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The Index of a Quantum Dynamical Semigroup*

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A numerical index is introduced for semigroups of completely positive maps of $\mathscr{B}(H)$ which generalizes the index of E_0-semigroups. It is shown that the index of a unital completely positive semigroup agrees with the index of its dilation to an E_0-semigroup, provided that the dilation is *minimal*. © 1997 Academic Press

1. INTRODUCTION

We introduce a numerical index for semigroups $P = \{P_t : t \geqslant 0\}$ of normal completely positive maps on the algebra $\mathscr{B}(H)$ of all bounded operators on a separable Hilbert space H. This index is defined in terms of basic structures associated with P, and generalizes the index of E_0-semigroups. In the case where $P_t(\mathbf{1}) = \mathbf{1}$, $t \geqslant 0$, we show that the index of P agrees with the index of its minimal dilation to an E_0-semigroup.

The key ingredients are the existence of the covariance function (Theorem 2.6), the relation between units of P and units of its minimal dilation (Theorem 3.6), and the mapping of covariance functions (Corollary 4.8). No examples are discussed here, but another paper is in preparation [5].

1. THE METRIC OPERATOR SPACE OF A COMPLETELY POSITIVE MAP

We consider the real vector space of all normal linear maps L of $\mathscr{B}(H)$ into itself which are *symmetric* in the sense that $L(x^*) = L(x)^*$, $x \in \mathscr{B}(H)$. For two such maps L_1, L_2 we write $L_1 \leqslant L_2$ if the difference $L_2 - L_1$ is completely positive. Every operator $a \in \mathscr{B}(H)$ gives rise to an elementary completely positive map Ω_a by way of

$$\Omega_a(x) = axa^*, \qquad x \in \mathscr{B}(H).$$

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DEFINITION 1.1. For every completely positive map P on $\mathscr{B}(H)$ we write $\mathscr{E}_P$ for the set of all operators $a \in \mathscr{B}(H)$ for which there is a positive constant k such that

$$\Omega_a \leqslant kP.$$

In this section we collect some elementary observations which imply that $\mathscr{E}_P$ is a vector space inheriting a natural inner product with respect to which it is a complex Hilbert space. Thus, every normal completely positive map is associated with a Hilbert space of operators which, as we will see, "implement" the mapping. The properties of these Hilbert spaces of operators will be fundamental to our methods in the sequel.

Because of Stinespring's theorem, every normal completely positive map P of $\mathscr{B}(H)$ into itself can be represented in the form

$$P(x) = V^*\pi(x)\, V, \qquad x \in \mathscr{B}(H), \tag{1.2}$$

where π is a representation of $\mathscr{B}(H)$ on some Hilbert space H_π and $V: H \to H_\pi$ is a bounded operator. We may always assume that the pair (V, π) is *minimal* in the sense that H_π is spanned by the set of vectors $\{\pi(x)\, V\xi : x \in \mathscr{B}(H), \xi \in H\}$, and in that case we have $\pi(\mathbf{1}) = \mathbf{1}$ and $V^*V = P(\mathbf{1})$. Two minimal pairs (V, π) and $(\tilde{V}, \tilde{\pi})$ for P are *equivalent* in the sense that there is a (necessarily unique) unitary operator $W: H_\pi \to H_{\tilde{\pi}}$ such that

$$WV = \tilde{V}, \tag{1.3a}$$

and

$$W\pi(x) = \tilde{\pi}(x)\, W, \qquad x \in \mathscr{B}(H). \tag{1.3.b}$$

Now since P is normal, the representation π occurring in any minimal pair (V, π) is necessarily a normal representation of $\mathscr{B}(H)$ and is therefore unitarily equivalent to a representation of the form

$$\pi(x) = x \oplus x \oplus \cdots,$$

acting on a direct sum H^n of n copies of H, n being a cardinal number which is countable because H is separable. Thus we may always assume that a minimal pair (V, π) consists of a representation of this form and that $V: H \to H^n$ has the form

$$V\xi = (v_1^*\xi, v_2^*\xi, \ldots),$$

where $v_1, v_2, \ldots$ is a sequence of bounded operators on H. Notice that the components of V are the adjoints of the operators v_k; this is essential in order for the operator multiplication to be properly related to the spaces $\mathscr{E}_P$ associated with completely positive maps P (see Theorem 1.12). After unravelling the formula (1.2) one finds that these operators satisfy

$$P(x) = \sum_{n \geq 1} v_n x v_n^*,$$

the sum on the right converging weakly because of the condition

$$\|v_1^* \xi\|^2 + \|v_2^* \xi\|^2 + \cdots = \|V\xi\|^2 < \infty, \qquad \xi \in H. \tag{1.4}$$

Finally, the minimality condition on (V, π) implies that the only operator c in the commutant of $\pi(\mathscr{B}(H))$ satisfying $cV = 0$ is $c = 0$. Considering the matrix representation of operators in the commutant of $\pi(\mathscr{B}(H))$, we find that this condition translates into the somewhat more concrete "linear independence" condition on the sequence $v_1, v_2, ...$:

$$(\lambda_1, \lambda_2, ...) \in \ell^2, \sum_k \lambda_k v_k = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = 0. \tag{1.5}$$

Notice that the series in (1.5) is strongly convergent, since it represents the composition of $V^*: H^n \to H$ with the operator $\xi \in H \mapsto (\lambda_1 \xi, \lambda_2 \xi, ...) \in H^n$. Conversely, if we start with an arbitrary sequence $v_1, v_2, ...$ of operators in $\mathscr{B}(H)$ for which (1.4) and the "linear independence" condition (1.5) are satisfied, then

$$P(x) = \sum_k v_k^* x v_k$$

defines a normal completely positive linear map on $\mathscr{B}(H)$. If we define $V: H \to H^n$ and $\pi: \mathscr{B}(H) \to \mathscr{B}(H^n)$ by

$$V\xi = (v_1 \xi, v_2 \xi, ...), \tag{1.6.a}$$

$$\pi(x) = x \oplus x \oplus \cdots, \tag{1.6.b}$$

then (V, π) is a minimal Stinespring pair (V, π) for P.

We now reformulate these observations in a coordinate-free form which is more useful for our purposes below.

Proposition 1.7. *Let $P(x) = V^* \pi(x) V$ be a minimal Stinespring representation for a normal completely positive map P of $\mathscr{B}(H)$, and let*

$$\mathscr{S} = \{T \in \mathscr{B}(H, H_\pi) : Tx = \pi(x) T, x \in \mathscr{B}(H)\}$$

be the intertwining space for π *and the identity representation. For any two operators* $T_1, T_2 \in \mathcal{S}$, $T_2^* T_1$ *is a scalar multiple of the identity of* $\mathcal{B}(H)$, *and*

$$\langle T_1, T_2 \rangle \, \mathbf{1} = T_2^* T_1$$

defines an inner product on $\mathcal{S}$ *with respect to which it is a Hilbert space in which the operator norm coincides with the Hilbert space norm.*

The linear mapping $T \in \mathcal{S} \to V^* T \in \mathcal{B}(H)$ *is injective and has range* $\mathcal{E}_P$. $\mathcal{E}_P$ *is a Hilbert space with respect to the inner product defined by pushing forward the inner product of* $\mathcal{S}$.

Proof. We merely sketch the argument, which is part of the folklore of representation theory. The first paragraph is completely straightforward. For example, if $T_1, T_2 \in \mathcal{S}$ then $T_2^* T_1$ must be a scalar multiple of the identity on H because for every $x \in \mathcal{B}(H)$ we have

$$T_2^* T_1 x = T_2^* \pi(x) \, T_1 = x T_2^* T_1.$$

Now let a be an operator of the form $a = V^* T$, $T \in \mathcal{S}$. We claim that a belongs to $\mathcal{E}_P$. Indeed, for every $x \in \mathcal{B}(H)$ we have

$$\Omega_a(x) = axa^* = V^* T x T^* V = V^* \pi(x) \, TT^* V.$$

Since TT^* is a bounded positive operator in the commutant of $\pi(\mathcal{B}(H))$, the operator $C = (\|T\|^2 \, \mathbf{1} - TT^*)^{1/2}$ is positive, commutes with $\pi(\mathcal{B}(H))$, and the preceding formula implies that the operator mapping

$$x \in \mathcal{B}(H) \mapsto \|T\|^2 \, P(x) - \Omega_a(x) = V^* C \pi(x) \, CV$$

is completely positive. Hence $a \in \mathcal{E}_P$.

The map $T \to V^* T$ is injective because it is linear, and because if an operator $T \in \mathcal{S}$ satisfies $V^* T = 0$ then for every $x \in \mathcal{B}(H)$ and every $\xi \in H$

$$T^* \pi(x) \, V\xi = x T^* V \xi = x (V^* T)^* \, \xi = 0.$$

Hence $T^* = 0$ because H_π is spanned by $\pi(\mathcal{B}(H)) \, H$, hence $T = 0$.

Finally, let a be an arbitrary element in $\mathcal{E}_P$ and choose a positive constant k such that $\Omega_a \leqslant kP$. We may find an operator $T \in \mathcal{S}$ which maps to a as follows. For any $n \geqslant 1$, any operators $x_1, x_2, ..., x_n \in \mathcal{B}(H)$ and any vectors $\xi_1, \xi_2, ..., \xi_n \in H$ we have

$$\begin{aligned} \left\| \sum_{k=1}^{n} x_k a^* \xi_k \right\|^2 &= \sum_{j,k=1}^{n} \langle \Omega_a(x_k^* x_j) \, \xi_j, \xi_k \rangle \leqslant k \sum_{k,j=1}^{n} \langle V^* \pi(x_k^* x_j) \, V\xi_j, \xi_k \rangle \\ &= k \left\| \sum_{k=1}^{n} \pi(x_k) \, V\xi_k \right\|^2. \end{aligned}$$

Thus there is a unique bounded operator $L: H_\pi = [\pi(\mathcal{B}(H)\, VH] \to H$ which satisfies $L(\pi(x)\, V\xi) = xa^*\xi$ for every $x \in \mathcal{B}(H)$, $\xi \in H$. Taking $T = L^*$ we find that $T \in \mathcal{S}$ and $a^* = T^*V$, hence $a = V^*T$. ∎

Remark 1.8. To reiterate, the inner product in $\mathcal{E}_P$ is defined as follows. Pick $a, b \in \mathcal{E}_P$. Then there are unique operators $S, T \in \mathcal{S}$ such that $a = V^*S$, $b = V^*T$, and $\langle a, b \rangle$ is defined by

$$\langle a, b \rangle\, \mathbf{1} = T^*S.$$

In more concrete terms, choose a minimal Stinespring representation $P(x) = V^*\pi(x)\, V$ where π is a representation on H^n and $V: H \to H^n$ is of the form $V\xi = (v_1\xi, v_2\xi, ...)$, the sequence of operators $v_1, v_2, ... \in \mathcal{B}(H)$ satisfying conditions (1.4) and (1.5). Then $\{v_1^*, v_2^*, ...\}$ is an orthonormal basis for the Hilbert space structure of $\mathcal{E}_P$ and thus $\mathcal{E}_P$ consists precisely of all operators a of the form

$$a = a(\lambda) = \lambda_1 v_1^* + \lambda_2 v_2^* + \cdots,$$

where $\lambda = (\lambda_1, \lambda_2, ...)$ is an arbitrary sequence in ℓ^2. The sequence $\lambda \in \ell^2$ is uniquely determined by the operator $a(\lambda)$, and the inner product in $\mathcal{E}_P$ satisfies

$$\langle a(\lambda), a(\mu) \rangle = \sum_{k \geqslant 1} \lambda_k \bar{\mu}_k.$$

DEFINITION 1.9. A *metric operator space* is a pair $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ consisting of a complex linear subspace $\mathcal{E}$ of $\mathcal{B}(H)$ together with an inner product $u, v \in \mathcal{E} \mapsto \langle u, v \rangle \in \mathbb{C}$ with respect to which $\mathcal{E}$ is a separable Hilbert space which has the following property: if $e_1, e_2, ...$ is an orthonormal basis for $\mathcal{E}$ then for any $\xi \in H$ we have

$$\|e_1^*\xi\|^2 + \|e_2^*\xi\|^2 + \cdots < \infty. \tag{1.10}$$

Remarks. The above discussion shows how, starting with a normal completely positive map P of $\mathcal{B}(H)$ into itself, we associate with P in an invariant way a metric operator space $\mathcal{E}_P$. This metric operator space has the property that if we pick an arbitrary orthonormal basis $e_1, e_2, ...$ for $\mathcal{E}$ then we recover the map P as

$$P(x) = \sum_k e_k x e_k^*, \qquad x \in \mathcal{B}(H), \tag{1.11}$$

the sum on the right being independent of the particular choice of basis. Conversely, starting with an arbitrary metric operator space $\mathcal{E}$ we may define a unique completely positive map P by the formula (1.11), and thus *we have a bijective correspondence $P \leftrightarrow \mathcal{E}$ between normal completely positive maps and metric operator spaces.*

Metric operator spaces offer several advantages over the Stinespring representation in describing normal completely positive maps on $\mathcal{B}(H)$, and it is appropriate to briefly discuss these issues here. For example, suppose we start with such a map P with metric operator space $\mathcal{E}$. We may use the inner product on $\mathcal{E}$ to define an inner product on the tensor product of vector spaces $\mathcal{E} \odot H$, and after completion we obtain a Hilbert space $\mathcal{E} \otimes H$. The natural multiplication map $M: \mathcal{E} \odot H \to H$ defined by $M(v \otimes \xi) = v\xi$ extends uniquely to a bounded operator from $\mathcal{E} \otimes H$ to H, which we denote by the same letter M. To see that, choose an orthonormal basis $e_1, e_2, \ldots$ for $\mathcal{E}$ and define a (necessarily bounded) operator $V: H \to \mathcal{E} \otimes H$ by

$$V\xi = \sum_k e_k \otimes e_k^* \xi.$$

A direct computation then shows that

$$\langle M(v \otimes \xi), \eta \rangle_H = \langle v \otimes \xi, V\eta \rangle_{\mathcal{E} \otimes H},$$

and hence $M = V^*$. In particular, the operator V is independent of the particular choice of basis, and represents "comultiplication". Moreover, if we define a normal representation $\pi: \mathcal{B}(H) \to \mathcal{B}(\mathcal{E} \otimes H)$ by

$$\pi(x) = \mathbf{1}_{\mathcal{E}} \otimes x,$$

then one finds that (V, π) is a *minimal* Stinespring representation for P. We conclude that with every normal completely positive map P there is a natural way of picking out a concrete minimal Stinespring pair (V, π) for P: one computes the metric operator space $\mathcal{E}$ associated with P, takes $V: H \to \mathcal{E} \otimes H$ to be comultiplication and takes π as above.

More significantly, notice that the Stinespring representation of normal completely positive maps does not behave well with respect to composition. For example, if we have two such maps $P_k: \mathcal{B}(H) \to \mathcal{B}(H)$, and we consider their respective minimal Stinespring pairs (V_k, π_k), then there is no natural way to combine (V_1, π_1) with (V_2, π_2) to obtain a Stinespring pair for the composition $P_1 P_2$, much less a minimal one. The description of such maps in terms of metric operator spaces is designed to deal efficiently with compositions. The following result implies that the metric operator space of $P_1 P_2$ is spanned (as a Hilbert space) by the set of all operator products $\mathcal{E}_1 \mathcal{E}_2$, $\mathcal{E}_k$ denoting the space associated with P_k. As we will see in the sequel, this is a critical feature when dealing with semigroups.

THEOREM 1.12. *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be metric operator spaces with corresponding completely positive maps $P_k = P_{\mathscr{E}_k}$, and let $P_1 P_2$ denote the composition. Let $\mathscr{E}_1 \otimes \mathscr{E}_2$ be the tensor product of Hilbert spaces. Then $\mathscr{E}_{P_1 P_2}$ contains the set of all operator products $\{uv : u \in \mathscr{E}_1, v \in \mathscr{E}_2\}$ and there is a unique bounded linear operator $M: \mathscr{E}_1 \otimes \mathscr{E}_2 \to \mathscr{E}_{P_1 P_2}$ satisfying*

$$M(u \otimes v) = uv, \qquad u \in \mathscr{E}_1, v \in \mathscr{E}_2. \tag{1.13}$$

The adjoint of M is an isometry

$$M^*: \mathscr{E}_{P_1 P_2} \hookrightarrow \mathscr{E}_1 \otimes \mathscr{E}_2$$

whose range is a (perhaps proper) closed subspace of $\mathscr{E}_1 \otimes \mathscr{E}_2$.

Remarks. We refer to the adjoint M^* of the operator M defined by (1.13) as *comultiplication*. Since comultiplication is an isometry, it follows that the range of the multiplication operator M is all of $\mathscr{E}_{P_1 P_2}$, and hence $\mathscr{E}_{P_1 P_2}$ is spanned by the set of products $\mathscr{E}_1 \mathscr{E}_2$.

Theorem 1.12 asserts that comultiplication gives rise to a natural identification of $\mathscr{E}_{P_1 P_2}$ with a closed subspace of $\mathscr{E}_1 \otimes \mathscr{E}_2$. Equivalently, the polar decomposition of the multiplication operator M has the form $M = UQ$, where $Q \in \mathscr{B}(\mathscr{E}_1 \otimes \mathscr{E}_2)$ is the projection onto this subspace and $U \in \mathscr{B}(\mathscr{E}_1 \otimes \mathscr{E}_2, \mathscr{E}_{P_1 P_2})$ is a partial isometry with $U^*U = Q$, $UU^* = \mathbf{1}_{\mathscr{E}_{P_1 P_2}}$.

Proof of Theorem 1.12. We find a Stinespring representation of $P_1 P_2$ in terms of $\mathscr{E}_1$ and $\mathscr{E}_2$ as follows. Consider the Hilbert space $K = \mathscr{E}_1 \otimes \mathscr{E}_2 \otimes H$, and the representation π of $\mathscr{B}(H)$ on K defined by

$$\pi(x) = \mathbf{1}_{\mathscr{E}_1} \otimes \mathbf{1}_{\mathscr{E}_2} \otimes x.$$

Choose an orthonormal basis $u_1, u_2, \ldots$ for $\mathscr{E}_1$ (resp. $v_1, v_2, \ldots$ for $\mathscr{E}_2$) and define an operator $V: H \to K$ by

$$V\xi = \sum_{i,j} u_i \otimes v_j \otimes v_j^* u_i^* \xi, \qquad \xi \in H.$$

It is clear that V is bounded, since

$$\| V\xi \|^2 = \sum_{i,j} \|v_j^* u_i^* \xi\|^2 = \sum_{i,j} \langle u_i v_j v_j^* u_i^* \xi, \xi \rangle = \langle P_1(P_2(\mathbf{1}))\, \xi, \xi \rangle,$$

and in fact $V^*V = P_1 P_2(\mathbf{1})$. A similar calculation shows that

$$V^* \pi(x)\, V = P_1 P_2(x), \qquad x \in \mathscr{B}(H).$$

However, (V, π) is not necessarily a *minimal* Stinespring pair. In order to arrange minimality, consider the subspace $K_0 \subseteq K$ defined by

$$K_0 = [\pi(x)\, V\xi : x \in \mathscr{B}(H), \xi \in H].$$

Since K_0 is invariant under the range of π its production belongs to the commutant

$$\pi(\mathscr{B}(H))' = \mathscr{B}(\mathscr{E}_1 \otimes \mathscr{E}_2) \otimes \mathbf{1}_H,$$

and hence there is a unique projection $Q \in \mathscr{B}(\mathscr{E}_1 \otimes \mathscr{E}_2)$ such that

$$P_{K_0} = Q \otimes \mathbf{1}_H.$$

The corresponding subrepresentation π_0 obtained by restricting π to K_0 gives rise to a minimal Stinespring pair (V, π_0) for $P_1 P_2$.

In order to calculate the metric operator space $\mathscr{E}_{P_1 P_2}$ we use Proposition 1.7 as follows. Notice that for every $\zeta \in \mathscr{E}_1 \otimes \mathscr{E}_2$ we can define a bounded operator $X_\zeta : H \to K$ by

$$X_\zeta \xi = \zeta \otimes \xi, \qquad \xi \in H.$$

It is clear that $X_\zeta a = \pi(a)\, X_\zeta$ for every $a \in \mathscr{B}(H)$, and moreover every bounded operator $X \in \mathscr{B}(H, K)$ satisfying $Xa = \pi(a)\, X$, $a \in \mathscr{B}(H)$, has the form $X = X_\zeta$ for a unique $\zeta \in \mathscr{E}_1 \otimes \mathscr{E}_2$. The range of X_ζ is contained in $K_0 = (Q \otimes \mathbf{1}_H)\, K$ if and only if ζ belongs to the range of Q. Thus the intertwining space

$$\{X \in \mathscr{B}(H, K_0) : Xa = \pi_0(a)\, X, a \in \mathscr{B}(H)\}$$

for π_0 is $\{X_\zeta : \zeta \in Q(\mathscr{E}_1 \otimes \mathscr{E}_2)\}$.

Now by Proposition 1.7, we have

$$\mathscr{E}_{P_1 P_2} = \{V^* X_\zeta : \zeta \in Q(\mathscr{E}_1 \otimes \mathscr{E}_2)\},$$

and the inner product of two operators $T_k = V^* X_{\zeta_k}$, $k = 1, 2$ in $\mathscr{E}_{P_1 P_2}$ is given by

$$\langle T_1, T_2 \rangle_{\mathscr{E}_{P_1 P_2}} \mathbf{1}_H = X_{\zeta_2}^* X_{\zeta_1} = \langle \zeta_1, \zeta_2 \rangle\, \mathbf{1}_H,$$

$\zeta_k \in Q(\mathscr{E}_1 \otimes \mathscr{E}_2)$.

Accordingly, we have defined a unitary operator $U : Q(\mathscr{E}_1 \otimes \mathscr{E}_2) \to \mathscr{E}_{P_1 P_2}$ by

$$U\zeta = V^* X_\zeta.$$

It remains to show that the bounded operator $M: \mathscr{E}_1 \otimes \mathscr{E}_2 \to \mathscr{E}_{P_1 P_2}$ defined by

$$M = UQ$$

represents multiplication in the sense that $M(u \otimes v) = uv$ for any $u \in \mathscr{E}_1$ and $v \in \mathscr{E}_2$. To see that, write

$$M(u \otimes v) = UQ(u \otimes v) = V^* X_{Q(u \otimes v)} = V^*(Q \otimes \mathbf{1}_H)\, X_{u \otimes v} = V^* X_{u \otimes v},$$

the last equality following from the fact that $Q \otimes \mathbf{1}_H V = P_{K_0} V = V$. Thus for $\xi, \eta \in H$ we have

$$\begin{aligned} &\langle M(u \otimes v)\, \xi, \eta \rangle \\ &\quad = \langle V^* X_{u \otimes v} \xi, \eta \rangle = \sum_{i,j} \langle u \otimes v \otimes \xi, u_i \otimes v_j \otimes v_j^* u_i^* \eta \rangle \\ &\quad = \sum_{i,j} \langle u, u_i \rangle \langle v, v_j \rangle \langle \xi, v_j^* u_i^* \eta \rangle = \sum_{i,j} \langle u, u_i \rangle \langle v, v_j \rangle \langle u_i v_j \xi, \eta \rangle. \end{aligned}$$

The term on the right is $\langle uv\xi, \eta \rangle$ because $\sum_i \langle u, u_i \rangle\, u_i = u$ and $\sum_j \langle v, v_j \rangle\, v_j = v$. ∎

Remark 1.14. Finally, we call attention to the special case in which P is a normal *-endomorphism, that is, a normal completely positive map for which $P(xy) = P(x)\, P(y)$ for all x, y. We do not assume that $P(\mathbf{1}) = \mathbf{1}$, but of course $P(\mathbf{1})$ must be a self-adjoint projection. In this case a minimal Stinespring representation $P = V^* \pi V$ is given by the pair (V, π), where V is the orthogonal projection of H onto $H_0 = P(\mathbf{1})\, H$ and $\pi(x)$ is the restriction of $P(x)$ to the invariant subspace H_0. In this case a straightforward computation shows that $\mathscr{E}_P$ reduces to the intertwining space

$$\mathscr{E}_P = \{ T \in \mathscr{B}(H) \colon P(x)\, T = Tx,\ x \in \mathscr{B}(H) \},$$

and that the inner product on $\mathscr{E}_P$ is defined by

$$\langle T_1, T_2 \rangle\, \mathbf{1} = T_2^* T_1, \qquad T_1, T_2 \in \mathscr{E}_P.$$

2. NUMERICAL INDEX

Let H be a separable Hilbert space and let $P = \{ P_t \colon t \geqslant 0 \}$ be a semigroup of normal completely positive maps of $\mathscr{B}(H)$ into itself which is continuous in the sense that for every $x \in \mathscr{B}(H)$ and every pair of vectors $\xi, \eta \in \mathscr{B}(H)$, the function $t \in [0, \infty) \mapsto \langle P_t(x)\, \xi, \eta \rangle$ is continuous. We do not assume that P_t preserves the unit, nor even that $\|P_t\| \leqslant 1$, but we do require that P_0 be the identity map; equivalently,

$$\lim_{t \to 0} \langle P_t(x)\, \xi, \eta \rangle = \langle x\xi, \eta \rangle, \qquad x \in \mathscr{B}(H),\ \xi, \eta \in H.$$

We refer to such a semigroup as a *CP semigroup*. A CP semigroup P is called *unital* if $P_t(\mathbf{1}) = \mathbf{1}$ for every $t \geqslant 0$, and *contractive* if $\|P_t\| = \|P_t(\mathbf{1})\| \leqslant 1$ for every $t \geqslant 0$.

In this section we introduce a numerical index for arbitrary CP semigroups which generalizes the definition of index of E_0-semigroups [1]. While the definition and Theorem 2.6 below are very general, the reader should keep in mind that we are primarily interested in the case of *unital* CP semigroups.

Definition 2.1. Let P be a CP semigroup acting on $\mathscr{B}(H)$. A *unit* of P is a semigroup $T = \{T_t : t \geqslant 0\}$ of bounded operators on H which is strongly continuous in the sense that

$$\lim_{t \to 0} \|T_t \xi - \xi\| = 0, \qquad \xi \in H,$$

and for which there is a real constant k such that for every $t > 0$, the operator mapping $\Omega_t(x) = T_t x T_t^*$ satisfies

$$\Omega_t \leqslant e^{kt} P_t.$$

Remark 2.2. We write $\mathscr{U}_P$ for the set of all units of P, and it will be convenient to denote the metric operator spaces $\mathscr{E}_{P_t}$ associated with the individual completely positive maps P_t with the notation $\mathscr{E}_P(t)$, $t \geqslant 0$. Notice that an operator semigroup $T = \{T_t : t \geqslant 0\}$ belongs to $\mathscr{U}_P$ if and only if (a) $T_t \in \mathscr{E}_P(t)$ for every $t > 0$ and (b) the Hilbert space norms $\langle T_t, T_t \rangle$ of these elements of $\mathscr{E}_P(t)$ satisfy the growth condition

$$\langle T_t, T_t \rangle \leqslant e^{kt}, \qquad t > 0. \tag{2.3}$$

Of course, every E_0-semigroup qualifies as a CP semigroup, and in this case remark (1.14) implies that Definition 2.1 agrees with the definition of unit for an E_0-semigroup given in [1]. The only issue here is the growth condition (2.3), which is not part of the definition of unit for an E_0-semigroup. However, if $T = \{T_t : t > 0\}$ is a unit for an E_0-semigroup $P = \{P_t : t \geqslant 0\}$ then we have

$$\langle T_t, T_t \rangle = e^{tc(T, T)},$$

where $c\colon \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C}$ is the covariance function defined in [1], and hence the growth condition (2.3) is *automatic* for E_0-semigroups.

Since there exist E_0-semigroups with no units whatsoever [9] we must allow for the possibility that a CP semigroup may have no units. However, assuming that P is a CP semigroup for which $\mathcal{U}_P \neq \emptyset$, we define a numerical index $d_*(P)$ in the following way. Choose $S, T \in \mathcal{U}_P$. Then for every $t > 0$ the operators S_t, T_t belong to the Hilbert space $\mathcal{E}_P(t)$ and we may consider their inner product

$$\langle S_t, T_t \rangle \in \mathbb{C}. \tag{2.4}$$

Notice that while the inner products (2.4) are computed in different Hilbert spaces $\mathcal{E}_P(t)$, there is no ambiguity in this notation so long as the variable t is displayed. We remark too that while neither semigroup S nor T can be the zero semigroup, it can certainly happen that $T_t = 0$ for certain positive values of t, and once T_t is zero for some particular value of t then it is zero for all larger t as well. However, strong continuity at $t = 0$ implies that for sufficiently small t, both operators S_t and T_t are nonzero. But even in this case, there is no obvious guarantee that the inner product $\langle S_t, T_t \rangle$ is nonzero.

Now fix $t > 0$ and choose $S, T \in \mathcal{E}_P(t)$. For each finite partition

$$\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$$

of the interval $[0, t]$ we define

$$f_{\mathcal{P}}(S, T; t) = \prod_{k=1}^{n} \langle S_{t_k - t_{k-1}}, T_{t_k - t_{k-1}} \rangle. \tag{2.5}$$

If we consider the set of partitions of $[0, t]$ as an increasing directed set in the usual way then (2.5) defines a net of complex numbers. The definition of index depends on the following result, which will be proved later in this section.

Theorem 2.6. *Let $P = \{P_t \colon t \geqslant 0\}$ be a CP semigroup acting on $\mathcal{B}(H)$, let S and T be units of P, and define $f_{\mathcal{P}}(S, T; t)$ as in (2.5). Then there is a (necessarily unique) complex number c such that*

$$\lim_{\mathcal{P}} f_{\mathcal{P}}(S, T; t) = e^{ct}$$

for every $t > 0$.

We postpone the proof of Theorem 2.6 in order to discuss its immediate consequences. We will write $c_P(S, T)$ for the constant c of Theorem 2.6. Thus we have defined a bivariate function

$$c_P\colon \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C},$$

which will be called the *covariance function* of the CP semigroup P.

PROPOSITION 2.7. *The covariance function is conditionally positive definite in the sense that if* $T_1, T_2, ..., T_n \in \mathcal{U}_P$ *and* $\lambda_1, \lambda_1, ..., \lambda_n$ *are complex numbers satisfying* $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$, *then*

$$\sum_{j, k=1}^{n} \lambda_j \bar{\lambda}_k c_P(T_j, T_k) \geqslant 0.$$

Proof. It suffices to show that for every fixed $t > 0$, the function

$$S, T \mapsto e^{tc_P(S, T)}$$

is positive definite [8]. Now for every positive λ, $S, T \mapsto \langle S_\lambda, T_\lambda \rangle$ is obviously a positive definite function. Since a finite pointwise product of positive definite functions is a positive definite function, it follows that for each partition $\mathcal{P}$ of $[0, t]$ the function

$$S, T \mapsto f_{\mathcal{P}}(S, T; t)$$

of (2.5) is positive definite. Finally, since the limit of a pointwise convergent net of positive definite functions is positive definite, we conclude from Theorem 2.6 that the function

$$e^{tc_P(S, T)} = \lim_{\mathcal{P}} f_{\mathcal{P}}(S, T; t)$$

must be a positive definite of S and T. ∎

Now suppose that $P = \{P_t : t \geqslant 0\}$ is a CP semigroup for which $\mathcal{U}_P \neq \emptyset$. We may construct a Hilbert space H_P out of the conditionally postive definite function $c_P\colon \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C}$ in the same way as for E_0-semigroups. More explicitly, on the vector space V consisiting of all finitely nonzero functions $f\colon \mathcal{U}_P \to \mathbb{C}$ satisfying

$$\sum_{T \in \mathcal{U}_P} f(T) = 0,$$

one defines a positive semidefinite sesquilinear form

$$\langle f, g\rangle = \sum_{S,\, T \in \mathcal{U}_P} f(S)\, \overline{g(T)}\, c_P(S, T),$$

and the Hilbert space H_P is obtained by completing the inner product space V/N, where N is the subspace

$$N = \{ f \in V : \langle f, f\rangle = 0\}.$$

We define the *index* of P as the dimension of this Hilbert space

$$d_*(P) = \dim(H_P).$$

For the principal class of examples in which P is a *unital* CP semigroup Corollary 4.8 below together with [1, Proposition 5.2] implies that H_P must be separable, so that $d_*(P)$ must take one of the values 0, 1, 2, ..., $\aleph_0$.

The exceptional case in which $\mathcal{U}_P = \varnothing$ is handled in the same way as for E_0-semigroups; in that event we define

$$d_*(P) = 2^{\aleph_0}$$

to be the cardinality of the continuum. This convention of choosing an uncountable value for the index in the exceptional case where there are no units allows for the unrestricted validity of the addition formula for tensor products

$$d_*(P \otimes Q) = d_*(P) + d_*(P)$$

in the same way it does for E_0-semigroups.

Proof of Theorem 2.6. Let $T_k = \{T_k(t) : t \geqslant 0\}$, $k = 1, 2$, be units of a fixed CP semigroup P. Because each unit T must satisfy a growth condition of the form $\langle T(t), T(t)\rangle \leqslant e^{ct}$, $t > 0$, we may rescale T_1 and T_2 with a factor of the form $e^{-c_k t}$ to achieve

$$\langle T_k(t), T_k(t)\rangle \leqslant 1, \qquad t > 0. \tag{2.8}$$

Notice that this rescaling does not affect either the existence of the limit of Theorem 2.6 or the exponential nature of its value, so it suffices to prove 2.6 in the presence of the normalization (2.8).

For each partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[0, t]$ we consider the 2×2 matrix $A_{\mathcal{P}}(t)$ whose ijth term is given by

$$f_{\mathcal{P}}(T_i, T_j; t) = \prod_{k=1}^{n} \langle T_i(t_k - t_{k-1}), T_j(t_k - t_{k-1})\rangle. \tag{2.9}$$

(2.8) implies that $|f_{\mathscr{P}}(T_i, T_j, ; t)| \leqslant 1$; thus we have a uniform bound

$$\|A_{\mathscr{P}}(t)\| \leqslant \operatorname{trace}(A_{\mathscr{P}}(t)^* A_{\mathscr{P}}(t))^{1/2} \leqslant 2.$$

As in the proof of Proposition (2.7), each function $f_{\mathscr{P}}(\cdot, \cdot; t)$ is positive definite; hence $A_{\mathscr{P}}(t)$ is a positive matrix. We claim that in fact

$$\mathscr{P}_1 \subseteq \mathscr{P}_2 \Rightarrow A_{\mathscr{P}_1}(t) \leqslant A_{\mathscr{P}_2}(t). \tag{2.10}$$

To see that, it is enough to consider the case where $\mathscr{P}_2$ is obtained by adjoining a single point λ to $\mathscr{P}_1 = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. Suppose that $t_{k-1} < \lambda < t_k$ for k between 1 and n. Note that $f_{\mathscr{P}_2}(T_i, T_j; t)$ is obtained from $f_{\mathscr{P}_1}(T_i, T_j; t)$ by replacing the kth term $\alpha_{ij} = \langle T_i(t_k - t_{k-1}), T_j(t_k - t_{k-1})\rangle$ in the product (2.9) with the term

$$\beta_{ij} = \langle T_i(\lambda - t_{k-1}), T_j(\lambda - t_{k-1})\rangle \langle T_i(t_k - \lambda), T_j(t_k - \lambda)\rangle.$$

Thus, the ijth term of $A_{\mathscr{P}_2}(t) - A_{\mathscr{P}_1}(t)$ has the form $(\beta_{ij} - \alpha_{ij})\, \gamma_{ij}$, where the 2×2 matrix (γ_{ij}) is positive. Since the Schur product of two positive matrices is positive, it suffices to show that $(\beta_{ij} - \alpha_{ij})$ is a positive 2×2 matrix. Now for any two complex numbers λ_1, λ_2 we have

$$\begin{aligned}
&\sum_{i,j=1}^{2} \lambda_i \bar{\lambda}_j \beta_{ij} - \sum_{i,j=1}^{2} \lambda_i \bar{\lambda}_j \alpha_{ij} \\
&\quad = \sum_{i,j=1}^{2} \lambda_i \bar{\lambda}_j \langle T_i(\lambda - t_{k-1}), T_j(\lambda - t_{k-1})\rangle \langle T_i(t_k - \lambda), T_j(t_k - \lambda)\rangle \\
&\qquad - \sum_{i,j=1}^{2} \lambda_i \bar{\lambda}_j \langle T_i(t_k - t_{k-1}), T_j(t_k - t_{k-1})\rangle \\
&\quad = \left\| \sum_i \lambda_i T_j(\lambda - t_{k-1})) \otimes T_i(t_k - \lambda) \right\|^2 - \left\| \sum_i \lambda_i T_i(t_k - t_{k-1}) \right\|^2.
\end{aligned}$$

Because of the semigroup property we have $T_i(t_k - t_{k-1}) = T_i(\lambda - t_{k-1})$ $T_i(t_k - \lambda)$. Thus the last term of the preceding formula is nonnegative because of Theorem 1.12, which implies that multiplication

$$M\colon \mathscr{E}_P(\lambda - t_{k-1}) \otimes \mathscr{E}_P(t_k - \lambda) \to \mathscr{E}_P(t_k - t_{k-1})$$

is a contraction. This establishes (2.10).

Since for fixed $t > 0$, $\mathscr{P} \mapsto A_{\mathscr{P}}(t)$ is a uniformly bounded increasing net of positive operators, conventional wisdom implies that there is a unique positive operator $B(t) \in M_2(\mathbb{C})$ such that

$$B(t) = \lim_{\mathscr{P}} A_{\mathscr{P}}(t).$$

Letting $b_{ij}(t)$ be the ijth entry of $B(t)$ we have the required limit (2.6),

$$b_{ij}(t) = \lim_{\mathscr{P}} f_{\mathscr{P}}(T_i, T_j; t). \tag{2.11}$$

It remains to show that the functions b_{ij} have the form

$$b_{ij}(t) = e^{tc_{ij}}, \qquad t > 0, \tag{2.12}$$

for some 2×2 matrix (c_{ij}). Now every pair $\mathscr{P}$, $\mathscr{Q}$ consisting of finite partitions of $[0, s]$ and $[0, t]$ respectively gives rise to a partition of $[0, s+t]$, simply by first listing the elements of $\mathscr{P}$ and then listing the elements of $s + \mathscr{Q}$. This construction gives all partitions of $[0, s+t]$ which contain the point s. Since the latter is a cofinal subset of all finite partitions of $[0, s+t]$ it follows from (2.11) that we have

$$b_{ij}(s+t) = b_{ij}(s)\, b_{ij}(t), \qquad s, t > 0.$$

Thus to prove (2.12) it is enough to show that the functions b_{ij} extend continuously to the origin in the following sense

$$\lim_{t \to 0+} b_{ij}(t) = 1.$$

The latter is an immediate consequence of the following two results.

LEMMA 2.14. *For $i, j = 1$ or 2 and $t > 0$ we have*

$$|b_{ij}(t) - \langle T_i(t), T_j(t) \rangle|^2 \leqslant (1 - \langle T_i(t), T_i(t) \rangle)(1 - \langle T_j(t), T_j(t) \rangle).$$

LEMMA 2.15. *For $i, j = 1$ or 2 we have*

$$\lim_{t \to 0+} \langle T_i(t), T_j(t) \rangle = 1.$$

Proof of Lemma 2.14. Fix $t > 0$. Because of (2.11), it suffices to show that for every i and j and every finite partition $\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[0, t]$, we have

$$\begin{aligned} &|f_{\mathscr{P}}(T_i, T_j; t) - \langle T_i(t), T_j(t) \rangle|^2 \\ &\quad \leqslant (1 - \langle T_i(t), T_i(t) \rangle)(1 - \langle T_j(t), T_j(t) \rangle). \end{aligned} \tag{2.16}$$

Consider the vectors $u_i \in \mathscr{E}_P(t_1 - t_0) \otimes \cdots \otimes \mathscr{E}_P(t_n - t_{n-1})$ defined by

$$u_i = T_i(t_1 - t_0) \otimes \cdots \otimes T_i(t_n - t_{n-1}),$$

$i=1, 2$. Notice that because of (2.8) we have $\|u_i\| \leqslant 1$ for $i=1, 2$, and

$$f_{\mathscr{P}}(T_i, T_j; t) = \langle u_i, u_j \rangle.$$

By an obvious induction using nothing more than the associative law, Theorem 1.12 implies that there is a unique multiplication operator

$$M: \mathscr{E}_P(t_1 - t_0) \otimes \cdots \otimes \mathscr{E}_P(t_n - t_{n-1}) \to \mathscr{E}_P(t)$$

satisfying $M(v_1 \otimes \cdots \otimes v_n) = v_1 v_2 \cdots v_n$, and moreover that $\|M\| \leqslant 1$. Noting that $Mu_i = T_i(t)$ and using $\|M\| \leqslant 1$ we have

$$\begin{aligned} |f_{\mathscr{P}}(T_i, T_j; t) - \langle T_i(t), T_j(t) \rangle| &= |\langle u_i, u_j \rangle - \langle Mu_i, Mu_j \rangle| \\ &= |\langle (\mathbf{1} - M^*M)\, u_i, u_j \rangle| \\ &\leqslant \|(\mathbf{1} - M^*M)^{1/2}\, u_i\| \cdot \|(\mathbf{1} - M^*M)^{1/2}\, u_j\|. \end{aligned}$$

Since

$$\begin{aligned} \|(\mathbf{1} - M^*M)^{1/2}\, u_j\|^2 = \langle (\mathbf{1} - M^*M)\, uj, u_j \rangle &= \|u_j\|^2 - \|Mu_j\|^2 \\ &\leqslant 1 - \|Mu_j\|^2 = 1 - \langle T_j(t), T_j(t) \rangle, \end{aligned}$$

the estimate of Lemma 2.14 follows. ∎

Proof of Lemma 2.15. We show first that for every unit $T \in \mathscr{U}_P$,

$$\lim_{t \to 0+} \langle T(t), T(t) \rangle = 1. \tag{2.17}$$

Indeed, since units must satisfy a growth condition of the form $\langle T(t), T(t) \rangle \leqslant e^{kt}$ it suffices to show that

$$1 \leqslant \liminf_{t \to 0+} \langle T(t), T(t) \rangle. \tag{2.18}$$

Now for every $t > 0$ the map

$$x \in \mathscr{B}(H) \mapsto \langle T(t), T(t) \rangle\, P_t(x) - T(t)\, xT(t)^*$$

is completely positive; taking $x = \mathbf{1}$ we find that for every unit vector $\xi \in H$

$$\|T(t)^*\, \xi\|^2 = \langle T(t)\, T(t)^*\, \xi, \xi \rangle \leqslant \langle T(t), T(t) \rangle \langle P_t(\mathbf{1})\, \xi, \xi \rangle.$$

As $t \to 0+$, $\langle P_t(\mathbf{1})\, \xi, \xi \rangle$ tends to $\langle \mathbf{1}\xi, \xi \rangle = 1$, and since $T(t)^*\, \xi$ tends to ξ in the norm of H we have $\|T(t)^*\, \xi\| \to 1$. (2.18) follows.

Now let $T_1, T_2 \in \mathscr{U}_P$. Because each unit satisfies a growth condition of the form (2.3) and since we can replace each $T_j(t)$ by $e^{-k_j t} T_j(t)$ without affecting the conclusion of Lemma 2.15, it suffices to prove Lemma 2.15 for units T_1, T_2 satisfying $\langle T_j(t), T_j(t)\rangle \leqslant 1$ for all $t>0$. Fix such a pair T_1, T_2, fix $t>0$, and set

$$u = T_1(t), \qquad v = \langle T_1(t), T_1(t)\rangle\, T_2(t) - \langle T_2(t), T_1(t)\rangle\, T_1(t). \tag{2.19}$$

u and v are orthogonal elements of $\mathscr{E}_P(t)$. We claim that for any two orthogonal elements $u, v \in \mathscr{E}_P(t)$ we have

$$\langle u, u\rangle\, vv^* \leqslant \langle v, v\rangle (\langle u, u\rangle\, P_t(\mathbf{1}) - uu^*). \tag{2.20}$$

Indeed, (2.20) is trivial if either u or v is 0, so we assume that both are non-zero. In this case, put

$$u_0 = \langle u, u\rangle^{-1/2}\, u, \qquad v_0 = \langle v, v\rangle^{-1/2}\, v.$$

Then $\{u_0, v_0\}$ is part of an orthonormal basis for $\mathscr{E}_P(t)$, hence the map

$$x \mapsto P_t(x) - u_0 x u_0^* - v_0 x v_0^*$$

is completely positive. Taking $x = \mathbf{1}$ we find that

$$v_0 v_0^* \leqslant P_t(\mathbf{1}) - v_0 v_0^*,$$

and (2.20) follows after multiplying through by $\langle u, u\rangle \langle v, v\rangle$.

For u and v as in (2.19), the inequality (2.20) implies that for every unit vector $\xi \in H$,

$$\langle T_1(t), T_1(t)\rangle\, \|\langle T_1(t), T_1(t)\rangle\, T_2(t)^*\, \xi - \langle T_1(t), T_2(t)\rangle\, T_1(t)^*\, \xi\|^2$$
$$\leqslant \langle v, v\rangle (\langle T_1(t), T_1(t)\rangle \langle P_t(\mathbf{1})\, \xi, \xi\rangle - \|T_1(t)^*\, \xi\|^2).$$

Notice that $\langle v, v\rangle \leqslant 4$. Indeed, since $\|T_j(t)\|_{\mathscr{E}_P(t)} \leqslant \langle T_j(t), T_j(t)\rangle^{1/2} \leqslant 1$ we have

$$\langle v, v\rangle = \|\langle T_1(t), T_1(t)\rangle\, T_2(t) - \langle T_2(t), T_1(t)\rangle\, T_1(t)\|^2_{\mathscr{E}_P(t)} \leqslant 4.$$

Thus the preceding inequality implies that

$$\|\langle T_1(t), T_1(t)\rangle\, T_2(t)^*\, \xi - \langle T_1(t), T_2(t)\rangle\, T_1(t)^*\, \xi\|^2 \tag{2.21}$$

is dominated by a term of the form

$$\frac{4}{\langle T_1(t), T_1(t)\rangle} (\langle T_1(t), T_1(t)\rangle \langle P_t(\mathbf{1})\, \xi, \xi\rangle - \|T_1(t)^*\, \xi\|^2). \tag{2.22}$$

As $t \to 0+$, the expression in (2.22) tends to zero because of (2.17) and the fact that both $\langle P_t(\mathbf{1})\, \xi, \xi \rangle$ and $\|T_1(t)^* \,\xi\|^2$ tend to $\|\xi\|^2 = 1$. Thus the term in (2.21) tends to zero as $t \to 0+$. Taking note of (2.17) once again, we conclude that

$$\lim_{t \to 0+} \|T_2(t)^* \,\xi - \langle T_1(t), T_2(t) \rangle \, T_1(t)^* \,\xi\| = 0.$$

Writing

$$\begin{aligned} |1 - \langle T_1(t), T_2(t) \rangle| &= \|\xi - \langle T_1(t), T_2(t) \rangle \,\xi\| \\ &\leqslant \|\xi - T_2(t)^* \,\xi\| + |\langle T_1(t), T_2(t) \rangle| \cdot \|\xi - T_1(t)^* \,\xi\| \\ &\quad + \|T_2(t)^* \,\xi - \langle T_1(t), T_2(t) \rangle \, T_1(t)^* \,\xi\|, \end{aligned}$$

and noting that each of the three terms on the right tends to zero as $t \to 0+$, we obtain

$$\lim_{t \to 0+} |1 - \langle T_1(t), T_2(t) \rangle| = 0$$

as required for Lemma 2.15. ∎

That also completes the proof of Theorem 2.6. ∎

3. LIFTING UNITS

Let $\alpha = \{\alpha_t : t \geqslant 0\}$ be an E_0-semigroup acting on $M = \mathscr{B}(H)$, H separable. α can be compressed to *certain* hereditary subalgebras $M_0 = p_0 M p_0$ of M so as to give a CP semigroup P acting on $M_0 \cong \mathscr{B}(p_0 H)$. In this section we show that the units of α map naturally to those of P, and in the case where α is minimal over P we show that this map is a bijection (Theorem 3.6).

A projection $p_0 \in M$ is said to be *increasing* if $\alpha_t(p_0) \geqslant p_0$ for every $t \geqslant 0$. In this case we obtain a CP semigroup $P = \{P_t : t \geqslant 0\}$ acting on $M_0 = p_0 M p_0$ by way of

$$P_t(x) = p_0 \alpha_t(x)\, p_0, \qquad t \geqslant 0, x \in M_0.$$

P is called a *compression* of α and α is called a *dilation* of P. It is possible for P itself to be an E_0-semigroup, that is to say $P_t(xy) = P_t(x)\, P_t(y)$ for every $x, y \in M_0$, $t \geqslant 0$. In this case we call P a multiplicative compression of α. Finally, α is said to be *minimal* over P if there are no intermediate multiplicative compressions; more explicitly, there should exist no increasing projection $q \in M$ for which (a) $q \geqslant p_0$ and (b) the compression of α to qMq is multiplicative, other than $q = \mathbf{1}$.

The issue of minimality over P merits some discussion (for full details see [3]). The condition

$$\alpha_t(p_0) \uparrow \mathbf{1}_H, \qquad \text{as} \quad t \to \infty,$$

is necessary, but *not* sufficient for minimality. There are a number of equivalent additional conditions that guarantee minimality, and the one we require is formulated as follows. For every $t>0$, let q_t be the projection onto the subspace $[\alpha_t(M)\, p_0 H]$. q_t obviously belongs to the commutant of $\alpha_t(M)$. For every fixed $t>0$ and every partition $\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[0, t]$, we set

$$q_{\mathscr{P}, t} = q_{t_1} \alpha_{t_1}(q_{t_2 - t_1})\, \alpha_{t_2}(q_{t_3 - t_2}) \cdots \alpha_{t_{n-1}}(q_{t_n - t_{n-1}}). \tag{3.1}$$

It is shown in [3, Proposition 3.4] that $q_{\mathscr{P}, t}$ is a projection in the commutant of $\alpha_t(M)$ and that

$$\mathscr{P}_1 \subseteq \mathscr{P}_2 \Rightarrow q_{\mathscr{P}_1, t} \leqslant q_{\mathscr{P}_2, t}.$$

Thus the strong limit

$$\bar{q}_t = \lim_{t = \to \infty} q_{\mathscr{P}, t}$$

exists for every $t>0$ and the resulting family of projections $\{\bar{q}_t \in \alpha_t(M)' : t > 0\}$ satisfies the cocycle equation

$$\bar{q}_{s+t} = \bar{q}_s \alpha_t(\bar{q}_t), \qquad s, t > 0$$

as well as a natural continuity condition. Moreover, it was shown in [3] that α is minimal over P iff the following two conditions are satisfied

$$\alpha_t(p_0) \uparrow \mathbf{1}, \qquad \text{as} \quad t \to \infty, \tag{3.2.1}$$

$$\bar{q}_t = \mathbf{1}, \qquad \text{for every} \quad t > 0. \tag{3.2.2}$$

The purpose of this section is to show how the units of α are related to the units of P in the case where α is minimal over P. More precisely, let $\mathscr{E}_\alpha = \{\mathscr{E}_\alpha(t) : t > 0\}$ be the product system of α. Thus $\mathscr{E}_\alpha(t)$ is the intertwining space

$$\mathscr{E}_\alpha(t) = \{T \in \mathscr{B}(H) : \alpha_t(x)\, T = Tx,\ x \in \mathscr{B}(H)\}$$

which becomes a separable Hilbert space with respect to the inner product defined by

$$\langle S, T \rangle\, \mathbf{1} = T^* S, \qquad S, T \in \mathscr{E}_\alpha(t).$$

PROPOSITION 3.3. *For every* $t>0$ *and every operator* $T \in \mathscr{E}_\alpha(t)$, *the subspace* $p_0 H$ *is invariant under the adjoint* T^*. *The operator* $S \in \mathscr{B}(p_0 H)$ *defined by*

$$S^* = T^* \restriction_{p_0 H}$$

belongs to the space $\mathscr{E}_P(t)$ *and satisfies*

$$\langle S, S \rangle_{\mathscr{E}_P(t)} \leqslant \langle T, T \rangle_{\mathscr{E}_\alpha(t)}. \tag{3.4}$$

Proof. The proof is a straightforward consequence of the fact that p_0 is an increasing projection. Indeed, if we choose an orthonormal basis $\{v_1, v_2, ...\}$ for $\mathscr{E}_\alpha(t)$ then we have

$$\sum_k v_k(\mathbf{1} - p_0)\, v_k^* = \alpha_t(\mathbf{1} - p_0) = \mathbf{1} - \alpha_t(p_0) \leqslant \mathbf{1} - p_0.$$

It follows that $v_k(\mathbf{1} - p_0)\, p_k^* \leqslant \mathbf{1} - p_0$ for every k, hence v_k leaves the orthogonal complement of $p_0 H$ invariant for every k, and hence $v_k^* p_0 H \subseteq p_0 H$. Since the linear span of the $\{v_k\}$ is dense in $\mathscr{E}_\alpha(t)$ in the operator norm, the assertion $\mathscr{E}_\alpha(t)^*\, p_0 H \subseteq p_0 H$ follows.

Let S be the indicated operator in $\mathscr{B}(p_0 H)$. Since α is an E_0-semigroup the Hilbert space norm of an element of $\mathscr{E}_\alpha(t)$ coincides with its operator norm. Thus, in order to show that $S \in \mathscr{E}_P(t)$ and satisfies the inequality (3.4), it suffices to show that the operator mapping L of $\mathscr{B}(p_0 H)$ defined by

$$L(x) = \|T\|^2\, P(x) - SxS^*$$

is completely positive. Now by definition of S see that for every $x \in p_0 M p_0$ we have

$$SxS^* = p_0 T x T^* p_0 = p_0 \alpha_t(x)\, TT^* p_0.$$

Since TT^* is a positive operator of norm $\|T\|^2$ in the commutant of $\alpha_t(M)$ it follows that $C = (\|T\|^2\, \mathbf{1} - TT^*)^{1/2}$ is a positive operator in the commutant of $\alpha_t(M)$, hence

$$L(x) = \|T\|^2\, p_0 \alpha_t(x)\, p_0 - p_0 \alpha_t(x)\, TT^* p_0 = p_0 C \alpha_t(x)\, C p_0$$

is obviously a completely positive mapping of $p_0 M p_0$ into itself. ∎

Proposition (3.3) implies that there is a natural mapping of the units of α to the units of P, defined as follows. In this concrete setting we may consider a unit of α to be a strongly continuous semigroup $T = \{T(t) : t \geqslant 0\}$ of operators in M satisfying

$$\alpha_t(x)\, T(t) = T(t)\, x, \qquad x \in M.$$

Choose such a T, and for every $t>0$ let $S(t)\in\mathscr{B}(p_0H)$ be the operator defined by

$$S(t)^* = T(t)^* \upharpoonright p_0 H. \tag{3.5}$$

It is obvious that $S=\{S(t)\colon t>0\}$ is a strongly continuous semigroup of bounded operators on p_0H for which $S(t)\to\mathbf{1}$ strongly as $t\to 0+$, and we have $S(t)=p_0T(t)\,p_0=p_0T(t)$ for every t. Proposition (3.3) implies that $S(t)$ belongs to $\mathscr{E}_P(t)$ for every $t>0$ and moreover

$$\langle S(t), S(t)\rangle_{\mathscr{E}_{P(t)}} \leqslant \|T(t)\|^2.$$

Because T is a unit of α we must have

$$\|T(t)\|^2 = e^{tC(T,\,T)}, \qquad t>0,$$

where $C\colon \mathscr{U}_\alpha\times\mathscr{U}_\alpha\to\mathbb{C}$ is the covariance function of α, and thus S is a unit of P.

THEOREM 3.6. *Suppose that α is minimal over P. Then the function $\theta\colon\mathscr{U}_\alpha\to\mathscr{U}_P$ defined by $\theta(T)=S$ is a bijection.*

Proof. In order to show that θ is one-to-one, fix $T_1, T_2\in\mathscr{U}_\alpha$ such that $\theta(T_1)=\theta(T_2)$. Thus $T_1(t)^*\upharpoonright_{p_0H}=T_2(t)^*\upharpoonright_{p_0H}$, for every $t>0$. Noting that $\alpha_t(x)\,T_k(t)=T_k(t)\,x$ it follows that for every $x\in M$ and $\xi\in p_0H$ we have

$$T_1^*(t)\,\alpha_t(x)\,\xi = xT_1(t)^*\,\xi = xT_2(t)^*\,\xi = T_2^*(t)\,\alpha_t(x)\,\xi.$$

Letting q_t be the projection on the subspace $[\alpha_t(M)\,p_0H]$ and taking adjoints, the preceding formula implies that

$$q_tT_1(t)=q_tT_2(t), \qquad t>0.$$

Note too that the preceding formula implies that for every $0<s<t$ we have

$$q_s\alpha_s(q_{t-s})\,T_1(t)=q_s\alpha_s(q_{t-s})\,T_2(t). \tag{3.7}$$

Indeed, the left side of (3.7) can be written

$$\begin{aligned} q_s\alpha_s(q_{t-s})\,T_1(s)\,T_1(t-s) &= q_sT_1(s)\,q_{t-s}T_1(t-s) = q_sT_2(s)\,q_{t-s}T_2(t-s)\\ &= q_s\alpha_s(q_{t-s})\,T_2(s)\,T_2(t-s) \end{aligned}$$

and (3.7) follows. By an obvious induction argument, it follows similarly that if $\mathscr{P}=\{0=t_0<t_1<\cdots<t_n=t\}$ is any finite partition of the interval $[0, t]$ and if $q_{\mathscr{P},t}$ is defined as in the discussion above, then we have

$$q_{\mathscr{P},t}T_1(t)=q_{\mathscr{P},t}T_2(t).$$

Because of the minimality condition (3.2.2) we may take the limit on $\mathscr{P}$ to obtain $T_1(t)=T_2(t)$.

In order to show that θ is surjective following:

Lemma 3.8. *Let $S=\{S(t): t\geqslant 0\}$ be a unit of P and for every $t>0$ let q_t be the projection onto $[\alpha_t(M)\,p_0H]$.*

Then for every $t>0$ there is a unique operator $v_t\in\mathscr{E}_\alpha(t)$ satisfying the two conditions $q_tv_t=v_t$, and $v_t^*\restriction p_0H=S_t^*$. Moreover, there is a real constant k such that $\|v_t\|\leqslant e^{kt}$ for every $t>0$.

Proof. Let $S=\{S(t): t\geqslant 0\}$ be a semigroup of bounded operators on $\mathscr{B}(p_0H)$ and let k be a real number with the property that for every $t>0$,

$$z\in\mathscr{B}(p_0H)\mapsto e^{kt}P_t(z)-S(t)\,zS(t)^* \tag{3.9}$$

is a completely positive map. Let $x_1, x_2, ..., x_n$ be a set of operators in the larger von Neumann algebra $M=\mathscr{B}(H)$ and choose vectors $\xi_1, \xi_2, ..., \xi_n\in p_0H$. We claim

$$\left\|\sum_{k=1}^{n} x_kS(t)^*\,\xi_k\right\|^2\leqslant e^{kt}\left\|\sum_{k=1}^{n}\alpha_t(x_k)\,\xi_k\right\|^2. \tag{3.10}$$

Indeed, the left side of (3.10) is

$$\sum_{k,j=1}^{n}\langle x_kS(t)^*\,\xi_k, x_jS(t)^*\,\xi_j\rangle$$

$$=\sum_{k,j=1}^{n}\langle S(t)\,p_0x_j^*x_kp_0S(t)^*\,\xi_k, \xi_j\rangle. \tag{3.11}$$

Since the $n\times n$ matrix (a_{jk}) defined by $a_{jk}=p_0x_j^*x_kp_0$ is a positive operator matrix with entries from p_0Mp_0, (3.9) implies that the right side of (3.11) is dominated by

$$e^{kt}\sum_{k,j=1}^{n}\langle\alpha_t(p_0x_j^*x_kp_0)\,\xi_k, \xi_j\rangle=e^{kt}\left\|\sum_{k=1}^{n}\alpha_t(x_kp_0)\,\xi_k\right\|^2.$$

Since p_0 is an increasing projection and $\xi_k \in p_0 H$, we can write $\alpha_t(x_k p_0)\, \xi_k = \alpha_t(x_k)\, \alpha_t(p_0)\, \xi_k = \alpha_t(x_k)\, \xi_k$ for each $k = 1, 2, ..., n$, and hence the right side of the previous formula becomes

$$e^{kt} \left\| \sum_{k=1}^{n} \alpha_t(x_k)\, \xi_k \right\|^2.$$

The inequality (3.10) follows.

From (3.10) it follows that there is a unique operator $v_t \in \mathscr{B}(H)$, having norm at most $e^{kt/2}$, and which satisfies

$$v_t^* \alpha_t(x)\, \xi = x S(t)^*\, \xi, \qquad x \in \mathscr{B}(H), \quad \xi \in p_0 H, \quad \text{and} \tag{3.12.1}$$

$$v_t^* = v_t^* q_t. \tag{3.12.2}$$

We claim that $v_t \in \mathscr{E}_\alpha(t)$ or equivalently, that

$$v_t^* \alpha_t(x) = x v_t^*, \qquad x \in \mathscr{B}(H). \tag{3.13}$$

Indeed, because of (3.12.2) we have $v_t^* \alpha_t(x) = v_t^* q_t \alpha_t(x) = v_t^* \alpha_t(x)\, q_t$, and similarly $x v_t^* = x v_t^* q_t$. Thus it suffices to show that the operators on both sides of (3.13) agree on vectors in $q_t H = [\alpha_t(M)\, p_0 H]$. If such a vector has the form $\eta = \alpha_t(y)\, \xi$ with $y \in \mathscr{B}(H)$ and $\xi \in p_0 H$ then we have

$$v_t^* \alpha_t(x)\, \eta = v_t^* \alpha_t(x)\, \alpha_t(y)\, \xi = v_t^* \alpha_t(xy)\, \xi = xy S(t)^*\, \xi = x v_t^* \alpha_t(y)\, \xi,$$

and (3.13) follows because such vectors η span the range of q_t.

This proves the existence assertion of Lemma 3.8. For uniqueness, let $w_t \in \mathscr{E}_\alpha(t)$ satisfy $q_t w_t = w_t$ and $w_t^* \restriction_{p_0 H} = S(t)^*$. Then for any vector η of the form $\eta = \alpha_t(x)\, \xi$, $x \in \mathscr{B}(H)$, $\xi \in p_0 H$ we have

$$w_t^* \eta = w_t^* \alpha_t(x)\, \xi = x w_t^* \xi = x S(t)^*\, \xi = v_t^* \eta,$$

so that w_t^* and v_t^* agree on $[\alpha_t(M)\, p_0 H] = q_t H$, and hence $w_t^* = w_t^* q_t$ and $v_t^* = v_t^* q_t$ agree. ∎

To complete the proof of Theorem 3.6, choose a unit $S = \{S_t : t \geqslant 0\}$ for P and let $\{v_t : t > 0\}$ be the family of operators defined by Lemma 3.8. This family of operators is certainly a section of the product system of α, but it is not a unit because it does not satisfy the semigroup property $v_{s+t} = v_s v_t$. In order to obtain a unit from this family $\{v_t : t > 0\}$ we carry out the following construction.

Fix $t > 0$. For every finite partition $\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[0, t]$, consider the operator

$$v_{\mathscr{P}, t} = v_{t_1 - t_0} v_{t_2 - t_1} \cdots v_{t_n - t_{n-1}}.$$

It is clear that $v_{\mathcal{P},t}$ belongs to $\mathcal{E}_\alpha(t)$, and because of the growth condition $\|v_s\| \leqslant e^{ks}$ for all positive s we have

$$\|v_{\mathcal{P},t}\| \leqslant e^{kt}.$$

Thus $\mathcal{P} \mapsto v_{\mathcal{P},t}$ defines a bounded net of operators belonging to the σ-weakly closed operator space $\mathcal{E}_\alpha(t)$. We will show next that this net converges weakly. The resulting limit

$$T_t = \lim_{\mathcal{P}} v_{\mathcal{P},t}$$

will satisfy the semigroup property $T_{s+t} = T_s T_t$, but since the net of finite partitions is uncountable, continuity (or even measurability) in t is not immediate. We then give a separate argument which guarantees that $\{T_t : t > 0\}$ is strongly continuous, and that the unit of α that it defines maps to S as required.

LEMMA 3.14. *For every $t > 0$ and every finite partition $\mathcal{P}$ of $[0, t]$, let $q_{\mathcal{P},t}$ be the projection defined in* (3.1). *Then for every pair of partitions satisfying $\mathcal{P}_1 \subseteq \mathcal{P}_2$ we have*

$$q_{\mathcal{P}_1} v_{\mathcal{P}_2,t} = v_{\mathcal{P}_1,t}.$$

Remark 3.15. We have already seen that the net of projections $\mathcal{P} \mapsto q_{\mathcal{P},t}$ is increasing in $\mathcal{P}$ and by minimality of α over P this net of projections has limit $\mathbf{1}$ for every fixed $t > 0$. Thus the coherence condition asserted in Lemma 3.14, together with the fact that $\|v_{\mathcal{P},t}\| \leqslant e^{kt}$, implies that the net of adjoint operators

$$\mathcal{P} \mapsto (v_{\mathcal{P},t})^*$$

must converge in the *strong* operator topology. In particular, the weak limit

$$T_t = \lim_{\mathcal{P}} v_{\mathcal{P},t}$$

exists for every t and defines an element of $\mathcal{E}_\alpha(t)$.

Proof of Lemma 3.14. We claim first that for every $s, t > 0$ we have

$$q_{s+t} v_s v_t = v_{s+t}. \tag{3.16}$$

Indeed, because of the uniqueness assertion of Lemma (3.8), it suffices to show that the operator $w = q_{s+t} v_s v_t$ belongs to $\mathcal{E}_\alpha(s+t)$ and satisfies $w^* \restriction_{p_0 H} = S(s+t)^*$. The first assertion is obvious because $v_s v_t \in \mathcal{E}_\alpha(s+t)$

and q_{s+t} commutes with $\alpha_{s+t}(M)$. To see that w^* restricts to $S(s+t)^*$ on $p_0 H$, choose $\xi \in p_0 H$ and note that

$$w^* \xi = v_t^* v_s^* q_{s+t} \xi = v_t^* v_s^* \xi = v_t^* S(s)^* \xi = S(t)^* S(s)^* \xi = S(s+t)^* \xi.$$

Thus (3.16) is established.

In order to prove Lemma (3.14), it is enough to consider the case where $\mathscr{P}_2$ is obtained from $\mathscr{P}_1 = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ by adjoining to it a single point τ, say

$$t_k < \tau < t_{k+1}$$

for some $k = 0, 1, ..., n-1$. Now by (3.16) we see that

$$q_{t_{k+1}-t_k} v_{\tau - t_k} v_{t_{k+1}-\tau} = v_{t_{k+1}-t_k},$$

and if we make this substitution for $v_{t_{k+1}-t_k}$ in the formula

$$v_{\mathscr{P}_1, t} = v_{t_1 - t_0} \cdots v_{t_{k+1}-t_k} \cdots v_{t_n - t_n - 1}$$

we obtain

$$\begin{aligned} v_{\mathscr{P}_1, t} &= v_{t_1 - t_0} \cdots v_{t_k - t_{k-1}} (q_{t_{k+1}-t_k} v_{\tau - t_k} v_{t_{k+1}-\tau})\, v_{t_{k+2}-t_{k+1}} \cdots v_{t_n - t_{n-1}} \\ &= (q_{t_1 - t_0} v_{t_1 - t_0}) \cdots (q_{t_{k+1}-t_k} v_{\tau - t_k} v_{t_{k+1}-\tau}) \cdots (q_{t_n - t_{n-1}} v_{t_n - t_{n-1}}). \end{aligned}$$

If we now move each of the "q" terms to the left, using the relation $v_s x = \alpha_s(x)\, v_s$, $x \in \mathscr{B}(H)$, that last expression on the right becomes

$$q_{t_1 - t_0} \alpha_{t_1}(q_{t_2 - t_1}) \cdots \alpha_{t_{n-1}}(q_{t_n - t_{n-1}})\, v_{t_1 - t_0} \cdots v_{\tau - t_k} v_{t_{k+1}-\tau} \cdots v_{t_n - t_{n-1}},$$

which is $q_{\mathscr{P}_1, t} v_{\mathscr{P}_2, t}$, as required in Lemma 3.14. ∎

It follows from Remark 3.15 that we have *strong* convergence of the net of adjoints

$$T_t^* = \lim_{\mathscr{P}} \, (v_{\mathscr{P}, t})^*$$

for every positive t. Since multiplication is strongly continuous on bounded sets we obtain $T_t^* T_s^*$ as a strong double limit

$$T_t^* T_s^* = \lim_{\mathscr{P}_1, \mathscr{P}_2} \, (v_{\mathscr{P}_1, t})^* \, (v_{\mathscr{P}_2, s})^* = \lim_{\mathscr{P}_1, \mathscr{P}_2} \, (v_{\mathscr{P}_1, s} v_{\mathscr{P}_2, t})^*.$$

Taking adjoints, we have the following weak convergence

$$T_s T_t = \lim_{\mathscr{P}_1, \mathscr{P}_2} \, v_{\mathscr{P}_1, s} v_{\mathscr{P}_2, t} = \lim_{\mathscr{P}_1, \mathscr{P}_2} \, v_{\mathscr{P}_1 \cup (s + \mathscr{P}_2), s+t},$$

where $\mathcal{P}_1 \cup (s+\mathcal{P}_2)$ denotes the partition of $[0, s+t]$ obtained by first listing the elements of $\mathcal{P}_1$ and then listing the elements of $s+\mathcal{P}_2$. Since the right side is a limit over a cofinal subnet of partitions of the interval $[0, s+t]$, we conclude that $T_s T_t = T_{s+t}$ for every positive s, t.

We claim next that $T_t^* \restriction_{p_0 H} = S(t)^*$. To see this, notice that since v_s^* restricts to $S(s)^*$ for every positive s and $\{S(s): s \geqslant 0\}$ is a semigroup, it follows that $(v_{\mathcal{P},t})^*$ restricts to $S(t)^*$ for every $t>0$. The claim follows because the net $(v_{\mathcal{P},t})^*$ converges weakly to T_t^*.

Finally, we show that the semigroup $\{T_t^*: t>0\}$ is strongly continuous; that is, we will show that

$$\lim \|T_t^* \xi - \xi\| = 0, \tag{3.17}$$

for every $\xi \in H$. Indeed, (3.17) is certainly true in case $\xi \in p_0 H$, because T_t^* restricts to $S(t)^*$ and S is a continuous semigroup of operators on $p_0 H$. Let K denote the set of all vectors $\xi \in H$ for which (3.17) holds. K is clearly a closed subspace of H which contains $p_0 H$. We assert now that for every $s>0$,

$$\alpha_s(M)\, K \subseteq K. \tag{3.18}$$

Indeed, if $s>0$ and $x \in M = \mathcal{B}(H)$, then for sufficiently small positive t we have $t<s$ and hence

$$T_t^* \alpha_s(x)\, \xi = \alpha_{s-t}(x)\, v_t^* \xi.$$

So if $\xi \in K$ then

$$\begin{aligned}
\|T_t^* \alpha_s(x)\, \xi - \alpha_s(x)\, \xi\| &= \|\alpha_{s-t}(x)\, v_t^* \xi - \alpha_s(x)\, \xi\| \\
&\leqslant \|\alpha_{s-t}(x)\, v_t^* \xi - \alpha_{s-t}(x)\, \xi\| + \|\alpha_{s-t}(x)\, \xi - \alpha_s(x)\, \xi\| \\
&\leqslant \|x\| \cdot \|v_t^* \xi - \xi\| + \|\alpha_{s-t}(x)\, \xi - \alpha_s(x)\, \xi\|.
\end{aligned}$$

Both terms on the right tend to 0 with t because $\xi \in K$ and α is a (continuous) E_0-semigroup. Thus K contains every vector of the form $\alpha_s(x)\, p_0 \xi$, where $x \in \mathcal{B}(H)$ and $\xi \in H$ are arbitrary, and s is an arbitrary positive number. Allowing s to tend to zero we find that $\alpha_s(x)$ tends strongly to x, and hence

$$K \supseteq [\mathcal{B}(H)\, p_0 H] = H.$$

Thus $\{T_t: t>0\}$ is strongly continuous.

It follows that $u = \{T_t: t>0\}$ is a unit of α for which $\theta(u) = S$, and the proof of Theorem 3.6 is complete. ∎

Remark 3.18. Notice that the semigroup $T = \{T_t : t > 0\} \in \mathcal{U}_\alpha$ defined by

$$T_t = \lim_{\mathscr{P}} v_{\mathscr{P}, t}, \qquad t > 0$$

projects as follows relative to any finite partition $\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0, t]$:

$$q_{\mathscr{P}, t} T_t = v_{\mathscr{P}, t} = v_{t_1 - t_0} v_{t_2 - t_1} \cdots v_{t_n - t_{n-1}}. \tag{3.19}$$

4. THE COVARIANCE FUNCTION OF A CP SEMIGROUP

Let $P = \{P_t : t \geqslant 0\}$ be a *unital* CP semigroup acting on $\mathscr{B}(H_0)$. By a theorem of B. V. R. Bhat, there is a Hilbert space H containing H_0 and an E_0-semigroup $\alpha = \{\alpha_t : t \geqslant 0\}$ acting on $\mathscr{B}(H)$ such that the projection p_0 onto H_0 is increasing for α and P is obtained by compressing α to $p_0 \mathscr{B}(H)\, p_0 \cong \mathscr{B}(p_0 H)$ as we have described above [6.7]. Moreover, one may also arrange (by passing to a suitable intermediate E_0-semigroup if necessary) that α is minimal over P [3]. Finally, any two minimal dilations of P are conjugate.

The purpose of this section is to calculate the covariance function

$$c_P : \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C}$$

of P in terms of the covariance function

$$c_\alpha : \mathcal{U}_\alpha \times \mathcal{U}_\alpha \to \mathbb{C}$$

of α when α is the minimal dilation of P. Indeed, letting $\theta : \mathcal{U}_\alpha \to \mathcal{U}_P$ be the bijection defined by Theorem 3.6, we will show that

$$c_P(\theta(u_1), \theta(u_2)) = c_\alpha(u_1, u_2). \tag{4.1}$$

Once one has (4.1), it is apparent that the bijection θ gives rise to a natural unitary operator from the Hilbert space associated with $(\mathcal{U}_\alpha, c_\alpha)$ onto that associated with $(\mathcal{U}_P, c_P)$, and in particular, these two Hilbert spaces have the same dimension. Hence, *the numerical index $d_*(P)$ of P must agree with the numerical index $d_*(\alpha)$ of its minimal dilation α.*

For every $t > 0$ and every partition $\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_n\}$ let $q_{\mathscr{P}, t}$ be the projection defined by (3.1). Since $q_{\mathscr{P}, t}$ belongs to the commutant of $\alpha_t(\mathscr{B}(H))$ it follows that

$$q_{\mathscr{P}, t} \mathscr{E}_\alpha(t) \subseteq \mathscr{E}_\alpha(t).$$

Thus we may consider the left multiplication operator

$$Q_{\mathscr{P},t}\colon x \in \mathscr{E}_\alpha(t) \mapsto q_{\mathscr{P},t}x \in \mathscr{E}_\alpha(t)$$

as a bounded operator on the Hilbert space $\mathscr{E}_\alpha(t)$. $Q_{\mathscr{P},t}$ is a self-adjoint projection in $\mathscr{B}(\mathscr{E}_\alpha(t))$.

PROPOSITION 4.2. *The projections $Q_{\mathscr{P},t} \in \mathscr{B}(\mathscr{E}_\alpha(t))$ are increasing in the variable $\mathscr{P}$ and*

$$\lim_{\mathscr{P}} \|Q_{\mathscr{P},t}x - x\|_{\mathscr{E}_\alpha(t)} = 0, \qquad x \in \mathscr{E}_\alpha(t).$$

Proof. These assertions are a simple consequence of the definition of the inner product $\langle \cdot, \cdot \rangle$ in $\mathscr{E}_\alpha(t)$:

$$\langle S, T \rangle \, \mathbf{1} = T^*S, \qquad S, T \in \mathscr{E}_\alpha(t).$$

Indeed, if $\mathscr{P}_1$ and $\mathscr{P}_2$ are two finite partitions of $[0, t]$ satisfying $\mathscr{P}_1 \subseteq \mathscr{P}_2$, then for every operator $T \in \mathscr{E}_\alpha(t)$ we have

$$\langle Q_{\mathscr{P}_1,t}T, T \rangle \, \mathbf{1}_H = T^*q_{\mathscr{P}_1,t}T \leqslant T^*q_{\mathscr{P}_2,t}T = \langle Q_{\mathscr{P}_2,t}T, T \rangle \, \mathbf{1}_H,$$

hence $Q_{\mathscr{P}_1,t} \leqslant Q_{\mathscr{P}_2,t}$. Similarly, the fact that the net $\mathscr{P} \mapsto Q_{\mathscr{P},t} \in \mathscr{B}(\mathscr{E}_\alpha(t))$ converges to the identity of $\mathscr{B}(\mathscr{E}_\alpha(t))$ follows immediately from (3.2.2). ∎

In Section 2, the covariance function of a CP semigroup P is defined in terms of limits of certain finite products of complex numbers of the form

$$\langle S_1(t), S_2(t) \rangle_{\mathscr{E}_{P(t)}} = \langle S_1(t), S_2(t) \rangle.$$

We now show how these products are expressed in terms of α.

THEOREM 4.3. *Let S_1 and S_2 be two units of a unital CP semigroup P. Let α be its minimal dilation to an E_0-semigroup and let T_1, T_2 be the unique units of α satisfying $\theta(T_k) = S_k$, $k = 1, 2$.*

Then for every $t > 0$ and every finite partition $\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[0, t]$, we have

$$\prod_{k=1}^{n} \langle S_1(t_k - t_{k-1}), S_2(t_k - t_{k-1}) \rangle = \langle Q_{\mathscr{P},t}T_1(t), T_2(t) \rangle,$$

the inner product on the right being relative to the Hilbert space $\mathscr{E}_\alpha(t)$.

Proof. For each t, let q_t be the projection onto the subspace $[\alpha_t(\mathscr{B}(H)\, p_0 H]$. Lemma 3.8 guarantees that there is a unique pair of operators $v_1(t), v_2(t) \in \mathscr{E}_\alpha(t)$ satisfying

$$q_t v_k(t) = v_k(t), \tag{4.4.1}$$

$$S_k(t)^* = v_k(t)^* \upharpoonright p_0 H, \tag{4.4.2}$$

for every $t > 0$. (4.4.3) implies that $S_k(t) = p_0 v_k(t)$.

We claim that

$$\langle S_1(t), S_2(t)\rangle_{\mathscr{E}_P(t)} = \langle v_1(t), v_2(t)\rangle_{\mathscr{E}_\alpha(t)}. \tag{4.5}$$

To see this we appeal to Proposition 1.7, which expresses the inner product of $\mathscr{E}_P(t)$ in terms of the *minimal* Stinespring dilation of the completely positive map P_t. We obtain such a dilation

$$P_t(x) = V^* \pi_t(x)\, V, \qquad x \in \mathscr{B}(p_0 H)$$

as follows.

For every $x \in \mathscr{B}(p_0 H)$ let $\pi_t(x)$ be the restriction of $\alpha_t(x p_0)$ to the invariant subspace $K = [\alpha_t(p_0 \mathscr{B}(H)\, p_0)\, p_0 H]$, and let V be the inclusion map of $p_0 H$ into K. Then since P_t is the compression of α_t to $\mathscr{B}(p_0 H)$ we see that

$$P_t(x) = V^* \pi_t(x)\, V, \qquad x \in \mathscr{B}(p_0 H),$$

and the latter is obviously a minimal Stinespring representation for P_t. Letting q_t be the projection on $[\alpha_t(\mathscr{B}(H))\, p_0 H]$, we claim first that

$$K = \alpha_t(p_0)\, q_t H. \tag{4.6}$$

Indeed, the two projections $\alpha_t(p_0)$ and q_t must commute because q_t belongs to the commutant of $\alpha_t(\mathscr{B}(H))$, and

$$K = [\alpha_t(p_0 \mathscr{B}(H)\, p_0)\, p_0 H] = [\alpha_t(p_0)\, \alpha_t(\mathscr{B}(H))\, p_0 H] = \alpha_t(p_0)\, q_t H.$$

For $k = 1, 2$ we claim that the operator

$$X_k = v_k(t) \upharpoonright_{p_0 H}$$

maps $p_0 H$ into K and satisfies

$$X_k x = \pi_t(x)\, X_k, \qquad x \in \mathscr{B}(p_0 H).$$

For that, note that since $v_k(t)$ belongs to $\mathscr{E}_\alpha(t)$ and satisfies (4.4.1) we have

$$X_k p_0 = v_k(t)\, p_0 = q_t v_k(t)\, p_0 = q_t \alpha_t(p_0)\, v_k p_0 = q_t \alpha_t(p_0)\, X_k p_0,$$

and hence (4.5) implies that $X_k p_0 H \subseteq K$. Similarly, for any operator x in $\mathscr{B}(p_0 H)$ we have $X_k x = X_k x p_0 = \alpha_t(x p_0)\, X_k = \pi_t(x)\, X_k$.

Finally, because of (4.4.2) we find that

$$S_k(t) = V^* X_k, \qquad k = 1, 2.$$

According to Proposition 1.7, the inner product $\langle S_1(t), S_2(t)\rangle$ is defined by

$$\langle S_1(t), S_2(t)\rangle\, \mathbf{1}_{p_0 H} = X_2^* X_1. \tag{4.7}$$

We compute the right side of (4.7). Since $v_k(t) \in \mathscr{E}_\alpha(t)$ it follows that

$$v_2(t)^*\, v_1(t) = \langle v_1(t), v_2(t)\rangle_{\mathscr{E}_\alpha(t)}\, \mathbf{1}_H,$$

and thus for $\xi, \eta \in p_0 H$,

$$\langle X_1 \xi, X_2 \eta\rangle = \langle v_1(t)\, \xi, v_2(t)\, \eta\rangle = \langle v_1(t), v_2(t)\rangle_{\mathscr{E}_\alpha(t)}\, \langle \xi, \eta\rangle.$$

It follows that

$$X_2^* X_1 = \langle v_1(t), v_2(t)\rangle_{\mathscr{E}_\alpha(t)}\, \mathbf{1}_{p_0 H},$$

and (4.5) follows.

Finally, letting $\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ be a finite partition of $[0, t]$ we find that

$$\begin{aligned} &\prod_{k=1}^{n} \langle S_1(t_k - t_{k-1}), S_2(t_k - t_{k-1})\rangle \\ &\quad = \prod_{k=1}^{n} \langle v_1(t_k - t_{k-1}), v_2(t_k - t_{k-1})\rangle \\ &\quad = \langle v_1(t_1 - t_0) \cdots v_1(t_n - t_{n-1}), v_2(t_1 - t_0) \cdots v_2(t_n - t_{n-1})\rangle_{\mathscr{E}_\alpha(t)}. \end{aligned}$$

Utilizing (3.19), the last term on the right of the above formula is

$$\langle q_{\mathscr{P}, t} T_1(t), q_{\mathscr{P}, t} T_2(t)\rangle_{\mathscr{E}_\alpha(t)} = \langle Q_{\mathscr{P}, t} T_1(t), T_2(t)\rangle_{\mathscr{E}_\alpha(t)},$$

and Theorem 4.3 follows. ∎

COROLLARY 4.8. *Let P be a unital CP semigroup with minimal dilation α, and let $\theta\colon \mathscr{U}_\alpha \to \mathscr{U}_P$ be the bijection of Theorem* 3.6. *Then for any two units u_1, u_2 of α we have*

$$c_P(\theta(u_1), \theta(u_2)) = c_\alpha(u_1, u_2).$$

Proof. Let $S_i = \theta(u_i) \in \mathcal{U}_P$, $i = 1, 2$. It is enough to show that

$$e^{tc_P(S_1, S_2)} = e^{tc_\alpha(u_1, u_2)}$$

for every $t > 0$. Now Theorem 4.3 implies that

$$\begin{aligned} e^{tc_P(S_1, S_2)} &= \lim_{\mathscr{P}} \prod_{k=1}^{n} \langle S_1(t_k - t_{k-1}), S_2(t_k - t_{k-1}) \rangle \\ &= \lim_{\mathscr{P}} \langle Q_{\mathscr{P}, t} u_1(t), u_2(t) \rangle_{\mathscr{E}_\alpha(t)}. \end{aligned}$$

On the other hand, Proposition 4.2 implies that the net of projections $Q_{\mathscr{P}, t} \in \mathscr{B}(\mathscr{E}_\alpha(t))$ increases with $\mathscr{P}$ to the identity operator of $\mathscr{B}(\mathscr{E}_\alpha(t))$. Hence

$$\lim_{\mathscr{P}} \langle Q_{\mathscr{P}, t} u_1(t), u_2(t) \rangle_{\mathscr{E}_\alpha(t)} = \langle u_1(t), u_2(t) \rangle_{\mathscr{E}_\alpha(t)}.$$

By definition of the covariance function of α [1] we have

$$\langle u_1(t), u_2(t) \rangle_{\mathscr{E}_\alpha(t)} = e^{tc_\alpha(u_1, u_2)},$$

as required. ▮

With Corollary 4.8 in hand, the remarks at the beginning of this section imply the following,

THEOREM 4.9. *Let P be a CP semigroup and let α be its minimal dilation to an E_0-semigroup. Then*

$$d_*(P) = d_*(\alpha).$$

Remark 4.9. If we are given two CP semigroups P and Q acting respectively on $\mathscr{B}(H)$ and $\mathscr{B}(K)$, then there is a natural CP semigroup $P \otimes Q$ acting on $\mathscr{B}(H \otimes K)$. For each $t \geqslant 0$, $(P \otimes Q)_t$ is defined uniquely by its action on elementary tensors via

$$(P \otimes Q)_t \colon x \otimes y \mapsto P_t(x) \otimes Q_t(y), \qquad x \in \mathscr{B}(H),\ y \in \mathscr{B}(K).$$

Now suppose that P and Q are unital CP semigroups. Using the minimality criteria developed in [3], it is quite easy to see that if α and β are respectively minimal dilations of P, Q to E_0-semigroups acting on $\mathscr{B}(\tilde{H})$, $\mathscr{B}(\tilde{K})$ where $\tilde{H} \supseteq H$ and $\tilde{K} \supseteq K$, then $\alpha \otimes \beta$ is a minimal dilation of the tensor product $P \otimes Q$ to an E_0-semigroup acting on $\mathscr{B}(\tilde{H} \otimes \tilde{K})$.

Thus, from Theorem 4.9 together with (a) Bhat's theorem [6, 7] on the existence of E_0-semigroup dilations of CP semigroups and (b) the addition formula for the index of E_0-semigroups [2], we deduce:

Corollary 4.10. *If P and Q are unital CP semigroups then*

$$d_*(P \otimes Q) = d_*(P) + d_*(Q).$$

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