitely many nonzero terms. There is an interaction (U, M) whose past and future states ω_-, ω_+ have eigenvalue lists Λ_- and Λ_+, and whose past and future E_0-semigroups are both cocycle perturbations of the CAR/CCR flow of index n.*

Remarks. Theorem A is established in Sect. 3. We conjecture that the finiteness hypothesis of Theorem A can be dropped.

Theorem A gives examples of interactions, but it provides no information about whether or not these interactions are nontrivial. We will show that this is the case whenever the eigenvalue lists of ω_- and ω_+ are different. That conclusion depends on the following, which is the main result of this paper (and which applies to interactions with arbitrary...i.e., not necessarily finitely nonzero...eigenvalue lists).

Theorem B. *Let (U, M) be an interaction with past and future states ω_- and ω_+, and let $\bar{\omega}_-$ and $\bar{\omega}_+$ denote their extensions to γ-invariant states of $\mathcal{A}$. Then*

$$\|\bar{\omega}_- - \bar{\omega}_+\| \geq \|\Lambda(\omega_- \otimes \omega_-) - \Lambda(\omega_+ \otimes \omega_+)\|.$$

Remarks. Theorem B is proved in Sect. 4. Notice the tensor product of states on the right. For example, $\Lambda(\omega_- \otimes \omega_-)$ is obtained from the eigenvalue list $\Lambda(\omega_-) = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ of ω_- by rearranging the doubly infinite sequence of all products $\lambda_i \lambda_j$, $i, j = 1, 2, \dots$ into decreasing order. It can be an unpleasant combinatorial chore to calculate $\Lambda(\omega_- \otimes \omega_-)$ even when $\Lambda(\omega_-)$ is relatively simple and finitely nonzero; but we also show in Sect. 4 that if A and B are two positive trace class operators such that $\Lambda(A \otimes A) = \Lambda(B \otimes B)$, then $\Lambda(A) = \Lambda(B)$. Thus we may conclude

Corollary 0.4. *Let (U, M), ω_-, ω_+ be as in Theorem B, and let Λ_- and Λ_+ be the eigenvalue lists of ω_- and ω_+ respectively. If $\Lambda_- \neq \Lambda_+$, then the interaction is nontrivial.*

The following implies that "strong" interactions exist.

Corollary 0.5. *Let $n = 1, 2, \dots, \infty$ and choose $\epsilon > 0$. There is an interaction (U, M) having past and future states ω_-, ω_+, such that α^- and α^+ are cocycle perturbations of the CAR/CCR flow of index n, for which*

$$\|\bar{\omega}_- - \bar{\omega}_+\| \geq 2 - \epsilon.$$

Theorem B depends on a more general result concerning the asymptotic behavior of eigenvalue lists, which may be of some interest on its own. Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $\mathcal{B}(H)$, which is *pure* in the sense defined above. The commutants $N_t = \alpha_t(\mathcal{B}(H))'$ are type I subfactors which increase with t, and because of purity their union is strongly dense in $\mathcal{B}(H)$. Let ρ be a normal state of $\mathcal{B}(H)$. We require the following information concerning the behavior of the eigenvalue lists of the restrictions $\rho \restriction_{N_t}$ for large t.

Theorem C. *Let α be a pure E_0-semigroup acting on $\mathcal{B}(H)$, which has a normal invariant state ω. Then for every normal state ρ of $\mathcal{B}(H)$ we have*

$$\lim_{t\to\infty} \|\Lambda(\rho \restriction_{\alpha_t(M)'}) - \Lambda(\rho \otimes \omega)\| = 0.$$

Remarks. One might expect that since the N_t increase to $\mathcal{B}(H)$, the restriction of a normal state to N_t should look like ρ itself when t is large. Indeed, if the invariant state ω is a vector state then its only nonzero eigenvalue is 1 and $\Lambda(\rho \otimes \omega) = \Lambda(\rho)$; in this case Theorem C implies that the restriction of ρ to N_t has almost the same list as ρ when t is large. On the other hand, if ω is not a vector state then $\Lambda(\rho \otimes \omega)$ is very different from $\Lambda(\rho)$, and Theorem C shows that this intuition is wrong.

We also remark that Theorem C is itself a special case of a more general result that is independent of the theory of E_0-semigroups (see [9]).

1. Existence of Dynamics

Flows on spaces are described infinitesimally by vector fields. Flows on Hilbert spaces (that is to say, one-parameter unitary groups) are described infinitesimally by unbounded self-adjoint operators. In practice, one is usually presented with a symmetric operator A that is not known to be self-adjoint (much like being presented with a differential equation that is not known to possess solutions for all time), and one wants to know if there is a one-parameter unitary group that can be associated with it. Precisely, one wants to know if A can be *extended* to a self-adjoint operator.

This problem of the existence of dynamics was solved by von Neumann as follows. Every densely defined symmetric operator A has an adjoint A^* with dense domain $\mathcal{D}^*$, and using A^* one defines two *deficiency spaces* $\mathcal{E}_-$, $\mathcal{E}_+$ by

$$\mathcal{E}_\pm = \{\xi \in \mathcal{D}^* : A^*\xi = \pm i\xi\}.$$

von Neumann's result is that A has self-adjoint extensions iff $\dim \mathcal{E}_- = \dim \mathcal{E}_+$ (see [15, Sect. XII.4]). Moreover, when $\mathcal{E}_-$ and $\mathcal{E}_+$ have the same dimension, von Neumann showed that for every unitary operator from $\mathcal{E}_-$ to $\mathcal{E}_+$ there is an associated self-adjoint extension of A. The purpose of this section is to establish an analogous result which locates the obstruction to the existence of dynamics for pairs of E_0-semigroups of the simplest kind (Corollary 1.1 below). That is based on the following more general result.

Let M be a type I subfactor of $\mathcal{B}(H)$, and let α, β be two E_0-semigroups acting, respectively, on M and its commutant M'. We want to examine conditions under which there is a one-parameter unitary group $U = \{U_t : t \in \mathbb{R}\}$ acting on H whose associated automorphism group $\gamma_t(A) = U_t A U_t^*$ has α as its past and β as its future in the sense that

$$\gamma_{-t} \restriction_M = \alpha_t, \quad \gamma_t \restriction_{M'} = \beta_t, \quad t \geq 0. \tag{1.1}$$

The following result asserts that there is such a unitary group U if and only if the product systems of α and β are *anti*-isomorphic.

Theorem. *Let $E^\alpha = \{E^\alpha(t) : t > 0\}$ and $E^\beta = \{E^\beta(t) : t > 0\}$ be the respective product systems of α and β,*

$$E^\alpha(t) = \{x \in M : \alpha_t(y)x = xy, \quad y \in M\},$$
$$E^\beta(t) = \{x' \in M' : \beta_t(y')x' = x'y', \quad y' \in M'\},$$

and assume that there is a one-parameter unitary group $U = \{U_t : t \in \mathbb{R}\}$ whose associated automorphism group satisfies (1.1). Then E^α and E^β are anti-isomorphic. Indeed, for every $t > 0$ we have $U_t E^\alpha(t) = E^\beta(t)$, and the map $\theta : E^\alpha \to E^\beta$ defined by

$$\theta(v) = U_t v, \quad v \in E^\alpha(t), \quad t > 0, \tag{1.2}$$

is an anti-isomorphism of product systems (i.e., it is a Borel-measurable map which is unitary on fibers, and which satisfies $\theta(vw) = \theta(w)\theta(v)$ for every $v \in E^\alpha(s)$, $w \in E^\alpha(t)$, $s, t > 0$).

Conversely, if $\theta : E^\alpha \to E^\beta$ is any anti-isomorphism of product systems, then for every $t > 0$ there is a unique unitary operator $U_t \in \mathcal{B}(H)$ which satisfies (1.2) for every $v \in E^\alpha(t)$. $\{U_t : t > 0\}$ is a strongly continuous semigroup of unitary operators tending strongly to the identity as $t \to 0+$, and its natural extension to a one-parameter unitary group gives rise to an automorphism group γ which satisfies (1.1).

Proof. Assume that $\gamma_t(A) = U_t A U_t^*$, $t \in \mathbb{R}$ satisfies (1.1). Fix $t > 0$. We claim first that $U_t E^\alpha(t) \subseteq M'$. Indeed, if $x \in M$ then for every $v \in E^\alpha(t)$ we have

$$xU_t v = U_t \gamma_{-t}(x)v = U_t \alpha_t(x)v = U_t vx.$$

Next, we claim that $U_t E^\alpha(t) \subseteq E^\beta(t)$. For $v \in E^\alpha(t)$, the preceding shows that $U_t v \in M'$, so it suffices to show that $\beta_t(y)U_t v = U_t vy$ for every $y \in M'$. For that, write

$$\beta_t(y)U_t v = \gamma_t(y)U_t v = U_t y U_t^* U_t v = U_t yv = U_t vy,$$

the last equality because $v \in M$ commutes with $y \in M'$.

Next, note that $E^\beta(t) \subseteq U_t E^\alpha(t)$. Choosing $w \in E^\beta(t)$, set $v = U_t^* w$. Note that $v \in M$ because for every $y \in M'$ we have

$$yv = yU_t^* w = U_t^* \gamma_t(y)w = U_t^* \beta_t(y)w = U_t^* wy = vy.$$

Note next that the element $v = U_t^* w \in M$ actually belongs to $E^\alpha(t)$. Indeed, for every $x \in M$ we have

$$\alpha_t(x)v = \alpha_t(x)U_t^* w.$$

Since γ_{-t} restricts to α_t on M, we have $\gamma_t(\alpha_t(x)) = x$ and the right side can be written

$$U_t^* \gamma_t(\alpha_t(x)) = U_t^* xw = U_t^* wx = vx.$$

The above shows that for every $t > 0$ we have a linear map $\theta_t : E^\alpha(t) \to E^\beta(t)$ defined by $\theta_t(v) = U_t v$. By assembling these maps we get a Borel-measurable map $\theta : E^\alpha \to E^\beta$ which is linear on fibers. Notice that θ_t is actually unitary, since for $v_1, v_2 \in E^\alpha(t)$ we have

$$\langle v_1, v_2 \rangle \mathbf{1} = v_2^* v_1 = (U_t v_2)^*(U_t v_1) = \theta(v_2)^* \theta(v_1) = \langle \theta(v_1), \theta(v_2) \rangle \mathbf{1}.$$

Finally, θ is an anti-isomorphism, because for $v \in E^\alpha(s)$, $w \in E^\alpha(t)$ we have

$$\theta(vw) = U_{s+t} vw = U_t(U_s v)w = U_t \theta(v)w = U_t w\theta(v) = \theta(w)\theta(v).$$

To prove the converse, fix an anti-isomorphism $\theta : E^\alpha \to E^\beta$. For every $t > 0$ pick an orthonormal basis $e_1(t), e_2(t) \ldots$ for $E^\alpha(t)$ (we will have to choose more carefully presently...but for the moment we choose an arbitrary orthonormal basis for each fiber space). For every $t > 0$ define an operator $U_t \in \mathcal{B}(H)$ by

$$U_t = \sum_{n=1}^{\infty} \theta(e_n(t)) e_n(t)^*.$$

One checks easily that $U_t U_t^* = U_t^* U_t = \mathbf{1}$, hence U_t is unitary. U_t also satisfies (1.2), for if $v \in E^\alpha(t)$ then we have $e_n(t)^* v = \langle v, e_n(t) \rangle \mathbf{1}$ and hence

$$U_t v = \sum_{n=1}^{\infty} \langle v, e_n(t) \rangle \, \theta(e_n(t)) = \theta(\sum_n \langle v, e_n(t) \rangle \, e_n(t)) = \theta(v).$$

Note too that since the ranges of the operators in $E^\alpha(t)$ span H, any operator U_t that satisfies (1.2) is determined uniquely. In particular, U_t does not depend on the choice of orthonormal basis $\{e_n(t)\}$ for $E^\alpha(t)$.

We may choose the orthonormal basis $\{e_n(t)\}$ so that each section $t \mapsto e_n(t) \in E^\alpha(t)$ is Borel measurable (because of the measurability axiom of product systems [2, Property 1.8 (iii)]), and once this is done we find that the function $t \in (0, \infty) \mapsto U_t \in \mathcal{B}(H)$ is Borel measurable.

We claim next that $\{U_t : t > 0\}$ is a semigroup. Indeed, if $w \in E^\alpha(s)$, $v \in E^\alpha(t)$, then since $\theta(v) \in M'$ commutes with $w \in M$ we have

$$U_s U_t vw = U_s \theta(v) w = U_s w \theta(v) = \theta(w)\theta(v) = \theta(vw) = U_{s+t} vw.$$

Since $E^\alpha(s+t)$ is spanned by such product vw and since $E^\alpha(s+t)H$ spans H, we conclude that $U_s U_t = U_{s+t}$.

At this point, we use the measurability proposition [2, Prop. 2.5 (ii)] (stated there for the more general case of cocycles) to conclude that a) U_t is strongly continuous in t for $t > 0$, and b) U_t tends strongly to $\mathbf{1}$ as $t \to 0+$. Now extend U in the obvious way to obtain a strongly continuous one-parameter unitary group acting on H.

Let $\gamma_t(A) = U_t A U_t^*$, $A \in \mathcal{B}(H)$, $t \in \mathbb{R}$. It remains to show that for every $t > 0$ we have $\gamma_{-t} \restriction_M = \alpha_t$ and $\gamma_t \restriction_{M'} = \beta_t$.

Choose $x \in M$. To show that $\gamma_{-t}(x) = \alpha_t(x)$, it suffices to show that $\gamma_{-t}(x)v = \alpha_t(x)v$ for every $v \in E^\alpha(t)$ (because H is spanned by the ranges of the operators in $E^\alpha(t)$). But for such a v we have

$$\gamma_{-t}(x)v = U_{-t} x U_t v = U_{-t} x \theta(v) = U_{-t} \theta(v) x = vx = \alpha_t(x)v.$$

Choose $y \in M'$. To show that $\gamma_t(y) = \beta_t(y)$ it suffices to show that $\gamma_t(y)w = \beta_t(y)w$ for all $w \in E^\beta(t)$. For such a w we have $w = \theta(v) = U_t v$ for some $v \in E^\alpha(t)$, hence

$$\gamma_t(y)w = U_t y U_t^* U_t v = U_t yv = U_t vy = wy = \beta_t(y)w,$$

and the proof is complete. □

We view the following result as a counterpart for noncommutative dynamics of von Neumann's theorem on the existence of self-adjoint extensions of symmetric operators in terms of deficiency indices.

Corollary 1.1. *Let α and β be two E_0-semigroups, acting on $\mathcal{B}(H)$ and $\mathcal{B}(K)$ respectively, each of which is a cocycle perturbation of a CCR/CAR flow. There is a one-parameter group of automorphisms of $\mathcal{B}(H \otimes K)$ which satisfies the condition of (1.1) if, and only if, α and β have the same numerical index.*

Proof. Consider the type I subfactor M of $\mathcal{B}(H \otimes K)$ defined by

$$M = \mathcal{B}(H) \otimes \mathbf{1}_K .$$

We have $M' = \mathbf{1}_H \otimes \mathcal{B}(K)$, and α (resp. β) is conjugate to the action on M (resp. M') defined by $A \otimes \mathbf{1}_K \mapsto \alpha_t(A) \otimes \mathbf{1}_K$ (resp. $\mathbf{1}_H \otimes B \mapsto \mathbf{1}_H \otimes \beta_t(B)$), $t \geq 0$.

Now the product system of any CAR/CCR flow is anti-isomorphic to itself. This follows, for example, from the structural results on divisible product systems of [2, Sect. 6]. Alternately, one can simply write down explicit anti-automorphisms of the product systems described on pp. 12–14 of [2]. Since the structure of the product system of any E_0-semigroup is stable under cocycle perturbations, the same is true of cocycle perturbations of CAR/CCR flows.

The preceding theorem implies that there is a one-parameter group of automorphisms $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ of $\mathcal{B}(H \otimes K)$ satisfying

$$\gamma_{-t}(A \otimes \mathbf{1}_K) = \alpha_t(A) \otimes \mathbf{1}_K, \qquad \gamma_t(\mathbf{1}_H \otimes B) = \mathbf{1}_H \otimes \beta_t(B)$$

for every $t \geq 0$ iff the product systems E^α and E_β are anti-isomorphic. The preceding paragraph shows that this is true iff E_α and E_β are isomorphic; and since α and β are simply cocycle perturbations of CAR/CCR flows, the latter holds iff α and β have the same numerical index. □

Corollary 1.2. *Let α and β be two pure E_0-semigroups which are cocycle-conjugate to the CAR/CCR flow of index $n = 1, 2, \ldots, \infty$. Then there is a history (U, M) whose past and future semigroups are conjugate, respectively, to α and β.*

Remarks on the existence and nonexistence of dynamics: general case. It is natural to ask if every E_0-semigroup α can represent both the past and future of some history. More precisely, is there a history whose past and future E_0-semigroups are both conjugate to cocycle perturbations of α? This is certainly the case for the CAR/CCR flows, by Corollary 1.2.

But in general the answer can be no. We have recently received a manuscript of Boris Tsirelson [34] in which examples of product systems are constructed which are *not* anti-isomorphic to themselves. It is shown in [3] that every abstract product system is isomorphic to the product system of some E_0-semigroup. It follows that there are E_0-semigroups α whose product systems are not anti-isomorphic themselves. Since cocycle perturbations of E_0-semigroups must have isomorphic product systems, the theorem proved above implies that such E_0-semigroups (i.e., those whose product systems are not anti-isomorphic to themselves) cannot serve as both the past and future of any history.

On the other hand, for every E_0-semigroup α acting on $\mathcal{B}(H)$, which is pure in the sense that $\cap_t \alpha_t(\mathcal{B}(H)) = \mathbb{C} \cdot \mathbf{1}$, there is a history whose past is conjugate to α. To see why this is so, let E be its product system. Let E^* be the product system opposite to E (E^* is defined as the same structure as E except for multiplication, and in E^* multiplication is defined by reversing the multiplication of E). Since E^* is a product system, the results of [3] imply that there is an E_0-semigroup β, acting on $\mathcal{B}(K)$, whose product system is isomorphic to E^* and therefore anti-isomorphic to E. We may conclude from the

theorem proved above that there is a one parameter group of automorphisms γ acting on $\mathcal{B}(H \otimes K)$ which satisfies (1.1) by having $\alpha \otimes \mathbf{1}$ (acting on $\mathcal{B}(H) \otimes \mathbf{1}$) as its past and $\mathbf{1} \otimes \beta$ (acting on $\mathbf{1} \otimes \mathcal{B}(K)$) as its future.

2. Eigenvalue Lists of Normal States

In this section we emphasize the importance of the "eigenvalue list" invariant that can be associated with normal states of type I factors, and we summarize its basic properties. An *eigenvalue list* is a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \dots$ of nonnegative real numbers satisfying $\sum_n \lambda_n < \infty$. If $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ and $\Lambda' = \{\lambda'_1 \geq \lambda'_2 \geq \dots\}$ are two such lists we write

$$\|\Lambda - \Lambda'\| = \sum_{n=1}^{\infty} |\lambda_n - \lambda'_n|$$

for the ℓ^1-distance from Λ to Λ', thereby making the space of all eigenvalue lists into a complete metric space.

Let A be a positive trace class operator acting on a separable Hilbert space H. The positive eigenvalues of A (counting multiplicity) can be arranged in decreasing order, and if there are only finitely many nonzero eigenvalues then we extend the list by appending zeros in the obvious way. This defines the eigenvalue list $\Lambda(A)$ of A. Notice that even when H is finite dimensional, $\Lambda(A)$ is an infinite list.

The following basic properties of eigenvalue lists will be used repeatedly.

Proposition 2.1.

2.1.1 *For every positive trace class operator A we have $\Lambda(A) = \Lambda(A \oplus 0_\infty)$, 0_∞ denoting the infinite dimensional zero operator.*

2.1.2 *For positive trace class operators A and B, $\Lambda(A) = \Lambda(B)$ iff $A \oplus 0_\infty$ is unitarily equivalent to $B \oplus 0_\infty$.*

2.1.3 *If L is any Hilbert-Schmidt operator from a Hilbert space H_1 to a Hilbert space H_2, then $\Lambda(L^*L) = \Lambda(LL^*)$.*

2.1.4 *For positive trace class operators A, B we have $\Lambda(A) = \Lambda(B)$ iff*

$$\operatorname{trace}(A^n) = \operatorname{trace}(B^n) \quad \text{for every } n = 1, 2, \dots$$

Proof. The assertion (2.1.1) is obvious, and (2.1.2) follows after a routine application of the spectral theorem for self-adjoint compact operators.

Proof of (2.1.3). Let $K_1 \subseteq H_1$ be the initial space of L and let $K_2 = \overline{LK_1} \subseteq H_2$ be its closed range. The polar decomposition implies that $L^*L \restriction_{K_1}$ and $LL^* \restriction_{K_2}$ are unitarily equivalent. Hence $L^*L \oplus 0_\infty$ and $LL^* \oplus_\infty$ are unitarily equivalent and the assertion (2.1.3) follows from (2.1.2). □

Proof of (2.1.4). If $\Lambda(A) = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ then

$$\operatorname{trace}(A^n) = \sum_{k=1}^{\infty} \lambda_k^n, \quad n = 1, 2, \dots$$

Thus $\Lambda(A) = \Lambda(B)$ implies that $\operatorname{trace}(A^n) = \operatorname{trace}(B^n)$ for every $n \geq 1$.

Conversely, suppose that $\operatorname{trace}(A^n) = \operatorname{trace}(B^n)$ for every $n = 1, 2, \dots$ Choose a positive number M so large that the interval $[0, M]$ contains the spectra of both operators

A and B. The linear functional $f \mapsto \text{trace}(Af(A))$ defined on the commutative C^*-algebra $C[0, M]$ is positive, hence there is a unique finite positive measure μ_A defined on $[0, M]$ such that

$$\int_0^M f(x)\, d\mu_A(x) = \text{trace}(Af(A)), \quad f \in C[0, M].$$

The restriction of μ_A to $(0, M]$ is concentrated on $\sigma(A) \cap (0, M]$, and for every positive eigenvalue λ of A we have

$$\mu_A(\{\lambda\}) = \lambda \cdot \text{multiplicity of } \lambda.$$

Doing the same for the operator B, we find that by hypothesis

$$\int_0^M x^n\, d\mu_A(x) = \int_0^M x^n\, d\mu_B(x), \quad n = 0, 1, 2, \ldots,$$

and hence by the Weierstrass approximation theorem μ_A and μ_B define the same linear functional on $C[0, M]$. It follows that $\mu_A = \mu_B$, and the preceding observations lead us to conclude that $\Lambda(A) = \Lambda(B)$. □

We will also make use of the following classical result, originating in work of Hermann Weyl around 1912.

Proposition 2.2. *If A, B are positive trace class operators acting on the same Hilbert space H, then*

$$\|\Lambda(A) - \Lambda(B)\| \leq \text{trace}|A - B|.$$

Proof. A proof can be found in the appendix of [29]. □

Remarks. Notice that since $\Lambda(A)$ depends only on the unitary equivalence class of A, Proposition 2.2 actually implies that

$$\|\Lambda(A) - \Lambda(B)\| \leq \inf_{A', B'} \text{trace}|A' - B'|,$$

where A' (resp. B') ranges over all operators unitarily equivalent to A (resp. B). Indeed, though we do not require the fact, it is not hard to show that $\|\Lambda(A) - \Lambda(B)\|$ is exactly the distance (relative to the trace norm) from the unitary equivalence class of $A \oplus 0_\infty$ to the unitary equivalence class of $B \oplus 0_\infty$. Thus the eigenvalue list $\Lambda(A)$ provides a more-or-less complete invariant for classifying positive trace class operators up to unitary equivalence.

On the other hand, the eigenvalue list is also a subtle invariant. To illustrate the point, suppose that A has only two positive eigenvalues $3/4$ and $1/4$, and that B has only three positive eigenvalues $3/5, 1/5, 1/5$. The spectrum of $A \oplus B$ is the union of the spectra and the spectrum of $A \otimes B$ is the set of products of elements from the two spectra; however, both of these sets must be rearranged in decreasing order. Thus

$$\Lambda(A \oplus B) = \{3/4, 3/5, 1/4, 1/5, 1/5, 0, \ldots\},$$
$$\Lambda(A \otimes B) = \{9/20, 3/20, 3/20, 3/20, 1/20, 1/20, 0, \ldots\}.$$

Notice that A has only eigenvalues of multiplicity 1, B has eigenvalues of multiplicities 1 and 2, but that $A \otimes B$ has an eigenvalue of "peculiar" multiplicity 3. In the case of

larger spectra, the relation between say $\Lambda(A \otimes B)$ and the individual lists $\Lambda(A)$ and $\Lambda(B)$ depends in a complex way on the relative sizes of eigenvalues, and the problem of rearranging the set of products into decreasing order can be a difficult combinatorial chore.

Turning now to normal states, let M be a type I_n factor, $n = 1, 2, \ldots, \infty$ (one can assume without essential loss that M is concretely represented as a subfactor of $\mathcal{B}(H)$ for some Hilbert space H), and let ρ be a normal state of M. There is a Hilbert space K of dimension n such that M is isomorphic as a $*$-algebra to $\mathcal{B}(K)$, and in this case any such $*$-isomorphism must be isometric and normal. Thus we may identify ρ with a normal state of $\mathcal{B}(K)$, and consequently there is a positive operator $R \in \mathcal{B}(K)$ of trace 1 such that

$$\rho(T) = \text{trace}(RT), \quad T \in \mathcal{B}(K).$$

The *eigenvalue list* of ρ is defined by $\Lambda(\rho) = \Lambda(R)$. The preceding discussion leads immediately to the following.

Proposition 2.3.

2.3.1 *If ρ_1 and ρ_2 are normal states of type I factors M_1 and M_2, and if ρ_1 and ρ_2 are conjugate in the sense that there is a $*$-isomorphism θ of M_1 onto M_2 such that $\rho_2 \circ \theta = \rho_1$, then $\Lambda(\rho_1) = \Lambda(\rho_2)$.*

2.3.2 *If ρ_1 and ρ_2 are two normal states of a type I factor M, then*

$$\|\Lambda(\rho_1) - \Lambda(\rho_2)\| \leq \|\rho_1 - \rho_2\|.$$

Proof. The first assertion is apparent after we realize M_k as $\mathcal{B}(H_k)$, $k = 1, 2$, use the fact that a $*$-isomorphism of $\mathcal{B}(H_1)$ onto $\mathcal{B}(H_2)$ is implemented by a unitary operator from H_1 to H_2, and make use of (2.1.2). The second assertion is the inequality of Proposition 2.2. □

3. CP Semigroups and the Existence of Interactions

The corollary of Sect. 1 implies that any pair of pure E_0-semigroups α_-, α_+, which are both cocycle conjugate to the same CAR/CCR flow, can be assembled so as to obtain a history (U, M) whose past and future E_0-semigroups are conjugate to α_- and α_+. Moreover, if both α_- and α_+ have normal invariant states then (U, M) is in fact an interaction.

Thus we are led to ask what the possibilities are. More precisely, suppose we are given an eigenvalue list $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots\}$ with $\sum_n \lambda_n = 1$ and a nonnegative integer $n = 1, 2, \ldots, \infty$. Does there exist a cocycle perturbation α of the CAR/CCR flow of index n which is pure, and which leaves invariant a normal state whose eigenvalue list is Λ?

We do not know the answer in general, but we conjecture that it is yes. The purpose of this section is to provide an affirmative answer for the cases in which Λ has only a finite number of nonzero terms (Theorem A). This is essentially the main result of [7] (together with Corollary 1.1), and we merely summarize the main ideas so as to emphasize the role of dilation theory and semigroups of completely postive maps (sometimes called quantum dynamical semigroups) acting on matrix algebras, for such constructions.

Suppose that $\alpha = \{\alpha_t : t \geq 0\}$ is an E_0-semigroup acting on $\mathcal{B}(H)$, and assume further that there is a normal state ω of $\mathcal{B}(H)$ which is invariant, $\omega \circ \alpha_t = \omega$, $t \geq 0$. Letting Ω be the density operator of ω,

$$\omega(T) = \text{trace}(\Omega T), \quad T \in \mathcal{B}(H),$$

then the projection P on the closed range of Ω is the support projection of ω, i.e., the largest projection with the property that $\omega(P^\perp) = 0$. Using $\omega \circ \alpha_t = \omega$, we find that $\omega(\mathbf{1} - \alpha_t(P)) = \omega(\alpha_t(P^\perp)) = \omega(P^\perp) = 0$, hence $\mathbf{1} - \alpha_t(P) \leq \mathbf{1} - P$, hence

$$\alpha_t(P) \geq P, \quad t \geq 0. \tag{3.1}$$

The inequality (3.1) has the following consequence. If we identify $\mathcal{B}(PH)$ with the corner $P\mathcal{B}(H)P$, then for every $t \geq 0$ we can compress α_t so as to obtain a completely positive map ϕ_t on $\mathcal{B}(PH)$,

$$\phi_t(X) = P\alpha_t(X) \restriction_{PH}, \quad X \in P\mathcal{B}(H)P.$$

More significantly, because of (3.1) we have the semigroup property $\phi_s \circ \phi_t = \phi_{s+t}$, as one can easily verify using $P\alpha_s(A)P = P\alpha_s(PAP)P$ for $A \in \mathcal{B}(H)$. Thus we have defined a semigroup $\phi = \{\phi_t : t \geq 0\}$ of normal completely positive maps of $\mathcal{B}(PH)$ satisfying $\phi_t(\mathbf{1}) = \mathbf{1}$ for $t \geq 0$, together with the natural continuity property

$$\lim_{t \to t_0} \langle \phi_t(X)\xi, \eta \rangle = \langle \phi_{t_0}(X)\xi, \eta \rangle,$$

$\xi, \eta \in PH$, $X \in \mathcal{B}(PH)$.

We appear to have lost ground, in that we started with a semigroup of $*$-endomorphisms and now have merely a semigroup of completely positive maps. However, notice that the restriction of ω to $\mathcal{B}(PH) = P\mathcal{B}(H)P$ is a *faithful* normal state which is invariant under the action of ϕ, $\omega \circ \phi_t = \omega$, $t \geq 0$.

Notice too that in case there are only a finite number of positive eigenvalues in the list $\Lambda(\omega)$ then PH is finite dimensional, and thus $\phi = \{\phi_t : t \geq 0\}$ is a CP semigroup acting essentially on a *matrix algebra*, which leaves invariant a faithful state with prescribed eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. If α began life as a pure E_0-semigroup then ω is an *absorbing* state for ϕ in the sense that for every normal state ρ of $\mathcal{B}(PH)$,

$$\lim_{t \to \infty} \|\rho \circ \phi_t - \omega\| = 0. \tag{3.2}$$

Conversely and most significantly, if we can create a pair (ϕ, ω) satisfying the conditions of the preceding paragraph then it is possible to reconstruct a pair (α, ω) consisting of an E_0-semigroup α having an invariant normal state ω with the expected eigenvalue list by a "dilation" procedure which reverses the "compression" procedure we have described above. Moreover, if the CP semigroup ϕ has a bounded generator (as it will surely have in the case where PH is finite dimensional), then its dilation to an E_0-semigroup will be cocycle-conjugate to a *CAR/CCR* flow whose index can be calculated directly in terms of ϕ (the details can be found in [6] and [7]). The following summarizes the result of the construction of (ϕ, ω) for finite eigenvalue lists given in [7, Theorem 5.1].

Theorem 3.1. *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ be a list of positive numbers and let ω be a state of the matrix algebra $M_r(\mathbb{C})$ whose density operator has this ordered list of eigenvalues.*

There is a semigroup $\phi = \{\phi_t : t \geq 0\}$ of unital completely positive maps on $M_r(\mathbb{C})$ which leaves ω invariant, satisfies (3.2), and which can be dilated to a pure cocycle perturbation of a CAR/CCR flow having a normal invariant state whose eigenvalue list has exactly $\lambda_1 \geq \cdots \geq \lambda_r$ as its nonzero elements.

Theorem 3.3 leads to the following (see pp. 40–42 of [7]).

Corollary 3.2. *Let $n = 1, 2, \ldots, \infty$ and let $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots\}$ be an eigenvalue list which has only a finite number of nonzero terms. There is a cocycle perturbation α of the CAR/CCR flow of index n which is pure, and which has an invariant normal state with eigenvalue list Λ.*

Using Corollary 1.1 of, we deduce Theorem A of the introduction.

Theorem A. *Let $n = 1, 2, \ldots, \infty$ and let Λ_- and Λ_+ be two eigenvalue lists having only a finite number of nonzero terms. There is an interaction (U, M) whose past and future normal states ω_-, ω_+ have eigenvalue lists Λ_-, Λ_+ respectively, and whose past and future E_0-semigroups are cocycle conjugate to the CAR/CCR flow of index n.*

4. The Interaction Inequality

Theorem A provides many examples of interactions, but it says nothing about whether or not these interactions are nontrivial. For that we need the inequality of Theorem B of the introduction. The purpose of this section is to prove Theorem B and discuss its consequences for interactions. Theorem B is based on the following more general result about E_0-semigroups. An E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ is said to be *pure* if

$$\bigcap_{t \geq 0} \alpha_t(\mathcal{B}(H)) = \mathbb{C} \cdot \mathbf{1}.$$

Purity implies that for any two normal states ρ_1 and ρ_2,

$$\lim_{t \to \infty} \|\rho_1 \circ \alpha_t - \rho_2 \circ \alpha_t\| = 0,$$

see Proposition 1.1 of [7]. In particular, if there is a *normal* state ω which is invariant under α in the sense that $\omega \circ \alpha_t = \omega$ for every $t \geq 0$ then ω must be an *absorbing* state in the sense that for every normal state ρ of $\mathcal{B}(H)$ we have

$$\lim_{t \to \infty} \|\rho \circ \alpha_t - \omega\| = 0. \tag{4.1}$$

Thus, if a pure E_0-semigroup has a normal invariant state then it is unique, and in particular the eigenvalue list $\Lambda(\omega)$ of a normal invariant state ω provides a conjugacy invariant of pure E_0-semigroups.

Given a pure E_0-semigroup acting on $\mathcal{B}(H)$, the commutants $N_t = \alpha_t(\mathcal{B}(H))'$ are type I subfactors of $\mathcal{B}(H)$ which increase with t, and by purity their union is a strongly dense $*$-subalgebra of $\mathcal{B}(H)$. Let ρ be any normal state of $\mathcal{B}(H)$. Since N_t is a type I factor, the restriction of ρ to N_t has an eigenvalue list, defined as in Sect. 3. The following result shows how these eigenvalue lists behave for large t.

Theorem C. *Let $\alpha = \{\alpha_t : t \geq 0\}$ be a pure E_0-semigroup having a normal invariant state ω, and let N_t be the commutant $\alpha_t(\mathcal{B}(H))'$. Then for every normal state ρ of $\mathcal{B}(H)$ we have*

$$\lim_{t\to\infty} \|\Lambda(\rho \restriction_{N_t}) - \Lambda(\rho \otimes \omega)\| = 0.$$

The proof of Theorem C requires some preparation.

Lemma 4.1. *Let $\{A_i : i \in I\}$ be a net of positive trace class operators acting on a Hilbert space H and let B be a positive trace class operator such that* trace (A_i) = trace (B) *for every $i \in I$. Suppose there is a set $S \subseteq H$, having H as its closed linear span, such that*

$$\lim_i \langle A_i\xi, \eta\rangle = \langle B\xi, \eta\rangle\,, \quad \xi, \eta \in S.$$

Then trace$|A_i - B| \to 0$, *as $i \to \infty$.*

Proof. By Proposition 1.6 of [7] it suffices to show that

$$\lim_{i\to\infty} \text{trace}(A_i K) = \text{trace}(BK)$$

for every compact operator $K \in \mathcal{B}(H)$. The set $\mathcal{S}$ of compact operators K for which the assertion is true is a norm-closed linear space which contains all rank-one operators of the form $\zeta \mapsto \langle \zeta, \xi\rangle\, \eta$, with $\xi, \eta \in S$. Since S spans H, it follows that $\mathcal{S}$ is the space of all compact operators. □

The next three lemmas relate to the following situation. We are given a normal $*$-endomorphism α of $\mathcal{B}(H)$ satisfying $\alpha(\mathbf{1}) = \mathbf{1}$. Let $\mathcal{E}$ be the linear space of operators

$$\mathcal{E} = \{v \in \mathcal{B}(H) : \alpha(x)v = vx, \quad x \in \mathcal{B}(H)\}.$$

If u, v are any two elements of $\mathcal{E}$ then v^*u is a scalar multiple of the identity operator, and in fact $\mathcal{E}$ is a Hilbert space relative to the inner product defined on it by

$$v^*u = \langle u, v\rangle_{\mathcal{E}}\, \mathbf{1}.$$

For any orthonormal basis $v_1, v_2, \ldots$ of $\mathcal{E}$ we have

$$\alpha(x) = \sum_n v_n x v_n^*, \quad x \in \mathcal{B}(H).$$

Let ρ be a normal state of $\mathcal{B}(H)$. It is clear that $u, v \in \mathcal{E} \mapsto \rho(uv^*)$ defines a bounded sesquilinear form on the Hilbert space $\mathcal{E}$, hence by the Riesz lemma there is a unique bounded operator $A \in \mathcal{B}(\mathcal{E})$ such that

$$\langle Au, v\rangle_{\mathcal{E}} = \rho(uv^*), \quad u, v \in \mathcal{E}.$$

A is obviously a positive operator and in fact we have trace $A = 1$, since for any orthonormal basis $v_1, v_2, \ldots$ for $\mathcal{E}$,

$$\text{trace } A = \sum_n \langle Av_n, v_n\rangle = \sum_n \rho(v_n v_n^*) = \rho(\alpha(\mathbf{1})) = \rho(\mathbf{1}) = 1.$$

The following result shows how to compute the eigenvalue list of the restriction of ρ to the commutant of $\alpha(\mathcal{B}(H))$ in terms of this "correlation" operator A.

Lemma 4.2. *Let ρ be a normal state of $\mathcal{B}(H)$ and let A be the positive trace class operator on $\mathcal{E}$ defined by $\langle Au, v\rangle_{\mathcal{E}} = \rho(uv^*)$, $u, v \in \mathcal{E}$. Then*

$$\Lambda(\rho \restriction_{\alpha(\mathcal{B}(H))'}) = \Lambda(A).$$

Proof. By Proposition 2.3.1, it suffices to exhibit a normal $*$-isomorphism θ of $\mathcal{B}(\mathcal{E})$ onto $\alpha(\mathcal{B}(H))'$ with the property that

$$\rho(\theta(T)) = \text{trace}(AT), \quad T \in \mathcal{B}(\mathcal{E}). \tag{4.2}$$

Consider the tensor product of Hilbert spaces $\mathcal{E} \otimes H$. In order to define θ we claim first that there is a unique unitary operator $W : \mathcal{E} \otimes H \to H$ which satisfies $W(v \otimes \xi) = v\xi$, $v \in \mathcal{E}$, $\xi \in H$. Indeed, for $v, w \in \mathcal{E}$, $\xi, \eta \in H$ we have

$$\langle v\xi, w\eta\rangle_H = \left\langle w^* v\xi, \eta\right\rangle = \langle v, w\rangle_{\mathcal{E}} \langle \xi, \eta\rangle = \langle v \otimes \xi, w \otimes \eta\rangle_{\mathcal{E}\otimes H}.$$

It follows that there is a unique isometry $W : \mathcal{E} \otimes H \to H$ with the stated property. W is unitary because its range spans all of H (indeed, any vector ζ orthogonal to the range of W has the property $v^*\zeta = 0$ for every $v \in \mathcal{E}$, hence $\zeta = \alpha(\mathbf{1})\zeta = \sum_n v_n v_n^* \zeta = 0$).

For every $X \in \mathcal{B}(H)$ we have

$$W(\mathbf{1} \otimes X)v \otimes \xi = W(v \otimes X\xi) = vX\xi = \alpha(X)v\xi = \alpha(X)W(v \otimes \xi),$$

hence $W(\mathbf{1} \otimes X)W^* = \alpha(X)$. It follows that $\alpha(\mathcal{B}(H))' = W(\mathcal{B}(\mathcal{E}) \otimes \mathbf{1})W^*$, and thus we can define a $*$-isomorphism $\theta : \mathcal{B}(\mathcal{E}) \to \alpha(\mathcal{B}(H))'$ by $\theta(T) = W(T \otimes \mathbf{1})W^*$.

Writing $u \times \bar{v}$ for the rank-one operator on $\mathcal{E}$ defined by $u \times \bar{v} : w \mapsto \langle w, v\rangle_{\mathcal{E}}\, u$, we claim that

$$\theta(u \times \bar{v}) = uv^*, \quad \text{for every } u, v \in \mathcal{E}. \tag{4.3}$$

Indeed, if we pick a vector in H of the form $\eta = w\xi = W(w \otimes \xi)$, where $w \in \mathcal{E}$ and $\xi \in H$ then we have

$$\begin{aligned}\theta(u \times \bar{v})\eta = \theta(w \times \bar{v})W(w \otimes \xi) = W((u \times \bar{v}) \otimes \mathbf{1})w \otimes \xi = W((u \times \bar{v})w \otimes \xi)\\ = \langle w, v\rangle_{\mathcal{E}}\, W(u \otimes \xi) = \langle w, v\rangle_{\mathcal{E}}\, u\xi = uv^*w\xi = uv^*\eta,\end{aligned}$$

and (4.3) follows because H is spanned by all such vectors η.

Now for every rank-one operator $T = u \times \bar{v} \in \mathcal{B}(E)$ we have

$$\rho(\theta(T)) = \rho(\theta(u \times \bar{v})) = \rho(uv^*) = \langle Au, v\rangle_{\mathcal{E}} = \text{trace}(AT).$$

Formula (4.2) follows for finite rank $T \in \mathcal{B}(\mathcal{E})$ by taking linear combinations, and the general case follows by approximating an arbitrary operator $T \in \mathcal{B}(\mathcal{E})$ in the strong operator topology with finite dimensional compressions PTP, P ranging over an increasing sequence of finite dimensional projections with limit $\mathbf{1}$. □

The following formulas provide a key step.

Lemma 4.3. *Let α, $\mathcal{E}$ be as above, let ρ be a normal state of $\mathcal{B}(H)$ and let $R \in \mathcal{L}^1(H)$ be its density operator $\rho(X) = \text{trace}(RX)$, $X \in \mathcal{B}(H)$. Define a linear operator L from $\mathcal{E}$ into the Hilbert space $\mathcal{L}^2(H)$ of all Hilbert-Schmidt operators on H by $Lv = R^{1/2}v$, $v \in \mathcal{E}$. Then*

4.3.1 $\langle L^*Lu, v\rangle_{\mathcal{E}} = \rho(uv^*)$, $u, v \in \mathcal{E}$, *and*

4.3.2 *for all* $\xi_1, \xi_2, \eta_1, \eta_2 \in H$ *we have*

$$\left\langle LL^*(\xi_1 \times \bar{\xi}_2), \eta_1 \times \bar{\eta}_2 \right\rangle_{\mathcal{L}^2(H)} = \left\langle \alpha(\eta_2 \times \bar{\xi}_2) R^{1/2}\xi_1, R^{1/2}\eta_1 \right\rangle_H .$$

Proof of (4.3.1). Simply write

$$\left\langle L^*Lu, v \right\rangle_{\mathcal{E}} = \langle Lu, Lv \rangle_{\mathcal{L}^2(H)} = \left\langle R^{1/2}u, R^{1/2}v \right\rangle_{\mathcal{L}^2(H)} = \text{trace}(v^*Ru) = \rho(uv^*). \qquad \square$$

Proof of (4.3.2). We have

$$\left\langle LL^*(\xi_1 \times \bar{\xi}_2), \eta_1 \times \bar{\eta}_2 \right\rangle_{\mathcal{L}^2(H)} = \left\langle L^*(\xi_1 \times \bar{\xi}_2), L^*(\eta_1 \times \bar{\eta}_2) \right\rangle_{\mathcal{E}} . \tag{4.4}$$

Pick an orthonormal basis $v_1, v_2, \ldots$ for $\mathcal{E}$. Then the right side of (4.7) can be rewritten as follows:

$$\begin{aligned}
&\sum_n \left\langle L^*(\xi_1 \times \bar{\xi}_2), v_n \right\rangle_{\mathcal{E}} \left\langle v_n, L^*(\eta_1 \times \bar{\eta}_2) \right\rangle_{\mathcal{E}} \\
&\quad = \sum_n \left\langle \xi_1 \times \bar{\xi}_2, R^{1/2}v_n \right\rangle_{\mathcal{L}^2(H)} \left\langle R^{1/2}v_n, \eta_1 \times \bar{\eta}_2 \right\rangle_{\mathcal{L}^2(H)} \\
&\quad = \sum_n \text{trace}(v_n^* R^{1/2}\xi_1 \times \bar{\xi}_2)\text{trace}(R^{1/2}v_n\eta_2 \times \bar{\eta}_1) = \\
&\qquad \sum_n \left\langle v_n^* R^{1/2}\xi_1, \xi_2 \right\rangle_H \left\langle R^{1/2}v_n\eta_2, \eta_1 \right\rangle_H .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left\langle \alpha(\eta_2 \times \bar{\xi}_2) R^{1/2}\xi_1, R^{1/2}\eta_1 \right\rangle_H &= \sum_n \left\langle v_n(\eta_2 \times \bar{\xi}_2) v_n^* R^{1/2}\xi_1, R^{1/2}\eta_1 \right\rangle_H \\
&= \sum_n \left\langle (\eta_2 \times \bar{\xi}_2) v_n^* R^{1/2}\xi_1, v_n^* R^{1/2}\eta_1 \right\rangle_H \\
&= \sum_n \left\langle v_n^* R^{1/2}\xi_1, \xi_2 \right\rangle_H \left\langle \eta_2, v_n^* R^{1/2}\eta_1 \right\rangle_H ,
\end{aligned}$$

and the last expression agrees with the bottom line of the previous formula. $\square$

Lemma 4.4. *For a pair* A, B *of self-adjoint compact operators on* H*, let* $A \circ B$ *be the bounded operator defined on the Hilbert space* $\mathcal{L}^2(H)$ *of Hilbert-Schmidt operators by* $A \circ B(T) = ATB$*. Then* $A \circ B$ *is unitarily equivalent to* $A \otimes B \in \mathcal{B}(H \otimes H)$*.*

Proof. Pick orthonormal bases $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ for H consisting of eigenvectors of A and B, $Ae_n = \alpha_n e_n$, $Bf_n = \beta_n f_n$, $n = 1, 2, \ldots$ Letting $e_m \times \bar{f}_n$ be the rank-one operator $\zeta \mapsto \langle \zeta, f_n \rangle e_n$, then $\{e_m \times \bar{f}_n : m, n = 1, 2, \ldots\}$ is an orthonormal basis for $\mathcal{L}^2(H)$ and we have

$$A \circ B(e_m \times \bar{f}_n) = \alpha_m \beta_n e_m \times \bar{f}_n, \qquad m, n = 1, 2, \ldots$$

Thus the unitary operator $W : \mathcal{L}^2(H) \to H \otimes H$ defined by $W(e_m \times \bar{f}_n) = e_m \otimes f_n$, $m, n = 1, 2, \ldots$ satisfies $W(A \circ B)(e_m \times \bar{f}_n) = (A \otimes B)W(e_m \times \bar{f}_n)$ for every $m, n = 1, 2, \ldots$, and hence $W(A \circ B)W^* = A \otimes B$. $\square$

Proof of Theorem C. Let $R \in \mathcal{B}(H)$ be the density operator of the normal state ρ, $\text{trace}(RT) = \rho(T), T \in \mathcal{B}(H)$. For every $t > 0$ let $\mathcal{E}_t$ be the Hilbert space of intertwining operators associated with α_t,

$$\mathcal{E}_t = \{T \in \mathcal{B}(H) : \alpha_t(A)T = TA, \quad A \in \mathcal{B}(H)\},$$

and let $L_t : \mathcal{E}_t \to \mathcal{L}^2(H)$ be the operator of Lemma 3, $L_t v = R^{1/2}v$, $v \in \mathcal{E}_t$.

Lemma 4.3.1 implies that $\rho(uv^*) = \langle L_t^* L_t u, v\rangle_{\mathcal{E}}$, hence the correlation operator of $\rho \restriction_{\alpha_t(\mathcal{B}(H))'}$ is $L_t^* L_t$. By Lemma 4.2

$$\Lambda(L_t^* L_t) = \Lambda(\rho \restriction_{\alpha_t(\mathcal{B}(H))'}).$$

On the other hand, (2.1.3) implies that $\Lambda(L_t^* L_t) = \Lambda(L_t L_t^*)$. Thus it suffices to show that the eigenvalue lists of the operators $L_t L_t^* \in \mathcal{B}(\mathcal{L}^2(H))$ converge to $\Lambda(\rho \otimes \omega)$, as $t \to \infty$, in the metric of eigenvalue lists.

By (4.3.2) we have

$$\langle L_t L_t^*(\xi_1 \times \bar{\xi}_2), \eta_1 \times \bar{\eta}_2\rangle_{\mathcal{L}^2(H)} = \left\langle \alpha_t(\eta_2 \times \bar{\xi}_2) R^{1/2}\xi_1, R^{1/2}\eta_1 \right\rangle_H, \tag{4.5}$$

for all $\xi_1, \xi_2, \eta_1, \eta_2 \in H$. Now since α is pure, $\alpha_t(X)$ converges in the weak*-topology to $\omega(X)\mathbf{1}$ as $t \to \infty$ (indeed, for every normal state σ, $\sigma(\alpha_t(X))$ converges to $\omega(X) = \sigma(\omega(X)\mathbf{1})$, and the assertion follows because every element of the predual of $\mathcal{B}(H)$ is a linear combination of normal states). Thus if we take the limit on t in the right side of (4.4) we obtain

$$\begin{aligned}\lim_{t\to\infty} \left\langle \alpha_t(\eta_2 \times \bar{\xi}_2) R^{1/2}\xi_1, R^{1/2}\eta_1 \right\rangle_H &= \omega(\eta_2 \times \bar{\xi}_2) \left\langle R^{1/2}\xi_1, R^{1/2}\eta_1 \right\rangle_H \\ &= \langle \Omega\eta_2, \xi_2\rangle_H \langle R\xi_1, \eta_1\rangle_H,\end{aligned}$$

where Ω is the density operator of ω, $\omega(T) = \text{trace}(\Omega T)$, $T \in \mathcal{B}(H)$.

Let $R \circ \Omega$ be the operator on $\mathcal{L}^2(H)$ defined in Lemma 4.4, and notice that the right side of the preceding expression is $\langle R \circ \Omega(\xi_1 \times \bar{\xi}_2), \eta_1 \times \bar{\eta}_2\rangle_{\mathcal{L}^2(H)}$. Indeed, by definition of $R \circ \Omega$ we have $R \circ \Omega(\xi_1 \times \bar{\xi}_2) = R\xi_1 \times \overline{\Omega\xi_2}$, and

$$\begin{aligned}\langle R\xi_1 \times \overline{\Omega\xi_2}, \eta_1 \times \bar{\eta}_2\rangle_{\mathcal{L}^2(H)} &= \text{trace}(\eta_2 \times \bar{\eta}_1 \cdot R\xi_1 \times \overline{\Omega\xi_2}) \\ &= \langle R\xi_1, \eta_1\rangle_H \,\text{trace}(\eta_2 \times \overline{\omega\xi_2}) \\ &\quad \langle R\xi_1, \eta_1\rangle_H \langle \eta_2, \Omega\xi_2\rangle_H,\end{aligned}$$

which, as asserted, agrees with the right side of the previous expression.

Thus we have shown that

$$\lim_{t\to\infty} \langle L_t L_t^*(A), B\rangle_{\mathcal{L}^2(H)} = \langle R \circ \Omega(A), B\rangle_{\mathcal{L}^2(H)}$$

for rank-one operators $A, B \in \mathcal{L}^2(H)$. Now Lemma 4.4 implies that $R \circ \Omega$ is unitarily equivalent to $R \otimes \Omega \in \mathcal{B}(H \otimes H)$, and hence $R \circ \Omega$ is a positive trace class operator for which

$$\Lambda(R \circ \Omega) = \Lambda(R \otimes \Omega) = \Lambda(\rho \otimes \omega).$$

On the other hand, Lemma 4.1 implies that

$$\lim_{t\to\infty} \text{trace}|L_t L_t^* - R \circ \Omega| = 0.$$

By the inequality (2.3.2) we conclude that

$$\limsup_{t\to\infty} \|\Lambda(L_t L_t^*) - \Lambda(R\circ\Omega)\| \le \lim_{t\to\infty} \operatorname{trace}|L_t L_t^* - R\circ\Omega| = 0.$$

We have already seen that $\Lambda(R\circ\Omega) = \Lambda(\rho\otimes\omega)$, and that $\Lambda(L_t L_t^*) = \Lambda(\rho\restriction_{\alpha_t(\mathcal{B}(H))'})$. Thus Theorem C is proved. □

We now readily deduce the interaction inequality.

Theorem B. *Let (U, M) be an interaction with past and future states ω_- and ω_+, and let $\bar\omega_-$ and $\bar\omega_+$ be their natural extensions to the local C^*-algebra $\mathcal{A}$. Then*

$$\|\bar\omega_- - \bar\omega_+\| \ge \|\Lambda(\omega_-\otimes\omega_-) - \Lambda(\omega_+\otimes\omega_+)\|.$$

Proof. Fix $\epsilon > 0$. By Theorem C we can find $T > 0$ large enough so that for all $t > T$ we have

$$\|\Lambda(\omega_+\restriction_{\mathcal{A}_{[0,t]}}) - \Lambda(\omega_+\otimes\omega_+)\| \le \epsilon$$

as well as

$$\|\Lambda(\omega_-\restriction_{\mathcal{A}_{[-t,0]}}) - \Lambda(\omega_-\otimes\omega_-)\| \le \epsilon.$$

Now for $t \ge T$,

$$\begin{aligned}\|\bar\omega_+ - \bar\omega_-\| &= \|\bar\omega_+\circ\gamma_t - \bar\omega_-\circ\gamma_{-t}\| \ge \|\bar\omega_+\circ\gamma_t\restriction_{\mathcal{A}_{[-t,t]}} - \bar\omega_-\circ\gamma_{-t}\restriction_{\mathcal{A}_{[-t,t]}}\| \\ &= \|\omega_+\circ\gamma_t\restriction_{\mathcal{A}_{[-t,t]}} - \omega_-\circ\gamma_{-t}\restriction_{\mathcal{A}_{[-t,t]}}\|.\end{aligned} \tag{4.6}$$

Since γ_t gives rise to a $*$-isomorphism of $\mathcal{A}_{[-t,t]}$ onto $\mathcal{A}_{[0,2t]}$ while γ_{-t} gives rise to a $*$-isomorphism of $\mathcal{A}_{[-t,t]}$ onto $\mathcal{A}_{[-2t,0]}$, (2.3.1) implies that

$$\begin{aligned}\Lambda(\omega_+\circ\gamma_t\restriction_{\mathcal{A}_{[-t,t]}}) &= \Lambda(\omega_+\restriction_{\mathcal{A}_{[0,2t]}}), \quad \text{and} \\ \Lambda(\omega_-\circ\gamma_{-t}\restriction_{\mathcal{A}_{[-t,t]}}) &= \Lambda(\omega_-\restriction_{\mathcal{A}_{[-2t,0]}}).\end{aligned}$$

Thus by Proposition 2.3 the last term of (4.5) is at least

$$\|\Lambda(\omega_+\restriction_{\mathcal{A}_{[0,2t]}}) - \Lambda(\omega_-\restriction_{\mathcal{A}_{[-2t,0]}})\|,$$

which by our initial choice of T is at least

$$\|\Lambda(\omega_+\otimes\omega_+) - \Lambda(\omega_-\otimes\omega_-)\| - 2\epsilon.$$

Since ϵ is arbitrary, the asserted inequality follows. □

Corollary 4.5. *Let (U, M) be an interaction with past and future states ω_-, ω_+. If $\Lambda(\omega_-) \ne \Lambda(\omega_+)$, then the interaction is nontrivial.*

Proof. Contrapositively, suppose that the interaction is trivial and let Ω_- and Ω_+ be the respective density operators of ω_- and ω_+. Theorem B implies that $\Omega_-\otimes\Omega_-$ and $\Omega_+\otimes\Omega_+$ must have the same eigenvalue list. (2.1.4) of Proposition 2.1 implies that for every $n = 1, 2, \ldots$ we have

$$\operatorname{trace}(\Omega_-^n)^2 = \operatorname{trace}((\Omega_-\otimes\Omega_-)^n) = \operatorname{trace}((\Omega_+\otimes\Omega_+)^n) = \operatorname{trace}(\Omega_+^n)^2.$$

Taking the square root we find that $\operatorname{trace}(\Omega_-^n) = \operatorname{trace}(\Omega_+^n)$ for every $n = 1, 2, \ldots$ and another application of (2.1.4) leads to $\Lambda(\Omega_-) = \Lambda(\Omega_+)$. □

Corollary 4.6. *Let $n = 1, 2, \ldots, \infty$ and choose $\epsilon > 0$. There is an interaction (U, M) whose past and future E_0-semigroups are cocycle-conjugate to the CAR/CCR flow of index n such that*

$$\|\bar{\omega}_+ - \bar{\omega}_-\| \geq 2 - \epsilon.$$

Proof. Choose positive integers $p < q$ and consider the eigenvalue lists

$$\Lambda_- = \{1/p, 1/p, \ldots, 1/p, 0, 0, \ldots\},$$
$$\Lambda_+ = \{1/q, 1/q, \ldots, 1/q, 0, 0, \ldots\},$$

where $1/p$ is repeated p times and $1/q$ is repeated q times. Theorem A implies that there is an interaction (U, M) whose past and future E_0 semigroups are cocycle-conjugate to the CAR/CCR flow of index n, for which $\Lambda(\omega_-) = \Lambda_-$ and $\Lambda(\omega_+) = \Lambda_+$. By Theorem B,

$$\|\bar{\omega}_+ - \bar{\omega}_-\| \geq \|\Lambda(\omega_+ \otimes \omega_+) - \Lambda(\omega_- \otimes \omega_-)\|.$$

If we neglect zeros, the eigenvalue list of $\omega_- \otimes \omega_-$ consists of the single eigenvalue $1/p^2$, repeated p^2 times, and that of $\omega_+ \otimes \omega_+$ consists of $1/q^2$ repeated q^2 times. Thus

$$\|\Lambda(\omega_+ \otimes \omega_+) - \Lambda(\omega_- \otimes \omega_-)\| = p^2(1/p^2 - 1/q^2) + (q^2 - p^2)/q^2 = 2 - 2p^2/q^2,$$

and the inequality of Corollary 2 follows whenever q is larger than $p\sqrt{2/\epsilon}$. □

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