Contemporary Mathematics
Volume **62**, 1987

NONLINEAR STATES ON C*-ALGEBRAS

Dedicated to

Hans Borchers, Nico Hugenholtz, Richard Kadison, and Daniel Kastler,

in Honor of their Sixtieth Birthdays

William Arveson*

1. INTRODUCTION

Let A be a unital C*-algebra. A complex-valued function ϕ defined on A is said to be of *positive type* (or simply *positive* when there is no chance of confusion) if

(1.2) $$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \phi(a_j^* a_i) \geq 0$$

for every n = 1,2, ..., $\lambda_1, ..., \lambda_n \in \mathbb{C}$, $a_1, ..., a_n \in A$. If ϕ happens to be a linear functional then (1.1) simply asserts that ϕ is positive in the traditional sense: $\phi(a^*a) \geq 0$, $a \in A$.

ϕ is said to be *completely positive* if, for every $n \geq 1$ and every positive $n \times n$ matrix $(a_{ij}) \in M_n(A)$, one has

(1.2) $$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \phi(a_{ij}) \geq 0,$$

for every $\lambda_1, ..., \lambda_n \in \mathbb{C}$. Again, if ϕ is a linear functional then (1.2) is equivalent to the assertion $\phi(a^*a) \geq 0$, $a \in A$. For nonlinear functions, however, we will find that the second property is much

*This research was supported in part by the National Science Foundation grant DMS-83-02061.

stronger than the first. For example, if A is the one-dimensional C*-algebra $\mathbb{C}$, then

$$\phi(z) = |z|^{\alpha}$$

is a positive function for every positive real number α, while it is completely positive only when α is an even integer.

The purpose of this paper is to characterize completely positive functions defined on unital C*-algebras. These functions are the appropriate nonlinear counterparts of states, and they occur naturally in several rather diverse contexts. Since we do not take up applications in this paper, we want to at least mention three of the more important settings in which nonlinear states appear. We will deal more completely with applications in a subsequent paper.

The most familiar example is the determinant function, which we want to consider in a particular way. Let $A = M_n(\mathbb{C})$ be the C*-algebra of all $n \times n$ complex matrices, considered as the algebra of all operators on an n-dimensional Hilbert space $\mathcal{H}$. Let A^n denote the symmetric tensor product of n copies of A; i.e., A^n is the sub C*-algebra of the full tensor product $A^{\otimes n}$ spanned by all elementary tensors of the form

$$a^{(n)} = a \otimes a \otimes \cdots \otimes a, \quad a \in A.$$

The mapping π: $A \to A^n$ defined by $\pi(a) = a^{(n)}$ is a nonlinear function which preserves multiplication, units, and the *-operation. Moreover, the action of A on $\mathcal{H}$ gives rise to a natural action of A^n on the n^{th} exterior power $\Lambda^n \mathcal{H}$ of $\mathcal{H}$. $\Lambda^n \mathcal{H}$ is a one-dimensional Hilbert space, and if we choose a unit vector $\xi \in \Lambda^n \mathcal{H}$ then we have a familiar and useful formula for the determinant of an operator a in A:

$$\det a = \langle \pi(a)\xi, \xi \rangle .$$

We also remark that, in general, this map π: $A \to A^n$ is *completely positive* in the sense that if (a_{ij}) is a positive $k \times k$ matrix over A then $(\pi(a_{ij}))$ is a positive $k \times k$ matrix over A^n, for every $k = 1, 2, \ldots$.

More generally, let A, B be arbitrary C*-algebras and let $\pi: A \to B$ be a function from A into B satisfying $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^*) = \pi(x)^*$, $x,y \in A$. Then for every positive linear functional ρ on B the function

$$\phi(x) = \rho(\pi(x))$$

is positive in the sense of (1.1). If π is in addition a completely positive map in the above sense, then ϕ is a completely positive function. As one might expect from this observation, there is an intimate connection between completely positive operator maps (such as π) and nonlinear states (such as ϕ), which runs parallel to the relation existing between cyclic representations of C*-algebras and their associated vector states (cf. Section 2).

For reasons which will become clear presently, it is desirable, if not necessary, to consider complex-valued functions ϕ (and multiplicative *-preserving maps of C*-algebras) which are only defined on the unit ball of A: indeed, in the later sections it will be necessary to restrict the domain even further to the open unit ball,

$$\text{ball } A = \{ a \in A: \|a\| < 1\}.$$

The definitions of positivity and complete positivity for such functions

$$\phi: \text{ball } A \to B$$

still make sense, and it will be this latter class of functions, particularly in the case $B = \mathbb{C}$, that will be the object of study in this paper.

A second type of example has its origins in the mathematical foundations of quantum field theory. Here, one begins with a complex Hilbert space $\mathcal{H}$ and constructs from it two subspaces of the Fock space over $\mathcal{H}$:

$$\mathcal{H}_+ = \sum_{n=0}^{\infty} \vee^n \mathcal{H} \qquad \text{(for Bosons)}$$

and

$$\mathcal{H}_- = \sum_{n=0}^{\infty} \wedge^n \mathcal{H} \qquad \text{(for Fermions)},$$

where $\vee^n \mathcal{H}$ (resp. $\wedge^n \mathcal{H}$) denotes the symmetric (resp. antisymmetric) tensor product of n copies of $\mathcal{H}$. For each *contraction* $a \in \mathcal{B}(\mathcal{H})$,

$$\Gamma(a) = \sum_{n=0}^{\infty}{}^{\otimes} \; a^{(n)}$$

is a bounded operator on Fock space which leaves both $\mathcal{H}_+$ and $\mathcal{H}_-$ invariant. Thus we obtain two operators $\Gamma_+(a)$ and $\Gamma_-(a)$ by restricting $\Gamma(a)$ appropriately,

$$\Gamma_+(a) = \Gamma(a)\big|_{\mathcal{H}_+}$$

$$\Gamma_-(a) = \Gamma(a)\big|_{\mathcal{H}_-} .$$

Both of these "second quantization" mappings are nonlinear, they preserve multiplication and the *-operation, and they carry the identity operator on $\mathcal{H}$ to the respective identities on $\mathcal{H}_+$ and $\mathcal{H}_-$. Moreover, since Γ_+ and Γ_- are both completely positive, we may compose either of them with, say, a vector state to obtain a scalar-valued completely positive function defined on the closed unit ball of the operator algebra $\mathcal{B}(\mathcal{H})$.

Both Γ_+ and Γ_- map unitary operators to unitary operators. We want to point out, however, that it is important for the theory that these maps are defined on operators throughout the unit ball. To see why this is so, suppose U_t is a one-parameter unitary group acting on $\mathcal{H}$ with self-adjoint (unbounded) generator H:

$$U_t = e^{itH} , \qquad t \in \mathbb{R} .$$

Then $\{\Gamma_+(U_t): t \in \mathbb{R}\}$ is a one-parameter unitary group acting on $\mathcal{H}_+$, and so it has a self-adjoint generator $d\Gamma_+(H)$, defined via Stone's theorem by

$$\Gamma_+(e^{itH}) = e^{itd\Gamma_+(H)}, \quad t \in \mathbb{R}.$$

If H has positive spectrum then so does $d\Gamma_+(H)$; moreover, in this case the two self-adjoint contraction semigroups e^{-sH} and $e^{-sd\Gamma_+(H)}$, $s \geq 0$, can be related directly to each other by the formula

$$(1.3) \qquad \Gamma_+(e^{-sH}) = e^{-sd\Gamma_+(H)}, \quad s \geq 0.$$

The useful formula (1.3) requires, of course, that Γ_+ should be defined throughout the closed unit ball of $\mathcal{B}(\mathcal{H})$. A similar formula holds for the Fermion mapping Γ_-.

A third area in which completely positive functions occur naturally is the problem of classifying nonlinear stochastic filters in terms of (nonlinear) spectral invariants. This was the initial source of my own interest in completely positive functions, and in fact a significant portion of the results of this paper were obtained in 1976-78 in connection with that work. Perhaps it would be too much of a digression to describe the role of completely positive functions in nonlinear filtering here; the interested reader can find more detail in [1], [2], and [5].

The main results of this paper can be summarized as follows. As above, A denotes a unital C*-algebra and ball A denotes the open unit ball of A.

THEOREM A. *Let* ϕ: ball A $\to \mathbb{C}$ *be a bounded function. The following are equivalent:*

(i) *ϕ is completely positive.*

(ii) *ϕ is positive and real-analytic.*

(iii) *There is a positive linear functional ρ on the C*-algebra* $e^A \otimes e^{\bar{A}}$ *such that*

$$\phi(a) = \rho(\Gamma(a) \otimes \Gamma(\bar{a})), \quad \|a\| < 1.$$

The terminology of part (iii) requires more explanation. In this paper, C*-algebraic tensor products will always be taken with respect to the *largest* C*-crossnorm. Thus, A ⊗ B has the following universal property (which, of course, can be used to make an appropriate definition of this norm on the algebraic tensor product $A \odot B$):

If C is a third C-algebra and* $\pi: A \to C$, $\sigma: B \to C$ *are two morphisms satisfying*

$$\pi(a)\sigma(b) = \sigma(b)\pi(a), \quad a \in A, \ b \in B,$$

then there is a unique morphism $\rho: A \otimes B \to C$ *such that*

$$\rho(a \otimes b) = \pi(a)\pi(b), \quad a \in A, \ b \in B.$$

For each positive integer n, we may form the n-fold tensor product of C*-algebras

$$A^{\otimes n} = A \otimes A \otimes \cdots \otimes A .$$

A^n will denote the C*-subalgebra of $A^{\otimes n}$ generated by all elementary tensors of the form

$$a^{(n)} = a \otimes a \otimes \cdots \otimes a, \quad a \in A.$$

Equivalently, A^n is the C*-subalgebra of $A^{\otimes n}$ consisting of those elements which are left fixed under the natural action of the permutation group S_n on $A^{\otimes n}$. We define e^A to be the direct sum of C*-algebras

$$e^A = \sum_{n=0}^{\infty} A^n ,$$

where A^0 is defined as $\mathbb{C}$. e^A never has a unit. There is a natural mapping Γ of the open unit ball of A into the unit ball of e^A defined by

$$\Gamma(a) = 1 \oplus a \oplus a^{(2)} \oplus \cdots , \quad \|a\| < 1.$$

Γ is certainly not linear, but it is multiplicative, continuous with respect to the norm topologies, and carries adjoints to adjoints.

Finally, with every C*-algebra A there is associated a natural *conjugate* C*-algebra $\bar{A}$, which is defined as the same set of elements as A, having the same multiplication, addition, and norm, but whose scalar multiplication is conjugated. Thus, if $b \in \overline{A}$ and $\lambda \in \mathbb{C}$, then the scalar product of λ with b in $\overline{A}$ means $\overline{\lambda}b$, rather than λb. The identity mapping, considered as a function from A to $\overline{A}$, is an antilinear isometry which preserves multiplication and the *-operation. The image in $\overline{A}$ of an element $a \in A$ under this mapping will be written $\overline{a}$.

The C*-algebra $\overline{A}$ need not be isomorphic to A. However, the "transpose" mapping t of A to $\overline{A}$, defined by

$$a^t = \overline{a}^*, \quad a \in A,$$

is clearly a linear anti-isomorphism. Thus $\overline{A}$ is, in general, *-isomorphic to the C*-algebra A^0 opposite to A. In particular, any assertion about $\overline{A}$ is equivalent to a corresponding assertion about A^0; we have chosen to work with $\overline{A}$ rather than A^0 in this paper because it appears to be a more natural object when one is dealing with holomorphic and anti-holomorphic functions defined on A.

In any case, if $a \in$ ball A, then $\overline{a}$ belongs to ball $\overline{A}$ and hence $\Gamma(\overline{a})$ is an element of $e^{\overline{A}}$. We will sometimes write $\overline{\Gamma}$ for this natural map of ball A into $e^{\overline{A}}$:

$$\overline{\Gamma}(a) = \Gamma(\overline{a}), \qquad \|a\| < 1.$$

The map $\Gamma \otimes \overline{\Gamma}$: ball $A \to e^A \otimes e^{\overline{A}}$ of part (iii) of Theorem A is neither holomorphic nor anti-holomorphic, but it is easily seen to be real-analytic (cf. Section 3).

Now $e^A \otimes e^{\overline{A}}$ decomposes into a doubly infinite direct sum of C*-algebras

$$e^A \otimes e^{\overline{A}} = \sum_{m,n=0}^{\infty} A^m \otimes \overline{A}^n ,$$

which gives rise to a corresponding decomposition of $\Gamma \otimes \overline{\Gamma}$

$$\Gamma(a) \otimes \Gamma(\overline{a}) = \sum_{m,n=0}^{\infty} a^{(m)} \otimes \overline{a}^{(n)}.$$

For the same reason, every positive linear functional ρ on $e^A \otimes e^{\overline{A}}$

admits a unique decomposition

$$\rho = \sum_{m,n=0}^{\infty} \rho_{mn} ,$$

where ρ_{mn} is a positive linear functional on $A^m \otimes \bar{A}^n$, and of course we have

$$\|\rho\| = \sum_{m,n=0}^{\infty} \|\rho_{mn}\|.$$

Hence, Theorem A(iii) becomes the assertion that ϕ has a decomposition

$$\phi(a) = \sum_{m,n=0}^{\infty} \rho_{mn}(a^{(m)} \otimes \bar{a}^{(n)}), \qquad \|a\| < 1,$$

where ρ_{mn} is a positive linear functional on $A^m \otimes \bar{A}^n$, such that the summability condition

$$\sum_{m,n=0}^{\infty} \|\rho_{mn}\| < \infty$$

is satisfied.

In the particular case $A = \mathbb{C}$, this implies that every completely positive function ϕ defined on the open unit disc $\{|z| < 1\}$ has a power series expansion

$$\phi(z) = \sum_{m,n=0}^{\infty} a_{mn} z^m \bar{z}^n, \qquad |z| < 1,$$

where the coefficients a_{mn} are all nonnegative and satisfy

$$\sum_{m,n=0}^{\infty} a_{mn} < \infty .$$

We have recently learned that this latter representation of completely positive functions on the disc is closely related to some new results of T. Ando and M.-D. Choi which are as yet unpublished.

The proof of Theorem A is long, but the individual steps are not technically difficult. Because of the length of the argument, it seemed appropriate to organize certain of its components into separate sections. Section 2 deals with a generalization of the GNS construction appropriate for nonlinear states. Section 3 gives a characterization of those functions on A which can be "linearized" through the map $\Gamma \otimes \overline{\Gamma}$ to positive linear functionals on $e^A \otimes e^{\overline{A}}$. In Sections 4 and 5 we establish the smoothness and regularity properties of nonlinear states.

Because all of the applications we envisage involve unital C*-algebras and because the presence of a unit allows for considerably more streamlined arguments, we only deal with the unital case in this paper.

2. THE GNS CONSTRUCTION FOR NONLINEAR STATES

Throughout this section, A will denote a unital C*-algebra. By a **-representation* of ball A we mean an operator-valued mapping π: ball A $\to$ $L(H)$ of the open unit ball of A into the algebra of all bounded operators on a Hilbert space H satisfying

$$\text{(i)} \quad \sup_{\|x\|<1} \|\pi(x)\| < \infty,$$

$$\text{(2.1)} \qquad \text{(ii)} \quad \pi(xy) = \pi(x)\pi(y),$$

$$\text{(iii)} \quad \pi(x^*) = \pi(x)^*.$$

Notice that, in fact, we must have $\|\pi(x)\| \leq 1$ for all x in ball A. Indeed, if M denotes the value of the supremum in (2.1)(i), then we have

$$\|\pi(x)\| = \|\pi(x)^*\pi(x)\|^{1/2} = \|\pi(x^*x)\|^{1/2} = \|\pi((x^*x)^2)\|^{1/4}$$
$$= \cdots = \|\pi((x^*x)^{2^n})\|^{1/2^{n+1}} \leq M^{1/2^{n+1}},$$

for every n = 1,2, ..., and the assertion follows after taking the limit on n.

Notice also that if we choose a vector $\xi \in H$ and define a function ϕ: ball A $\to \mathbb{C}$ by

$$\phi(x) = \langle\pi(x)\xi,\xi\rangle, \quad \|x\| < 1,$$

then ϕ is a bounded function of positive type. The purpose of this section is to prove the following converse, to establish its counterpart for completely positive functions, and to draw out one or two consequences for later use.

THEOREM 2.2. *Let* ϕ: ball A $\to \mathbb{C}$ *be a bounded function which is positive in the sense of* (1.1). *Then there is a triple* $(\pi, \xi, \mathcal{H})$ *consisting of a *-representation* π *of* ball A *on a Hilbert space* $\mathcal{H}$ *and a vector* ξ *in* $\mathcal{H}$ *satisfying*

(i) $[\pi(x)\xi;\ \|x\| < 1] = \mathcal{H}$,

(ii) $\phi(x) = \langle\pi(x)\xi, \xi\rangle$, $\|x\| < 1$.

If $(\pi_1, \xi_1, \mathcal{H}_1)$ *and* $(\pi_2, \xi_2, \mathcal{H}_2)$ *are two such triples, then there is a unique unitary operator* U: $\mathcal{H}_1 \to \mathcal{H}_2$ *satisfying*

$$U\xi_1 = \xi_2,$$

$$U\pi_1(x) = \pi_2(x)U, \quad \|x\| < 1.$$

The proof of Theorem 2.2 is nontrivial because ball A does not contain the unit of A; the absence of a multiplicative unit presents anomalies that require some care.

We begin by collecting a result from the lore of dilation theory, essentially an improvement of a result in the appendix of ([7], also see [6], Thm. 1, p. 27). By a **-semigroup* we mean a semigroup Σ endowed with an involution $x \mapsto x^*$ (i.e., $x^{**} = x$) satisfying $(xy)^* = y^*x^*$, $x,y \in \Sigma$. A *unit* is an element e of Σ satisfying $ex = xe = x$ for all x. Units are unique, when they exist, and are self-adjoint. By a *representation* of a *-semigroup Σ we mean an operator-valued mapping π: $\Sigma \to \mathcal{L}(\mathcal{H})$ satisfying the three conditions of (2.1). Notice that the argument given above in the special case Σ = ball A actually implies that we have

$$\|\pi(x)\| \leq 1, \quad x \in \Sigma$$

for a general representation π of a general *-semigroup Σ.

LEMMA 2.3. *Let* Σ *be a *-semigroup with unit and let* ϕ *be a bounded complex-valued function on* Σ *satisfying*

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \phi(x_j^* x_i) \geq 0$$

for all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $x_1, \ldots, x_n \in \Sigma$, $n \geq 1$.

Then there is a triple (π, ξ, H) *consisting of a representation* π *of* Σ *on a Hilbert space* H *and a vector* $\xi \in H$ *satisfying*

$$[\pi(\Sigma)\xi] = H, \quad \textit{and}$$

$$\phi(x) = \langle \pi(x)\xi, \xi \rangle, \quad x \in \Sigma.$$

If (π_1, ξ_1, H_1) *and* (π_2, ξ_2, H_2) *are two such triples, then there is a unique unitary operator* U: $H_1 \to H_2$ *satisfying*

$$U\xi_1 = \xi_2, \quad \textit{and}$$

$$U\pi_1(x) = \pi_2(x)U, \quad x \in \Sigma.$$

PROOF. For the readers' convenience, we sketch the proof. The complex Banach space $\ell^1(\Sigma)$ becomes a unital Banach *-algebra relative to the multiplication

$$f * g(x) = \sum_{yz=x} f(y)g(z)$$

and the involution $f \longmapsto \breve{f}$, where

$$\breve{f}(x) = \overline{f(x^*)},$$

in which the function

$$u(x) = \begin{cases} 1, & x = e \\ 0, & x \neq e \end{cases}$$

serves as a unit. The given function ϕ belongs to $\ell^\infty(\Sigma)$, and therefore gives rise to a bounded linear functional Φ on $\ell^1(\Sigma)$ in the usual way:

$$\Phi(f) = \sum_x f(x)\phi(x).$$

It is a simple matter to check that the hypothesis on ϕ entails

$$\Phi(\tilde{f} * f) \geq 0$$

for all $f \in \ell^1(\Sigma)$, so that Φ is a positive linear functional. By the traditional GNS construction (cf. [3], or [8]) there is a unital *-representation π_1 of $\ell^1(\Sigma)$ on a Hilbert space H and a cyclic vector $\xi \in H$ such that

$$\Phi(f) = \langle \pi_1(f)\xi,\xi \rangle, \quad f \in \ell'(\Sigma).$$

If we define

$$\pi(x) = \pi_1(\delta_x), \quad x \in \Sigma,$$

where δ_x is the delta function with mass at x, then the relations

$$\tilde{\delta}_x = \delta_{x^*},$$

$$\delta_x \delta_y = \delta_{xy},$$

$$\delta_e = u$$

imply that π is a representation of Σ for which the triple (π,ξ, H) has the asserted properties.

The proof of uniqueness amounts to nothing more than checking inner products, and we omit it. □

We will also require the following elementary result, for which we have been unable to find a reference.

PROPOSITION 2.4. *Let ϕ be a bounded real-valued function on the open unit interval $0 < t < 1$ with the property that, for every $0 < s,t < 1$, the 2×2 matrix*

$$\begin{bmatrix} \phi(s^2) & \phi(st) \\ \phi(st) & \phi(t^2) \end{bmatrix}$$

is positive semidefinite. Then ϕ is monotone increasing and continuous.

REMARKS. Note first that the hypothesis on ϕ reduces to the assertion that ϕ is a nonnegative bounded function satisfying

$$\phi(\sqrt{st}) \leq \sqrt{\phi(s)\phi(t)}, \quad 0 < s,t < 1.$$

The open unit interval can be viewed as a *-semigroup relative to the usual multiplication and the trivial involution $t^* = t$, $t \in (0,1)$. Note that this *-semigroup has no unit. The hypothesis of Proposition 2.4 is, of course, the case $n = 2$ of the positivity condition

$$(2.5) \qquad \sum_{i,j=1}^{n} \lambda_i\bar{\lambda}_j\phi(t_it_j) \geq 0$$

appropriate for this *-semigroup. In particular, Proposition 2.4 implies that *every bounded positive function on the *-semigroup* (0,1) *is continuous and monotone increasing.*

Finally, we want to point out that this latter conclusion can fail for unbounded positive functions on (0,1). Indeed, for every positive real number α the function

$$\phi(t) = \alpha t + t^{-1}, \quad 0 < t < 1$$

satisfies the positivity condition (2.5), while if $\alpha > 1$ the function ϕ is neither increasing nor decreasing.

PROOF OF PROPOSITION 2.4. We show first that ϕ is increasing. Fix r, $0 < r < 1$. It clearly suffices to show that $\phi(rt) \leq \phi(t)$ for all $t \in (0,1)$.

For that, define $\psi: (0,1] \to \mathbb{C}$ by $\psi(t) = \phi(rt)$. Now for each $t \in (0,1]$, the 2×2 matrix

$$\begin{bmatrix} \psi(1) & \psi(t) \\ \psi(t) & \psi(t^2) \end{bmatrix}$$

is positive semidefinite because it has the form $(\phi(t_it_j))$ for $t_1 = r^{1/2}$, $t_2 = r^{1/2}t$. Thus $\psi(t)^2 \leq \psi(1)\psi(t^2)$. Since ϕ (and therefore ψ) is nonnegative, we have

$$\psi(t) \leq \psi(1)^{1/2}\psi(t^2)^{1/2}.$$

Iterating this inequality n times gives

$$\psi(t) \leqslant \psi(1)^{1/2+\dots+1/2^n}\, \psi(t^{2^n})^{1/2^n}.$$

If M is an upper bound for ϕ on (0,1), then the latter implies

$$\psi(t) \leqslant \psi(1)^{1/2+\dots+1/2^n}\, M^{1/2^n}, \tag{2.6}$$

and the required inequality, namely that $\psi(t) \leqslant \psi(1)$, follows by taking the limit of (2.6) as n tends to infinity.

To prove continuity of ϕ, we claim first that

$$\phi(t) - \phi(t-) \leqslant \phi(t+) - \phi(t),$$

for all $0 < t < 1$. Of course, both left and right limits $\phi(t-)$ and $\phi(t+)$ exist because ϕ has already been shown to be monotonic. This inequality is the same as

$$\phi(t) \leqslant \frac{1}{2}(\phi(t+) + \phi(t-)),$$

and so by the inequality of the arithmetic and geometric means, it suffices to show that

$$\phi(t) \leqslant (\phi(t+)\phi(t-))^{1/2}. \tag{2.7}$$

For that, fix $\delta > 0$ and put

$$t_1 = t^{1/2}(1 + \delta)$$

$$t_2 = t^{1/2}(1 + \delta)^{-1}.$$

Then for small δ, both t_1 and t_2 belong to (0,1) and so the 2×2 matrix

$$(\phi(t_i t_j)) = \begin{bmatrix} \phi(t(1+\delta)^2) & \phi(t) \\ \phi(t) & \phi(t(1+\delta)^{-2}) \end{bmatrix}$$

is positive. Thus we have

$$\phi(t)^2 \leqslant \phi(t(1+\delta)^2)\phi(t(1+\delta)^{-2}),$$

and hence we obtain (2.7) by allowing δ to decrease to zero.

Now to establish continuity of ϕ, consider the function ϕ_+: $(0,1) \to \mathbb{C}$ defined by $\phi_+(t) = \phi(t+)$. ϕ_+ is clearly bounded and continuous from the right. Moreover, since ϕ was already shown to be increasing, ϕ_+ is also increasing and its jumps occur at the same places as the jumps of ϕ. Thus it suffices to show that ϕ_+ is continuous.

Note that ϕ_+ must satisfy the same hypothesis as ϕ, because of the fact that

$$\phi_+(t) = \lim_{n\to\infty} \phi\left[t\left[1 + \frac{1}{n}\right]\right].$$

Thus by the arguments already given, we have

$$\phi_+(t) - \phi_+(t-) \leqslant \phi_+(t+) - \phi_+(t).$$

The right side of the preceding inequality is zero because ϕ_+ is right continuous, and hence $\phi_+(t) \leqslant \phi_+(t-)$. But since ϕ_+ is increasing we have $\phi_+(t-) \leqslant \phi_+(t)$, and hence

$$\phi_+(t) = \phi_+(t-).$$

This shows that ϕ_+ is left continuous as well as right continuous, completing the proof. □

Turning now to the proof of Theorem 2.2, let ϕ: ball $A \to \mathbb{C}$ be a bounded function of positive type. For each r, $0 < r < 1$, define ϕ_r on the *closed* unit ball of A by

$$\phi_r(a) = \phi(ra), \quad \|a\| \leqslant 1.$$

Since the closed unit ball of A is a *-semigroup with unit, we see from Lemma 2.3 that there is a representation π_r of $\{\|a\| \leqslant 1\}$ on a Hilbert space $\mathcal{H}_r$ and a vector $\xi_r \in \mathcal{H}_r$ such that

$$\phi_r(a) = \langle\pi_r(a)\xi_r,\xi_r\rangle, \quad \|a\| \leqslant 1,$$

and the set of vectors $\{\pi_r(a)\xi_r\colon \|a\| \leqslant 1\}$ spans $\mathcal{H}_r$.

For a fixed r in (0,1), we claim that

$$\lim_{t\uparrow 1} \pi_r(t1) = 1$$

in the strong operator topology. Since the operators $\{\pi_r(t1): 0 \leqslant t < 1\}$ are uniformly bounded, it suffices to show that

$$\lim_{t\uparrow 1} \|\pi_r(t1)\eta - \eta\| = 0,$$

for all η in the fundamental set of vectors of the form $\eta = \pi_r(a)\xi_r$, $\|a\| \leqslant 1$. But for such an $\eta = \pi_r(a)\xi_r$ we have

$$\begin{aligned} \text{(2.8)} \qquad \|\pi_r(t1)\eta - \eta\|^2 &= \|\pi_r(ta)\xi_r - \pi_r(a)\xi_r\|^2 \\ &= \langle\pi_r(t^2a^*a)\xi_r,\xi_r\rangle - 2\ \mathrm{Re}\langle\pi_r(ta^*a)\xi_r,\xi_r\rangle \\ &\quad + \langle\pi_r(a^*a)\xi_r,\xi_r\rangle \\ &= \phi(rt^2a^*a) - 2\ \mathrm{Re}\ \phi(rta^*a) + \phi(ra^*a). \end{aligned}$$

Now the function ψ: $(0,1) \rightarrow \mathbb{C}$ defined by

$$\psi(s) = \phi(sa^*a)$$

satisfies the hypothesis of Proposition 2.4 because ϕ is of positive type, and hence is continuous. So as t increases to 1, the last term in (2.8) tends to

$$\psi(r) - 2\ \mathrm{Re}\ \psi(r) + \psi(r) = 0.$$

and so we have the required conclusion

$$\lim_{t\uparrow 1} \|\pi_r(t1)\eta - \eta\| = 0.$$

We claim next that, for each $t \in (0,1]$, the operator $\pi_r(t1)$ has trivial kernel. Indeed, if $\eta \in H_r$ is such that $\pi_r(t1)\eta = 0$, then since π_r is a representation, we can write

$$\|\pi_r(t^{1/2}1)\eta\|^2 = <\pi_r(t^{1/2}1)\eta,\pi_r(t^{1/2}1)\eta>$$
$$= <\pi_r(t1)\eta,\eta> = 0.$$

It follows that $\pi_r(t^{1/2}1)\eta = 0$. Continuing in this way we obtain $\pi_r(t^{1/2n}1)\eta = 0$ for every $n = 1,2, ...,$ and so by the preceding paragraphs we conclude that

$$\eta = \lim_{n\to\infty} \pi_r(t^{1/2^n}\ 1)\eta = 0,$$

as asserted.

Still keeping $r \in (0,1)$ fixed, we now want to show that one can represent the positive functions ϕ_ρ, for $r < \rho < 1$, in terms of the representation π_r and certain vectors in the space H_r.

LEMMA 2.9. *For each $\rho \in (r,1)$, there is a unique vector $\zeta_\rho \in H_r$ such that*

$$\pi_r((r/\rho)^{1/2})\zeta_\rho = \xi_r\ .$$

Moreover, one has

$$\phi(\rho a) = <\pi_r(a)\zeta_\rho,\zeta_\rho>, \quad \|a\| \leq 1.$$

REMARK. For a positive scalar λ, $0 < \lambda < 1$, we will denote the operator $\pi_r(\lambda 1)$ more briefly as $\pi_r(\lambda)$.

PROOF OF LEMMA 2.9. We first construct the vector ζ_ρ. Fix r, ρ satisfying $0 < r \leq \rho < 1$. Utilizing the triple $(\pi_\rho,\xi_\rho, H_\rho)$ constructed from ϕ_ρ via Lemma 2.3, we can write

$$\phi(\rho a) = <\pi_\rho(a)\xi_\rho,\xi_\rho>\ , \quad \|a\| \leq 1.$$

Thus,

$$\phi_r(a) = \phi\left[\rho\left[\frac{r}{\rho}a\right]\right] = \phi_\rho\left[\frac{r}{\rho}\ a\right]$$
$$= <\pi_\rho\left[\frac{r}{\rho}\ a\right]\xi_\rho,\xi_\rho>$$

$$= \langle \pi_\rho\left(\left(\frac{r}{\rho}\right)^{1/2}\right)\pi_\rho(a)\pi_\rho\left(\left(\frac{r}{\rho}\right)^{1/2}\right)\xi_\rho,\xi_\rho\rangle$$

$$= \langle \pi_\rho(a)\pi_\rho\left(\left(\frac{r}{\rho}\right)^{1/2}\right)\xi_\rho, \pi_\rho\left(\left(\frac{r}{\rho}\right)^{1/2}\right)\xi_\rho\rangle .$$

This expression shows that the triple $(\pi_\rho, \pi_\rho((r/\rho)^{1/2})\xi_\rho, K)$, where K is the following subspace of H_ρ

$$K = \left[\pi_\rho(a)\pi_\rho\left(\left(\frac{r}{\rho}\right)^{1/2}\right)\xi_\rho: \|a\| \leq 1\right],$$

serves as well as the triple (π_r, ξ_r, H_r) to represent the positive function $\phi_r: \{\|a\| \leq 1\} \to \mathbb{C}$.

We claim next that $K = H_\rho$. Indeed, since $(r/\rho)^{1/2}1$ commutes with every element of A, the operator $\pi_\rho((r/\rho)^{1/2})$ must commute with the range of π_ρ. Hence

$$\text{(2.10)} \qquad K = \left[\pi_\rho\left(\left(\frac{r}{\rho}\right)^{1/2}\right)\pi_\rho(a)\xi_\rho: \|a\| \leq 1\right]$$

$$= \overline{\operatorname{ran} \pi_\rho((r/\rho)^{1/2})}$$

$$= \left\{\ker \pi_\rho\left(\left(\frac{r}{\rho}\right)^{1/2}\right)\right\}^\perp .$$

The argument preceding this lemma shows that $\pi_\rho((r/\rho)^{1/2})$ has trivial kernel, and so (2.10) implies that $K = H_\rho$, as asserted.

By the uniqueness assertion of Lemma 2.3, the triples (π_r, ξ_r, H_r) and $(\pi_\rho, \pi_\rho((r/\rho)^{1/2})\xi_\rho, H_\rho)$ are unitarily equivalent in the sense that there is a unitary operator U: $H_r \to H_\rho$ which carries ξ_r to $\pi((r/\rho)^{1/2})\xi_\rho$ and satisfies

$$U\pi_r = \pi_\rho U.$$

So if we now define the vector $\zeta_\rho \in H_r$ by

$$\zeta_\rho = U^{-1}\xi_\rho,$$

then we have

$$\pi_r\left[\left(\frac{r}{\rho}\right)^{1/2}\right]\zeta_\rho = U^{-1}\pi_\rho\left[\left(\frac{r}{\rho}\right)^{1/2}\right]\xi_\rho$$

$$= U^{-1}U\xi_\rho = \xi_\rho ,$$

as required.

ζ_ρ is clearly unique because, as we have already seen, $\pi_r((r/\rho)^{1/2})$ has trivial kernel.

Finally, for each $a \in A$, $\|a\| \leq 1$, we have

$$\phi_\rho(a) = \langle\pi_\rho(a)\xi_\rho,\xi_\rho\rangle = \langle U\pi_r(a)U^{-1}\xi_\rho,\xi_\rho\rangle$$
$$= \langle\pi_r(a)\zeta_\rho,\zeta_\rho\rangle ,$$

which completes the proof of Lemma 2.9. □

So for each ρ in the interval $r < \rho < 1$, we have a vector $\zeta_\rho \in \mathcal{H}_r$ as above. We claim next that the limit

$$\zeta = \lim_{\rho\uparrow 1} \zeta_\rho$$

exists (in the norm of $\mathcal{H}_r$). To see that, choose $\rho_1, \rho_2 \in (r,1)$. Then of course

(2.11) $$\|\zeta_{\rho_2}-\zeta_{\rho_1}\|^2 = \|\zeta_{\rho_2}\|^2 - 2\,\mathrm{Re}\,\langle\zeta_{\rho_2},\zeta_{\rho_1}\rangle + \|\zeta_{\rho_1}\|^2.$$

Now let us suppose that $\rho_1 < \rho_2$. Then we claim:

(2.12) $$\zeta_{\rho_1} = \pi_r\left[\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\right]\zeta_{\rho_2} .$$

Indeed, if we apply the operator $\pi_r((r/\rho_1)^{1/2})$ to both sides of (2.12), then on the left we obtain

$$\pi_r\left[\left(\frac{r}{\rho_1}\right)^{1/2}\right]\zeta_{\rho_1} ,$$

while on the right we have

$$\pi_r\left[\left(\frac{r}{\rho_1}\right)^{1/2}\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\right]\zeta_{\rho_2} = \pi_r\left[\left(\frac{r}{\rho_2}\right)^{1/2}\right]\zeta_{\rho_2} = \xi_r.$$

(2.12) follows because $\pi_r((r/\rho_2)^{1/2})$ has trivial kernel.

Substituting the formula (2.12) into (2.11), we obtain

$$\begin{aligned}\|\zeta_{\rho_2} - \zeta_{\rho_1}\|^2 &= \phi(\rho_2) - 2\ \mathrm{Re}<\pi_r\left[\left[\frac{\rho_1}{\rho_2}\right]^{1/2}\right]\zeta_{\rho_2},\zeta_{\rho_2}> + \phi(\rho_1) \\ &= \phi(\rho_2) - 2\phi\left[\rho_2\left[\frac{\rho_1}{\rho_2}\right]^{1/2}\right] + \phi(\rho_1) \\ &= \phi(\rho_2) - 2\phi((\rho_1\rho_2)^{1/2}) + \phi(\rho_1).\end{aligned}$$

Since ϕ is bounded and monotone increasing by Proposition 2.4, the right side of the latter expression tends to

$$\phi(1-) - 2\phi(1-) + \phi(1-) = 0,$$

as ρ_1 and ρ_2 tend to 1 from below. It follows that

$$\lim_{\rho_1,\rho_2\to 1-} \|\zeta_{\rho_2} - \zeta_{\rho_1}\| = 0,$$

proving the asserted existence of the strong limit

$$\zeta = \lim_{\rho\to 1-} \zeta_\rho .$$

To complete the proof of Theorem 2.2, it remains to show that (for any fixed $r \in (0,1)$) the triple (π_r,ζ, H_r) has the properties

(2.13) $$\phi(a) = <\pi_r(a)\zeta,\zeta>, \quad \|a\| < 1,$$

and

(2.14) $$[\pi_r(a)\zeta: \|a\| < 1] = H_r.$$

To this end, we claim first that

(2.15) $$\pi_r(\rho^{1/2})\zeta = \zeta_\rho ,$$

for every $\rho \in [r,1)$. Indeed, if σ tends to 1 from below then we know from arguments given above that

$$\pi_r\left(\left[\frac{\rho}{\sigma}\right]^{1/2}\right) \to \pi_r(\rho^{1/2})$$

in the strong operator topology, while at the same time,

$$\zeta_\sigma \to \zeta$$

in the norm of H_r. It follows that

$$\pi_r\left(\left[\frac{\rho}{\sigma}\right]^{1/2}\right)\zeta_\sigma \to \pi_r(\rho^{1/2})\zeta$$

in norm. On the other hand, for $\sigma > \rho$ the left side has the constant value ζ_ρ. (2.15) follows.

To prove (2.13), choose $a \in A$, $\|a\| < 1$, and choose ρ such that $\|a\| < \rho < 1$. Then by (2.15) we have

$$\begin{aligned} \langle\pi_r(a)\zeta,\zeta\rangle &= \langle\pi_r(\rho^{-1}a)\pi_r(\rho^{1/2})\zeta,\pi_r(\rho^{1/2})\zeta\rangle \\ &= \langle\pi_r(\rho^{-1}a)\zeta_\rho,\zeta_\rho\rangle \\ &= \phi_\rho(\rho^{-1}a) = \phi(a). \end{aligned}$$

Finally, the cyclicity condition (2.14) is straightforward, since for each $\rho \in (r,1)$ the set of vectors

$$\{\pi_r(a)\zeta\colon \|a\| < 1\}$$

contains the set

$$\{\pi_r(b)\pi_r(\rho^{1/2})\zeta\colon \|b\| \leqslant 1\} = \{\pi_r(b)\zeta_\rho\colon \|b\| \leqslant 1\},$$

which is fundamental in H_r because of the unitary equivalence of the triples (π_r,ζ_ρ, H_r) and $(\pi_\rho,\xi_\rho, H_\rho)$, together with the fact that

$$\{\pi_\rho(a)\xi_\rho\colon \|a\| < 1\}$$

is fundamental in H_ρ.

The proof of the uniqueness assertion of Theorem 2.2 is a routine matter of checking inner products, which we omit. □

We shall make use of the following consequence of Theorem 2.2 (and its counterpart for completely positive functions) at several points in the sequel.

COROLLARY. *Let* ϕ: ball A $\to \mathbb{C}$ *be a bounded positive function. Then the function*

$$\psi(a) = \phi(a) - \phi(0)$$

is also positive.

PROOF. Write

$$\phi(x) = \langle\pi(x)\xi,\xi\rangle, \quad \|x\| < 1,$$

where (π,ξ, H) is a triple as in Theorem 2.2. Since 0 is a self-adjoint idempotent in the *-semigroup ball A satisfying $0\cdot a = a\cdot 0 = 0$ for all $a \in$ ball A, it follows that $P = \pi(0)$ is a projection in the commutant of the range of π satisfying

$$P\pi(a) = \pi(a)P = P,$$

for all $\|a\| < 1$. Define ψ: ball A $\to \mathbb{C}$ by

$$\psi(x) = \langle\pi(x)(1 - P)\xi, (1 - P)\xi\rangle .$$

ψ is clearly of positive type, and we have

$$\begin{aligned}\psi(x) + \phi(0) &= \langle\pi(x)(1 - P)\xi, (1 - P)\xi\rangle + \langle P\xi,\xi\rangle \\ &= \langle\pi(x)(1 - P)\xi,\xi\rangle + \langle\pi(x)P\xi,\xi\rangle \\ &= \langle\pi(x)\xi,\xi\rangle = \phi(x),\end{aligned}$$

as required. □

Let A and B be two C*-algebras, and let π: ball A $\to$ B be a function. For every $n \geq 1$, we can define a function π_n: ball $M_n(A) \to M_n(B)$ by applying π element by element to matrices over A:

$$\pi_n: (a_{ij}) \longmapsto (\pi(a_{ij})).$$

DEFINITION 2.16. π is called *completely positive* if, for every $n \geqslant 1$, π_n carries positive elements of ball $M_n(A)$ into positive elements of $M_n(B)$.

In the case where the scalar-valued function ϕ: ball $A \to \mathbb{C}$ is completely positive, we can assert that its "GNS" representation π is also completely positive. Indeed, we have

THEOREM 2.17. *Let* ϕ: ball $A \to \mathbb{C}$ *be a bounded positive function, and let* $(\pi, \xi, \mathcal{H})$ *be a triple as in Theorem* 2.2. *Then* ϕ *is completely positive iff* π *is a completely positive function from* ball A *to* $L(\mathcal{H})$.

PROOF. Suppose ϕ is completely positive, and let (a_{ij}) be a positive $n \times n$ matrix in ball $M_n(A)$, $n \geqslant 1$. We have to show that if $\xi_1, \ldots, \xi_n$ are vectors in $\mathcal{H}$, then

$$\sum_{i,j=1}^{n} <\pi(a_{ij})\xi_j, \xi_i> \geqslant 0.$$

To see this, we may clearly assume that each ξ_j belongs to the dense set of $\mathcal{H}$ consisting of all linear combinations of vectors of the form $\pi(a)\xi$, $\|a\| < 1$. So assume that we have elements b_{jp} in ball A and $\lambda_{pj} \in \mathbb{C}$, $1 \leqslant j \leqslant n$, $1 \leqslant p \leqslant N$, such that

$$\xi_j = \sum_{p=1}^{N} \lambda_{p_j} \pi(b_{jp})\xi,$$

for $j = 1,2, \ldots, n$. Then

$$\sum_{i,j} <\pi(a_{ij})\xi_j, \xi_i> = \sum_{i,j,p,q} \lambda_{pj}\overline{\lambda}_{qi} <\pi(a_{ij})\pi(b_{jp})\xi, \pi(b_{iq})\xi> \tag{2.18}$$
$$= \sum \lambda_{pj}\overline{\lambda}_{qi} <\pi(b_{iq}^{*}a_{ij}b_{jp})\xi, \xi>$$
$$= \sum \lambda_{pj}\overline{\lambda}_{qi}\phi(b_{iq}^{*}a_{ij}b_{jp}).$$

Since the $nN \times nN$ matrix $(c_{\alpha\beta})$, defined for pairs $\alpha = (i,q)$, $\beta = (j,p)$, by

$$c_{(i,q),(j,p)} = b_{iq}^{*}a_{ij}b_{jp}$$

is positive semidefinite (this can be easily seen by expressing the $n \times n$ matrix (a_{ij}) as a product of $n \times n$ matrices D^*D, i.e.,

$$a_{ij} = \sum_{\ell=1}^{n} d_{\ell i}^* d_{\ell j} ,$$

and then checking the inequality $\Sigma\langle c_{\alpha\beta}\zeta_\beta,\zeta_\alpha\rangle \geq 0$ for arbitrary vectors $\zeta_\alpha \in H$), the complete positivity of ϕ implies that the last term in (2.18) is nonnegative.

The converse assertion of (2.17) amounts to nothing more than the observation that the composition of an (operator-valued) completely positive function with a (scalar-valued) completely positive function is completely positive. □

COROLLARY. *If* ϕ: ball A $\to \mathbb{C}$ *is a bounded completely positive function, then the function*

$$\psi(x) = \phi(x) - \phi(0), \qquad \|x\| < 1$$

is completely positive.

PROOF. The argument is a small variation on the proof of the corollary of Theorem 2.2, and is left for the reader. □

REMARK. Let r be a positive real number and let

$$\phi\colon \text{ball}_r A \to \mathbb{C}$$

be a bounded completely positive function. Then

$$\psi(x) = \phi(x) - \phi(0)$$

is a completely positive function on $\text{ball}_r A$. This follows from the preceding corollary by a simple change of scale. More explicitly, if

$$\phi_r\colon \text{ball } A \to \mathbb{C}$$

is defined on the open unit ball of A by

$$\phi_r(a) = \phi(ra),$$

then one observes that ϕ is completely positive iff ϕ_r is completely positive.

3. STATES OF $e^A \otimes e^{\bar{A}}$

Let A be a unital C*-algebra. The purpose of this section is to establish the following correspondence between real-analytic functions of positive type defined on the open unit ball of A and positive linear functionals on the C*-algebra $e^A \otimes e^{\bar{A}}$.

THEOREM 3.1. *Let ρ be a positive linear functional on $e^A \otimes e^{\bar{A}}$, and define a complex-valued function ϕ on* ball A *by*

$$\phi(a) = \rho(\Gamma(a) \otimes \Gamma(\bar{a})).$$

Then ϕ is bounded, real-analytic, and completely positive.

Conversely, if ϕ: ball A $\to \mathbb{C}$ *is a bounded real-analytic function of positive type, then there is a unique positive linear functional ρ on $e^A \otimes e^{\bar{A}}$ such that*

$$\phi(a) = \rho(\Gamma(a) \otimes \Gamma(\bar{a})), \quad \|a\| < 1.$$

REMARKS. Here, Γ is the map of ball A (resp. ball $\bar{A}$) into e^A (resp. $e^{\bar{A}}$) defined in Section 1,

$$\Gamma(a) = \sum_{n=0}^{\infty}{}^{\otimes} a^{(n)}, \quad \|a\| < 1.$$

We also remark that there is another description of states of $e^A \otimes e^{\bar{A}}$ in terms of bounded completely positive functions defined on ball A, which makes no reference to analyticity properties (cf. Theorem 5.1).

We begin by recalling a few basic results and terminology relating to analytic and holomorphic functions on infinite dimensional spaces. Let E be a complex Banach space and let B be a nonvoid open subset of E; for the purposes of this paper one may think of B as the open ball of radius $r > 0$ in a C*-algebra E. Let

$$f: B \to \mathbb{C}$$

be a complex-valued function and let n be a positive integer. f is called a (complex) homogeneous polynomial of degree n if there is a bounded complex multilinear form F, defined on the n-fold Cartesian product $E \times \cdots \times E$ of copies of E, such that

$$f(x) = F(x, ..., x), \quad x \in B.$$

If f does not vanish identically, then n is uniquely determined by f. By replacing F with its symmetrization $\breve{F}$, if necessary,

$$\breve{F}(x_1, ..., x_n) = \frac{1}{n!} \sum_{\pi \in S_n} F(x_{\pi(1)}, ..., x_{\pi(n)}),$$

we can clearly assume that F is a symmetric function in all of its variables. If $f \neq 0$, then in this case F is uniquely determined by f on all of $E \times \cdots \times E$ by a known polarization formula (e.g., see Proposition 3.2 below).

There is a similar definition of (real) homogeneous polynomials defined on real Banach spaces, which we will not reiterate here. In particular, if we think of the given complex Banach space E and the scalar field $\mathbb{C}$ as real Banach spaces, then we have a notion of (real) homogeneous polynomials $f: B \to \mathbb{C}$. Of course, it is important to keep the distinction in mind, and we will indicate this by systematically using the terminology *real* or *complex* when referring to complex-valued homogeneous polynomials.

For example, if $f: B \to \mathbb{C}$ is a nonzero complex homogeneous polynomial of degree n, then its complex conjugate $\bar{f}: B \to \mathbb{C}$, defined by $\bar{f}: x \mapsto \overline{f(x)}$, is not. However, both f and $\bar{f}$ are real homogeneous polynomials of degree n. In either case, a constant function is called a homogeneous polynomial of degree zero.

A function $f: B \to \mathbb{C}$ is called *real-analytic* (resp. *holomorphic*) if for every $x_0 \in B$ there is a $\delta > 0$ so that the ball of radius δ about x_0 is in B, and there is a sequence $f_0, f_1, ...$ of real (resp. complex) homogeneous polynomials such that f_n has degree n and

$$f(x_0 + x) = \sum_{n=0}^{\infty} f_n(x), \quad \|x\| < \delta,$$

the series on the right converging absolutely for $\|x\| < \delta$. In the case where B is the open unit ball of radius $r > 0$ in E and $x_0 = 0$, one actually has absolute convergence of the series representation

$$f(x) = \sum_{n=0}^{\infty} f_n(x),$$

for all $\|x\| < r$.

More generally, if $\check{E}$ is another complex Banach space and $f: B \to \check{E}$ is a given function, one says that f is real-analytic (resp. holomorphic) if for every bounded (complex-) linear functional ρ on $\check{E}$, the scalar valued function ρof has the corresponding property defined above. In this case, it is well-known that f can be locally represented as a norm-convergent vector power series

$$f(x_0 + x) = \sum_{n=0}^{\infty} f_n(x)$$

as above, in terms of vector-valued homogeneous polynomials $f: E \to \check{E}$; fortunately, we do not require the latter results in this paper.

Let $\mathcal{F}(E)$ be the complex vector space of all complex-valued functions defined on E. For each $h \in E$, let Δ_h be the difference operator on $\mathcal{F}(E)$ defined by

$$\Delta_h f(x) = f(x + h) - f(x), \quad x \in E.$$

For every $n \geq 1$ and every $f \in \mathcal{F}(E)$, we define a function $\Delta^n f$ of n+1 variables as follows:

$$\Delta^n f(x; h_1, ..., h_n) = (\Delta_{h_1} \Delta_{h_2} \cdots \Delta_{h_n} f)(x).$$

$\Delta^0 f$ is defined as f itself. The quantity $\Delta^n f$ plays a role analogous to the n^{th} order differential of a smooth function f, defined by

$$d^n f(x;h_1, ..., h_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} f(x + t_1 h_1 + \cdots + t_h h_n)\Big|_{t=0}.$$

In any case, it is clear that $\Delta^n f(x;h_1, ..., h_n)$ is a symmetric function in the n variables $h_1, ..., h_n$, simply because the difference operators Δ_h, $h \in E$, all commute with each other. The first few are given by:

$$\Delta^0 f(x) = f(x)$$

$$\Delta^1 f(x;h) = f(x + h) - f(x)$$

$$\Delta^2 f(x;h_1,h_2) = f(x+h_1+h_2) - f(x+h_1) - f(x+h_2) + f(x).$$

Finally, a real (resp. complex) polynomial is a finite linear combination of real (resp. complex) homogeneous polynomials. If

$$f(x) = \sum_{n=0}^{N} f_n(x),$$

where f_n is a homogeneous polynomial of degree n and $f_N \neq 0$, then as usual N is called the *degree* of f. We come now to the basic polarization formula for homogeneous polynomials and a characterization of polynomials in terms of the higher order difference operators Δ^n, n = 01,2, ...

PROPOSITION 3.2. *Let* n = 1,2, ... *and let* $F(x_1, ..., x_n)$ *be a symmetric real multilinear function of* n *variables on* E *and let* $f: E \to \mathbb{C}$ *be the associated homogeneous polynomial*

$$f(x) = F(x, ..., x), \quad x \in E.$$

Then

$$\Delta^n f(x;h_1, ..., h_n) = n!F(h_1, ..., h_n),$$

for all $x, h_1, ..., h_n$ *in* E.

A proof of Proposition 3.2 can be found in ([4], cf. formula (8), p. 322).

Notice that Proposition 3.2 implies that the same formula is valid for recovering a *complex* multilinear form from its associated complex homogeneus polynomial.

Note too that the right side of the above polarization formula is independent of x, and so $\Delta^{n+1}f = 0$ for every homogeneous polynomial f of degree n. It follows that if f is any real polynomial and n is an integer which exceeds the degree of f, then $\Delta^n f = 0$. The following result asserts the converse, and will be required in the sequel. We have been unable to find a reference to this result in the literature, and so we have included a proof for the convenience of the reader.

PROPOSITION 3.3. *Let* n *be a positive integer and let* $f: E \to \mathbb{C}$ *be a continuous function such that* $\Delta^{n+1}f = 0$.

(i) *There is a symmetric bounded real multilinear function* $F(x_1, \ldots, x_n)$ *of* n *variables on* E *such that*

$$\Delta^n f(x;h_1, \ldots, h_n) = F(h_1, \ldots, h_n)$$

for all $x, h_1, \ldots, h_n \in E$.

(ii) f *is a (real) polynomial of degree at most* n.

PROOF. The topology on E is that induced by the norm. However, the reader will note that the proof applies equally if f is merely weakly continuous, and in fact the conclusions remain valid under substantially weaker hypotheses.

In any case, suppose first that (i) is known to be valid for all n. Then (ii) follows by a straightforward inductive argument. For if f and F satisfy (i), and f_n: $B \to \mathbb{C}$ is defined to be the homogeneous polynomial

$$f_n(x) = \frac{1}{n!}F(x,x, \ldots, x).$$

then the polarization formula of Proposition 3.2 implies that

$$\Delta^n(f - f_n) = 0.$$

Thus by the induction hypothesis we see that $f - f_n$ is a polynomial of degree at most $n - 1$.

To prove (i), we again use induction on n. For $n = 1$, the hypothesis is $\Delta^2 f = 0$, or

$$f(x+h_1+h_2) - f(x+h_1) - f(x+h_2) + f(x) = 0,$$

for all $x, h_1, h_2 \in E$. Setting $x = 0$ in this equation yields

$$g(h_1 + h_2) = g(h_1) + g(h_2),$$

where g is the function $g(y) = f(y) - f(0)$. Thus g is a continuous homomorphism of the additive group of E into that of $\mathbb{C}$. Such a function is necessarily real-linear, and hence

$$f(x) = f(0) + g(x)$$

is a polynomial of degree at most 1.

Assume that $n \geq 2$ and that (i) is valid for $n - 1$. We are assuming that $\Delta^{n+1}f = 0$ so that if, for a fixed element h in E, we let

$$g_h(x) = \Delta_h f(x) = f(x + h) - f(x),$$

then $\Delta^n g_h = 0$ because

$$\begin{aligned}\Delta^n g_h(x;h_1, \ldots, h_n) &= (\Delta_{h_1} \cdots \Delta_{h_n} \Delta_h f)(x)\\ &= \Delta^{n+1} f(x;h_1, \ldots, h_n,h) = 0.\end{aligned}$$

By the induction hypothesis, there is a bounded symmetric real-multilinear function F_h of $n - 1$ variables such that

$$\Delta^{n-1} g_h(x;h_1, \ldots, h_{n-1}) = F_h(h_1, \ldots, h_{n-1})$$

for all x, $h_1, \ldots, h_{n-1}$ in E. Now the left side of this equation is

$$(\Delta_{h_1} \cdots \Delta_{h_{n-1}} \Delta_h f)(x) = \Delta^n f(x;h_1, \ldots, h_{n-1},h),$$

a symmetric function of all n variables $h_1, \ldots, h_{n-1},h$. Thus $F_h(h_1, \ldots, h_{n-1})$ too is symmetric in $h_1, \ldots, h_{n-1}h$; and since it is real-linear in each variable $h_1, \ldots, h_{n-1}$, it must also be real-linear in h. The same argument shows that it is (separately) continuous in h, as well as the other variables $h_1, \ldots, h_{n-1}$, and therefore it is a bounded symmetric real multilinear function in all n variables $h_1, \ldots, h_{n-1},h$. That completes the proof. □

With these preliminaries in hand, we now take up the proof of Theorem 3.1. Let ρ be a positive linear functional on $e^A \otimes e^{\overline{A}}$ and define ϕ: ball $A \to \mathbb{C}$ by

$$\phi(a) = \rho(\Gamma(a) \otimes \Gamma(\overline{a})), \quad \|a\| < 1.$$

The direct sum decomposition

$$e^A \otimes e^{\overline{A}} = \sum_{m,n=0}^{\infty} A^m \otimes \overline{A}^n$$

gives rise to a corresponding decomposition of ρ into positive linear functionals ρ_{mn}: $A^m \otimes \bar{A}^n \to \mathbb{C}$,

$$\rho = \sum_{m,n=0}^{\infty} \rho_{mn} ,$$

and we have

$$\|\rho\| = \Sigma \|\rho_{mn}\| .$$

Thus we have a decomposition of ϕ

(3.4) $$\phi(a) = \sum_{m,n=0}^{\infty} \phi_{mn}(a),$$

the sum on the right converging absolutely, where each ϕ_{mn} is defined by

$$\phi_{mn}(a) = \rho_{mn}(a^{(m)} \otimes \bar{a}^{(n)}), \quad \|a\| < 1.$$

Since ϕ_{mn} is clearly a real homogeneous polynomial of degree m+n, (3.4) shows that ϕ is real-analytic.

ϕ is bounded on ball A because for each $a \in A$, $\|a\| < 1$, we have

$$\|a^{(m)} \otimes \bar{a}^{(n)}\| \leqslant 1$$

for all $m,n \geqslant 0$ and thus

$$|\phi(a)| \leqslant \sum_{m,n} \|\rho_{mn}\| = \|\rho\|.$$

To show that ϕ is completely positive, it suffices to show that each ϕ_{mn} is completely positive. We may extend ρ_{mn} to a positive linear functional $\tilde{\rho}_{mn}$ on the full tensor product

$$A^{\otimes m} \otimes \bar{A}^{\otimes n}$$

without increasing its norm, and then we have a natural decomposition of ϕ_{mn} as

$$\phi_{mn} = \tilde{\rho}_{mn} \circ \Gamma_{mn} ,$$

where Γ_{mn}: ball A $\to A^{\otimes m} \otimes \bar{A}^{\otimes n}$ is the map $\Gamma_{mn}(a) = a^{\otimes m} \otimes \bar{a}^{\otimes m}$. $\tilde{\rho}_{mn}$ is a completely positive linear map of $A^{\otimes m} \otimes \bar{A}^{\otimes n}$ into $\mathbb{C}$, and thus it suffices to show that Γ_{mn} is completely positive. This follows from the following result, generalizing the fact that the Hadamard product of positive matrices is a positive matrix.

LEMMA 3.5. *Let* $A, B_1, \ldots, B_n$ *be* C*-*algebras and let*

$$\pi_i: \text{ball } A \to B_i$$

*be a completely positive mapping of *-semigroups,* $i = 1,2, \ldots, n$. *Then the tensor product of maps*

$$\pi_1 \otimes \cdots \otimes \pi_n: \text{ball } A \to B_1 \otimes \cdots \otimes B_n,$$

defined by $\pi_1 \otimes \cdots \otimes \pi_n(a) = \pi_1(a) \otimes \cdots \otimes \pi_n(a)$, *is a completely positive function.*

REMARK. Noting that both the inclusion map of ball A to A, and the map $a \mapsto \bar{a}$ of ball A to $\bar{A}$ are completely positive homomorphisms of *-semigroups, we conclude from Lemma 3.5 that each map Γ_{mn} is completely positive.

PROOF OF LEMMA 3.5. By an obvious induction argument, it suffices to prove the lemma for the case $n = 2$. Fix $k \geq 1$ and let (a_{ij}) be a positive element of $M_k(A)$. Define

$$c_{ij} = \pi_1(a_{ij}), \quad d_{ij} = \pi_2(a_{ij}).$$

Then (c_{ij}) (resp. (d_{ij})) is a positive element of $M_k(B_1)$ (resp. $M_k(B_2)$), and we have to show that the "Hadamard product"

$$(c_{ij} \otimes d_{ij}) \in M_k(B_1 \otimes B_2)$$

is positive. For that, it suffices to show that if $\sigma_i: B_i \to L(H)$, $i = 1,2$, is any pair of representations on the same Hilbert space H such that $\sigma_1(B_1)$ commutes with $\sigma_2(B_2)$, then the $k \times k$ operator matrix $(\sigma_1(b_{ij})\sigma_2(c_{ij}))$ is positive, considered as an operator on $\mathbb{C}^k \times H$.

For that, choose $\xi_1, \ldots, \xi_k \in H$, and consider the quantity

(3.6) $$\sum_{i,j=1}^{k} <\sigma_1(b_{ij})\sigma_2(c_{ij})\xi_j, \xi_i>.$$

To see that this is nonnegative, we find a k × k operator matrix (u_{ij}) in $M_k(\sigma_1(B_1))$ such that

$$\sigma_1(b_{ij}) = \sum_{p=1}^{k} u_{pi}^* u_{pj}, \quad 1 \leq i,j \leq k.$$

This is possible because $(\sigma_1(b_{ij}))$ is a positive k × k matrix over $\sigma_1(B_1)$. Similarly, we find $(v_{ij}) \in M_k(\sigma_2(B_2))$ such that

$$\sigma_2(c_{ij}) = \sum_{q=1}^{k} v_{qi}^* v_{qj}, \quad 1 \leq i,j \leq k.$$

Using the commutativity of $\sigma_1(B_1)$ and $\sigma_2(B_2)$, we can write

$$\begin{aligned} \sigma_1(b_{ij})\sigma_2(c_{ij}) &= \sum_{p,q} u_{pi}^* u_{pj} v_{qi}^* v_{qj} \\ &= \sum_{p,q} (u_{pi}v_{qi})^*(u_{pj}v_{qj}), \end{aligned}$$

and hence (3.6) becomes the expression

$$\sum_{p,q} \left\{ \sum_{i,j} <u_{pj}v_{qj}\xi_j, u_{pi}v_{qi}\xi_i> \right\} = \sum_{p,q} \left\| \sum_{j=1}^{k} u_{pj}v_{qj}\xi_j \right\|^2,$$

which is plainly nonnegative. □

Turning now to the proof of the converse assertion of Theorem 3.1, let

$$\phi\colon \text{ball } A \to \mathbb{C}$$

be a bounded real-analytic function of positive type. Let

(3.7) $$\phi(a) = \sum_{n=0}^{\infty} \phi_n(a), \quad \|a\| < 1$$

be the power series expansion of ϕ, where ϕ_n is a (real) homogeneous polynomial of degree n. We will first show that each coefficient function ϕ_n defines a bounded function of positive type on ball A. For that, we require

LEMMA 3.8. *Let* f: [0,1] → $\mathbb{C}$ *be a bounded function of positive type on the *-semigroup* $0 \leq t \leq 1$, *which is real analytic*:

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \quad 0 \leq t \leq 1.$$

Then

$$0 \leq a_n \leq \sup_{0 \leq t \leq 1} |f(t)|,$$

for every n = 0,1, ...

PROOF. It suffices to show that $a_n \geq 0$ for every n, for then we clearly have

$$\sum_{n=0}^{\infty} a_n = f(1) \leq \sup_{0 \leq t \leq 1} |f(t)|,$$

which implies the asserted upper bound on each a_n.

Clearly $a_0 = f(0)$ must be nonnegative. It suffices to show that the function g: $[0,1] \to \mathbb{C}$ defined by

$$g(t) = \begin{cases} f(t)-f(0)/t, & t > 0 \\ f'(0), & t = 0 \end{cases}$$

satisfies the same hypotheses as f. For the power series of g is

$$g(t) = \sum_{n=0}^{\infty} a_{n+1} t^n ,$$

which implies in these circumstances that $a_1 = g(0) \geq 0$; we can repeat the same argument to obtain $a_2 \geq 0$, and so on.

g is clearly bounded and real analytic on [0,1]. To show that g is of positive type, it suffices by continuity to show that

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j g(t_i t_j) \geq 0$$

for all $t_1, ..., t_n \in (0,1]$, $\lambda_1, ..., \lambda_n \in \mathbb{C}$, $n \geq 1$. Replacing λ_i with $t_i^{-1}\lambda_i$, we see that the latter is equivalent to showing that the function

$$tg(t) = f(t) - f(0)$$

is of positive type on $0 < t \leq 1$.

For that, consider the function $t \in [0,1] \longmapsto f(t) - f(0)$. Since [0,1] is a *-semigroup *with unit*, Lemma 2.3 shows that there is a triple (π, ξ, H) with properties stated there, for which

$$f(t) = \langle \pi(t)\xi, \xi \rangle, \quad 0 \leq t \leq 1.$$

The proof of the corollary of Theorem 2.2 applies verbatim here (replacing ball A with [0,1]) to show that the function

$$t \longmapsto f(t) - f(0)$$

is of positive type on [0,1], and so we have the desired conclusion. □

Let us now show that *the homogeneous polynomials* ϕ_n: A → $\mathbb{C}$ *defined by* (3.7) *are of positive type and bounded on* ball A. For that, fix $N \geq 1$, $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, and $a_1, \ldots, a_N \in$ ball A. We will show that each of the numbers

$$\alpha_n = \sum_{i,j=1}^{N} \lambda_i \bar{\lambda}_j \phi_n(a_j^* a_i)$$

is nonnegative, and is bounded by

$$\|\phi\|_\infty = \sup_{\|a\|<1} |\phi(a)| < \infty .$$

For the first conclusion, consider the power series in the real varible t given by

$$u(t) = \sum_{n=0}^{\infty} \alpha_n t^n .$$

This series converges uniformly on $0 \leq t \leq 1$ to the function

$$\sum_{i,j=1}^{N} \lambda_i \bar{\lambda}_j \phi(t a_j^* a_i).$$

Notice that u satisfies the hypotheses of Lemma 3.8. Indeed, if $t_1, \ldots, t_M \in [0,1]$ and $\mu_1, \ldots, \mu_M \in \mathbb{C}$, then we have

$$\begin{aligned} \sum_{p,q} \mu_p \bar{\mu}_q u(t_p t_q) &= \sum_{i,j,p,q,n} \mu_p \bar{\mu}_q \lambda_i \bar{\lambda}_j (t_p t_q)^n \phi_n(a_j^* a_i) \\ &= \sum_{i,j,p,q} \mu_p \lambda_i \overline{\mu_q \lambda_j}\, \phi((t_q a_j)^* t_p a_i), \end{aligned}$$

and the latter term is clearly nonnegative because ϕ is of positive type. By Lemma 3.8, we conclude that

$$\alpha_n = \sum_{i,j=1}^{N} \lambda_i \bar{\lambda}_j \phi_n(a_j^* a_i) \geq 0,$$

for every n, so that each ϕ_n is of positive type.

Moreover, taking N = 1 and $\lambda_1 = 1$, we see that for each n,

(3.9) $$0 \leq \phi_n(a^*a) \leq \sup_{0 \leq t \leq 1} \phi(ta^*a) \leq \|\phi\|_\infty$$

for all a ∈ ball A. To deduce from this that ϕ_n is bounded by $\|\phi\|_\infty$, choose b ∈ ball A and r so that $\|b\| < r < 1$. Then since the 2 × 2 matrix

$$\begin{bmatrix} r^2 1 & a \\ a^* & r^{-2}a^*a \end{bmatrix}$$

has the form $(c_j^* c_i)$, with $c_1 = r1$, $c_2 = r^{-1}a$, it follows that the 2 × 2 scalar matrix

$$\begin{bmatrix} \phi_n(r^2 1) & \phi_n(a) \\ \phi_n(a^*) & \phi_n(r^{-2}a^*a) \end{bmatrix}$$

is positive semidefinite. Therefore $\phi_n(a^*) = \overline{\phi_n(a)}$ and, moreover,

$$|\phi_n(a)|^2 = \phi_n(a^*)\phi_n(a) \leq \phi_n(r^2 1)\phi_n(r^{-2}a^*a).$$

By (3.9), the right side of this inequality is dominated by $\|\phi\|_\infty^2$, and thus we obtain the desired estimate on the bound of each ϕ_n:

$$\sup_{\|a\|<1} |\phi_n(a)| \leq \|\phi\|_\infty .$$

The next step in the proof of Theorem 3.1 is to show that, for each n ≥ 1, there is a positive *complex*-linear functional f_n on the C^*-algebra $(A \oplus \bar{A})^n$ such that

(3.10) $$\phi_n(a) = f_n((a \oplus \bar{a}) \otimes \cdots \otimes (a \oplus \bar{a})),$$

for all $a \in A$. For the proof of (3.10), we require

LEMMA 3.11. *Let* E *be a complex vector space and let* $f\colon E \to \mathbb{C}$ *be a real-linear functional. Then there is a unique complex-linear functional* $g\colon E \oplus \overline{E} \to \mathbb{C}$ *such that* $f(x) = g(x \oplus \overline{x})$, $x \in E$.

If E *is a Banach space and* f *is continuous, then so is* g.

PROOF (Uniqueness). Let $x \longmapsto \overline{x}$ denote the natural antilinear isomorphism of E onto $\overline{E}$, and let $d\colon E \to E \oplus \overline{E}$ be the real-linear map defined by

$$d(x) = x \oplus \overline{x}\,.$$

It suffices to show that

$$d(E) + id(E) = E \oplus \overline{E}\,.$$

The inclusion $\subseteq$ is obvious. For $\supseteq$, choose $x \in E$. Writing

$$\begin{aligned} x \oplus 0 &= \frac{1}{2}\,[x \oplus \overline{x} + x \oplus (-\overline{x})] \\ &= \frac{1}{2}\,(x \oplus \overline{x}) - \frac{i}{2}\,(ix) \oplus (-i\overline{x}) \\ &= d\left(\frac{1}{2}\,x\right) - id\left(\frac{i}{2}\,x\right), \end{aligned}$$

we see that $x \oplus 0$ belongs to $d(E) + id(E)$. Similarly,

$$\begin{aligned} 0 \oplus \overline{x} &= \frac{1}{2}\,[x \oplus \overline{x} - x \oplus (-\overline{x})] \\ &= \frac{1}{2}\,(x \oplus \overline{x}) + \frac{i}{2}\,(ix \oplus (-i\overline{x})) \\ &= d\left(\frac{1}{2}\,x\right) + id\left(\frac{i}{2}\,x\right) \end{aligned}$$

belongs to $d(E) + id(E)$, proving the assertion.

(Existence). Define a function $g\colon E \oplus \overline{E} \to \mathbb{C}$ by

$$g(x \oplus \overline{y}) = \frac{1}{2}\,(f(x) + f(y)) + \frac{i}{2}\,(f(iy) - f(ix)).$$

g is clearly real-linear, and g∘d = f. The reader may easily check that $g(iz) = ig(z)$, by the definition of g in terms of f. Thus g is complex-linear.

If E is a Banach space and f is continuous, the definition of g exhibits it as a sum of continuous functions, hence g is continuous. □

LEMMA 3.12. *Let* E *be a complex vector space and let* $f: E \to \mathbb{C}$ *be a (real) homogeneous polynomial of degree* $n \geq 1$. *Then there is a unique symmetric complex-multilinear form* $g(z_1, \ldots, z_n)$ *on* $(E \oplus \bar{E}) \times \cdots \times (E \oplus \bar{E})$ *such that*

$$f(x) = g(x \oplus \bar{x}, \ldots, x \oplus \bar{x}), \quad x \in E.$$

If E *is a Banach space and* f *is continuous, then* g *is bounded.*

PROOF (Existence). Utilizing the polarization formula of Proposition 3.2, we see that the function

$$F(x_1, \ldots, x_n) = \frac{1}{n!}(\Delta_{x_1} \cdots \Delta_{x_n} f)(0)$$

is a symmetric (real) multilinear form on $E \times \cdots \times E$ such that

$$f(x) = F(x, \ldots, x), \quad x \in E.$$

Arguing in each variable separately, we may apply Lemma 3.11 to obtain a complex multilinear symmetric form $g(z_1, \ldots, z_n)$ on $(E \oplus \bar{E}) \times \cdots \times (E \oplus \bar{E})$ such that

$$F(x_1, \ldots, x_n) = g(x_1 \oplus \bar{x}_1, \ldots, x_n \oplus \bar{x}_n),$$

for all $x_1, \ldots, x_n \in E$.

If E is a Banach space and f is continuous, then the definition of F shows it to be (jointly) continuous. The explicit formula for g in terms of F which is implied by the construction in the proof of Lemma 3.11 (and which can in fact be written down explicitly) shows that g is continuous, and therefore bounded.

(Uniqueness). Let $\breve{g}(z_1, \ldots, z_n)$ be another symmetric multilinear form on $(E \oplus \bar{E}) \times \cdots \times (E \oplus \bar{E})$ satisfying

$$\breve{g}(x \oplus \bar{x}, ..., x \oplus \bar{x}) = f(x), \quad x \in E.$$

Letting $F(x_1, ..., x_n)$ be the form on $E \times \cdots \times E$ defined as above, we infer that

$$(\Delta_{x_1 \oplus \bar{x}_1} \cdots \Delta_{x_n \oplus \bar{x}_n} \breve{g})(0) = (\Delta_{x_1} \cdots \Delta_{x_n} f)(0)$$
$$= F(x_1, ..., x_n),$$

for all $x_1, ..., x_n \in E$. The left side of this expression is simply $\breve{g}(x_1 \oplus \bar{x}_1, ..., x_n \oplus \bar{x}_n)$. Thus, if g denotes the multilinear form constructed above, then we have

$$(g - \breve{g})(x_1 \oplus \bar{x}_1, ..., x_n \oplus \bar{x}_n) = 0,$$

for all $x_1, ..., x_n \in E$. Using the uniqueness assertion of Lemma 3.11 and the fact that $g - \breve{g}$ is complex-linear, we may argue one variable at a time in the preceding equation to conclude that $g - \breve{g} = 0$. □

Returning now to the discussion preceding Lemma 3.11, we may find, for each $n \geq 1$, a bounded symmetric complex multilinear functional of n variables

$$g_n: (A \oplus \bar{A}) \times \cdots \times (A \oplus \bar{A}) \to \mathbb{C}$$

such that

$$\phi_n(a) = g_n(a \oplus \bar{a}, ..., a \oplus \bar{a}), \quad a \in A.$$

Letting B_n denote the completion of the n-fold symmetric tensor product of copies of $A \oplus \bar{A}$ in the *projective* cross norm, we see from general principles that there is a unique bounded (complex) linear functional f_n on B_n such that

$$g(z_1, ..., z_n) = f_n(z_1 \vee z_2 \vee \cdots \vee z_n),$$

for $z_1, ..., z_n \in A \oplus \bar{A}$. Here, we are using the notation $z_1 \vee \cdots \vee z_n$ to denote the projection of the elementary tensor

$$z_1 \otimes \cdots \otimes z_n \in (A \oplus \bar{A})^{\otimes n}$$

to the symmetric tensor product B_n, under the natural symmetrization operator:

$$z_1 \vee \cdots \vee z_n = \frac{1}{n!} \sum_{\pi \in S_n} z_{\pi(1)} \otimes \cdots \otimes z_{\pi(n)} .$$

Now $A \oplus \bar{A}$ is a unital C*-algebra, and therefore B_n is a unital Banach *-algebra relative to the operations defined by

$$(3.13) \qquad \begin{aligned} &(z_1 \vee \cdots \vee z_n)^* = z_1^* \vee \cdots \vee z_n^* \\ &(z_1 \vee \cdots \vee z_n)(w_1 \vee \cdots \vee w_n) = z_1 w_1 \vee \cdots \vee z_n w_n . \end{aligned}$$

Therefore, we may speak of positive linear functionals on the Banach *-algebra B_n, and we have

LEMMA 3.14. f_n *is a positive linear functional on* B_n.

PROOF. We claim first that the set of elements of B_n of the form

$$(x \oplus \bar{x}) \vee \cdots \vee (x \oplus \bar{x}),$$

where $x \in A$, spans B_n. To see this, let ℓ be a bounded linear functional on B_n such that

$$\ell((x \oplus \bar{x}) \vee \cdots \vee (x \oplus \bar{x})) = 0,$$

for all $x \in A$. Clearly

$$(z_1, \ldots, z_n) \longmapsto \ell(z_1 \vee \cdots \vee z_n)$$

defines a bounded symmetric complex multilinear functional on $(A \oplus \bar{A}) \times \cdots \times (A \oplus \bar{A})$ which vanishes identically on "diagonal" elements of the form $(x \oplus \bar{x}, x \oplus \bar{x}, \ldots, x \oplus \bar{x})$. The uniqueness assertion of Lemma 3.12 implies that this multilinear form is zero, and hence $\ell = 0$ because B_n is spanned by elements of the form $z_1 \vee \cdots \vee z_n$, $z_i \in A \oplus \bar{A}$, proving the claim.

Now f_n is a bounded linear functional on the *-algebra B_n, so to show that f_n is positive, it suffices to show that $f_n(\zeta^*\zeta) \geq 0$ for all ζ belonging to the dense set of all elements of the form

$$\zeta = \sum_{j=1}^{N} \lambda_j(x_j \oplus \bar{x}_j) \vee \cdots \vee (x_j \oplus \bar{x}_j),$$

where $\lambda_1, ..., \lambda_N \in \mathbb{C}$, $x_1, ..., x_n \in A$, $N \geq 1$. Fix such a ζ. Then

$$\zeta^*\zeta = \sum_{j,k=1}^{n} \bar{\lambda}_j\lambda_k(x_j^*x_k \oplus \bar{x}_j^*\bar{x}_k) \vee \cdots \vee (x_j^*x_k \oplus \bar{x}_j^*x_k).$$

Thus, using linearity of f_n and the relation existing between f_n and ϕ_n, we have

$$f_n(\zeta^*\zeta) = \sum_{j,k=1}^{n} \bar{\lambda}_j\lambda_k\phi_n(x_j^*x_k).$$

The latter is nonnegative because ϕ_n is of positive type. □

Now of course, B_n is not a C*-algebra. But it has an enveloping C*-algebra $C^*(B_n)$ ([3], [8]). Letting α_n: $B_n \to C^*(B_n)$ be the natural *-homomorphism, Lemma 3.14 implies that there is a unique *positive* linear functional ρ_n on $C^*(B_n)$ such that

(3.15) $$\rho_n \circ \alpha_n = f_n .$$

We now want to show that $C^*(B_n)$ is naturally identified with the C*-algebra $(A \oplus \bar{A})^n$, in such a way that Lemma 3.14 becomes the relation

(3.16) $$\rho_n(b^{(n)}) = f_n(b \vee b \vee \cdots \vee b), \quad b \in A \oplus \bar{A}.$$

In view of the relation existing between f_n and ϕ_n, (3.16) will then imply

(3.17) $$\phi_n(a) = \rho_n((a \oplus \bar{a})^{(n)}), \quad a \in A.$$

This will follow from

LEMMA 3.18. *Let* B *be a unital C*-algebra and let* B_n *be the completion of the symmetric tensor product of* n *copies of* B *in the projective cross norm. Make* B_n *into a Banach *-algebra by defining the involution and multiplication as in* (3.13).

Let B^n *be the C*-algebraic symmetric tensor product of* n *copies* *of* B. *Then the unique bounded linear map* β: $B_n \to B^n$, *defined on generators of* B_n *by*

$$\beta(z^{(n)}) = z^{(n)}, \quad z \in B,$$

*is a *-homomorphism of Banach *-algebras. It extends uniquely to a *-isomorphism of* $C^*(B_n)$ *onto* B^n.

REMARK. Let α: $B_n \to C^*(B_n)$ be the natural *-homomorphism, there is a unique morphism of C*-algebras $\tilde{\beta}$: $C^*(B_n) \to B^n$ such that the diagram

$$\text{(3.19)} \qquad \begin{array}{ccc} B_n & \xrightarrow{\ \beta\ } & B^n \\ {\scriptstyle \alpha}\downarrow & \nearrow_{\tilde{\beta}} & \\ C^*(B_n) & & \end{array}$$

commutes. The essential content of Lemma 3.18 is that $\tilde{\beta}$ is an isomorphism.

PROOF OF LEMMA 3.18. The proof is quite straightforward, and we merely sketch the details for the case n = 2. Since the bilinear map of B × B into B^2 defined by

$$(z_1, z_2) \longmapsto z_1 \vee z_2$$

is bounded and symmetric, it follows from general properties of projective tensor products that there is a bounded linear map β: $B_2 \to B^2$ such that

$$\beta(z_1 \vee z_2) = z_1 \vee z_2\,,$$

$z_i \in B$. β is uniquely determined by its action on elements of the form $z^{(2)} = z \vee z$ because these span B_2. The definitions of multiplication and involution in B_2 lead directly to the fact that β is a *-homomorphism.

Let $\tilde{\beta}$: $C^*(B_n) \to B^n$ be the *-homomorphism which completes the diagram (3.19). $\tilde{\beta}$ clearly has dense range, and we only need to show that $\tilde{\beta}$ is injective.

To do that, it suffices to show that for every positive linear functional ρ on $C^*(B_2)$, there is a positive linear functional σ on B^2 such that

$$\sigma = \rho \circ \breve{\beta}.$$

To prove that, fix ρ. Then

$$(z_1, z_2) \longmapsto \rho(z_1 \vee z_2)$$

is a bounded bilinear form on $B \times B$ and so there is a bounded linear functional $\breve{\rho}$ on the completion $B \hat{\otimes} B$ of the *full* algebraic tensor product, in the projective cross norm, such that

$$\breve{\rho}(z_1 \otimes z_2) = \rho(z_1 \vee z_2), \quad z_i \in B.$$

We can make $B \hat{\otimes} B$ into a Banach *-algebra in a natural way, by putting $(z_1 \otimes z_2)^* = z_1^* \otimes z_2^*$ and $(z_1 \otimes z_2)(w_1 \otimes w_2) = z_1 w_1 \otimes z_2 w_2$, and then we have

$$\breve{\rho}(\zeta^*\zeta) \geqslant 0 \quad \textit{for all} \quad \zeta \in B \hat{\otimes} B.$$

By the usual GNS construction, we find a representation π of $B \hat{\otimes} B$ on a Hilbert space $\mathcal{H}$ and a vector $\xi \in \mathcal{H}$ such that

$$\breve{\rho}(\zeta) = \langle \pi(\zeta)\xi, \xi \rangle, \quad \zeta \in B \hat{\otimes} B.$$

Define $\pi_i: B \to \mathcal{H}$ by

$$\pi_1(b) = \pi(b \otimes 1)$$

$$\pi_2(b) = \pi(1 \otimes b), \quad b \in B.$$

Then $\pi_1(B)$ and $\pi_2(B)$ are mutually commuting C*-algebras of operators on $\mathcal{H}$. So by the universal properties of the largest C*-*algebraic* cross norm, there is a unique representation $\breve{\pi}: B \otimes B \to \mathcal{L}(\mathcal{H})$ such that

$$\breve{\pi}(b_1 \otimes b_2) = \pi_1(b_1)\pi_2(b_2).$$

Thus, we can define a positive linear functional σ on B^2 by

$$\sigma(w) = \langle\breve{\pi}(w)\xi,\xi\rangle, \quad w \in B^2.$$

The formula $\sigma = \rho \circ \breve{\beta}$ is immediate, for we have for each $z \in B$,

$$\begin{aligned}\rho \circ \breve{\beta}(z^{(2)}) &= \langle\pi(z \otimes z)\xi,\xi\rangle \\ &= \langle\pi_1(z)\pi_2(z)\xi,\xi\rangle \\ &= \langle\breve{\pi}(z \otimes z)\xi,\xi\rangle \\ &= \sigma(z^{(2)}).\end{aligned}$$

That finishes the proof. □

To summarize, we have shown that if ϕ: ball $A \to \mathbb{C}$ is a bounded real-analytic function of positive type, and if

$$\phi(a) = \sum_{n=0}^{\infty} \phi_n(a), \quad \|a\| < 1,$$

is the power series expansion of ϕ into (real) homogeneous polynomials ϕ_n of degree n, then there are positive linear functionals

$$\rho_n : (A \oplus \bar{A})^n \to \mathbb{C}$$

such that

$$\phi_n(a) = \rho_n((a \oplus \bar{a})^{(n)}), \quad \|a\| < 1. \tag{3.20}$$

It only remains to get proper estimates on the norms $\|\rho_n\|$. In fact, we will show that

$$\sum_{n=0}^{\infty} \|\rho_n\| \leq \|\phi\|_\infty . \tag{3.21}$$

Assuming (3.21) for the moment, notice that the representation required by Theorem 3.1 follows. Indeed, the direct sum decomposition of C*-algebras

$$(A \oplus \bar{A})^n = \sum_{p+q=n} A^p \otimes \bar{A}^q$$

induces a corresponding decomposition

$$(3.22) \qquad \rho_n = \sum_{p+q=n} \rho_{pq}$$

where ρ_{pq} is a positive linear functional on $A^p \otimes \bar{A}^q$, in which

$$\|\rho_n\| = \sum_{p+q=n} \|\rho_{pq}\|.$$

So by (3.21), we will have

$$\sum_{p,q=0}^{\infty} \|\rho_{pq}\| \leqslant \|\phi\|_\infty .$$

Therefore, we may define a positive linear functional ρ on $e^A \otimes e^A$ by

$$\rho = \sum_{p,q=0}^{\infty} \rho_{pq} ,$$

and formulas (3.20) and (3.22) combine to give thc required representation of ϕ:

$$\begin{aligned}\phi(a) &= \sum_{p,q=0}^{\infty} \rho_{pq}(a^{(p)} \otimes \bar{a}^{(q)}) \\ &= \rho(\Gamma(a) \otimes \Gamma(\bar{a})), \qquad \|a\| < 1.\end{aligned}$$

The proof of (3.21) is quite simple. For each $r \in [0,1)$, we have

$$\|\phi\|_\infty \geqslant \phi(r1) = \sum_{n=0}^{\infty} \phi_n(r \cdot 1) = \sum_{n=0}^{\infty} \rho_n(e_{r,n}),$$

where $e_{r,n}$ is the positive element of $(A \oplus \bar{A})^n$ given by

$$e_{r,n} = (r1 \oplus \overline{r1})^{(n)}.$$

For fixed n, the net of operators $e_{r,n}$ increases as r increases to 1, and converges in norm to $(1 \oplus \bar{1})^{(n)}$, the identity of $(A \oplus \bar{A})^n$. Since ρ_n is a positive linear functional, the net

$$r \in [0,1) \longmapsto \rho_n(e_{r,n})$$

increases to $\|\rho_n\|$ as r increases to 1. So by the monotone convergence theorem, we have

$$\lim_{r\uparrow 1} \phi(r1) = \sum_{n=0}^{\infty} \lim_{r\uparrow 1} \rho_n(e_{r,n})$$

$$= \sum_{n=0}^{\infty} \|\rho_n\| ,$$

from which (3.21) follows. $\square$

The only part of Theorem 3.1 that has not yet been proved is the uniqueness assertion; that will follow from:

LEMMA 3.23. $e^A \otimes e^{\bar{A}}$ *is the closed linear span of* $\{\Gamma(a) \otimes \Gamma(\bar{a}): \|a\| < 1\}$.

PROOF. Let ρ be a bounded linear functional on $e^A \otimes e^{\bar{A}}$ such that

$$\rho(\Gamma(a) \otimes \Gamma(\bar{a})) = 0, \quad \|a\| < 1.$$

We have to show that $\rho = 0$.

If we realize $e^A \otimes e^{\bar{A}}$ as the direct sum of C*-algebras

$$e^A \otimes e^{\bar{A}} = \sum_{n=0}^{\infty} (A \oplus \bar{A})^n,$$

then we have a corresponding decomposition of ρ and $\Gamma \otimes \bar{\Gamma}$:

$$\rho = \sum_{n=0}^{\infty} \rho_n,$$

$$\Gamma(a) \otimes \Gamma(\bar{a}) = \sum_{n=0}^{\infty} (a \oplus \bar{a})^{(n)}, \quad \|a\| < 1.$$

Thus

$$\sum_{n=0}^{\infty} \rho_n((a \oplus \bar{a})^{(n)}) = \rho(a) = 0, \quad \|a\| < 1.$$

Fixing a in ball A and $-1 \leqslant t \leqslant 1$, the above implies

$$\sum_{n=0}^{\infty} t^n \rho_n((a \oplus \bar{a})^{(n)}) = \rho(ta) = 0.$$

the left side being an absolutely convergent power series in t. Thus we must have

$$\rho_n((a \oplus \bar{a})^{(n)}) = 0, \quad \|a\| < 1,$$

for every $n = 0,1,2, \dots$.

From the uniqueness assertion of Lemma 3.12 we conclude that $\rho_n = 0$ for all n, and finally

$$\rho = \sum_{n=0}^{\infty} \rho_n = 0,$$

as required. □

4. CONTINUITY OF COMPLETELY POSITIVE FUNCTIONS

The purpose of this section is to show that completely positive functions are continuous. As it turns out, we need only consider functions which are defined on all of the given C*-algebra; however, the reader should note that the conclusion is true in the more general case of bounded completely positive functions defined on the open unit ball (cf. Theorem 5.1).

THEOREM 4.1. *Let* A *be a unital* C**-algebra and let* ϕ: A $\to \mathbb{C}$ *be a completely positive function. Then* ϕ is continuous.

We have organized the proof into a series of simple lemmas. We continue to use the somewhat abbreviated notation $\phi(\lambda)$ for the value of ϕ at a scalar multiple $\lambda 1$, $\lambda \in \mathbb{C}$, of the identity of A.

LEMMA 4.2. *For every* $r \geq 0$,

$$\sup_{\|a\| \leq r} |\phi(a)| = \phi(r).$$

PROOF. The nontrivial inequality $\leq$ follows from the observation that if an element $a \in A$ satisfies $\|a\| \leq r$, then the 2×2 operator matrix

$$\begin{bmatrix} r1 & a \\ a^* & r1 \end{bmatrix}$$

is positive and therefore

$$\begin{bmatrix} \phi(r) & \phi(a) \\ \phi(a^*) & \phi(r) \end{bmatrix}$$

is a positive element of $M_2(\mathbb{C})$. Therefore $\phi(a^*) = \overline{\phi(a)}$, $\phi(r) \geq 0$ and $\phi(r)^2 \geq \phi(a)\phi(a^*) = |\phi(a)|^2$, from which Lemma 4.2 follows. □

LEMMA 4.3. *For every positive element* $h \in A$, *the function* $\Delta_h\phi$: $A \to \mathbb{C}$ *defined by*

$$\Delta_h\phi(a) = \phi(a + h) - \phi(a)$$

is completely positive.

PROOF. Let $n \geq 1$, let (a_{ij}) be a positive element of $M_n(A)$, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. We have to show that

$$\sum_{i,j=1}^{n} \lambda_i\bar{\lambda}_j(\phi(a_{ij} + h) - \phi(a_{ij})) \geq 0.$$

Consider the function u: $A \to \mathbb{C}$ defined by

$$u(b) = \sum_{i,j=1}^{n} \lambda_i\bar{\lambda}_j\phi(a_{ij} + b).$$

Then the desired inequality is simply $u(h) \geq u(0)$. By the corollary of Theorem 2.17 and the subsequent remark, it suffices to show that u is completely positive and bounded on some open ball of radius $r > \|h\|$.

To see that, fix $r > \|h\|$, and let (b_{ij}) be a positive element of $M_m(A)$, $m \geq 1$, satisfying $\|(b_{ij})\| < r$. We have, for each $\mu_1, \ldots, \mu_m \in \mathbb{C}$,

$$\sum_{p,q} \mu_p\bar{\mu}_q u(b_{pq}) = \sum_{i,j,p,q} \lambda_i\mu_p\bar{\lambda}_j\bar{\mu}_q\phi(a_{ij} + b_{pq}),$$

which is nonnegative by the completely positivity of ϕ, because it has the form

$$\sum_{\alpha,\beta} \nu_\alpha\bar{\nu}_\beta\phi(c_{\alpha\beta}),$$

where α and β range over all ordered pairs

$$\{(i,p)\colon 1 \leqslant i \leqslant n,\ 1 \leqslant p \leqslant m\},$$

where $\nu_{(i,p)} = \lambda_i \mu_p$, and where $(c_{\alpha\beta})$ is the positive mn × mn matrix whose entries are given by

$$c_{(i,p),(j,q)} = a_{ij} + b_{pq}\,.$$

This shows that u is completely positive on $\mathrm{ball}_r A$ for every r > 0. Lemma 4.2 now implies the necessary boundedness condition, and so we have the desired conclusion. □

LEMMA 4.4. *Let* a, h *belong to* A, *with* h *positive and* a ≠ 0. *Then*

$$|\phi(a + h) - \phi(a)| \leqslant \phi(\|a\| + \|h\|) - \phi(\|a\|).$$

PROOF. In the notation of Lemma 4.3, the function $x \longmapsto \Delta_h\phi(x)$ is completely positive. So by Lemma 4.2 we have

$$|\Delta_h\phi(a)| \leqslant \Delta_h\phi(\|a\|) = \phi(\|a\| + h) - \phi(\|a\|).$$

The same argument, applied to the function $x \longmapsto \phi(\|a\| + x) - \phi(\|a\|)$, shows that the right side of the preceding inequality is dominated by

$$\phi(\|a\| + \|h\|) - \phi(\|a\|),$$

as required. □

We also require the following result, essentially a corollary of Proposition 2.4.

LEMMA 4.5. *Let* $u\colon (0,\infty) \to \mathbb{R}^+$ *be a function satisfying*

(i) $\sup\limits_{0 \leqslant t \leqslant T} u(t) < \infty$, *for all* T > 0, *and*

(ii) $\begin{bmatrix} u(s^2) & u(st) \\ u(st) & u(t^2) \end{bmatrix} \geqslant 0$, *for every* s, t > 0.

Then u *is monotone increasing and continuous.*

PROOF. For each T > 0, define a function u_T on the open unit interval (0,1) by $u_T(s) = u(Ts)$. The reader may easily check that u_T satisfies the hypotheses of Proposition 2.4, and hence u_T is increasing and continuous on (0,1). Since T is arbitrary, this implies the conclusion of Lemma 4.5. □

LEMMA 4.6. *Let* $a,b \in A$, $a \neq 0$. *Then*

$$|\phi(b) - \phi(a)| \leq 2(\phi(\|a\| + 2\|b - a\|) - \phi(\|a\|)).$$

PROOF. Assume first that the difference $b - a$ is self-adjoint. Then we can write

$$b - a = h - k,$$

where h and k are positive elements of A satisfying $\|h\| \leq \|b-a\|$, $\|k\| \leq \|b-a\|$. We have

$$|\phi(b) - \phi(a)| = |\phi(a-k+h) - \phi(a)|$$

$$\leq |\phi(a-k+h) - \phi(a-K)| + |\phi(a) - \phi(a-k)|.$$

It suffices to show that each of the terms on the right is dominated by the quantity

$$\phi(\|a\| + 2\|b - a\|) - \phi(\|a\|).$$

Indeed, Lemma 4.4 implies that

$$|\phi(a-k+h) - \phi(a-k)| \leq \phi(\|a-k\| + \|h\|) - \phi(\|a-k\|).$$

If we now consider the function $u: (0,\infty) \to \mathbb{R}$ defined by

$$u(t) = \phi(t + \|h\|) - \phi(t),$$

we see that u can be written in the form

$$u(t) = \Delta_{\|h\|1}\phi(t),$$

and so by Lemma 4.3, u is the restriction to $\{\lambda 1: \lambda > 0\}$ of a completely positive function on A. In particular, u satisfies the hypotheses of Lemma 4.5, and so is monotone increasing on $(0,\infty)$. Thus

$$\begin{aligned}\phi(\|a-k\| + \|h\|) - \phi(\|a-k\|) &= u(\|a-k\|)\\ &\leq u(\|a\| + \|k\|)\\ &\leq u(\|a\| + \|b-a\|)\\ &= \phi(\|a\|+\|a-b\|+\|h\|) - \phi(\|a\|+\|b-a\|)\\ &\leq \phi(\|a\|+2\|b-a\|) - \phi(\|a\|+\|b-a\|)\\ &\leq \phi(\|a\|+2\|b-a\|) - \phi(\|a\|).\end{aligned}$$

A similar argument shows that the second term can be estimated as follows:

$$\begin{aligned}|\phi(a) - \phi(a-k)| &= |\phi(a-k+k) - \phi(a-k)|\\ &\leq \phi(\|a-k\| + \|k\|) - \phi(\|a-k\|)\\ &\leq \phi(\|a\| + \|b-a\| + k) - \phi(\|a\| + \|b-a\|),\end{aligned}$$

where the last inequality uses monotonicity of $\Delta_{\|k\|}\phi$ and the fact that $\|a-k\| \leq \|a\| + \|b-a\|$. The last term is dominated by

$$\phi(\|a\|+2\|b-a\|) - \phi(\|a\|+\|b-a\|) \leq \phi(\|a\|+2\|b-a\|)-\phi(\|a\|),$$

as required.

For the general case where $b - a$ is not necessarily self-adjont, consider the elements $x,y \in M_2(A)$ defined by

$$x = \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix}.$$

Fix a vector $\xi \in \mathbb{C}^2$ and consider the completely positive function ψ: $M_2(A) \to \mathbb{C}$ defined by

$$\psi(z) = \langle\phi_2(z)\xi,\xi\rangle ,$$

where ϕ_2: $M_2(A) \to M_2(\mathbb{C})$ is the natural map obtained by applying ϕ element-by-element to 2×2 matries over A.

Because both x and y are self-adjoint, the preceding argument implies

$$(4.7) \qquad |\psi(y) - \psi(x)| \leq 2(\psi(\|x\| + 2\|y-x\|) - \psi(\|x\|)).$$

Since for a scalar multiple t1 of the identity in $M_2(A)$ we have $\psi(t) = \phi(t)\|\xi\|^2$, the right side of the preceding inequality is

$$2 \|\xi\|^2 (\phi(\|x\| + 2\|y-x\|) - \phi\|x\|))$$
$$= 2\|\xi\|^2(\phi\|a\| + 2\|b-a\|) - \phi(\|a\|)).$$

Taking the supremum over all unit vectors $\xi \in \mathbb{C}^2$ in the inequality (4.7) therefore yields

$$\left\| \begin{bmatrix} 0 & \phi(b) \\ \phi(b^*) & 0 \end{bmatrix} - \begin{bmatrix} 0 & \phi(a) \\ \phi(a^*) & 0 \end{bmatrix} \right\|$$
$$\leq 2(\phi(\|a\| + 2\|b-a\|) - \phi(\|a\|)).$$

The left side of this latter inequality is just $|\phi(b) - \phi(a)|$, and so the proof is complete. □

To prove Theorem 4.1, choose an element $x \in A$, and let x_n be a sequence of elements of A which converges in norm to x, such that $x_n \neq x$ for every n.

If $x \neq 0$, we may apply the estimate of Lemma 4.6 directly (taking $b = x_n$, $a = x$) to obtain

$$|\phi(x_n) - \phi(x)| \leq 2(\phi(\|x\| + 2\|x_n-x\|) - \phi(\|x\|)).$$

By Lemma 4.5, the right side tends to zero as $n \to \infty$, and hence $\phi(x_n) \to \phi(x)$.

If $x = 0$, we take $a = x_n$ and $b = x = 0$ in Lemma 4.6 to obtain

$$|\phi(0) - \phi(x_n)| \leq 2(\phi(\|x_n\| + 2\|x_n\|) - \phi(\|x_n\|))$$
$$= 2(\phi(3\|x_n\|) - \phi(\|x_n\|)).$$

By Lemma 4.5, the function $t \in (0,\infty) \longmapsto \phi(t)$ is monotone increasing and, of course, bounded on intervals of the form $0 < t \leq T$. So as $n \to \infty$, $\|x_n\|$ tends to zero and we have

$$\phi(3\|x_n\|) - \phi(\|x_n\|) \to \phi(0+) - \phi(0+) = 0.$$

This implies $\phi(x_n) \to \phi(0)$, as required. □

5. STRUCTURE OF COMPLETELY POSITIVE FUNCTIONS

In this section we will prove the following

THEOREM 5.1. *Let* A *be a unital* C*-*algebra and let*

$$\phi\colon \text{ball } A \to \mathbb{C}$$

be a bounded completely positive function. Then there is a positive linear functional ρ *on* $e^A \otimes e^{\bar{A}}$ *such that*

$$\phi(a) = \rho(\Gamma(a) \otimes \Gamma(\bar{a})), \qquad \|a\| < 1.$$

REMARKS. ρ is, of course, unique by the results of Section 3. This result, together with Theorem 3.1 imply Theorem A of the introduction.

We also want to point out that it is essential for the conclusion of Theorem 5.1 that the domain of ϕ be the *open* unit ball of A rather than its closure, since it is easy to give examples of bounded completely positive functions

$$\phi\colon \{\|a\| \leq 1\} \to \mathbb{C}$$

which are not even continuous. For example, take $A = \mathbb{C}$ and put

$$\phi(z) = \begin{cases} z, & \text{if } |z| = 1 \\ 0, & \text{if } |z| < 1. \end{cases}$$

ϕ is clearly bounded on $\{|z| \leq 1\}$, and it is completely positive because it is the pointwise limit over $\{|z| \leq 1\}$ of the sequence $\phi_1, \phi_2, \ldots$, where

$$\phi_n(z) = z^{n+1}\bar{z}^n.$$

Note that each ϕ_n is completely positive by Theorem 3.1.

The proof of Theorem 5.1 proceeds in two steps. We first reduce the problem to the case of certain extremal completely positive functions. We then show that such an extremal completely positive function must be a (continuous) real homogeneous polynomial. The result will then follow from the analysis of Section 3.

For $r > 0$, we will consider the set $CP_r(A)$ of all completely positive functions ϕ: ball $A \to \mathbb{C}$ satisfying

$$\sup_{\|a\|<1} |\phi(a)| \leq r.$$

$CP_r(A)$ is clearly a convex set of complex valued functions in the Cartesian product

$$\{|z| \leq r\}^{\text{ball } A}$$

which is closed in the topology of pointwise convergence. By the Tychonov theorem, $CP_r(A)$ is a compact convex set (in a locally convex topological vector space).

LEMMA 5.2. *It suffices to prove Theorem* 5.1 *for functions* ϕ *which are extreme points of* $CP_1(A)$.

PROOF. By scaling in an obvious way, Theorem 5.1 is immediately reduced to the case where the given completely positive function ϕ belongs to $CP_1(A)$.

Let P be the space of all positive linear functionals ρ on $e^A \otimes \overline{e^A}$ such that $\|\rho\| \leq 1$. P is convex and compact in its relative weak* topology. Moreover, since the map $\rho \in P \longmapsto \tilde{\rho} \in CP_1(A)$ defined by

$$\tilde{\rho}(a) = \rho(\Gamma(a) \otimes \Gamma(\bar{a})), \quad \|a\| < 1$$

is clearly a weak*-continuous homomorphism of the respective affine structures, its range is a compact convex subset of $CP_1(A)$. Since $CP_1(A)$ is the closed convex hull of its extreme points (Krein-Milman theorem), we may conclude that $\rho \mapsto \tilde{\rho}$ is surjective provided we can show that every extreme point of $CP_1(A)$ has the form $\tilde{\rho}$ for some $\rho \in P$. □

Throughout the remainder of this section, ϕ *will denote an extreme point of* $CP_1^{\cdot}(A)$. We emphasize that ϕ is necessarily continuous, by Theorem 4.1.

LEMMA 5.3. *If* ϕ *is not a constant, then there is a positive real number* α *such that*

$$\phi(ta) = t^{\alpha}\phi(a)$$

for all $a \in \text{ball } A$, $0 < t \leq 1$.

PROOF. Let (π, ξ, H) be a triple obtained from ϕ via the "GNS" construction of Theorem 2.2, so that

$$\phi(a) = \langle \pi(a)\xi, \xi \rangle, \quad \|a\| < 1.$$

We claim first that the self-adjoint family of operators

$$S = \{\pi(a) : \|a\| < 1\}$$

has trivial commutant. By a familiar argument involving the spectral theorem, it suffices to show that the only projections in S' are 0 and 1.

So assume $E = E^*$ is a projection in S', $0 \neq E \neq 1$. Note that $E\xi \neq 0$. For if $E\xi = 0$, then

$$E\pi(a)\xi = \pi(a)E\xi = 0,$$

for all $\|a\| < 1$, and hence $E = 0$ on $H = [\pi(a)\xi : \|a\| < 1]$, a contradiction. In a similar way, we have $(1 - E)\xi \neq 0$.

Putting

$$\theta = \|E\xi\|^2$$

$$\phi_1(a) = \|e\xi\|^{-2}<\pi(a)E\xi,E\xi>$$

$$\phi_2(a) = \|(1-E)\xi\|^{-2}<\pi(a)(1-E)\xi,(1-E)\xi>$$

we see that $0 < \theta < 1$, both ϕ_1 and ϕ_2 belong to $CP_1(A)$ (because π: ball $A \rightarrow L(H)$ is completely positive by Theorem 2.17), and

$$\phi = \theta\phi_1 + (1 - \theta)\phi_2.$$

By extremality, we must have $\phi_1 = \phi$, and hence

(5.4) $$<\pi(a)E\xi,E\xi> = \|E\xi\|^2<\pi(a)\xi,\xi>,$$

for all $a \in$ ball A. Since π is a *-homomorphism of *-semigroups and since $\{\pi(a)\xi: \|a\| < 1\}$ spans H, (5.4) implies that

$$<E\eta,\zeta> = \|E\xi\|^2<\eta,\zeta>$$

for all $\eta,\zeta \in H$, and hence

$$E = \|E\xi\|^2 1 = \theta 1.$$

Since $0 < \theta < 1$, this contradicts the fact that E is a projection.

Now fix t, $0 < t < 1$. Since $\pi(t)\pi(a) = \pi(ta) = \pi(a)\pi(t)$ for all $\|a\| < 1$, we see from the preceding that there must be a scalar $c(t) \in \mathbb{C}$ such that

$$\pi(t) = c(t)1, \qquad 0 < t < 1.$$

$\pi(t)$ is a positive operator because $\pi(t^{1/2})$ is self-adjoint and we have $\pi(t) = \pi(t^{1/2})^2$, and so $c(t)$ is nonnegative.

Notice that we have

(5.5) $$\phi(ta) = c(t)\phi(a),$$

for $\|a\| < 1$, $0 < t < 1$. Indeed, the left side is

$$<\pi(ta)\xi,\xi> = <\pi(t)\pi(a)\xi,\xi> = c(t)<\pi(a)\xi,\xi> = c(t)\phi(a).$$

Note next that the function c is continuous. Indeed, since ϕ is not identically zero we can find a ϵ ball A such that $\phi(a) \neq 0$, and thus

$$c(t) = \phi(ta)\phi(a)^{-1}$$

is continuous by Theorem 4.1.

Notice that the formula (5.5) also implies $c(st) = c(s)c(t)$ for $0 < s,t < 1$. The only continuous functions satisfying this functional equation on (0,1) are the zero function (which is impossible in this case because $\phi \not\equiv 0$), and functions of the form

$$c(t) = t^{\alpha}$$

for some $\alpha \geqslant 0$ (this is easily seen after making the change of variables $t = e^{-x}$, $x > 0$).

We note, finally, that the case $\alpha = 0$ cannot occur here; for that would imply that $\phi(a) = \phi(ta)$ for all $a \in$ ball A and all $t \in (0,1)$ which, by continuity of ϕ at 0, implies that $\phi(a) = \phi(0)$ for all a. □

This homogeneity property allows us to extend ϕ to a completely positive function defined on all of A. While it is not necessary to make use of this extension of ϕ, the subsequent arguments in this section become substantially simpler with it.

LEMMA 5.6. *There is a completely positive function* $\breve{\phi}$: $A \to \mathbb{C}$ *such that*

(i) $\breve{\phi}(ta) = t^{\alpha}\breve{\phi}(a)$, $t > 0$, $a \in A$

(ii) $\breve{\phi}(a) = \phi(a)$, $\|a\| < 1$.

PROOF. If $a \in A$ is nonzero, define

$$\breve{\phi}(a) = (2\|a\|)^{\alpha}\phi((2\|a\|)^{-1}a).$$

$\breve{\phi}(0)$ is defined as $\phi(0)$. We leave it for the reader to supply the routine (though tedious) verification that $\breve{\phi}$ has the asserted properties (i) and (ii) and is completely positive. □

Thus, the proof of Theorem 5.1 has been reduced to proving the following assertion:

ASSERTION 5.7. *Let* ϕ: A $\to \mathbb{C}$ *be a nonconstant completely positive function which satisfies*

$$\phi(ta) = t^{\alpha}\phi(a)$$

for all $a \in A$, $t > 0$, *and some* $\alpha > 0$. *Then there is a positive linear functional* ρ *on* $e^{A} \otimes e^{\overline{A}}$ *such that*

$$\phi(a) = \rho(\Gamma(a) \otimes \Gamma(\overline{a})), \quad \|a\| < 1.$$

Actually, we will prove that α is an integer and ϕ is a (real) homogeneous polynomial of degree α; the conclusion will then follow from the results of Section 3.

First, we require an appropriate notion of complete positivity for multivariate functions. Let $A_1, \ldots, A_n$ be C*-algebras and let

$$\phi: A_1 \times \cdots \times A_n \to \mathbb{C}$$

be a function of n operator variables. ϕ is said to be *completely positive* if the associated function $\breve{\phi}$, defined on the direct sum $A_1 \oplus \cdots \oplus A_n$ by

$$\breve{\phi}(a_1 \oplus \cdots \oplus a_n) = \phi(a_1, \ldots, a_n),$$

is completely positive.

Recalling the "Δ" notation of Section 3, we have

LEMMA 5.8. *Let* ϕ: A $\to \mathbb{C}$ *be a completely positive function. Then for each* $n \geq 1$, *the function*

$$\Delta^n\phi(x;h_1, \ldots, h_n) = \Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n}\phi(x)$$

is a completely positive function of the $n + 1$ *variables* $x, h_1, \ldots, h_n$.

PROOF. For each $k \geq 1$, let

$$\phi_k: M_k(A) \to M_k(\mathbb{C})$$

be the function obtained by applying ϕ element by element to matrices over A. Since ϕ is completely positive and bounded on

$\text{ball}_r A$ for every $r > 0$, the same is true of ϕ_k for every $k \geq 1$.

For each $h \in M_k(A)$, we define $\Delta_h \phi_k$: $M_k(A) \to M_k(\mathbb{C})$ by

$$\Delta_h \phi_k(x) = \phi_k(x + h) - \phi_k(x).$$

We have to show that if $x, h_1, \ldots, h_n$ are positive elements in $M_k(A)$, then

$$\Delta_{h_1} \Delta_{h_2} \cdots \Delta_{h_n} \phi_k(x)$$

is a positive element in $M_k(\mathbb{C})$; i.e.,

$$<\phi_{h_1} \cdots \Delta_{h_n} \phi(x)\xi, \xi> \geq 0,$$

for every vector $\xi \in \mathbb{C}^k$. To see that, fix ξ and define λ: $M_k(A) \to \mathbb{C}$

$$\lambda(y) = <\phi_k(y)\xi, \xi>.$$

By the corollary of Theorem 2.17, the function

$$y \in M_k(A) \longmapsto \Delta_{h_n} \lambda(y)$$

is a completely positive function on $M_k(A)$. Repeating the argument, we see that

$$y \in M_k(A) \longmapsto \Delta_{h_{n-1}} \Delta_{h_n} \lambda(y)$$

is completely positive. After more repetitions we obtain that

$$y \in M_k(A) \longmapsto \Delta_{h_1} \Delta_{h_2} \cdots \Delta_{h_n} \lambda(y)$$

is completely positive. Taking $y = x$, we see in particular that

$$<\Delta_{h_1} \cdots \Delta_{h_n} \phi_k(x)\xi, \xi> = \Delta_{h_1} \cdots \Delta_{h_n} \lambda(x) \geq 0,$$

as required. □

We are now in a position to prove Assertion 5.7. We claim first that α is an integer. For that, consider the function u: $(0,\infty) \to [0,\infty)$ defined by

$$u(t) = \phi(t1).$$

Because of the homogeneity condition on ϕ, we have that

$$u(t) = t^{\alpha}\phi(1), \tag{5.9}$$

for all $t > 0$. Note that $\phi(1)$ cannot be zero, for this would imply that $\phi(t1) = 0$ for all positive t, and hence $\phi \equiv 0$ by Lemma 4.2, contradicting the fact that ϕ is not a constant.

For $h \in (0,\infty)$, define the function $\Delta_h u$ by

$$\Delta_h u(t) = u(t + h) - u(t).$$

Lemma 5.8 implies that if $h_1, ..., h_n \geqslant 0$, then all higher order differences

$$\Delta_{h_1}\Delta_{h_2} \cdots \Delta_{h_n} u$$

are nonnegative functions. u is clearly a smooth function (by (5.9)) and so all of its derivatives must be nonnegative. Thus, the numbers

$$\alpha\phi(1), \quad \alpha(\alpha - 1)\phi(1), \quad \alpha(\alpha - 1)(\alpha - 2)\phi(1), \ ...$$

are all nonnegative. Since $\phi(1) > 0$, this can only happen if α is a positive integer, and the claim is proved.

We now claim that ϕ is a (real) homogeneous polynomial of degree α. To see that, consider the function ψ, defined on the direct sum of $\alpha + 2$ copies of A, by

$$\psi(x \oplus h_1 \oplus \cdots \oplus h_{\alpha+1}) = \Delta_{h_1} \cdots \Delta_{h_{\alpha+1}} \phi(x).$$

Now since the restriction of ϕ to the positive scalars of A is a homogeneous polynomial of degree α, it follows that ψ vanishes on all elements $x \oplus h_1 \oplus \cdots \oplus h_{\alpha+1}$ for which $x, h_1, ..., h_{\alpha+1}$ are all positive scalars. In particular, ψ vanishes on every positive scalar multiple of the identity of its domain $A \oplus \cdots \oplus A$ (α+2 times). Since ψ is completely positive by Lemma 5.8, we conclude from

Lemma 4.2 that ψ vanishes identically.

In other words, $\Delta^{\alpha=1}\phi = 0$. By Proposition 3.3, ϕ must be a (real) polynomial of degree at most α

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_\alpha ,$$

where ϕ_k: A $\rightarrow \mathbb{C}$ is a (real) homogeneous polynomial of degree k. The condition $\phi(ta) = t^\alpha \phi(a)$ clearly implies that the ϕ_k's must vanish for $k = 0,1, ..., \alpha-1$, and hence ϕ is a homogeneous (real) polynomial of degree α.

The assertion 5.7 is now an immediate consequence of these facts, together with Theorem 3.1.

REFERENCES

1. Arveson, W., "Spectral theory for nonlinear random processes," in *Symposi a Matematica* - 20, Instituto Nazionale di alta Matematica, Academic Press, London, 1976, pp. 531-537.

2. Arveson, W., "A spectral theorem for non-linear operators," *Bull. Amer. Math. Soc.* **82** (1976), 511-513.

3. Dixmier, J., *Les C*-algèbres et leurs représentations*, Gauthier Villars, Paris, 1964.

4. Grothendieck, A., "La théorie de Fredholm," *Bull. Soc. Math. France* **84** (1956), 319-384.

5. Jorgensen, P. E. T., "Spectral representations of unbounded non-linear operators on Hilbert spaces," *Pac. J. Math.* **111**:1 (1984), 93-104.

6. Mlak, W., "Dilations of Hilbert space operators (General theory)" *Dissertationes Mathematicae* **153** (1978).

7. Pedersen, G. K., *C*-algebras and their Automorphism Groups*, Academic Press, Longon, 1979.

8. Riesz, F. and Sz.-Nagy, B., *Functional Analysis*, New York, 1955.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720