Proceedings of Symposia in Pure Mathematics
Volume **51** (1990), Part 1

The Spectral C^*-Algebra of an E_0-Semigroup

WILLIAM ARVESON

Introduction. An *E_0-semigroup* is a one-parameter semigroup $\{\alpha_t : t \geq 0\}$ of normal *-endomorphisms of the algebra $\mathscr{B}(H)$ of all bounded operators on a separable Hilbert space H, satisfying $\alpha_t(1) = 1$ for $t \geq 0$, and such that $\langle \alpha_t(A)\xi, \eta \rangle$ is continuous in t for fixed $A \in \mathscr{B}(H)$ and ξ, $\eta \in H$. E_0-semigroups were introduced by Powers [**10**], and their theory has been undergoing development by Powers [**11**], Powers and Robinson [**13**], Powers and Price [**12**], Price [**16**], and the author [**1, 2, 4, 5**].

E_0-semigroups occur naturally in a number of ways. For example, if we are given a one-parameter unitary group $\{U_t : t \in \mathbf{R}\}$ acting on a separable Hilbert space H, and a type-I subfactor M of $\mathscr{B}(H)$ which is *invariant* in the sense that $U_t M U_t^*$ is contained in M for every $t \geq 0$, then we obtain *two* semigroups α, β acting respectively on M and its commutant M' by

$$\alpha_t(A) = U_t A U_t^*, \qquad A \in M,\ t \geq 0,$$
$$\beta_t(B) = U_t^* B U_t, \qquad B \in M',\ t \geq 0.$$

If we realize the type-I factors M, M' as $\mathscr{B}(K)$, $\mathscr{B}(K')$, respectively, then of course α and β are seen to be E_0-semigroups. More specifically, if one is given a system of local observables which is acted upon by the inhomogeneous Lorentz group in such a way that the Haag-Kastler axioms are satisfied ([**9**, p. 99]), then it is a simple matter to write down nontrivial examples of pairs $\{U_t\}$, M satisfying the above conditions. Some more elementary constructions of E_0-semigroups are described in [**3**] and [**10**].

It is correct to think of E_0-semigroups as quantized versions of semigroups of isometries [**3**], but one must not push this analogy too far. For example, by the Wold decomposition, every semigroup of isometries $\{U_t : t \geq 0\}$ acting on a Hilbert space decomposes uniquely into a direct sum

$$U_t = V_t \oplus W_t,$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L40; Secondary 81E05.
Research supported in part by NSF Grant DMS-86-00375.
This paper is in final form and will not be submitted for publication elsewhere.

where W is a semigroup of unitary operators and V is the direct sum of a number n of copies of the natural semigroup of shifts acting on $L^2(0,\infty)$. The pair (n, W) completely determines U to unitary equivalence in the sense that if

$$U'_t = V'_t \oplus W'_t$$

is another semigroup of isometries decomposed in the above way, then U is unitarily equivalent to U' iff $n = n'$ and W is unitarily equivalent to W'.

This might lead one to suspect that the problem of classifying E_0-semigroups up to conjugacy should be similar in that (a) a given E_0-semigroup should decompose into a tensor product of a "pure" E_0-semigroup with a semigroup of *-automorphisms, and (b) the pure E_0-semigroups should have a relatively simple classification to within conjugacy (an E_0-semigroup α is said to be *pure* if the family of von Neumann algebras $M_t = \alpha_t(\mathscr{B}(H))$ decreases to $\mathbf{C}1$ as t tends to infinity). It is interesting that these naive assertions are both entirely wrong. It is known, for example, that there exist E_0-semigroups α having the property that $\bigcap\{\alpha_t(\mathscr{B}(H)) \colon t \geq 0\}$ is a factor of type II or III. This implies that nothing like (a) can be true. More significantly, Powers has shown that, in addition to the natural examples of pure E_0-semigroups, there is an enormous variety of others which are not conjugate (or even cocycle-conjugate) to these elementary ones **[11]**.

These remarks show that, unlike the theory of one-parameter groups of automorphisms of $\mathscr{B}(H)$, the theory of semigroups of endomorphisms of $\mathscr{B}(H)$ is rather subtle. In **[3]**, we summarized the results of some recent work on an index theory appropriate for E_0-semigroups. Those results relate to the problem of classifying E_0-semigroups up to cocycle-conjugacy. The purpose of this paper is to summarize progress on *spectral* invariants of E_0-semigroups, with emphasis on problems that remain open. Here, the appropriate notion of spectrum is a noncommutative topological space, namely a separable C^*-algebra which has the "correct" representation theory.

1. Preliminaries. Our approach to the theory of E_0-semigroups is based on the notion of continuous tensor product systems (product systems, for short). A *product system* is a measurable family

$$p \colon E \to (0,\infty)$$

of nonzero separable Hilbert spaces over the open interval $(0,\infty)$, on which there is defined an associative multiplication which *acts like tensoring*. This means that the operation is measurable and bilinear on fiber spaces, and such that for each s, $t > 0$, the fiber space $E_{s+t} = p^{p-1}(s+t)$ is spanned by all products $\{xy \colon x \in E_s, y \in E_t\}$, and we have

$$\langle xy, x'y' \rangle = \langle x, x' \rangle \langle y, y' \rangle,$$

for all x, x' in E_s and all y, y' in E_t. We will write $\{E_t \colon t > 0\}$, or simply E, for a product system $p \colon E \to (0,\infty)$.

Every E_0-semigroup is associated with a canonical product system. Indeed, let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $\mathscr{B}(H)$. For every $t > 0$, let E_t be the complex vector space of operators defined by

$$E_t = \{A \in \mathscr{B}(H)\colon \alpha_t(T)A = AT, T \in \mathscr{B}(H)\}.$$

We may define an inner product in E_t by the formula

$$B^*A = \langle A, B\rangle 1, \qquad A, B \in E_t,$$

and this defines a family of Hilbert spaces. The total space of this family is the set of ordered pairs $E = \{(t, A)\colon A \in E_t, t > 0\}$, and the projection

$$p\colon E \to (0, \infty)$$

is defined by $p(t, A) = t$. If we use operator multiplication to define a binary operation in E in the natural way by

$$(s, A)(t, B) = (s + t, AB), \qquad A \in E_s,\ B \in E_t,$$

then E becomes a product system (cf. [**1**] or [**3**] for more detail).

Notice that for this *particular* product system, we can define a natural operator-valued mapping $\phi\colon E \to \mathscr{B}(H)$, namely

$$\phi((t, A)) = A, \qquad (t, A) \in E.$$

The map ϕ is an *essential representation* of E, as defined below in (2.1). Thus, an E_0-semigroup acting on $\mathscr{B}(H)$ gives rise to a pair (E, ϕ) consisting of a product system E and an essential representation $\phi\colon E \to \mathscr{B}(H)$. Conversely, if one is given a product system E and an essential representation of E on a separable Hilbert space, then it is not hard to write down an E_0-semigroup which is associated to the pair (E, ϕ) as above (see [**1**, Proposition 2.7]).

In this way, one can obtain information about the general problem of classifying E_0-semigroups by analyzing the structure of their associated product systems, and by seeking to understand the representation theory of product systems.

2. The spectral C^*-algebra. Let E be a product system. By a *representation* of E we mean a weakly measurable operator-valued function $\phi\colon E \to \mathscr{B}(H)$ having the following properties:

$$\text{(2.1)} \qquad \begin{array}{ll} \text{(i)} & \phi(xy) = \phi(x)\phi(y), \quad \text{for all } x, y \text{ in } E, \\ \text{(ii)} & \phi(y)^*\phi(x) = \langle x, y\rangle 1, \quad \text{for all } x, y \text{ in } E_t \text{ and every } t > 0. \end{array}$$

Antirepresentations are defined similarly, except that the order of the factors on the right of (2.1) part (i) is reversed. (2.1) part (ii) implies that the restriction of ϕ to every fiber E_t, $t > 0$, is a linear isometry from the Hilbert space E_t to $\mathscr{B}(H)$ ([**1**, Section 1]). For every $t > 0$, we have a subspace H_t of H defined by

$$\text{(2.2)} \qquad H_t = [\phi(x)\xi\colon x \in E_t,\ \xi \in H].$$

These subspaces are decreasing in t, and their union is dense in H ([**1**, Corollary of Proposition 2.4]). ϕ is called *singular* or *essential* according as $\bigcap\{H_t : t > 0\} = \{0\}$, or $H_t = H$ for every $t > 0$. This terminology differs slightly from previous usage (in [**1**] and [**3**], essential representations were called *nonsingular*), and is somewhat more convenient.

We now introduce a C^*-algebra $C^*(E)$ [**4**], which plays the role of the spectrum of the product system E in the sense that the (separable) representations of E correspond precisely to the (separable) *-representations of $C^*(E)$. Let $L^2(E)$ be the Hilbert space of all square-integrable sections of E. The inner product in $L^2(E)$ is given by

$$\langle f, g\rangle = \int_0^\infty \langle f(t), g(t)\rangle \, dt.$$

We have a direct integral decomposition of $L^2(E)$ over the measure space $((0, \infty)\, dt)$,

$$L^2(E) = \int^\oplus E_t \, dt, \tag{2.3}$$

which shows that $L^2(E)$ is a continuous analogue of the full Fock space over an infinite dimensional one-particle space [**1**]. In particular, for every v in E, we can define left and right creation operators $l(v)$, $r(v)$ on $L^2(E)$ by

$$l(v)\xi(x) = \begin{cases} v\xi(x-t), & \text{if } x > t, \\ 0, & \text{if } 0 < x \le t, \end{cases}$$

$$r(v)\xi(x) = \begin{cases} \xi(x-t)v, & \text{if } x > t, \\ 0, & \text{if } 0 < x \le t \end{cases}$$

for ξ in $L^2(E)$. l is a singular representation of E, and r is a singular antirepresentation. The two sets of operators $l(E)$ and $r(E)$ mutually commute (though of course neither $l(E)$ nor $r(E)$ is a commutative set of operators), but $l(E)^*$ does not commute with $r(E)$. Indeed, both $l(E) \bigcup l(E)^*$ and $r(E) \bigcup r(E)^*$ are irreducible sets of operators ([**4**, Theorem 5.2]). l (resp. r) is called the *regular representation* (resp. *regular antirepresentation*) of E.

More generally, if $\phi : E \to \mathscr{B}(H)$ is an arbitrary representation or antirepresentation of E, and f belongs to the Banach space $L^1(E)$ of all integrable sections of E, then the weak integral

$$\int_0^\infty \phi(f(t))\, dt$$

defines a bounded operator on H. This integral defines a linear mapping of $L^1(E)$ into $\mathscr{B}(H)$ of norm of, at most, one, which we will denote by the same letter ϕ. One may verify that for f, g in $L^1(E)$, we have

$$\phi(f)\phi(g) = \phi(f * g),$$

where $f * g$ denotes the *convolution* of f and g,

$$f * g(x) = \int_0^x f(t) g(x-t)\, dt, \qquad x > 0. \tag{2.4}$$

The multiplication defined on $L^1(E)$ by (2.4) makes $L^1(E)$ into a Banach algebra, and ϕ: $L^1(E) \to \mathscr{B}(H)$ is a contractive homomorphism of $L^1(E)$ onto a nonselfadjoint algebra of operators, whose norm-closure in $\mathscr{B}(H)$ is a separable Banach algebra. If one starts with an antirepresentation ϕ: $E \to \mathscr{B}(H)$, then this integration process obviously produces a contractive antihomomorphism of Banach algebras ϕ: $L^1(E) \to \mathscr{B}(H)$.

Applying this to the left and right regular representations, we see that for f in $L^1(E)$, $l(f)$, and $r(f)$ are respectively left and right convolution by f:

$$(l(f)\xi)(x) = \int_0^x f(t)\xi(x-t)\,dt,$$
$$(r(f)\xi)(x) = \int_0^x \xi(x-t)f(t)\,dt,$$

for every ξ in $L^2(E)$. Moreover, using (2.1) part (ii), it is rather easy to show that for every pair f, g of functions in $L^1(E)$, there are functions h_1, h_2 in $L^1(E)$ such that

$$l(g)^*l(f) = l(h_1) + l(h_2)^*;$$

indeed, a straightforward computation allows one to write down explicit formulas for h_1 and h_2 in terms of f and g. It follows that the norm-closed linear span

$$C^*(E) = \text{span}\{l(f)l(g)^* : f, g \in L^1(E)\} \tag{2.5}$$

is a separable C^*-algebra.

DEFINITION 2.6. The C^*-algebra defined by (2.5) is called the spectral C^*-algebra of E.

The reader may note that Definition 2.6 is simpler and considerably more concrete than the definition of $C^*(E)$ given in [**4**]. On the other hand, it is not clear at all that the C^*-algebra defined by 2.5 has the correct representation theory. The fact that it does follows from the results of [**4**] and [**5**]. For the reader's convenience, we indicate how the proof of this basic universal property can be dug out of those two references.

THEOREM 2.7. *Let E be a product system. For every representation ϕ of E on a separable Hilbert space H, there is a unique $*$-representation π of $C^*(E)$ on H satisfying*

$$\pi(l(f)l(g)^*) = \phi(f)\phi(g)^*, \tag{2.8}$$

for every f, g in $L^1(E)$. π is necessarily nondegenerate, and $\phi(E)$ and $\pi(C^*(E))$ generate the same von Neumann algebra. Conversely, every nondegenerate representation π of $C^*(E)$ on a separable Hilbert space has the form (2.8) *for a unique representation ϕ of E.*

PROOF. Let $\mathscr{A}$ denote the C^*-algebra defined in ([**4**, Definition 2.12]); and for every f, g in $L^1(E)$, let $f \otimes \bar{g}$ be the element of $\mathscr{A}$ defined in the discussion preceding ([**4**, Proposition 2.13]). Theorem 2.16 of [**4**] asserts that for every separable representation ϕ of E, there is a unique *-representation π of $\mathscr{A}$ which satisfies the analogue of (2.8):

$$\pi(f \otimes \bar{g}) = \phi(f)\phi(g)^*, \qquad f, g \in L^1(E). \tag{2.9}$$

Conversely, every separable nondegenerate representation π of $\mathscr{A}$ is related to a unique representation ϕ of E by the formula (2.9) ([**4**, Corollary 2 of Theorem 3.4]). Thus, the desired universal property holds for $\mathscr{A}$.

Applying this universal property to the left regular representation $l: E \to \mathscr{B}(L^1(E))$, we obtain a *-representation λ of $\mathscr{A}$ on $L^2(E)$ such that

$$\lambda(f \otimes \bar{g}) = l(f)l(g)^*, \qquad f, g \in L^1(E).$$

By ([**5**, Corollary 3 of Theorem 3.1]), λ is a faithful representation of $\mathscr{A}$. It follows that $C^*(E)$ inherits the required universal property of $\mathscr{A}$ through λ. $\square$

Evans has shown that the Cuntz C^*-algebra O_∞ is isomorphic to the C^*-algebra generated by all left creation operators acting on the full Fock space $\mathscr{F}(H)$ over an infinite dimensional (separable) one-particle space H [**8**]. Thus, the spectral C^*-algebras $C^*(E)$ are properly thought of as continuous analogues of O_∞. They are unitless, separable, nuclear C^*-algebras which are, in most cases, simple ([**4**, Theorem 4.1, and Corollary 2 of Theorem 8.2]). Our proof of simplicity does not work for product systems which contain no "units"; nevertheless, we conjecture that, in general, *$C^*(E)$ is simple for every nontrivial product system E.*

Very little is known about the classification of these spectral C^*-algebras. In more detail, let $\theta: E \to F$ be an isomorphism of product systems. This means that θ is a measurable bijection which preserves multiplication and restricts to a unitary operator on each fiber space E_t, $t > 0$. The set $\operatorname{aut}(E)$ of all automorphisms of a given product system E is obviously a group. It is possible to compute $\operatorname{aut}(E)$ very explicitly for the simplest product systems E, and in those cases there is a natural topology on $\operatorname{aut}(E)$, making it into a Polish group which is often *locally compact* ([**1**, Theorem 8.8]). This group involves the canonical commutation relations in an essential way. The structure of $\operatorname{aut}(E)$ for general product systems E is unknown.

Due to the functorial nature of the construction of $C^*(E)$, every isomorphism $\theta: E \to F$ of product systems induces an isomorphism of C^*-algebras $\hat{\theta}: C^*(E) \to C^*(F)$. More explicitly, θ induces a unitary operator U_θ from $L^2(E)$ to $L^2(F)$ by way of

$$U_\theta \xi(t) = \theta(\xi(t)), \qquad \xi \in L^2(E), \quad t > 0.$$

$\hat{\theta}$ is the corresponding spatial isomorphism of $\mathscr{B}(L^2(E))$ to $\mathscr{B}(L^2(F))$,

$$\hat{\theta}(A) = U_\theta A U_\theta^*, \qquad A \in \mathscr{B}(L^2(E)).$$

Notice that $\hat{\theta}$ carries a generator $l(f_1)l(f_2)^*$ of $C^*(E)$ to the generator of $C^*(F)$ given by $l(\tilde{f}_1)l(\tilde{f}_2)^*$ where, for f in $L^1(E)$, $\tilde{f}$ is the element of $L^1(F)$

given by

$$\tilde{f}(t) = \theta(f(t)), \qquad t > 0. \tag{2.10}$$

Isomorphisms of $C^*(E)$ onto $C^*(F)$ having this form $\hat{\theta}$ for some isomorphism of product systems $\theta: E \to F$ are called *quasifree isomorphisms*.

As we have said, little is known about the classification theory of these C^*-algebras. For example, there is a particularly simple sequence of "standard" product systems $E_1, E_2, \ldots, E_\infty$ (cf. [**1**]), which are obtained by quantizing semigroups of unilateral shifts of various multiplicities. These are the product systems associated with CAR flows [**10**] and CCR flows [**3**]. We conjecture that $C^*(E_m)$ *is not isomorphic to* $C^*(E_n)$ *when* $m \neq n$. All we know about this problem is that, because of the dimension invariant of product systems [**1, 3**], E_m cannot be isomorphic to E_n unless $m = n$. This tells us that there cannot exist a *quasifree* isomorphism from $C^*(E_m)$ to $C^*(E_n)$ when $m \neq n$, but, of course, one does not know how to relate arbitrary isomorphisms to quasifree isomorphisms. More generally, it is conceivable that for *arbitrary* product systems E, $C^*(E)$ is a complete isomorphism invariant of E.

For a fixed product system E, the map $\theta \to U_\theta$ is a unitary representation of the group $\operatorname{aut}(E)$ on $L^2(E)$, and therefore $\operatorname{aut}(E)$ acts naturally on $C^*(E)$ as a group of spatially implemented automorphisms. There is a distinguished one-parameter subgroup of the center of $\operatorname{aut}(E)$, which will be called the *gauge group*. A real number λ acts on each fiber space E_t, $t > 0$, as follows,

$$\lambda(v) = e^{it\lambda}v, \qquad v \in E_t.$$

The gauge group defines a one-parameter unitary group $\{U_\lambda : \lambda \in \mathbf{R}\}$, which acts on $L^2(E)$ via

$$U_\lambda \xi(t) = e^{it\lambda}\xi(t), \qquad \xi \in L^2(E), \quad t > 0.$$

Notice that, in contrast to the gauge group associated with Cuntz algebras, this one-parameter unitary group is aperiodic. Its generator N, defined via Stone's theorem by

$$U_\lambda = e^{i\lambda N}, \qquad \lambda \in \mathbf{R}$$

is called the *number operator*. The direct integral decomposition (2.3) diagonalizes N, and shows that N has continuous (Lebesgue) spectrum on the interval $[0, \infty)$ and uniformly infinite multiplicity whenever E is nontrivial.

The action of the gauge group on $C^*(E)$ gives rise to a *C^*-dynamical system*. It is known that this C^*-dynamical system is $\mathbf{R}$-simple for arbitrary nontrivial product systems E; i.e., there are no nontrivial closed ideals in $C^*(E)$ which are invariant under the action of the gauge group ([**4**, Theorem 7.1 et seq]). This result suggests that, if one wants to prove that spectral C^*-algebras are always simple, one might begin by getting a better understanding of this C^*-dynamical system.

3. The left and right C^*-algebras. Let E be a product system. We have defined $C^*(E)$ as a separable C^*-algebra associated with the left regular representation of E. There is a corresponding C^*-algebra associated with the right regular antirepresentation of E. For this section, let us denote these two C^*-algebras by $\mathscr{A}$ and $\mathscr{B}$:

$$\begin{aligned} \mathscr{A} &= \operatorname{span}\{l(f)l(g)^*: f, g \in L^1(E)\}, \\ \mathscr{B} &= \operatorname{span}\{r(f)r(g)^*: f, g \in L^1(E)\}. \end{aligned} \tag{3.1}$$

If G is a nontrivial locally compact group, then the C^*-algebras associated with the left regular representation and right regular antirepresentation mutually commute, and hence are never irreducible. Here, the facts are rather different. In general, both $\mathscr{A}$ and $\mathscr{B}$ are irreducible C^*-algebras, and neither contains any nonzero compact operators ([**4**, Theorem 6.1]). On the other hand, $\mathscr{A}$ and $\mathscr{B}$ do not commute; rather, they commute modulo compacts ([**4**, cf. concluding remarks in Section 6]).

Let $\mathscr{E}$ be the C^*-algebra generated by all products AB, and $A \in \mathscr{A}$ and $B \in \mathscr{B}$. By the preceding remarks, $\mathscr{E}$ must contain the C^*-algebra $\mathscr{K}$ of all compact operators on $L^2(E)$. Moreover, recalling that $\mathscr{A}$ is nuclear, the quotient $\mathscr{E}/\mathscr{K}$ is a homomorphic image of the spatial tensor product $\mathscr{A} \otimes \mathscr{B}$. It is known that this homomorphism is an isomorphism whenever the product system E has a "unit" ([**4**, Theorem 8.9]). Therefore, in most cases, at least, we have an exact sequence of C^*-algebras:

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{A} \otimes \mathscr{B} \to 0. \tag{3.2}$$

Little is known about this extension or its significance in the theory of E_0-semigroups. It is surely not split, though at the moment we do not have a proof of even that.

If one carries out these constructions in the case of the trivial product system $E = Z$, one obtains the Wiener-Hopf extension. Indeed, in this case the operators on $L^2(0, \infty)$ given by left and right convolution by an integrable function are identical, and thus $\mathscr{A} = \mathscr{B} = \mathscr{E}$. The extension corresponding to (3.2) reduces to

$$0 \to \mathscr{K} \to \mathscr{E} \to C_0(\mathbf{R}) \to 0.$$

This suggests, perhaps, that one should view (3.2) as a counterpart of the Wiener-Hopf extension appropriate for the theory of E_0-semigroups.

4. Singular states. The purpose of this section and the next is to show how the spectral C^*-algebras introduced in Section 2 can be used to obtain information about E_0-semigroups and the product systems associated with them. To motivate the discussion, we recall some familiar results about semigroups of isometries acting on Hilbert spaces.

Let $U = \{U_t: t \geq 0\}$ be a strongly continuous semigroup of isometries acting on a Hilbert space H. The Wold decomposition asserts that U decomposes uniquely into a direct sum

$$U_t = V_t \oplus W_t, \qquad t \geq 0, \tag{4.1}$$

where W is a semigroup of unitary operators and V is a semigroup of isometries which is *pure* in the sense that

$$\bigcap\{V_tH: t \geq 0\} = \{0\}.$$

Moreover, this pure summand V is unitarily equivalent to the direct sum of a number n of copies of the *semigroup of shifts* $S = \{S_t: t \geq 0\}$, which acts on $L^2(0, \infty)$ by way of

$$S_tf(x) = \begin{cases} f(x-t) & \text{if } x > t, \\ 0, & \text{if } 0 < x < t. \end{cases}$$

The cardinal n is an invariant of U up to unitary equivalence. Moreover, the decomposition (4.1) is *central* in that the two projections associated with the decomposition belong to the center of the von Neumann algebra generated by $\{U_t\}$.

Semigroups of isometries correspond to representations of the trivial product system Z (cf. Introduction of [**5**]). More generally, it is rather easy to show that there is a corresponding decomposition for representations of arbitrary product systems E. Indeed, every representation $\phi: E \to \mathscr{B}(H)$ decomposes uniquely into a direct sum

$$\phi = \phi_s \oplus \phi_e, \tag{4.2}$$

where ϕ_s is a *singular* representation of E, and ϕ_e is an *essential* representation ([**1**, Proposition 1.14]). The decomposition (4.2) is *central* in that the two projections associated with it belong to the center of the von Neumann algebra generated by $\phi(E)$.

The essential summand ϕ_e is analogous to the unitary summand W in (4.1). In general, it is the essential representations of product systems that correspond to E_0-semigroups; for while every representation of a product system E on a Hilbert space H will give rise to a continuous semigroup of *-endomorphisms $\alpha = \{\alpha_t: t \geq 0\}$ acting on $\mathscr{B}(H)$, one has $\alpha_t(1) = 1$ for all $t \geq 0$ iff the representation of E is essential ([**1**, Section 2]).

Similarly, the singular summand ϕ_s of (4.2) is analogous to the pure summand V of (4.1). Since every pure semigroup of isometries is a direct sum of copies of the semigroup of shifts, one might be led to ask, *Is every singular representation of a product system E unitarily equivalent to a direct sum of copies of the left regular representation of E*? Perhaps it is not surprising that the answer is no, but more interestingly, it is almost yes. We will discuss singular representations and the corresponding states of the spectral C^*-algebra in this section. Essential representations will be taken up in Section 5.

Notice that there is a decomposition like (4.2) for representations of the spectral C^*-algebra of a product system E. Indeed, if $\pi: C^*(E) \to \mathscr{B}(H)$ is any (separable nondegenerate) representation, and $\phi: E \to \mathscr{B}(H)$ is the representation of E associated with π by Theorem 2.7,

$$\pi(l(f)l(g)^*) = \phi(f)\phi(g)^*, \qquad f, g \in L^1(E), \tag{4.3}$$

we will say that π is *singular* or *essential* according as ϕ has the corresponding property. In general, the decomposition (4.2) gives rise to a decomposition of an arbitrary representation π of $C^*(E)$ into a central direct sum

$$\pi = \pi_s \oplus \pi_e, \tag{4.4}$$

where π_s is singular and π_e is essential. The decomposition (4.4) is induced by a central projection in the enveloping von Neumann algebra $C^*(E)^{**}$ of $C^*(E)$.

Similarly, a bounded linear functional ρ on $C^*(E)$ is called *singular* or *essential* according as the cyclic representation $\pi_{|\rho|}$, associated with the positive part $|\rho|$ of the polar decomposition of ρ by way of the GNS construction, is singular or essential. The set $\mathcal{S}$ (resp. $\mathcal{E}$) of all singular (resp. essential) elements of $C^*(E)^*$ is a Banach space as well as an order ideal in the dual of $C^*(E)$. Corresponding to (4.4), every ρ in $C^*(E)^*$ decomposes uniquely into a sum $\rho_s + \rho_e$, where $\rho_s \in \mathcal{S}$, $\rho_e \in \mathcal{E}$, and we have

$$\|\rho\| = \|\rho_s\| + \|\rho_e\|.$$

Thus, $C^*(E)^* = \mathcal{S} \oplus \mathcal{E}$.

There are two descriptions of the space $\mathcal{S}$. The first gives rather precise information about the structure of $\mathcal{S}$ as a Banach space and is basically the main result of ([**5**, Corollary 1 of Theorem 3.1]). Here, we discuss only the second, which shows how the elements of $\mathcal{S}$ are related to the regular representation of $C^*(E)$ (Theorem 4.6 below).

Let E be a nontrivial product system, and consider the Hilbert space $L^2(E)$. We will consider a certain operator algebra acting on $L^2(E)$, which has the structure of a C^*-algebraic inductive limit of type-I factors. Associated with the *right* regular representation $r: E \to \mathcal{B}(L^2(E))$, we have a continuous semigroup $\beta = \{\beta_t : t \geq 0\}$ of *-endomorphisms defined as follows: For $t > 0$, choose an orthonormal basis $\{e_1(t), e_2(t), \dots\}$ for E_t, and define

$$\beta_t(A) = \sum_{n=1}^{\infty} r(e_n(t)) A r(e_n(t))^*. \tag{4.5}$$

For $t = 0$, β_t is defined as the identity map of $\beta(L^2(E))$. The fact that β has the asserted properties follows from ([**1**, Proposition 2.5]). Of course, β is not an E_0-semigroup, because the projections $P_t = \beta_t(1)$ decrease to zero as $t \to \infty$.

For every $t \geq 0$, let M_t denote the von Neumann algebra $\beta_t(M_0)$, where M_0 is taken as $\mathcal{B}(L^2(E))$. We have $M_s \supseteq M_t$ if $s \leq t$, and of course the unit of M_t is $\beta_t(1)$. Let $\mathcal{M}$ be the norm-closure of the union $\bigcup\{M_t : t \geq 0\}$. $\mathcal{M}$ is the C^*-algebraic inductive limit of a sequence of type-I factors $\mathcal{M}_n = M_{1/n}$, $n = 1, 2, \dots$, where the embedding of $\mathcal{M}_n$ in M_{n+1} is normal but is not unit preserving. $\mathcal{M}$ itself has no unit, and is weakly dense in $\mathcal{B}(L^2(E))$.

Let f be a bounded linear functional on $\mathcal{M}$. f is called *locally normal* if, for every $t \geq 0$, the restriction of f to the von Neumann algebra M_t is normal;

f is called *normal* if there is a (necessarily unique) trace-class operator T on $L^2(E)$ such that

$$f(A) = \text{trace}(TA), \qquad A \in \mathscr{M}.$$

Every normal element of $\mathscr{M}^*$ is obviously locally normal, and it is rather easy to show that the set of locally normal (resp. normal) linear functionals on $\mathscr{M}$ is a norm-closed linear subspace of $\mathscr{M}^*$ which is an *order ideal* in the sense that if f is a positive locally normal (resp. normal) functional and g satisfies $0 \leq g \leq f$, then g is locally normal (resp. normal).

THEOREM 4.6. *$\mathscr{M}$ contains $C^*(E)$. Moreover, the restriction map*

$$f \varepsilon \mathscr{M}^* \to f|_{C^*(E)}$$

defines an isometric order isomorphism of the space of locally normal elements of $\mathscr{M}^*$ onto the space $\mathscr{S}$ of singular elements of $C^*(E)^*$.

This is a consequence of ([**5**, see Section 5]). Now let ρ be a positive linear functional on $C^*(E)$, let π be the cyclic representation of $C^*(E)$ associated with ρ via the GNS construction, and let ϕ be the representation of E associated with π as in (4.3). Theorem 4.6, together with the preceding remarks, implies that π is a singular representation iff ρ extends to a locally normal state of $\mathscr{M}$. Similarly, one can see easily that ϕ is a direct sum of copies of the regular representation l iff ρ extends to a normal state of $\mathscr{M}$. Thus, the question we have asked above is equivalent to asking if every locally normal state of $\mathscr{M}$ is normal. Now, once one has an appropriate description of the Banach space of locally normal elements of $\mathscr{M}^*$, it is not hard to construct examples of nonnormal locally normal states of $\mathscr{M}$ ([**5**, Proposition 4.3]). Thus, we may conclude that *every nontrivial product system has singular representations which are not multiples of the regular representation.*

Another significant consequence of these results is that every singular representation of a nontrivial product system E can be *approximated* in a particular way by multiples of the regular representation of E ([**5**, Corollaries 2 and 4 of Theorem 3.1]). This provides a key element in the proof of the universal property of $C^*(E)$ (cf. Theorem 2.7).

5. Essential states. We have seen that every E_0-semigroup acting on $\mathscr{B}(H)$ gives rise to a product system E together with an essential representation $\phi: E \to \mathscr{B}(H)$. It is natural to ask if every product system arises in this way from an E_0-semigroup; equivalently, *does every product system have an essential representation*? It was shown in the appendix of [**1**] that this is true for product systems which possess units, but not all product systems have that property [**11**]. Indeed, the problem of constructing essential representations of arbitrary product systems has been recalcitrant. In general, one can construct the regular representation of any product system, but that representation is always singular and is therefore *not* associated with an E_0-semigroup.

The purpose of this section is to indicate how one can make use of the spectral C^*-algebra of a product system to construct essential representations.

We will sketch the ideas and state the central result (Theorem 5.6); the details will appear in **[6]**.

Let E be a nontrivial product system, which will be fixed throughout this section. We will show how the cone $\mathscr{E}^+$ of *essential* positive linear functionals on the spectral C^*-algebra

$$C^*(E) = \operatorname{span}\{l(f)l(g)^*: f, g \in L^1(E)\}$$

can be identified with a cone of "invariant weights" on the von Neumann algebra $M = \mathscr{B}(L^2(E))$. Since it is rather easy to show that such invariant weights always exist (see 5.2 below), we arrive at the desired conclusion that $\mathscr{E}^+ \neq \{0\}$.

There are several equivalent definitions of normal weights on von Neumann algebras. For our purposes, a *weight* on M is a function defined on positive operators $\omega: M^+ \to [0, +\infty]$, which has a representation of the form

$$\omega(A) = \sum_i \langle A\xi_i, \xi_i \rangle,$$

$\{\xi_i\}$ being a fixed family of vectors in the underlying Hilbert space. Let $\beta = \{\beta_t : t \geq 0\}$ be the semigroup associated with the right regular antirepresentation of E on $L^2(E)$ as in (4.5), and let $P_t = 1 - \beta_t(1)$, $t \geq 0$. We are interested in the partially ordered cone $\mathscr{W}_\beta$ of all such weights ω which are *invariant* and *semifinite* in the sense that

$$(5.1)\qquad \begin{aligned} &\text{(i)}\quad \omega(\beta_t(A)) = \omega(A), \qquad A \in M^+, \quad \textit{and} \\ &\text{(ii)}\quad \omega(P_t) < \infty, \quad \textit{for every } t > 0. \end{aligned}$$

The following result implies that $\mathscr{W}_\beta$ is always nontrivial.

THEOREM 5.2. *Let α be a semigroup of endomorphisms of $\mathscr{B}(H)$ such that $\alpha_t(1) \neq 1$ for every $t > 0$. Then there is an invariant weight ω for α such that*

$$\omega(1 - \alpha_t(1)) = t, \quad \textit{for every } t \geq 0.$$

PROOF. We will sketch the proof; the reader should have no difficulty supplying the details.

Fix $T > 0$. The projection $1 - \alpha_T(1)$ is nonzero, and thus we may select a normal state μ_0 of $\mathscr{B}(L^2(E))$ such that $\mu_0(1 - \alpha_T(1)) = 1$. Since α_T is unitarily equivalent to a direct sum of copies of the identity representation of $\mathscr{B}(L^2(E))$, we may also find an isometry V such that

$$(5.3)\qquad VA = \alpha_T(A)V, \qquad A \in \mathscr{B}(L^2(E)).$$

For each $n \geq 1$, define a normal state μ_n by

$$\mu_n(A) = \mu_0(V^{*n}AV^n), \qquad A \in \mathscr{B}(L^2(E)).$$

By its definition, μ_0 annihilates the range of α_T, and, because of the commutation relation 5.3, we have $\mu_n(\alpha_T(A)) = \mu_{n-1}(A)$ for every A and every

$n > 0$. Hence, μ_n lives on the projection $\alpha_{nT}(1) - \alpha_{(n+1)T}(1)$, and

$$\mu = \sum_{n=0}^{\infty} \mu_n \tag{5.4}$$

defines a normal weight on $\mathscr{B}(L^2(E))$ which is invariant under α_T and has the value 1 on the projection $1 - \alpha_T(1)$. Therefore, the integral

$$\tilde{\mu}(A) = \int_0^T \mu(\alpha_t(A))\,dt, \qquad A \in \mathscr{B}(L^2(E))^+$$

defines a normal weight which is invariant under the entire semigroup $\alpha = \{\alpha_t : t \geq 0\}$, and which is nonzero and finite on $1 - \alpha_T(1)$.

The function $u: [0,\infty) \to [0,\infty]$ defined by $u(t) = \tilde{\mu}(1 - \alpha_t(1))$ is lower semicontinuous, not identically zero, and satisfies $u(s+t) = u(s)+u(t)$ because of the invariance of $\tilde{\mu}$ under the semigroup α. Hence, there is a positive constant c such that $u(t) = ct$. The required weight is $\omega = c^{-1}\tilde{\mu}$. □

We will now show how every weight $\omega \in \mathscr{W}_\beta$ gives rise to a positive linear functional $d\omega$ on $C^*(E)$. In order to define this map, we require a new description of $C^*(E)$ in terms of the generator of the semigroup β. Let δ be the generator of β, defined by

$$\delta(A) = \lim_{t\to 0} t^{-1}(A - \beta_t(A)).$$

(The limit is taken in the strong operator topology, and the domain of δ is the set of all operators A for which the above strong limit exists **[10]**.) Let $\mathscr{A}$ denote the set of all operators A in the domain of δ with the property that $\delta(A)$ is a *compact* operator having *bounded support* in the sense that

$$\beta_t(1)\delta(A) = \delta(A)\beta_t(1) = 0$$

for sufficiently large $t = t(A) > 0$. It is not hard to show that *$\mathscr{A}$ is a selfadjoint algebra of operators whose norm-closure is precisely $C^*(E)$.*

Fix ω in $\mathscr{W}_\beta$ and choose $t > 0$. Since $\omega(1 - \beta_t(1))$ is finite, we may extend ω naturally to a linear functional on the algebra of all bounded operators B which are supported in $1 - \beta_t(1)$, in the sense that

$$B = (1 - \beta_t(1))B(1 - \beta_t(1)).$$

This extension is a positive linear functional of norm $\omega(1 - \beta_t(1))$. In particular, for every A in $\mathscr{A}$, the operator $\delta(A)$ is supported in such a projection, and thus we can make sense out of $\omega(\delta(A))$. This defines a linear functional $d\omega$ on $\mathscr{A}$,

$$d\omega(A) = \omega(\delta(A)), \qquad A \in \mathscr{A}. \tag{5.5}$$

Significantly, $d\omega$ is bounded, and in fact we have the following rather explicit description of $\mathscr{E}^+$.

THEOREM 5.6. *The map $\omega \to d\omega$ is an affine order isomorphism of $\mathscr{W}_\beta$ onto the cone $\mathscr{E}^+$ of all essential positive linear functionals on $C^*(E)$.*

The norm of the linear functional $d\omega \in C^*(E)^*$ can be expressed in terms of ω as follows: For a given element ω of $\mathscr{W}_\beta$, the last part of the proof of Theorem 5.2 shows that there is a nonnegative constant c such that

$$\omega(1 - \beta_t(1)) = ct, \qquad t \geq 0.$$

Then we have $\|d\omega\| = c$.

COROLLARY 5.7. *For every product system E, there is an E_0-semigroup whose canonical product system is isomorphic to E.*

I cannot resist describing an application of these results about essential representations which completes the proof of a result asserted by Powers and Robinson **[13]** about the existence of extensions of E_0-semigroups to one-parameter automorphism groups.

THEOREM 5.8. *Let $\alpha = \{\alpha_t \colon t \geq 0\}$ be an E_0-semigroup acting on a type-I factor M having a separable predual. Then there is a faithful normal nondegenerate representation π of M on a separable Hilbert space H, and a strongly continuous one-parameter unitary group $U = \{U_t \colon t \in \mathbf{R}\}$ acting on H such that*

$$\pi(\alpha_t(A)) = U_t \pi(A) U_t^*, \qquad A \in M, \quad t \geq 0.$$

PROOF. We may assume that $M = \mathscr{B}(K)$, where K is a separable (infinite dimensional) Hilbert space. We will show that there is a strongly continuous one-parameter unitary group $U = \{U_t \colon t \in \mathbf{R}\}$ acting on $H = K \otimes K$ such that

$$U_t(A \otimes 1)U_t^* = \alpha_t(A) \otimes 1$$

for every $t \geq 0$, $A \in \mathscr{B}(K)$. Theorem 5.8 will then follow by taking $\pi(A) = A \otimes 1$.

Let E be the product system of α and let $\phi \colon E \to \mathscr{B}(K)$ be the canonical representation of E associated with α. Let F be the product system opposite to E. This means that $F = E$ as a measurable family of Hilbert spaces, but that in F, multiplication is reversed: Thus, for x, y in F, xy is defined as yx. It is apparent that F is a product system, and we may consider the identity map of E to be an anti-isomorphism $\omega \colon E \to F$.

By Corollary 5.7, F has an essential representation $\phi \colon F \to \mathscr{B}(K)$ on a separable infinite dimensional Hilbert space, which we may take as K. Therefore,

$$\psi(x) = \phi(\omega(x)), \qquad x \in E$$

defines an essential *antirepresentation* of E on K. We can now simply write down the required one-parameter unitary group. For every $t > 0$, choose an orthonormal basis $\{e_1(t), e_2(t), \dots\}$ for E_t, and put

$$U_t = \sum_{n=1}^{\infty} \phi(e_n(t)) \otimes \psi(e_n(t))^*.$$

For $t < 0$, let $U_{|t|}^*$, and put $U_0 = 1$. It is not hard to verify that for each $t > 0$, U_t does not depend on the particular choice of basis $\{e_n(t): n = 1, 2, \dots\}$, and that in fact $\{U_t\}$ is a strongly continuous one-parameter unitary group having the asserted properties. The details can be found in the proof of Theorem 3.4 of [**1**]. □

REFERENCES

1. W. Arveson, *Continuous analogues of Fock space*, Mem. Amer. Math. Soc. no. 409, vol. 80, July 1989.
2. ———, *An addition formula for the index of semigroups of endomorphisms of* $\mathscr{B}(H)$, Pac. J. Math. **137**, no. 1 (1989), 19–36.
3. ———, *Quantizing the Fredholm index*, Proceedings of the Great Plains Operator Theory Seminar, 1988 (to appear).
4. ———, *Continuous Analogues of Fock space* II: *the spectral* C^**-algebra*, J. Funct. Anal. (to appear).
5. ———, *Continuous Analogues of Fock space* III: *singular states*, J. Operator Theory (to appear).
6. ———, *Continuous analogues of Fock space* IV: *Essential states,* Acta Math. (to appear).
7. J. Cuntz, *Simple* C^**-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.
8. D. Evans, *On* O_n, Publ. Res. Inst. Math. Sci., Kyoto Univ., **16** (1980), 915–927.
9. J. Glimm and A. Jaffe, *Quantum physics*, Second Ed., Springer-Verlag, New York, 1987.
10. R. T. Powers, *An index theory for semigroups of endomorphisms of* $B(H)$ *and type II factors*, Can. J. Math. **XL** 1 (1988), 86–114.
11. ———, *A nonspatial continuous semigroup of* $*$*-endomorphisms of* $\mathscr{B}(H)$, Publ. Res. Inst. Math. Sci., Kyoto Univ. (to appear).
12. R. T. Powers and G. Price, *Continuous spatial semigroups of* $*$*-endomorphisms of* $\mathscr{B}(H)$, preprint.
13. R. T. Powers and D. W. Robinson, *An index for continuous semigroups of endomorphisms of* $\mathscr{B}(H)$, J. Funct. Anal. (to appear).
14. G. Price, *Shifts on type II factors*, Can. J. Math. **XXXIX** 2 (1987), 492–511.
15. ———, *Shifts of integer index on the hyperfinite II factor*, Pac. J. Math. (to appear).
16. ———, *The* C^**-algebras generated by pairs of semigroups of isometries satisfying certain commutation relations*, preprint.

UNIVERSITY OF CALIFORNIA, BERKELEY
arveson@math.berkeley.edu