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# Pure $E_0$-Semigroups and Absorbing States*

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**Abstract:** An $E_0$-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ is called *pure* if its tail von Neumann algebra is trivial in the sense that

$$\cap_t \alpha_t(\mathcal{B}(H)) = \mathbb{C}\mathbf{1}.$$

We determine all pure $E_0$-semigroups which have a *weakly continuous* invariant state $\omega$ and which are minimal in an appropriate sense. In such cases the dynamics of the state space must stabilize as follows: for every normal state $\rho$ of $\mathcal{B}(H)$ there is convergence to equilibrium in the trace norm

$$\lim_{t\to\infty} \|\rho \circ \alpha_t - \omega\| = 0.$$

A normal state $\omega$ with this property is called an *absorbing* state for $\alpha$.

Such $E_0$-semigroups must be cocycle perturbations of $CAR/CCR$ flows, and we develop systematic methods for constructing those perturbations which have absorbing states with prescribed finite eigenvalue lists.

## Introduction

An $E_0$-semigroup is a semigroup of normal $*$-endomorphisms $\alpha = \{\alpha_t : t \geq 0\}$ of the algebra $\mathcal{B}(H)$ of all bounded operators on a separable Hilbert space, which satisfies $\alpha_t(\mathbf{1}) = \mathbf{1}$ and the natural continuity property

$$\lim_{t\to 0} \langle \alpha_t(x)\xi, \eta\rangle = \langle x\xi, \eta\rangle, \qquad x \in \mathcal{B}(H), \quad \xi, \eta \in H.$$

There is a sequence of $E_0$-semigroups $\alpha^n$, $n = 1, 2, \ldots, \infty$ that can be constructed using the *natural* irreducible representations of either the canonical anticommutation relations

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or the canonical commutation relations. These $E_0$-semigroups are called $CAR/CCR$ flows. They occupy a position in the category of $E_0$-semigroups roughly analogous to that of the unilateral shifts (of various multiplicities) in the category of isometries on Hilbert space.

This paper addresses the perturbation theory of $CAR/CCR$ flows. A *cocycle perturbation* of an $E_0$-semigroup $\alpha$ is an $E_0$-semigroup $\beta$ which is related to $\alpha$ by way of

$$\beta_t(x) = U_t\alpha_t(x)U_t^*, \qquad x \in \mathcal{B}(H), t \geq 0$$

where $\{U_t : t \geq 0\}$ is a strongly continuous family of unitary operators in $\mathcal{B}(H)$ which satisfies the cocycle equation

$$U_{s+t} = U_s\alpha_s(U_t), \qquad s, t \geq 0,$$

We are interested in cocycle perturbations $\beta$ of the $CAR/CCR$ flows whose dynamics “stabilize" in that there should exist a normal state $\omega$ which is absorbing in the sense that for every normal state $\rho$ we have

$$\lim_{t\to\infty} \|\rho \circ \beta_t - \omega\| = 0. \tag{0.1}$$

It is obvious that when an absorbing state exists it is invariant under the action of $\beta$, and is in fact the unique normal $\beta$-invariant state. Physicists refer to the property (0.1) as *return to equilibrium*, while in ergodic theory the corresponding property is called *mixing*.

Every normal state $\omega$ of $\mathcal{B}(H)$ has a unique *eigenvalue list*, that is, a finite or infinite sequence of positive numbers $\lambda_1, \lambda_2, \dots$ which is decreasing ($\lambda_k \geq \lambda_{k+1}$, $k \geq 1$) and which has the property that for some orthonormal set $\xi_1, \xi_2, \dots$ in $H$ we have

$$\omega(x) = \sum_k \lambda_k \langle x\xi_k, \xi_k\rangle .$$

Clearly $\lambda_1 + \lambda_2 + \cdots = 1$, and of course there may be a finite number of repetitions of a given element in the eigenvalue list. The *set* $\{\lambda_k : k \geq 1\} \cup \{0\}$ determined by the eigenvalue list is the spectrum of the density operator of $\omega$. The eigenvalue list is finite iff $\omega$ is continuous in the weak operator topology of $\mathcal{B}(H)$.

If $\beta$ has an absorbing state $\omega$ then it is obvious from (0.1) that the eigenvalue list of $\omega$ contains all of the information that could be obtained from the dynamics of expectation values observed over the long term. Thus it is natural to ask what the possibilities are, and how one finds absorbing states for cocycle perturbations of the simplest $E_0$-semigroups.

In this paper we will be concerned with *pure* $E_0$-semigroups, i.e., $E_0$-semigroups $\beta$ with the property that the tail von Neumann algebra is trivial,

$$\cap_t\, \beta_t(\mathcal{B}(H)) = \mathbb{C}\mathbf{1}. \tag{0.2}$$

After discussing the relationship between purity and the existence of absorbing states in general, we take up the analysis of *weakly continuous* absorbing states, and we obtain more or less complete information about how to construct them. Those results are applied in Sect. 5 to establish the following

**Theorem A.** *Let $\alpha^n$ be the $CAR/CCR$ flow of index $n$, $1 \leq n \leq \infty$, and let $\lambda_1, \dots, \lambda_r$ be a finite decreasing sequence of positive numbers summing to 1. Then there is a cocycle perturbation $\beta$ of $\alpha^n$ which has an absorbing state $\omega$ with eigenvalue list $\lambda_1, \dots, \lambda_r$.*

*If $n \leq r^2 - 1$ (and in this event $r \geq 2$) then one can arrange that $\beta$ is minimal over the support projection of $\omega$.*

*Conversely, if $r \geq 2$ and $\beta$ is any $E_0$-semigroup which has an absorbing state $\omega$ with eigenvalue list $\lambda_1, \dots, \lambda_r$, and which is minimal over the support projection of $\omega$, then $\beta$ is conjugate to a cocycle perturbation of $\alpha^n$ for some $n$, $1 \leq n \leq r^2 - 1$.*

*Remarks.* The assertions about minimality relate to dilation theory. If $\omega$ is an invariant normal state for an $E_0$-semigroup $\beta$ then the support projection $p$ of $\omega$ is *increasing* in the sense that

$$\beta_t(p) \geq p, \qquad t \geq 0,$$

(see the discussion following Proposition 2.4). It follows that the family of completely positive linear maps $P = \{P_t : t \geq 0\}$ defined on the hereditary subalgebra $p\mathcal{B}(H)p \cong \mathcal{B}(pH)$ by

$$P_t(x) = p\beta_t(x)p, \qquad x \in p\mathcal{B}(H)p, t \geq 0$$

is in fact a semigroup of completely positive maps. The minimality assertions of the second and third paragraphs mean that $\beta$ is a minimal dilation of $P$ in the sense of [2].

If $\beta$ is not a minimal dilation of $P$ then there is a projection $q \geq p$ satisfying $\beta_t(q) = q$ for every $t \geq 0$ and such that the compression of $\beta$ to the hereditary subalgebra defined by $q$ is a minimal dilation of $P$ (see [2]). Thus we may conclude that $E_0$-semigroups having absorbing states with *finite* eigenvalue lists $\lambda_1, \dots, \lambda_r$, $r \geq 2$ are always associated with perturbations of $CAR/CCR$ flows.

*Remarks.* In [12], Powers constructed a new class of examples of $E_0$-semigroups. Such an $E_0$-semigroup $\alpha$ has the property (0.2) and moreover, there is a unit vector $\xi \in H$ such that the pure state $\omega(x) = \langle x\xi, \xi \rangle$ is invariant under the action of $\alpha$; indeed $\omega$ is an absorbing state.

In [9], Bratteli, Jorgensen and Price took up the construction of pure invariant states for single endomorphisms $\alpha$ of $\mathcal{B}(H)$ satisfying the discrete counterpart of (0.2),

$$\cap_n \alpha^n(\mathcal{B}(H)) = \mathbb{C} \cdot \mathbf{1},$$

and they obtain a (non-smooth) paramaterization of such states. While both of these results clearly bear some relation to the problems taken up below, we are concerned here with absorbing states that are *not* pure. Indeed, Theorem A has little content for eigenvalue lists of length 1, and the dilation theory associated with a pure invariant state is trivial.

Finally, it is appropriate to comment briefly on terminology. A semigroup of isometries $U = \{U_t : t \geq 0\}$ acting on a Hilbert space $H$ is traditionally called *pure* if

$$\cap_{t>0} U_t H = \{0\}.$$

A familiar theorem in operator theory asserts that every pure semigroup of isometries is unitarily equivalent to a direct sum of copies of the *shift* semigroup $S = \{S_t : t \geq 0\}$, which acts on the Hilbert space $L^2[0, \infty)$ by way of

$$S_t f(x) = \begin{cases} f(x-t), & x > t \\ 0, & 0 \leq x \leq t. \end{cases}$$

In the theory of $E_0$-semigroups, the proper analogue of the *shift* of mulitipicity $n = 1, 2, \dots, \infty$ is the $CAR/CCR$ flow of index $n$. There is no theorem in $E_0$-semigroup theory analogous to the one cited above for semigroups of isometries. Indeed, the work of Powers [12, 13] implies that there are $E_0$-semigroups $\alpha$ having the property (0.2) which are not cocycle conjugate to $CAR/CCR$ flows. Thus we have elected to use the term *pure* for an $E_0$-semigroup satisfying the condition (0.2), and we reserve the term *shift* for the $CAR/CCR$ flows.

## 1. Purity and Absorbing States

In this section we collect some basic observations about pure $E_0$-semigroups acting on von Neumann algebras. An $E_0$-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on a von Neumann algebra $M$ is called *pure* if the intersection $\cap_t \alpha_t(M)$ reduces to the scalar multiples of the identity. The following result characterizes purity in terms of the action of $\alpha$ on the predual of $M$.

**Proposition 1.1.** *Let $\alpha = \{\alpha_t : t \geq 0\}$ be an $E_0$-semigroup acting on a von Neumann algebra $M$. Then $\cap_t \alpha_t(M) = \mathbb{C}\mathbf{1}$ iff for every pair of normal states $\rho_1, \rho_2$ of $M$ we have*

$$\lim_{t\to\infty} \|\rho_1 \circ \alpha_t - \rho_2 \circ \alpha_t\| = 0. \tag{1.1.1}$$

*Proof.* We write $M_\infty$ for the von Neumann subalgebra $\cap_t \alpha_t(M)$. Assume first that $\alpha$ satisfies condition (1.1.1). To show that $M_\infty \subseteq \mathbb{C}\mathbf{1}$ it suffices to show that for every normal linear functional $\lambda \in M_*$ satisfying $\lambda(\mathbf{1}) = 0$, we have $\lambda(M_\infty) = \{0\}$. Choose such a $\lambda$ and let $\lambda = \lambda_1 + i\lambda_2$ be its Cartesian decomposition, where $\lambda_k(z^*) = \overline{\lambda_k(z)}$, $k = 1, 2$. Since $\lambda_k(\mathbf{1}) = 0$, it suffices to prove the assertion for self-adjoint elements $\lambda$ in the predual of $M$.

Now by the Hahn decomposition, every self-adjoint element of the predual of $M$ which annihilates the identity operator is a scalar multiple of the difference of two normal states. Thus, after rescaling, we can assume that there are normal states $\rho_1, \rho_2$ of $M$ such that $\lambda = \rho_1 - \rho_2$, and have to show that $\rho_1(x) = \rho_2(x)$ for every element $x \in M_\infty$. Since the restriction of each $\alpha_t$ to $M_\infty$ is obviously a $*$-automorphism of $M_\infty$, we can find a family of operators $x_t \in M_\infty$ such that $\alpha_t(x_t) = x$ for every $t \geq 0$. We have $\|x_t\| = \|\alpha_t(x_t)\| = \|x\|$ for every $t$ and hence

$$|\rho_1(x) - \rho_2(x)| = |(\rho_1 \circ \alpha_t - \rho_2 \circ \alpha_t)(x_t)| \leq \|\rho_1 \circ \alpha_t - \rho_2 \circ \alpha_t\| \cdot \|x\|$$

for every $t$. By hypothesis the right side tends to 0 with $t$, and we have the desired conclusion $\lambda(x) = \rho_1(x) - \rho_2(x) = 0$.

For the converse, let $\rho$ be an arbitrary normal linear functional on $M$. We claim that

$$\lim_{t\to\infty} \|\rho \circ \alpha_t\| = \|\rho \restriction_{M_\infty}\|. \tag{1.2}$$

For this, we note first that

$$\|\rho \circ \alpha_t\| = \|\rho \restriction_{\alpha_t(M)}\|. \tag{1.3}$$

Indeed, the inequality $\leq$ follows from the fact that for every $x \in M$,

$$|\rho(\alpha_t(x))| \leq \|\rho \restriction_{\alpha_t(M)}\| \cdot \|\alpha_t(x)\|.$$

While on the other hand, if $x \in \alpha_t(M)$ is an element of norm 1 for which

$$|\rho(x)| = \|\rho \restriction_{\alpha_t(M)}\|,$$

then we may find $x_0 \in M$ with $x = \alpha_t(x_0)$. Noting that $\|x_0\| = \|x\|$ because $\alpha_t$ is an isometry, we have

$$\|\rho \restriction_{\alpha_t(M)} \| = |\rho(x)| = |\rho \circ \alpha_t(x_0)| \leq \|\rho \circ \alpha_t\|.$$

Thus, (1.2) is equivalent to the assertion

$$\lim_{t\to\infty} \|\rho \restriction_{\alpha_t(M)} \| = \|\rho \restriction_{M_\infty} \|. \tag{1.4}$$

Since the range of $\alpha_t$ is a von Neumann subalgebra of $M$, we may deduce (1.4) from general principles. Indeed, if $M_t$, $t \geq 0$ is a decreasing family of weak$^*$-closed linear subspaces of the dual of a Banach space $E$ having intersection $M_\infty$, and $\rho$ is a weak$^*$-continuous linear functional on $E'$, then by a standard argument using weak$^*$-compactness of the unit ball of $E'$ we find that the norms $\|\rho \restriction_{M_t} \|$ must decrease to $\|\rho \restriction_{M_\infty} \|$.

Assuming now that $M_\infty = \mathbb{C}\mathbf{1}$, let $\rho_1$ and $\rho_2$ be normal states of $M$ and let $\lambda = \rho_1 - \rho_2$. Then the restriction of $\lambda$ to $M_\infty$ vanishes, so by (1.2) we have

$$\lim_{t\to\infty} \|\rho_1 \circ \alpha_t - \rho_2 \circ \alpha_t\| = \lim_{t\to\infty} \|\lambda \circ \alpha_t\| = 0,$$

as required. □

**Definition 1.5.** *Let $\alpha = \{\alpha_t : t \geq 0\}$ be an $E_0$-semigroup acting on a von Neumann algebra $M$. An absorbing state for $\alpha$ is a normal state $\omega$ on $M$ such that for every normal state $\rho$,*

$$\lim_{t\to\infty} \|\rho \circ \alpha_t - \omega\| = 0.$$

*Remarks.* An absorbing state $\omega$ is obviously *invariant* in the sense that $\omega \circ \alpha_t = \omega$, $t \geq 0$, and in fact is the *unique* normal invariant state. Pure absorbing states for $E_0$-semigroups acting on $\mathcal{B}(H)$ were introduced by Powers [13] in his work in $E_0$-semigroups of type $II$. Powers' definition differs somewhat from Definition 1.5, in that he requires only weak convergence to $\omega$

$$\lim_{t\to\infty} \rho(\alpha_t(x)) = \omega(x), \qquad x \in \mathcal{B}(H),$$

for every normal state $\rho$. But as the following observation shows, the two definitions are in fact equivalent.

**Proposition 1.6.** *Let $\{\rho_i : i \in I\}$ be a net of normal states of $M = \mathcal{B}(H)$ and let $\omega$ be a normal state such that*

$$\lim_i \rho_i(x) = \omega(x), \tag{1.6.1}$$

*for every compact operator $x$. Then $\lim_i \|\rho_i - \omega\| = 0$.*

*Proof.* Choose $\epsilon > 0$. Since $\omega$ is a normal state we can find a finite rank projection $p$ such that

$$\omega(p) \geq 1 - \epsilon. \tag{1.7}$$

Since $pMp \cong \mathcal{B}(pH)$ is a finite dimensional space of finite-rank operators, (1.6.1) implies that we have norm convergence

$$\lim_i \|\rho_i \restriction_{pMp} - \omega \restriction_{pMp} \| = 0,$$

and hence

$$\sup_{x\in M, \|x\|\leq 1} |\rho_i(pxp) - \omega(pxp)| \to 0, \tag{1.8}$$

as $i \to \infty$. Now in general, we have

$$\|\rho_i - \omega\| \leq \sup_{\|x\|\leq 1} |\rho_i(pxp) - \omega(pxp)| + \sup_{\|x\|\leq 1} |\rho_i(x - pxp)| + \sup_{\|x\|\leq 1} |\omega(x - pxp)|.$$

By (1.8), the first term on the right tends to 0 as $i \to \infty$, and we can estimate the second and third terms as follows. Writing $x - pxp = (\mathbf{1} - p)x + px(\mathbf{1} - p)$, we find from the Schwarz inequality that

$$|\rho_i((\mathbf{1} - p)x)|^2 \leq \rho_i(\mathbf{1} - p)\rho_i(x^*x) \leq (1 - \rho_i(p))\|x\|^2,$$

and hence

$$|\rho_i((\mathbf{1} - p)x)| \leq (1 - \rho_i(p))^{1/2}\|x\|.$$

Similarly,

$$|\rho_i(px(\mathbf{1} - p))| \leq (1 - \rho_i(p))^{1/2}\|px\| \leq (1 - \rho_i(p))^{1/2}\|x\|.$$

It follows that

$$\sup_{\|x\|\leq 1} |\rho_i(x - pxp)| \leq 2(1 - \rho_i(p))^{1/2}.$$

Since $1 - \rho_i(p)$ tends to $1 - \omega(p) \leq \epsilon$ as $i \to \infty$, it follows that

$$\limsup_{i\to\infty} \sup_{\|x\|\leq 1} |\rho_i(x - pxp)| \leq 2\epsilon^{1/2}.$$

Similar estimates show that

$$\sup_{\|x\|\leq 1} |\omega(x - pxp)| \leq 2\epsilon^{1/2}.$$

Using (1.8), we conclude that

$$\limsup_{i\to\infty} \|\rho_i - \omega\| \leq 4\epsilon^{1/2}$$

and (1.6.1) follows because $\epsilon$ is arbitrary $\square$

*Remarks.* Suppose that $\alpha = \{\alpha_t : t \geq 0\}$ is a pure $E_0$-semigroup acting on an arbitrary von Neumann algebra $M$, and that $\omega$ is a normal state of $M$ which is invariant under $\alpha$. Then for every normal state $\rho$, Proposition 1.1 implies that

$$\lim_{t\to\infty} \|\rho \circ \alpha_t - \omega\| = \lim_{t\to\infty} \|\rho \circ \alpha_t - \omega \circ \alpha_t\| = 0,$$

hence $\omega$ is an absorbing state. Conversely, if an $E_0$-semigroup $\alpha$ has an absorbing state, then by Proposition 1.1 $\alpha$ must be a pure $E_0$-semigroup. Thus we have the following description of the relationship between absorbing states and pure $E_0$-semigroups:

**Proposition 1.9.** *Let $\alpha = \{\alpha_t : t \geq 0\}$ be an $E_0$-semigroup acting on a von Neumann algebra $M$ which has a normal invariant state $\omega$. Then $\alpha$ is pure if and only if $\omega$ is an absorbing state.*

*Remarks.* Every abelian semigroup is amenable. Thus one can make use of a Banach limit on the additive semigroup of nonnegative reals to average any $E_0$-semigroup in the point-weak operator topology to show that there is a state of $\mathcal{B}(H)$ which is invariant under the action of the $E_0$-semigroup. However, invariant states constructed by such devices tend to be singular. Indeed, the results of [6] show that there are pure $E_0$-semigroups (acting on $\mathcal{B}(H)$) which do not have normal invariant states.

## 2. Pure CP Semigroups

**Definition 2.1.** *A CP semigroup is a semigroup $P = \{P_t : t \geq 0\}$ of normal completely positive maps of $\mathcal{B}(H)$ which satisfies the natural continuity property*

$$\lim_{t\to 0+} \langle P_t(x)\xi, \eta\rangle = \langle x\xi, \eta\rangle , \qquad x \in \mathcal{B}(H), \xi, \eta \in H.$$

*$P$ is called unital if $P_t(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$.*

*A unital CP semigroup $P$ is said to be pure if, for every pair of normal states $\rho_1, \rho_2$ of $\mathcal{B}(H)$ we have*

$$\lim_{t\to\infty} \|\rho_1 \circ P_t - \rho_2 \circ P_t\| = 0.$$

Notice that pure CP semigroups are required to be unital. Unital CP semigroups are often called *quantum dynamical semigroups* in the mathematical physics literature. The purpose of this section is to briefly discuss the relationship between pure CP semigroups and pure $E_0$-semigroups. This relationship is not bijective, but it is close enough to being so that results in one category usually have immediate implications for the other.

For example, suppose that $P$ is a pure CP semigroup acting on $\mathcal{B}(H)$. A recent dilation theorem of B. V. R. Bhat [7, 8] implies that there is a Hilbert space $K \supseteq H$ and an $E_0$-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(K)$ which is a *dilation* of $P$ in the following sense. Letting $p_0 \in \mathcal{B}(K)$ be the projection onto the subspace $H$ and identifying $\mathcal{B}(H)$ with the hereditary subalgebra $M_0 = p_0\mathcal{B}(K)p_0$ of $\mathcal{B}(K)$, then we have

$$\alpha_t(p_0) \geq p_0, \qquad \text{and} \tag{2.2.1}$$
$$P_t(x) = p_0\alpha_t(x)p_0, \qquad x \in M_0 \tag{2.2.2}$$

for every $t \geq 0$. Because of (2.2.1), the operator

$$p_\infty = \lim_{t\to\infty} \alpha_t(p_0)$$

exists as a strong limit of projections, and is therefore a projection fixed under the action of $\alpha$. By compressing $\alpha$ to the hereditary subalgebra $p_\infty\mathcal{B}(K)p_\infty$ if necessary, we can assume that $K = p_\infty K$ and hence that

$$\alpha_t(p_0) \uparrow \mathbf{1}_K, \qquad \text{as } t \to \infty. \tag{2.3}$$

When (2.3) is satisfied we will say that $\alpha$ is a *dilation* of $P$.

Dilations in this sense are not unique. In order to obtain uniqueness (up to conjugacy), one must in general compress $\alpha$ to a smaller hereditary subalgebra of $\mathcal{B}(K)$. Once that is done $\alpha$ is called a *minimal* dilation of $P$. The issue of minimality is a subtle one, and we will not have to be very specific about its nature here (see [2] for more detail). For our purposes, it is enough to know that every dilation can be compressed uniquely to a minimal dilation, and that minimal dilations are unique up to conjugacy. Moreover, nonminimal dilations of a given CP semigroup exist in profusion. For example, the trivial CP semigroup acting on $\mathbb{C}$ has many dilations to nontrivial, nonconjugate $E_0$-semigroups [13]. The following result implies that all such dilations are pure.

**Proposition 2.4.** *Let $P = \{P_t : t \geq 0\}$ be a pure CP semigroup acting on $\mathcal{B}(H)$. Then every dilation of $P$ to an $E_0$-semigroup is pure.*

*Proof.* Let $\alpha$ be a dilation of $P$ which acts on $\mathcal{B}(K)$, $K$ being a Hilbert space containing $H$. Letting $p_0 \in \mathcal{B}(K)$ be the projection on $H$, then by (2.3) we see that the subspaces

$$K_t = \alpha_t(p_0)K$$

increase with $t$ and their union is dense in $K$. If we let $\mathcal{N}_t$ denote the set of all normal states $\rho$ of $\mathcal{B}(K)$ which can be represented in the form

$$\rho(x) = \sum_k \langle x\xi_k, \xi_k\rangle \,,$$

with vectors $\xi_1, \xi_2, \ldots \in K_t$, then the sets $\mathcal{N}_t$ increase with $t$ and their union is *norm*-dense in the space of all normal states of $\mathcal{B}(K)$.

Using this observation together with Proposition 1.1, it is enough to show that for every $t > 0$ and every pair of normal states $\rho_1, \rho_2 \in \mathcal{N}_t$, we have

$$\lim_{s\to\infty} \|\rho_1 \circ \alpha_s - \rho_2 \circ \alpha_s\| = 0. \tag{2.5}$$

To prove (2.5), fix $t > 0$ and choose $s > t$. We claim that for $k = 1, 2$ and $x \in \mathcal{B}(K)$ we have

$$\rho_k(\alpha_s(x)) = \rho_k(\alpha_t(P_{s-t}(p_0 x p_0))). \tag{2.6}$$

Indeed, since $p_0 \le \alpha_{s-t}(p_0)$ we have

$$P_{s-t}(p_0xp_0) = p_0\alpha_{s-t}(p_0xp_0) = p_0\alpha_{s-t}(p_0)\alpha_{s-t}(x)\alpha_{s-t}(p_0)p_0 = p_0\alpha_{s-t}(x)p_0,$$

so that

$$\alpha_t(P_{s-t}(p_0xp_0))) = \alpha_t(p_0\alpha_{s-t}(x)p_0) = \alpha_t(p_0)\alpha_s(x)\alpha_t(p_0).$$

Hence the right side of (2.6) can be written

$$\rho_k(\alpha_t(p_0)\alpha_s(x)\alpha_t(p_0)).$$

Since $\rho_k$ belongs to $\mathcal{N}_t$ we must have $\rho_k(\alpha_t(p_0)z\alpha_t(p_0)) = \rho_k(z)$ for every $z \in \mathcal{B}(K)$, and (2.6) follows.

Letting $\sigma_k$ be the restriction of $\rho_k \circ \alpha_t$ to $M_0 = p_0\mathcal{B}(K)p_0$ we find that for every $x \in \mathcal{B}(K)$,

$$|\rho_1(\alpha_s(x)) - \rho_2(\alpha_s(x))| = |\sigma_1(P_{s-t}(p_0xp_0) - \sigma_2(P_{s-t}(p_0xp_0)|.$$

Thus

$$\|\rho_1 \circ \alpha_s - \rho_2 \circ \alpha_s\| = \|\sigma_1 \circ P_{s-t} - \sigma_2 \circ P_{s-t}\|$$

must tend to 0 as $s$ tends to $\infty$, and (2.5) follows. $\square$

Suppose now that we start with a pure $E_0$-semigroup acting on $\mathcal{B}(H)$. It is not always possible to locate a CP semigroup as a compression of $\alpha$ because we know of no general method for locating a projection $p_0 \in \mathcal{B}(H)$ satisfying $\alpha_t(p_0) \ge p_0$ for every $t$. However, if $\alpha$ has an invariant normal state $\omega$, then the support projection of $\omega$ provides such a projection $p_0$. To see that, simply notice that $\omega \circ \alpha_t(\mathbf{1} - p_0) = \omega(\mathbf{1} - p_0) = 0$, hence $\alpha_t(\mathbf{1} - p_0) \le \mathbf{1} - p_0$, hence $\alpha_t(p_0) \ge p_0$.

Given such a projection $p_0$, we can compress $\alpha$ to obtain a family of normal completely positive maps $P = \{P_t : t \ge 0\}$ of $\mathcal{B}(p_0H) \cong p_0\mathcal{B}(H)p_0$ by way of

$$P_t(x) = p_0\alpha_t(x)p_0, \qquad t \ge 0, x \in p_0\mathcal{B}(H)p_0. \tag{2.7}$$

The fact that $\alpha_t(p_0) \ge p_0$ insures that $P$ is in fact a CP semigroup. The following summarizes these remarks.

**Proposition 2.8.** *Suppose that $\alpha$ is a pure $E_0$-semigroup acting on $\mathcal{B}(H)$ and $\omega$ is a normal $\alpha$-invariant state with support projection $p_0$. Then the CP semigroup P defined by (2.7) is pure, and the restriction $\omega_0$ of $\omega$ to $p_0\mathcal{B}(H)p_0 \cong \mathcal{B}(p_0H)$ is a faithful normal $P$-invariant state which is absorbing in the sense that for every normal state $\rho$ of $\mathcal{B}(p_0H)$,*

$$\lim_{t\to\infty} \|\rho \circ P_t - \omega_0\| = 0.$$

*If $\omega$ is weakly continuous and not a pure state of $\mathcal{B}(H)$, then $P$ may be considered a CP semigroup acting on a matrix algebra $M_n(\mathbb{C})$, $n = 2, 3, \ldots$.*

The preceding discussion shows the extent to which the theory of pure $E_0$-semigroups having an absorbing state can be reduced to the theory of CP semigroups having a *faithful* absorbing state. While the latter problem is an attractive one in general, we still lack tools that are appropriate for arbitrary invariant normal states. The following sections address the case of weakly continuous invariant states.

## 3. Perturbations and Invariant States

In order to describe the pure CP semigroups acting on matrix algebras we must first obtain information about invariant states. More precisely, given a *faithful* state $\omega$ on a matrix algebra $M = M_N(\mathbb{C})$, $N = 2, 3, \ldots$, we want to identify the unital CP semigroups $P = \{P_t : t \geq 0\}$ that leave $\omega$ invariant in the sense that

$$\omega \circ P_t = \omega, \qquad t \geq 0.$$

It is not obvious that such semigroups exist when $\omega$ is not a tracial state. In this section we characterize the generators of such semigroups up to perturbations (Theorem 3.8) and we give explicit examples in Corollary 3.16. In general, the generator $L$ of a CP semigroup has a decomposition of the form

$$L(x) = P(x) + kx + xk^*, \qquad x \in M, \tag{3.1}$$

where $P$ is a completely positive map on $M$ and $k \in M$ [10]. The associated semigroup $\{\exp tL : t \geq 0\}$ is unital iff

$$L(\mathbf{1}) = 0 \tag{3.2}$$

and it leaves $\omega$ invariant iff

$$\omega \circ L = 0. \tag{3.3}$$

It is easy to satisfy (3.2), but less easy to satisfy both (3.2) and (3.3). Indeed, setting $x = \mathbf{1}$ in (3.1) we find that (3.2) holds iff $k$ has a Cartesian decomposition

$$k = -1/2P(\mathbf{1}) + \ell,$$

where $\ell$ is an element of $M$ satisfying $\ell^* = -\ell$. In this case (3.1) becomes

$$L(x) = P(x) - 1/2(P(\mathbf{1})x + xP(\mathbf{1})) + [\ell, x]. \tag{3.4}$$

There is a natural decomposition of this operator corresponding to the Cartesian decomposition of $k$:

$$L(x) = L_0(x) + [\ell, x],$$

where $L_0$ is the "unperturbed" part of $L$,

$$L_0(x) = P(x) - 1/2(P(\mathbf{1})x + xP(\mathbf{1})). \tag{3.5}$$

Notice that both $L_0$ and $L$ generate unital CP semigroups, and because of (3.3) the semigroup generated by $L$ leaves $\omega$ invariant. If $\omega$ is not a trace then the unperturbed CP semigroup $\exp tL_0$ need not leave $\omega$ invariant (see Proposition 3.18). Thus we are led to seek *perturbations* of $L_0$ which solve both Eqs. (3.2) and (3.3).

In order to discuss this issue in more concrete terms, let $\Omega$ be the density matrix of the state $\omega$,

$$\omega(x) = \text{trace}(\Omega x), \qquad x \in M.$$

Since $\omega$ is faithful, $\Omega$ is a positive invertible operator. More generally, we identify the dual $M'$ of $M$ with $M$ itself in the usual way, the isomorphism $a \in M \mapsto \omega_a \in M'$ being defined by

$$\omega_a(x) = \text{trace}(ax), \qquad x \in M.$$

For every linear map $L : M \to M$ the dual map $L_*$, defined on $M'$ by $L_*(\rho) = \rho \circ L$, becomes

$$\text{trace}(L_*(y)x) = \text{trace}(yL(x)), \qquad x, y \in M.$$

Now a linear map $L : M \to M$ satisfies $\omega \circ L = 0$ iff its dual satisfies $L_*(\Omega) = 0$. If we choose a completely positive map $P : M \to M$ and define $L_0$ as in (3.5), then we seek a skew-adjoint operator $\ell \in M$ satisfying the operator equation

$$L_{0*}(\Omega) = \ell\Omega - \Omega\ell. \tag{3.6}$$

It is not always possible to solve (3.6). But if a solution $\ell_0$ exists then there are infinitely many, the most general one having the form $\ell = \ell_0 + k$, $k$ being a skew-adjoint operator commuting with $\Omega$.

We will show that (3.6) is solvable iff $P$ satisfies a certain symmetry requirement. The symmetry involves an involution # and is described as follows. For every linear map $L : M \to M$, let $L^\# : M \to M$ be the linear map

$$L^\#(x) = \Omega^{-1/2}L_*(\Omega^{1/2}x\Omega^{1/2})\Omega^{-1/2}. \tag{3.7}$$

For our purposes, the important properties of the operation $L \mapsto L^\#$ are summarized as follows.

**Proposition.** *$L \mapsto L^\#$ is a linear isomorphism satisfying $L^{\#\#} = L$, and if $L$ is completely positive then so is $L^\#$.*

*Sketch of proof.* The argument is completely straightforward. A direct computation shows that

$$(L^\#)_*(x) = \Omega^{1/2}L(\Omega^{-1/2}x\Omega^{-1/2})\Omega^{1/2},$$

from which $L^{\#\#} = L$ is immediate. The fact that # preserves complete positivity follows from the fact that if $P$ is a completely positive map then so is $P_*$. $\square$

**Theorem 3.8.** *Let $\omega$ be a faithful state on a matrix algebra $M$, let $Q : M \to M$ be a completely positive linear map, and define $Q^\#$ by (3.7). Then the following are equivalent.*

*(i) There is a unital CP semigroup $P = \{P_t : t \geq 0\}$ which leaves $\omega$ invariant and whose generator has the form*

$$L(x) = Q(x) + kx + xk^*$$

*for some $k \in M$.*

*(ii) For every minimal spectral projection $e$ of $\Omega$ we have $eQ(\mathbf{1})e = eQ^{\#}(\mathbf{1})e$.*

Our proof of Theorem 3.8 is based on the following general result. Let $A$ be the centralizer algebra of $\omega$,

$$A = \{a \in M : \omega(ax) = \omega(xa), x \in M\}.$$

If we consider the spectral decomposition of $\Omega$,

$$\Omega = \sum_{k=1}^{r} \lambda_k e_k,$$

where $e_1, \dots, e_r$ are the minimal spectral projections of $\Omega$ and $0 < \lambda_1 < \dots < \lambda_r$ are the distinct eigenvalues, then $A$ is the commutant of $\{\Omega\}$ and hence

$$A = \{a \in M : ae_k = e_k a, 1 \leq k \leq r\}.$$

$A$ is a direct sum of full matrix algebras, and the restriction of $\omega$ to $A$ is a faithful tracial state. The natural conditional expectation $E_A : M \to A$ is given by

$$E_A(x) = \sum_k e_k x e_k, \qquad x \in M.$$

The following result implies that the solvability of Eq. (3.6) depends only on the compression of $L$ to the centralizer algebra $A$.

**Lemma 3.9.** *Let $\omega$ be a faithful state of $M$ and let $L : M \to M$ be a linear map satisfying $L(x)^* = L(x^*)$, $x \in M$. The following are equivalent:*

*(i) There is a skew-adjoint operator $\ell \in M$ such that the perturbation $L'(x) = L(x) + [\ell, x]$ satisfies $\omega \circ L' = 0$.*

*(ii) The restriction of $\omega \circ L$ to $A$ vanishes.*

*More generally, setting $L_0 = E_A L E_A$, there is a perturbation $L'(x) = L(x) + [\ell, x]$ of the form (i) such that $\omega \circ L' = \omega \circ L_0$.*

*Proof.* (i)$\Longrightarrow$(ii) Suppose that $\ell$ is an operator in $M$ for which $\omega \circ L' = 0$, $L'$ being the operator of part (i). Since $\omega(\ell a - a\ell) = 0$ for all $a$ in the centralizer algebra we have

$$\omega(L(a)) = \omega(L(a) + [\ell, a]) = \omega \circ L'(a) = 0,$$

hence (ii).

We now prove the general assertion of the last sentence. Noting that $\omega \circ E_A = \omega$, we have

$$\omega(L_0(x)) = \omega(L(E_A(x))), \qquad x \in M,$$

and hence we must exhibit an operator $\ell \in M$ satisfying $\ell^* = -\ell$ and

$$\omega(L(x) + [\ell, x] - L(E_A(x))) = 0, \qquad x \in M.$$

After dualizing, the previous equation becomes

$$L_*(\Omega) - [\ell, \Omega] - E_A(L_*(\Omega)) = 0,$$

or

$$L_*(\Omega) - E_A(L_*(\Omega)) = \ell\Omega - \Omega\ell. \tag{3.10}$$

Let $T$ be the left side of (3.10). $T$ is a self-adjoint operator satisfying $E_A(T) = 0$. Thus if

$$\Omega = \sum_{k=1}^{r} \lambda_k e_k$$

is the spectral decomposition of $\Omega$ then we have $e_k T e_k = 0$ for all $k$. Set

$$\ell = \sum_{i \neq j} \frac{1}{\lambda_j - \lambda_i} e_i T e_j.$$

It is obvious that $\ell^* = -\ell$, and since $\Omega e_k = e_k \Omega = \lambda_k e_k$ for all $k$ we have

$$\begin{aligned} \Omega\ell &= \sum_{i \neq j} \frac{\lambda_i}{\lambda_j - \lambda_i} e_i T e_j, \\ \ell\Omega &= \sum_{i \neq j} \frac{\lambda_j}{\lambda_j - \lambda_i} e_i T e_j. \end{aligned}$$

Hence

$$\ell\Omega - \Omega\ell = \sum_{i \neq j} e_i T e_j = T,$$

as required.

The implication (ii)$\Longrightarrow$(i) follows immediately, for if $\omega \circ L(a) = 0$ for all $a \in A$, then because $\omega \circ E_A = \omega$ we have $\omega \circ L_0 = 0$. Thus the preceding argument gives a perturbation $L'$ of the form (i) satisfying $\omega \circ L' = \omega \circ L_0 = 0$ □

*Proof of Theorem 3.8.* Let $Q$ be a completely positive map and define $L : M \to M$ by

$$L(x) = Q(x) - 1/2(Q(\mathbf{1})x + xQ(\mathbf{1})).$$

The assertion (i) of Theorem 3.8 is equivalent to the existence of a skew-adjoint operator $\ell \in M$ such that

$$\omega(L(x) + [\ell, x]) = 0, \qquad x \in M. \tag{3.11}$$

By Lemma 3.9, the latter is equivalent to

$$\omega(L(a)) = 0, \qquad a \in A. \tag{3.12}$$

Thus we have to show that (3.12) is equivalent to the operator equation

$$E_A(Q(\mathbf{1})) = E_A(Q^\#(\mathbf{1})). \tag{3.13}$$

Looking first at (3.12), we have

$$\omega(L(a)) = \omega(Q(a)) - 1/2\omega(Q(\mathbf{1})a + aQ(\mathbf{1})).$$

Now since every element $a \in A$ commutes with $\Omega$ we have

$$\begin{aligned} 1/2\omega(Q(\mathbf{1})a + aQ(\mathbf{1})) &= 1/2\mathrm{trace}(\Omega Q(\mathbf{1})a + \Omega a Q(\mathbf{1})) \\ &= \mathrm{trace}(\Omega Q(\mathbf{1})a) = \omega(Q(\mathbf{1})a). \end{aligned}$$

Hence (3.12) asserts that

$$\omega(Q(a)) - \omega(Q(\mathbf{1})a) = 0, \qquad a \in A. \tag{3.14}$$

Writing

$$\begin{aligned}\omega(Q(a)) &= \mathrm{trace}(\Omega Q(a)) = \mathrm{trace}(Q_*(\Omega)a)\\ &= \mathrm{trace}(\Omega \cdot \Omega^{-1/2}Q_*(\Omega^{1/2} \cdot \Omega^{1/2})\Omega^{-1/2}a) = \omega(Q^\#(\mathbf{1})a),\end{aligned}$$

we rewrite (3.14) as

$$\omega((Q^\#(\mathbf{1}) - Q(\mathbf{1}))a) = 0, \qquad a \in A.$$

Since $\omega \circ E_A = \omega$ and $E_A(xa) = E_A(x)a$ for $a \in A$ the preceding formula becomes

$$\omega(E_A(Q^\#(\mathbf{1}) - Q(\mathbf{1}))a) = 0, \qquad a \in A.$$

Since $\omega \restriction_A$ is a faithful trace on $A$, the latter is equivalent to Eq. (3.13). $\square$

*Remark 3.15.* In the important case where $\omega$ is the tracial state on $M$ the density matrix of $\omega$ is a scalar, the map # reduces to the dual mapping $L^\# = L_*$, and $E_A$ is the identity map. In this case the criterion (ii) of Theorem 3.8 degenerates to $Q(\mathbf{1}) = Q_*(\mathbf{1})$. For example, if $Q$ has the form

$$Q(x) = \sum_{k=1}^{r} v_k x v_k^*,$$

where $v_1, v_2, \ldots, v_r \in M$, then condition (ii) becomes

$$\sum_{k=1}^{r} v_k v_k^* = \sum_{k=1}^{r} v_k^* v_k.$$

Moreover, when this condition is satisfied and $\omega$ is the tracial state no perturbation is necessary. One simply shows by a direct calculation that the mapping

$$L(x) = Q(x) - 1/2(Q(\mathbf{1})x + xQ(\mathbf{1}))$$

satisfies trace $\circ\, L = 0$ iff $Q(\mathbf{1}) = Q_*(\mathbf{1})$.

**Corollary 3.16.** *Let $\omega$ be a faithful state on $M$ with density matrix $\Omega$ and let $v_1, \ldots, v_r \in M$ satisfy*

$$\sum_{k=1}^{r} v_k v_k^* = \sum_{k=1}^{r} v_k^* v_k.$$

*Then there is a unital $\omega$-preserving CP semigroup whose generator has the form*

$$L(x) = \Omega^{-1/2}(\sum_{k=1}^{r} v_k x v_k^*)\Omega^{-1/2} + kx + xk^*$$

*for some operator $k \in M$.*

*Proof.* Let $Q$ be the completely positive map

$$Q(x) = \Omega^{-1/2}(\sum_{k=1}^{r} v_k x v_k^*)\Omega^{-1/2}.$$

By Theorem 3.8 it suffices to show that $Q^\#(\mathbf{1}) = Q(\mathbf{1})$. A direct computation shows that the dual of $Q$ is given by

$$Q_*(x) = \sum_{k-1}^{r} v_k^* \Omega^{-1/2} x \Omega^{-1/2} v_k$$

Hence

$$Q^\#(\mathbf{1}) = \Omega^{-1/2} Q_*(\Omega)\Omega^{-1/2} = \Omega^{-1/2}(\sum_{k=1}^{r} v_k^* v_k)\Omega^{-1/2}.$$

The right side is $Q(\mathbf{1})$ because of the hyposthesis on $v_1, \dots, v_r$. $\square$

*Remark 3.17. The necessity of perturbations.* In view of Remark 3.15 it is natural to ask if nontrivial perturbations are really necessary, and we conclude this section with some remarks concerning that issue. Suppose that $P$ is a normal completely positive map of $M$ and $L$ is the unperturbed generator

$$L(x) = P(x) - 1/2(P(\mathbf{1})x + xP(\mathbf{1})). \tag{3.18}$$

**Proposition 3.19.** *Let $\omega$ be a faithful state on $M = M_N(\mathbb{C})$ which is not a trace. Then there is an operator $L$ of the form (3.18) and a skew-adjoint operator $\ell \in M$ such that if $L'(x) = L(x) + [\ell, x]$ then $\omega \circ L \neq 0$ while $\omega \circ L' = 0$.*

*Proof.* Consider the spectral decomposition of the density matrix of $\omega$,

$$\Omega = \sum_{k=1}^{r} \lambda_k e_k.$$

We must have $r \geq 2$ because $\omega$ is not a trace. Choose a nonzero partial isometry $v$ satisfying $v^*v \leq e_1$ and $vv^* \leq e_2$. Since $\Omega$ is an invertible positive operator there is an $\epsilon > 0$ such that

$$\Omega' = \Omega + \epsilon(v + v^*)$$

is positive. Since the trace of $\Omega'$ is 1 we may consider the state $\omega'$ having density matrix $\Omega'$. Let $P$ be a normal completely positive map satisfying $P(\mathbf{1}) = \mathbf{1}$ and $\omega \circ P = \omega'$ (there are many such maps, the simplest one being $P(x) = \omega'(x)\mathbf{1}$), and define

$$L(x) = P(x) - x.$$

Then $\omega \circ L = \omega' - \omega \neq 0$.

On the other hand, since $P_*(\Omega) = \Omega'$ we have

$$P^\#(\mathbf{1}) = \Omega^{-1/2} P_*(\Omega)\Omega^{-1/2} = \Omega^{-1/2}\Omega'\Omega^{-1/2}.$$

Thus, letting

$$E_A(x) = \sum_{k=1}^{r} e_k x e_k$$

be the conditional expectation onto the centralizer algebra of $\omega$ and using $E_A(v) = E_A(v^*) = 0$, we have $E_A(\Omega') = E_A(\Omega)$. Hence

$$E_A(P^{\#}(\mathbf{1})) = \Omega^{-1/2} E_A(\Omega')\Omega^{-1/2} = \mathbf{1}.$$

From Theorem 3.8 we may conclude that there is a skew-adjoint operator $\ell$ such that the perturbation

$$L'(x) = L(x) + [\ell, x]$$

satisfies $\omega \circ L' = 0$ $\square$

## 4. Ergodicity and Purity

The purpose of this section is to give a concrete characterization of the generators of pure CP semigroups acting on matrix algebras, given that the CP semigroup has a faithful invariant state (Theorem 4.4).

**Definition 4.1.** *A unital CP semigroup $P = \{P_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ is called ergodic if the only operators $x$ satisfying $P_t(x) = x$ for every $t \geq 0$ are scalars.*

The set $\mathcal{A} = \{x \in \mathcal{B}(H) : P_t(x) = x, t \geq 0\}$ is obviously a weak$^*$-closed self-adjoint linear subspace of $\mathcal{B}(H)$ containing the identity. In general it need not be a von Neumann algebra, but as we will see presently, it is a von Neumann algebra in the cases of primary interest for our purposes here.

**Proposition 4.2.** *Every pure CP semigroup is ergodic.*

*Proof.* Suppose $P = \{P_t : t \geq 0\}$ is pure and $x$ is an operator satisfying $\|x\| \leq 1$ and $P_t(x) = x$ for every $t$. To show that $x$ must be a scalar multiple of $\mathbf{1}$ it suffices to show that for every normal linear functional $\rho$ on $\mathcal{B}(H)$ satisfying $\rho(\mathbf{1}) = 0$ we have $\rho(x) = 0$. Since any normal linear functional $\rho$ satisfying $\rho(\mathbf{1}) = 0$ can be decomposed into a sum of the form

$$\rho = b(\rho_1 - \rho_2) + ic(\rho_3 - \rho_4),$$

where $b$ and $c$ are real numbers and the $\rho_k$ are normal states, we conclude from the purity of $P$ that

$$\lim_{t\to\infty} \|\rho \circ P_t\| = 0.$$

Since $x$ is fixed under the action of $P$ we have

$$|\rho(x)| = |\rho(P_t(x))| \leq \|\rho \circ P_t\|$$

for every $t \geq 0$, from which $\rho(x) = 0$ follows. $\square$

**Proposition 4.3.** *Let $P = \{P_t : t \geq 0\}$ be a unital CP semigroup which leaves invariant some faithful normal state of $\mathcal{B}(H)$. Then*

$$\mathcal{A} = \{a \in \mathcal{B}(H) : P_t(a) = a, t \geq 0\}$$

*is a von Neumann algebra. Assuming further that $P$ has a bounded generator $L$ represented in the form*

$$L(x) = \sum_j v_j x v_j^* + kx + xk^* \tag{4.3.1}$$

*for operators $k, v_1, v_2, \ldots \in \mathcal{B}(H)$, then $\mathcal{A}$ is the commutant of the von Neumann algebra generated by $\{k, v_1, v_2, \ldots\}$.*

*Proof.* In view of the preceding remarks, the first paragraph will follow if we show that $\mathcal{A}$ is closed under operator multiplication. By polarization, it is enough to show that $a \in \mathcal{A} \Longrightarrow a^*a \in \mathcal{A}$. For each $a \in \mathcal{A}$ we have by the Schwarz inequality

$$a^*a = P_t(a)^*P_t(a) \leq P_t(a^*a)$$

for every $t \geq 0$. Letting $\omega$ be a faithful state invariant under $P$ we have $\omega(P_t(a^*a)-a^*a) = 0$, and hence $P_t(a^*a) = a^*a$. Thus $a^*a \in \mathcal{A}$.

Suppose now that $P$ has a bounded generator of the form (4.3.1), and let $\mathcal{B}$ be the $*$-algebra generated by $\{k, v_1, v_2, \dots\}$. Noting that $\mathcal{A} = \{x \in M : L(x) = 0\}$, we show that $\mathcal{A} = \mathcal{B}'$. If $x \in \mathcal{B}'$ then (4.3.1) becomes

$$L(x) = x(\sum_j v_j v_j^* + k + k^*) = xL(\mathbf{1}) = 0.$$

It follows that $\exp tL(x) = x$ for every $t$, hence $x \in \mathcal{A}$.

For the inclusion $\mathcal{A} \subseteq \mathcal{B}'$, we claim first that for every $a \in \mathcal{A}$,

$$[v_j, a] = v_j a - a v_j = 0, \qquad j = 1, 2, \dots.$$

Indeed, since $\mathbf{1}, a, a^*$, and $aa^*$ all belong to $\mathcal{A}$ and $L(\mathcal{A}) = \{0\}$, we have

$$L(aa^*) - aL(a^*) - L(a)a^* + aL(\mathbf{1})a^* = 0.$$

Substituting the formula (4.3.1) for $L$ in the above we find that the terms involving $k$ drop out and we are left with the formula

$$\sum_k [v_j, a][v_j, a]^* = -\sum_j [v_j, a][v_j^*, a^*] = 0.$$

It follows that $[v_j, a] = 0$ for every $k$. Replacing $a$ with $a^*$ we see that $a$ must commute with the self-adjoint set of operators $\{v_1, v_2, \dots, v_1^*, v_2^*, \dots\}$.

Now since $L(\mathbf{1}) = 0$, it follows from (4.3.1) that $\sum_j v_j v_j^* + k + k^* = 0$, and hence $k$ has Cartesian decomposition $k = -h + \ell$, where

$$h = 1/2 \sum_j v_j v_j^*$$

and $\ell$ is a skew-adjoint operator. Setting

$$L_0(x) = \sum_j v_j x v_j^* - hx - xh,$$

we have

$$L(x) = L_0(x) + [\ell, x],$$

and $L_0(\mathcal{A}) = \{0\}$ by what was just proved. Thus, for $a \in \mathcal{A}$,

$$[\ell, a] = L(a) = 0,$$

and hence $a$ must commute with $\ell$ as well. The inclusion $\mathcal{A} \subseteq \mathcal{B}'$ follows. $\square$

**Theorem 4.4.** *Let $P = \{P_t : t \geq 0\}$ be a unital CP semigroup acting on a matrix algebra $M = M_N(\mathbb{C})$, $N = 2, 3, \ldots$, which leaves invariant some faithful state $\omega$. Let*

$$L(x) = \sum_{j=1}^{r} v_j x v_j^* + kx + xk^*$$

*be the generator of $P$. Then the following are equivalent:*

*(i) $P$ is pure.*

*(ii) $P$ is ergodic.*

*(iii) The set of operators $\{k, k^*, v_1, \ldots, v_r, v_1^*, \ldots, v_r^*\}$ is irreducible.*

*Proof.* In view of Propositions 4.2 and 4.3, we need only prove the implication (ii)$\Longrightarrow$(i). Assuming that $P$ is ergodic, we consider its generator $L$ as an operator on the Hilbert space $L^2(M, \omega)$ with inner product

$$\langle x, y \rangle = \omega(y^* x), \qquad x, y \in M.$$

We have $L(\mathbf{1}) = 0$ because $P$ is unital, and $L^*(\mathbf{1}) = 0$ follows from the fact that $\omega \circ L = 0$, $L^*$ denoting the adjoint of $L \in \mathcal{B}(L^2(M, \omega))$. It follows that $\{\lambda\mathbf{1} : \lambda \in \mathbb{C}\}$ is a one-dimensional reducing subspace for $L$ and we can consider the restriction $L_0$ of $L$ to the subspace

$$H_0 = \{x \in L^2(M, \omega) : x \perp \mathbf{1}\} = \{x \in M : \omega(x) = 0\}.$$

We will show that

$$\lim_{t \to \infty} \| \exp tL_0 \| = 0, \tag{4.5}$$

$\| \cdot \|$ denoting the operator norm in $\mathcal{B}(H_0)$.

Notice that (4.5) implies that $P$ is pure with absorbing state $\omega$. Indeed, for any $x \in M$ we set $x_0 = x - \omega(x)\mathbf{1}$. Then $x_0 \in H_0$ and we may conclude from (4.5) that

$$\lim_{t \to \infty} P_t(x_0) = 0,$$

hence

$$\lim_{t \to \infty} P_t(x) = \omega(x)\mathbf{1},$$

and finally

$$\lim_{t \to \infty} \|\rho \circ P_t - \omega\| = 0$$

for every state $\rho$ of $M$ because $M$ is finite dimensional.

In order to prove (4.5), we note first that $\{\exp tL_0 : t \geq 0\}$ is a contraction semigroup acting on $H_0$. Indeed, $\exp tL$ is a contraction in $\mathcal{B}(L^2(M, \omega))$ for every $t$ by virtue of the inequality

$$\|P_t(x)\|^2_{L^2(M,\omega)} = \omega(P_t(x)^* P_t(x)) \leq \omega(P_t(x^* x)) = \omega(x^* x) = \|x\|^2_{L^2(M,\omega)},$$

and the restriction of $P_t$ to $H_0$ is $\exp tL_0$.

In particular, the spectrum of $L_0$ is contained in the left half plane

$$\sigma(L_0) \subseteq \{z \in \mathbb{C} : z + \bar{z} \leq 0\}.$$

We claim that $\sigma(L_0)$ contains no points on the imaginary axis $\{iy : y \in \mathbb{R}\}$. To see this, notice first that $0 \notin \sigma(L_0)$. Indeed, if $L(x) = L_0(x) = 0$ for $x \in H_0$ then $x$ must be a scalar multiple of $\mathbf{1}$ by ergodicity, and since $\omega(x) = 0$ we have $x = 0$.

Suppose now that $\alpha$ is a nonzero real number such that $i\alpha \in \sigma(L_0)$. Then there is an element $x \neq 0$ in $H_0$ for which $L(x) = i\alpha x$. Note first that $x$ is a scalar multiple of a unitary operator. Indeed, from the equation $L(x) = i\alpha x$ it follows that

$$P_t(x) = e^{i\alpha t}x \qquad \text{for every } t \geq 0,$$

hence

$$x^*x = P_t(x)^*P_t(x) \leq P_t(x^*x)$$

by the Schwarz inequality. Since $\omega(P_t(x^*x) - x^*x) = 0$ and $\omega$ is faithful we conclude that $P_t(x^*x) = x^*x$; so by ergodicity $x^*x$ must be a scalar multiple of $\mathbf{1}$. Thus $x$ must be proportional to an isometry in $M$.

We have located a unitary operator $u \in M$ such that $L_0(u) = i\alpha u$. Now we assert that $u$ must commute with the self-adjoint set of operators $\{v_1, \dots, v_r, v_1^*, \dots, v_r^*\}$. To see that we make use of the formula

$$L(xx^*) - xL(x)^* - L(x)x^* + xL(\mathbf{1})x^* = \sum_{j=1}^{r}[v_j, x][v_j, x]^* \tag{4.6}$$

(see the proof of Proposition 4.3). Setting $x = u$ we find that the left side of (4.6) is

$$-uL(u)^* - L(u)u^* = i\alpha\mathbf{1} - i\alpha\mathbf{1} = 0,$$

and hence

$$\sum_{j=1}^{r}[v_j, u][v_j, u]^* = 0,$$

from which we deduce that $[v_j, u] = 0$ for every $k$. Since $u$ is unitary the assertion follows.

Set

$$h = 1/2\sum_{j=1}^{r} v_j v_j^*.$$

Since $L(\mathbf{1}) = 0$ it follows that $k$ has Cartesian decomposition of the form $k = -h + \ell$, where $\ell^* = -\ell$, hence $L$ decomposes into a sum of the form

$$L(x) = L_0(x) + [\ell, x],$$

where

$$L_0(x) = \sum_{j=1}^{r} v_j x v_j^* - hx - xh.$$

By what we have just proved, $L_0(u) = uL_0(\mathbf{1}) = 0$. It follows that the equation $L(u) = i\alpha u$ reduces to

$$[\ell, u] = i\alpha u. \tag{4.7}$$

Now since $\ell$ is skew-adjoint, $v_s = e^{s\ell}$ defines a one-parameter group of unitary operators in $M$ and (4.7) implies that for every $s \in \mathbb{R}$ we have

$$v_s u v_s^* = e^{i\alpha s}u.$$

Since $x \mapsto v_s x v_s^*$ is a $*$-automorphism of $M$ for every $s \in \mathbb{R}$ it follows that the spectrum of $u$ must be invariant under all rotations of the unit circle of the form $\lambda \mapsto e^{i\alpha s}\lambda$, contradicting the fact that the spectrum of an $N \times N$ unitary matrix is a finite subset of $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$. This contradiction shows that $\sigma(L_0)$ cannot meet the imaginary axis.

We conclude that

$$\sigma(L_0) \subseteq \{z \in \mathbb{C} : z + \bar{z} < 0\},$$

and hence there is a positive number $\epsilon$ such that

$$\sigma(L_0) \subseteq \{z \in \mathbb{C} : z + \bar{z} < -2\epsilon\}. \tag{4.8}$$

Consider the operator $A = \exp L_0 \in \mathcal{B}(H_0)$. By the spectral mapping theorem the spectral radius of $A$ satisfies

$$\sup\{|e^z| : z \in \sigma(L_0)\} < e^{-\epsilon},$$

and hence there is a constant $c > 0$ such that

$$\|A^n\| \leq ce^{-n\epsilon}, \qquad n = 0, 1, 2, \ldots .$$

Letting $[t]$ denote the greatest integer not exceeding $t \geq 0$ we find that for every $t > 0$,

$$\| \exp tL_0\| \leq \| \exp [t]L_0\| = \|A^{[t]}\| \leq ce^{-[t]\epsilon},$$

and hence

$$\lim_{t\to\infty} \| \exp tL_0\| = 0,$$

as asserted. □

## 5. Applications

In [4], a numerical index $d_*(P)$ was introduced for arbitrary CP semigroups $P = \{P_t : t \geq 0\}$ acting on $\mathcal{B}(H)$. It was shown that for unital CP semigroups $P$, $d_*(P)$ is a nonnegative integer or $\infty = \aleph_0$, or $2^{\aleph_0}$, and in fact $d_*(P)$ agrees with the index of the minimal dilation of $P$ to an $E_0$-semigroup. In [5], $d_*(P)$ is calculated in all cases where the generator of $P$ is bounded, and in particular for CP semigroups acting on matrix algebras.

We will make use of this numerical index in the following result, from which we will deduce Theorem A.

**Theorem 5.1.** *Let $\omega$ be a faithful state of $M_r(\mathbb{C})$, $r \geq 2$, and let $n$ be a positive integer satisfying $n \leq r^2 - 1$. Then there is a pure CP semigroup $P = \{P_t : t \geq 0\}$ acting on $M_r(\mathbb{C})$ satisfying*

*(i)* $\omega \circ P_t = \omega$ *for every* $t \geq 0$, *and*

*(ii)* $d_*(P) = n$.

We have based the proof of Theorem 5.1 on the following result.

**Proposition 5.2.** *Suppose that $T$ is a non-scalar matrix in $M_r(\mathbb{C})$, $r \geq 2$, and let $\lambda = e^{2\pi i/r}$. Then there is a pair $u, v$ of unitary operators in $M_r(\mathbb{C})$ with the properties*

*5.2.1* $u^r = v^r = \mathbf{1}$,

*5.2.2* $vu = \lambda uv$,

*5.2.3* $\{T, u\}' = \mathbb{C} \cdot \mathbf{1}$.

*Proof of Proposition 5.2.* The assertion 5.2.3 is that the only operators commuting with both $u$ and $T$ are scalars. Let $H$ be an $r$-dimensional Hilbert space and identify $M_r(\mathbb{C})$ with $\mathcal{B}(H)$.

We claim first that there is an orthonormal basis $\xi_0, \xi_1, \dots, \xi_{r-1}$ for $H$ such that

$$\langle T\xi_0, \xi_k \rangle \neq 0, \qquad 1 \leq k \leq r-1. \tag{5.3}$$

Indeed, since $T$ is not a scalar there must be a unit vector $\xi_0 \in H$ which is not an eigenvector of $T$. Thus there is a complex number $a$ and a nonzero vector $\zeta$ orthogonal to $\xi_0$ such that

$$T\xi_0 = a\xi_0 + \zeta.$$

Let $c_1, c_2, \dots, c_{r-1}$ be any sequence of nonzero complex numbers satisfying

$$|c_1|^2 + |c_2|^2 + \cdots + |c_{r-1}|^2 = \|\zeta\|^2.$$

Since $\zeta \neq 0$ we can find an orthonormal basis $\xi_1, \xi_2, \dots, \xi_{r-1}$ for $[\xi_0]^\perp$ such that $\langle \zeta, \xi_k \rangle = c_k$ for $k = 1, 2, \dots, r-1$. For such a choice, the set $\{\xi_0, \xi_1, \dots, \xi_{r-1}\}$ is an orthonormal basis with the asserted property (5.3).

Now define $u, v \in \mathcal{B}(H)$ by

$$\begin{aligned} u\xi_k &= \lambda^{-k}\xi_k \qquad \text{and} \\ v\xi_k &= \xi_{k \dot{+} 1} \end{aligned}$$

for $0 \leq k \leq r-1$, where $\dot{+}$ denotes addition modulo $r$. It is obvious that $u$ and $v$ are unitary operators, and a straightforward computation shows that they satisfy formulas 5.2.1 and 5.2.2.

We claim now that if $B \in \mathcal{B}(H)$ satisfies $BT = TB$ and $Bu = uB$ then $B$ must be a scalar multiple of the identity. Indeed, from $Bu = uB$ and the fact that $u$ is a unitary operator with distinct eigenvalues, we find that each $\xi_k$ must be an eigenvector of both $B$ and $B^*$. Choosing $d_k \in \mathbb{C}$ such that $B\xi_k = d_k\xi_k$, then $B^*\xi_k = \bar{d}_k\xi_k$ and for each $k = 1, 2, \dots, r-1$ we have

$$d_0 \langle T\xi_0, \xi_k \rangle = \langle TB\xi_0, \xi_k \rangle = \langle BT\xi_0, \xi_k \rangle = \langle T\xi_0, B^*\xi_k \rangle = d_k \langle T\xi_0, \xi_k \rangle .$$

It follows that $(d_k - d_0) \langle T\xi_0, \xi_k \rangle = 0$ for $1 \leq k \leq r-1$. Because none of the inner products $\langle T\xi_0, \xi_k \rangle$ can be zero we conclude that $d_0 = d_1 = \cdots = d_{r-1}$. Thus $B = d_0 \cdot \mathbf{1}$, establishing Proposition 5.2. □

*Remarks* Let $\lambda$ be a primitive $r^{\text{th}}$ root of unity and let $u, v$ be two unitaries satisfying condition 5.2.1 and 5.2.2. Consider the family of $r^2$ unitary operators $\{w_{i,j} : 0 \leq i, j \leq r-1\}$ defined by

$$w_{i,j} = u^i v^j.$$

We may consider that the indices $i, j$ range over the abelian group $\mathbb{Z}/r\mathbb{Z}$, and with that convention the $w_{i,j}$ are seen to satisfy the commutation relations for this group

$$w_{i,j}w_{p,q} = \lambda^{jp} w_{i+p,j+q}, \tag{5.4}$$

$$w_{i,j}^* = \lambda^{ij} w_{-i,-j}, \tag{5.5}$$

where the operations $i + p$, $j + q$, $-i$, $-j$ are performed modulo $r$. Of course, we have $w_{0,0} = \mathbf{1}$. It follows from (5.4) and (5.5) that the set of operators $\{w_{i,j}\}$ satisfies

$$w_{i,j}w_{p,q}w_{i,j}^* = \lambda^{jp-qi} w_{p,q}.$$

This formula, together with the fact that $\lambda$ is a primitive $r^{\text{th}}$ root of unity, implies that

$$\operatorname{trace}(w_{p,q}) = 0, \qquad \text{for } 0 \leq p, q \leq r-1, \quad p+q > 0. \tag{5.6}$$

In particular, from (5.4)–(5.6) we see that relative to the inner product on $M_r(\mathbb{C})$ defined by the normalized trace, the set of operators $\{w_{i,j} : 0 \leq i, j \leq r-1\}$ is an orthonormal basis. Thus the $\{w_{i,j} : 0 \leq i, j \leq r-1\}$ are linearly independent.

*Proof of Theorem 5.1.* Assume first that $\omega$ is not the tracial state, and let $\Omega$ be its density matrix. Then $\Omega$ is not a scalar multiple of the identity and Proposition 5.2 provides a pair of unitary operators $u, v$ satisfying (5.2.1), (5.2.2) and (5.2.3) for $T = \Omega$. Define $w_{i,j} = u^i v^j$, $0 \leq i, j \leq r-1$. By the preceding remarks the set of $r^2 - 1$ unitary operators $\mathcal{S} = \{w_{i,j} : 0 \leq i, j \leq r-1, i+j > 0\}$ is linearly independent and consists of trace zero operators.

Choose $n$ satisfying $1 \leq n \leq r^2 - 1$ and let $v_1, v_2, \dots, v_n$ be any set of $n$ distinct elements of $\mathcal{S}$ such that $v_1 = w_{1,0} = u$. By (5.2.3) we have

$$\{\Omega, v_1\}' = \mathbb{C}\mathbf{1},$$

and hence

$$\{\Omega, v_1, v_2, \dots, v_n\}' = \mathbb{C}\mathbf{1}. \tag{5.7}$$

Consider the completely positive map of $M_r(\mathbb{C})$ defined by

$$Q(x) = \Omega^{-1/2}\Big(\sum_{k=1}^{n} v_k x v_k^*\Big)\Omega^{-1/2}.$$

Since the $v_k$ are unitary operators we have

$$\sum_{k=1}^{r} v_k v_k^* = \sum_{k=1}^{r} v_k^* v_k,$$

hence Corollary 3.16 implies that there is an operator $k \in M_r(\mathbb{C})$ such that

$$L(x) = Q(x) + kx + xk^*$$

generates a unital CP semigroup $P = \{P_t : t \geq 0\}$ satisfying $\omega \circ P_t = \omega$ for every $t \geq 0$. Because of (5.7), Theorem 4.4 implies that $P$ is a pure semigroup.

It remains to show that $d_*(P) = n$, and for that we appeal to the results of [5]. Consider the linear span

$$\mathcal{E} = \operatorname{span}\{\Omega^{-1/2}v_1, \Omega^{-1/2}v_2, \dots, \Omega^{-1/2}v_n\}.$$

We claim first that $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \{0\}$. Indeed, if this intersection were not trivial then we would have

$$\mathbf{1} = c_1\Omega^{-1/2}v_1 + \cdots + c_n\Omega^{-1/2}v_n$$

for some scalars $c_1, \dots, c_n$. Hence

$$\Omega^{1/2} = c_1 v_1 + \cdots + c_n v_n.$$

This is impossible because the left side has positive trace, while by (5.6) the right side has trace zero.

We can make $\mathcal{E}$ into a metric operator space [4, Definition 1.9] by declaring the linear basis $\Omega^{-1/2}v_1, \dots, \Omega^{-1/2}v_n$ to be an orthonormal basis, and once this is done we find that $\mathcal{E}$ is the metric operator space associated with the completely positive map $Q$. From [5, Theorem 2.3] we have $d_*(P) = \dim \mathcal{E} = n$, as required.

It remains to deal with the case where $\omega$ is the normalized trace on $M_r(\mathbb{C})$. That requires a small variation of the preceding argument. Choose an arbitrary operator $T \in M_r(\mathbb{C})$ so that $T$ is not a scalar and satisfies $T^* = -T$. Let $\lambda$ be a primitive $r^{\text{th}}$ root of unity and let $u$, $v$ be two unitary operators satisfying the three conditions of Proposition 5.2. Now we form the operators $w_{i,j}$ exactly as before, and obtain $n$ unitary operators $\{v_1, v_2, \dots, v_n\}$ by enumerating the elements of $\{w_{i,j} : 0 \leq i, j \leq r-1, i+j > 0\}$ in such a way that $v_1 = u$. Define an operator $L$ on $M_r(\mathbb{C})$ by

$$L(x) = \sum_{k=1}^{n} v_k x v_k^* - nx + [T, x].$$

Notice that $L(\mathbf{1}) = 0$ and, since we obviously have $\sum_k v_k v_k^* = \sum_k v_k^* v_k$, it follows that $\operatorname{trace}(L(x)) = 0$ for all $x \in M_r(\mathbb{C})$. Hence $L$ is the generator of a unital CP semigroup $P = \{P_t : t \geq 0\}$ which preserves the tracial state $\omega$.

Notice that $P$ is pure. Indeed, by (5.2.3) we have $\{v_1, T\}' = \mathbb{C}\mathbf{1}$, and hence the $*$-algebra generated by the set $\{v_1, \dots, v_n, T\}$ is irreducible. Theorem 4.4 implies that $P$ is a pure CP semigroup.

Finally, $d_*(P) = n$ follows exactly as in the non-tracial case already established. $\square$

We are now in position to prove Theorem A, as stated in the introduction. Let $r$ and $n$ be positive numbers with $r \geq 2$, and let $\lambda_1, \lambda_2, \dots, \lambda_r$ be a sequence of positive numbers summing to 1. We have to show that there is a cocycle perturbation of the $CAR/CCR$ flow of index $n$ which has an absorbing state with eigenvalue list $\lambda_1, \lambda_2, \dots, \lambda_r$.

We first consider the case in which $n \leq r^2 - 1$. Let $H_0$ be a Hilbert space of dimension $r$, and identify $M_r(\mathbb{C})$ with $\mathcal{B}(H_0)$. Choose an orthonormal basis $\xi_1, \xi_2, \dots, \xi_r$ for $H_0$ and let $\omega_0$ be the state of $\mathcal{B}(H_0)$ defined by

$$\omega_0(x) = \sum_{k=1}^{r} \lambda_k \langle x\xi_k, \xi_k \rangle .$$

Then $\omega_0$ is a faithful state on $\mathcal{B}(H_0)$ having eigenvalue list $\lambda_1, \lambda_2, \dots, \lambda_r$. By Theorem 5.1, there is a pure CP semigroup $P = \{P_t : t \geq 0\}$ acting on $\mathcal{B}(H_0)$ such that $\omega_0 \circ P_t = \omega_0$ for every $t \geq 0$. Using Bhat's dilation theorem [7, 8], there is a Hilbert space $H \supseteq H_0$ and an $E_0$-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ such that if we identify $\mathcal{B}(H_0)$ with the corner $p_0\mathcal{B}(H)p_0$ ($p_0$ denoting the projection of $H$ onto $H_0$), then we have $\alpha_t(p_0) \geq p_0$ for every $t \geq 0$ and for every $x \in \mathcal{B}(H_0)$,

$$P_t(x) = p_0\alpha_t(x)p_0, \qquad t \geq 0.$$

Using [2], we may assume that $\alpha$ is *minimal* over the projection $p_0$.

Now by Proposition 2.4, $\alpha$ is a pure $E_0$-semigroup. Moreover, if we define a normal state $\omega$ of $\mathcal{B}(H)$ by

$$\omega(x) = \omega_0(p_0 x p_0),$$

then $\omega$ must be invariant under $\alpha$. Indeed, since $\alpha_t(p_0) \geq p_0$ we have for every $x \in \mathcal{B}(H)$

$$p_0\alpha_t(x)p_0 = p_0\alpha_t(p_0 x p_0)p_0 = P_t(p_0 x p_0),$$

hence

$$\omega(\alpha_t(x)) = \omega_0(P_t(p_0 x p_0) = \omega_0(p_0 x p_0) = \omega(x),$$

as asserted. By the general discussion of Sect. 1 it follows that $\omega$ is an absorbing state, and of course the eigenvalue list of $\omega$ is the same as that for $\omega_0$, namely $\lambda_1, \lambda_2, \ldots, \lambda_r$. Thus it only remains to show that $\alpha$ is conjugate to a cocycle perturbation of the $CAR/CCR$ flow of index $n$. But by Corollary 4.21 of [5], $\alpha$ is cocycle conjugate to a $CAR/CCR$ flow of index $d_*(P) = n$, and the proof of this case is complete.

Suppose now that $n > r^2 - 1$. In this case, pick any positive integer $k \leq r^2 - 1$. By what was just proved, we can find a cocycle perturbation $\alpha$ of the $CAR/CCR$ flow of index $k$ which has an absorbing state $\omega$ having eigenvalue list $\lambda_1, \lambda_2, \ldots, \lambda_r$. Moreover, letting $p_0$ be the support projection of $\omega$ then $p_0$ has rank $r$ and if $P$ is the CP semigroup obtained by compressing $\alpha$ to $p_0\mathcal{B}(H)p_0$, then $P$ is a pure CP semigroup and $\alpha$ can be assumed to be the minimal dilation of $P$.

We will show how to use $\alpha$ to construct a *nonminimal* dilation $\beta$ of $P$ which is pure, conjugate to a cocycle perturbation of the $CAR/CCR$ flow of index $n$, and has an absorbing state with the same eigenvalue list. For that, let $m = n - k$ and let $\alpha^m$ be the $CAR/CCR$ flow of index $m$, acting on $\mathcal{B}(K)$. It is known that every $CAR/CCR$ flow has a pure absorbing state $\rho$ (the vacuum state) [13]. Thus letting $\zeta \in K$ be the vacuum vector then we have

$$\rho(x) = \langle x\zeta, \zeta \rangle \, .$$

If we write $[\zeta]$ for the rank-one projection defined by $\zeta$ then $\alpha_t^m([\zeta]) \geq [\zeta]$ for every $t \geq 0$ and in fact

$$\lim_{t\to\infty} \alpha_t^m([\zeta]) = \mathbf{1}_K. \tag{5.8}$$

Let $\beta$ be the $E_0$-semigroup defined on $\mathcal{B}(H \otimes K)$ by $\beta = \alpha \otimes \alpha^m$, i.e.,

$$\beta_t(x \otimes y) = \alpha_t(x) \otimes \alpha_t^m(y), \qquad x \in \mathcal{B}(H), y \in \mathcal{B}(K), t \geq 0.$$

$\beta$ is obviously a cocycle perturbation of the $CAR/CCR$ flow of index $n = k + m$. We will show that $\beta$ is a pure $E_0$-semigroup having an invariant state with eigenvalue list $\lambda_1, \lambda_2, \ldots, \lambda_r$.

To that end, consider the normal state $\omega'$ defined on $\mathcal{B}(H \otimes K)$ by

$$\omega' = \omega \otimes \rho.$$

Since $\rho$ is a vector state, $\omega'$ has the same eigenvalue list as $\omega$, namely $\lambda_1, \lambda_2, \ldots, \lambda_r$. Moreover, $\omega'$ is invariant under $\beta$ because $\omega$ (resp. $\rho$) is invariant under $\alpha$ (resp. $\alpha^m$). Thus it remains to show that $\beta$ is a pure $E_0$-semigroup.

For that, we appeal to Proposition 2.4 as follows. Let $q_0 = p_0 \otimes [v]$ be the support projection of $\omega'$. Then we have

$$\beta_t(q_0) = \alpha_t(p_0) \otimes \alpha_t^m([v]).$$

Since the projections $\alpha_t(p_0)$ (resp. $\alpha_t^m([v])$) increase with $t$ to $\mathbf{1}_H$ (resp. $\mathbf{1}_K$), it follows that $\beta_t(q_0) \geq q_0$ and

$$\lim_{t\to\infty} \beta_t(q_0) = \mathbf{1}_{H\otimes K}.$$

Thus if we let $Q = \{Q_t : t \geq 0\}$ be the CP semigroup obtained by compressing $\beta$ to the corner $q_0\mathcal{B}(H \otimes K)q_0$, it follows that $\beta$ is a (nonminimal) dilation of $Q$. Finally, since $[v]$ is one-dimensional, $Q$ is conjugate to the original CP semigroup $P$, and is therefore pure. By Proposition 2.4, we conclude that $\beta$ is a pure $E_0$-semigroup.

We have established all but the third paragraph of Theorem A, to which we now turn our attention. Let $r \geq 2$ be an integer and let $\beta$ be an $E_0$-semigroup acting on $\mathcal{B}(H)$, $H$ being a separable infinite dimensional Hilbert space, which has an absorbing state with eigenvalue list $\lambda_1, \lambda_2, \dots, \lambda_r$. Assuming that $\beta$ is minimal over the support projection $p_0$ of $\omega$, we have to show that $\beta$ is cocycle conjugate to a $CAR/CCR$ flow of index $n$, where $n$ is a positive integer not exceeding $r^2 - 1$.

Let $H_0 = p_0 H$ and let $P = \{P_t : t \geq 0\}$ be the CP semigroup obtained by compressing $\beta$ to the corner $p_0\mathcal{B}(H)p_0 \cong \mathcal{B}(H_0)$. Let $L$ be the generator of the semigroup $P$. By [4, 5] there is an operator $k \in \mathcal{B}(H_0)$ and a metric operator space $\mathcal{E} \subseteq \mathcal{B}(H_0)$ (possibly $\{0\}$) satisfying $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \{0\}$ and which give rise to $L$ as follows:

$$L(x) = \sum_{k=1}^{n} v_k x v_k^* + kx + xk^*, \qquad x \in \mathcal{B}(H_0), \tag{5.9}$$

$v_1, v_2, \dots, v_n$ denoting any orthonormal basis for $\mathcal{E}$. Since $\mathcal{E}$ is a proper subspace of the $r^2$-dimensional vector space $\mathcal{B}(H_0)$, the integer $n = \dim \mathcal{E}$ has possible values $0, 1, \dots, r^2 - 1$.

Note first that $n$ cannot be 0. For in that case (5.9) reduces to $L(x) = kx + xk^*$. Using the fact that $L(\mathbf{1}) = 0$, we find that $k$ must be a skew-adjoint operator for which $L(x) = [k, x]$, hence

$$P_t(x) = \exp tL(x) = e^{tk} x e^{-tk}$$

is a semigroup of $*$-automorphisms of $\mathcal{B}(H_0)$. Since $\beta$ is a minimal dilation of $P$ we must have $H = H_0$ and $\beta_t = P_t$ for every $t \geq 0$, contradicting the fact that $\beta$ is an $E_0$-semigroup acting on an infinite dimensional type $I$ factor.

Thus $1 \leq n \leq r^2 - 1$. Theorem 2.3 of [5] implies that the index of $P$ is given by $d_*(P) = \dim \mathcal{E} = n$, and by [4] Theorem 4.9 we have $d_*(\beta) = d_*(P) = n$. $\beta$ must be completely spatial by [5] Theorem 4.8, and finally by the classification results of [1] (Corollary of Proposition 7.2) every completely spatial $E_0$-semigroup is conjugate to a cocycle perturbation of a $CAR/CCR$ flow. That completes the proof of Theorem A.

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