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Markov Operators and OS-Positive Processes

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We study factorization problems for Markov operators between probability spaces. The factorable operators are shown to be a σ-weakly closed cone, and examples of non-factorable operators are constructed using C^*-algebraic methods. These ideas are applied to the study of OS-positive processes in discrete time, and some examples of OS-positive processes are constructed which cannot be extended to Markov processes.

INTRODUCTION

The motivation for the work behind this paper arose in connection with certain mathematical problems of Euclidean quantum field theory. These problems concern the stochastic processes that give rise to quantum systems, i.e., processes obeying Osterwalder–Schrader positivity (we will follow conventional usage in referring to such processes as OS-positive). In this paper we consider OS-positive processes in discrete time and resolve a problem corresponding to an open problem concerning continuous time processes, namely that of whether or not such processes can be "extended" to Markov processes. We introduce a method of constructing an extensive class of OS-positive processes which allows us to show that, here at least,

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the answer is no. Our methods and perspective are distinctly operator-theoretic, and we feel that some of the results may have interest in a broader context.

With that in mind, we have divided the paper into two parts. Part one concerns order-preserving operators on L^2 spaces, and is written more from the perspective of abstract operator theory. Part two concerns the application of these ideas to OS-positive processes.

OS-positive processes in continuous time had their origins in the Euclidean approach to constructive quantum field theory. A principal and simplifying feature of this approach is that, by analytically continuing certain functions to imaginary time, noncommutative C^*-algebras of local observables correspond to commuting families of random variables on a probability space (see Sect. 2.1). On the other hand, our construction of OS-positive processes makes essential use of noncommutative C^*-algebras. Thus it appears ironic that, if one wants to understand the structure of discrete time OS-positive processes, one must apparently utilize noncommutative techniques.

Since the processes discussed in this paper *are* on discrete time, they cannot give examples of quantum fields. We do not know at this point if these methods can be adapted to continuous time. This problem and others are discussed in somewhat more detail at the end of Section 2.3.

Finally, I want to thank Abel Klein for some enlightening conversations about OS-positive processes, and Edward Nelson for a helpful letter at an early stage of this work.

As for terminology, we follow the conventions of operator theory; Hilbert spaces are complex, inner products are antilinear in the second variable, and operators are bounded unless otherwise specified.

I. Markov Operators

We discuss Markov operators between the L^2 spaces associated with probability spaces. In Section 1.1 we present a canonical factorization of such operators which exists and is unique in general. Section 1.2 concerns the problem of factoring a given order-preserving operator in the form B^*B, where B is an order-preserving operator. We show that the factorable operators form a cone which is closed in the σ-weak operator topology, and we characterize that cone. In Section 1.3 we construct a class of examples of positive semidefinite Markov operators which are not factorable.

1.1. *The Canonical Decomposition*

We will be primarily concerned with complex Hilbert spaces which have been "coordinatized" so that they appear as the space $L^2(X, \mu)$ associated with a probability space (X, μ). By a *Markov operator* we mean a bounded linear operator

$$A:\quad L^2(X, \mu) \to L^2(Y, \nu)$$

between such spaces which is *order-preserving* in the sense that

$$f \geqslant 0 \quad \text{a.e. } (d\mu) \Rightarrow Af \geqslant 0 \quad \text{a.e. } (d\nu),$$

and which possesses the two normalizing properties

$$A1 = 1, \qquad A^*1 = 1,$$

where 1 denotes the unit constant function in the appropriate L^2 space.

Markov operators are continuous, infinite-dimensional generalizations of rectangular doubly stochastic matrices. More explicitly, let $[a_{ij}]$ be an $m \times n$ matrix of complex numbers, considered as a linear operator from $\mathbb{C}^n$ to $\mathbb{C}^m$. If we coordinatize the two spaces of column vectors as $L^2(X, \mu)$ and $L^2(Y, \nu)$, where

$$X = \{1, 2, ..., n\}, \qquad \mu(j) = 1/n, \quad 1 \leqslant j \leqslant n,$$
$$Y = \{1, 2, ..., m\}, \qquad \nu(k) = 1/m, \quad 1 \leqslant k \leqslant m,$$

then A is a Markov operator if and only if each entry a_{ij} is nonnegative and we have both conditions

$$\sum_{j=1}^{n} a_{ij} = 1, \qquad 1 \leqslant i \leqslant m$$
$$\sum_{i=1}^{m} a_{ij} = 1, \qquad 1 \leqslant j \leqslant n.$$

Such operators have been widely studied, and we have made no effort to compile an adequate bibliography. However, we do want to acknowledge the influence of two papers of Nelson on our thinking at an early stage of this research [12, 13]. We also want to point out that while Markov operators are related to the integral operators of Halmos and Sunder [6], there are some significant differences. They are more general than integral operators because their "kernels" are measures rather than functions, and they are more special in that their kernels are always nonnegative. In particular, the methods we use here are quite different from the methods of [6]. Markov operators are pseudo integral operators in the sense of [2], but again the problems discussed here call for methods different from those of [2].

There are two basic types of Markov operators: those arising from measure-preserving transformations and those arising from conditional expectations. The purpose of this section is to show that every Markov operator can be written as a product of these two basic ones and, most significantly, that this decomposition is *unique*. We first collect a few well-known properties of Markov operators.

PROPOSITION 1.1.1. *Let* $A: L^2(X, \mu) \to L^2(Y, \nu)$ *be a Markov operator. We have*

(i) $A(\bar{f}) = \overline{Af}$ *for every* $f \in L^2(X, \mu)$.

(ii) $A^*: L^2(Y, \nu) \to L^2(X, \mu)$ *is a Markov operator.*

(iii) *A induces a contraction from* $L^p(X, \mu)$ *to* $L^p(Y, \nu)$ *for every p,* $1 \leqslant p \leqslant +\infty$. *For* $p = \infty$, *the operator is weak*-continuous.*

(iv) (*Schwarz inequality*) $|Af|^2 \leqslant A(|f|^2)$ *a.e.* $(d\nu)$ *for every* $f \in L^2(X, \mu)$.

We omit the proof of (i), (ii), (iii); a generalization of (iv) is proved later in 1.2.3. The following is also known (see [13], for example), but since it is so basic to what follows we have included a proof for the reader's convenience.

PROPOSITION 1.1.2. *A Markov operator* $A: L^2(X, \mu) \to L^2(Y, \nu)$ *is an isometry if and only if the restriction of A to* L^∞ *is multiplicative.*

Proof. First, assume that A preserves the L^2 norm. By the Schwarz inequality we have

$$A(|f|^2) - |Af|^2 \geqslant 0$$

for each $f \in L^\infty(X, \mu)$. The integral of this nonnegative function over Y must be zero because A preserves the L^2 norm. Thus $A(|f|^2) = |Af|^2$. Considering the sesquilinear form defined on $L^\infty(X, \mu) \times L^\infty(X, \mu)$ by

$$B(f, g) = A(f\bar{g}) - Af\,\overline{Ag},$$

we have $B(f, f) = 0$ for all f and therefore B vanishes identically by the usual polarization argument. This shows that A is multiplicative on L^∞.

Conversely, if A is multiplicative on L^∞, then for each bounded function f in $L^2(X, \mu)$ we have

$$\begin{aligned} \|Af\|^2 &= \int_Y |Af|^2\, d\nu = \int_Y A(|f|^2)\, d\nu \\ &= \langle A(|f|^2), 1 \rangle = \langle |f|^2, A^*(1) \rangle \\ &= \langle |f|^2, 1 \rangle = \|f\|^2. \end{aligned}$$

The conclusion follows because L^∞ is dense in L^2. ∎

The standard example of a Markov isometry $A: L^2(X, \mu) \to L^2(Y, \nu)$ uses a measure-preserving transformation $T: Y \to X$. This means that T is a (measurable) function for which we have $\mu(T^{-1}E) = \nu(E)$ for every (measurable) subset $E \subseteq Y$. T is *not* required to be one-to-one. A good example is the measure-preserving transformation

$$X = Y = \{z \in \mathbb{C}: |z| = 1\},$$
$$\mu = \nu = \text{Lebesgue measure}, \tag{1.1.3}$$
$$Tz = z^2.$$

In general, every measure-preserving transformation $T: Y \to X$ gives rise to a Markov operator $A: L^2(X, \mu) \to L^2(Y, \nu)$ by

$$Af(y) = f(Ty). \tag{1.1.4}$$

In fact, Proposition 1.1.2 implies that (1.1.4) is the most general example of a Markov isometry from $L^2(X, \mu)$ to $L^2(Y, \nu)$, at least when one has reasonable Borel structures on X and Y (e.g., it is enough to assume that X and Y are standard [3]). In a similar way, for reasonable measure spaces the most general Markov unitary operator $A: L^2(X, \mu) \to L^2(Y, \nu)$ arises from a measure-preserving isomorphism $T: Y \to X$. It will not be necessary to make use of this realization of Markov isometries in terms of point mappings in the sequel, but we shall refer to it occasionally for descriptive purposes.

Let A be the Markov isometry defined by (1.1.4). The adjoint of A is easily described. Let $\mathscr{B}_0$ be the σ-field of all subsets of Y of the form $T^{-1}(E)$, where E is a Borel set in X, and let $\mathscr{M} \subseteq L^2(Y, \nu)$ be the space of all $\mathscr{B}_0$-measurable functions in $L^2(Y, \nu)$. Then the projection of $L^2(Y, \nu)$ onto $\mathscr{M}$ is just the conditional expectation $E(\cdot \mid \mathscr{B}_0)$, and $\mathscr{M}$ is precisely the range of the isometry A. Thus A^* has as its polar decomposition

$$A^* = A^{-1}E(\cdot \mid \mathscr{B}_0),$$

where $A^{-1}: \mathscr{M} \to L^2(X, \mu)$ denotes the inverse of the unitary operator

$$A: L^2(X, \mu) \to \mathscr{M}.$$

Let (X_1, μ_1), (X_2, μ_2) be probability spaces. We can construct a Markov operator from $L^2(X_1, \mu_1)$ to $L^2(X_2, \mu_2)$ out of a pair of Markov isometries as follows. Let (Y, ν) be a third probability space and, for each $i = 1, 2$, let

$$V_i: \quad L^2(X_i, \mu_i) \to L^2(Y, \nu)$$

be a Markov isometry. The pair (V_1, V_2) is called *minimal* if the set of functions

$$V_1(L^\infty(X_1, \mu_1)) \cup V_2(L^2(X_2, \mu_2))$$

generates $L^\infty(Y, \nu)$ as a weak*-closed algebra. In any case,

$$A = V_2^* V_1$$

defines a Markov operator from $L^2(X_1, \mu_1)$ to $L^2(X_2, \mu_2)$. Following is the main result of this section, which asserts that every Markov operator arises in this way from an essentially unique pair (V_1, V_2).

THEOREM 1.1.5. (i) *For every Markov operator*

$$A\colon L^2(X_1, \mu_1) \to L^2(X_2, \mu_2)$$

there is a probability space (Y, ν) and a minimal pair (V_1, V_2) of Markov isometries

$$V_i\colon \quad L^2(X_i, \mu_i) \to L^2(Y, \nu)$$

such that $A = V_2^* V_1$.

(ii) *If $\tilde{V}_i\colon L^2(X_i, \mu_i) \to L^2(\tilde{Y}, \tilde{\nu})$ is another minimal pair for which $A = \tilde{V}_2^* \tilde{V}_1$, then there is a unique Markov unitary operator*

$$W\colon L^2(Y, \nu) \to L^2(\tilde{Y}, \tilde{\nu})$$

satisfying $WV_i = \tilde{V}_i$, $i = 1, 2$.

To prove (i) we require a lemma about bilinear forms on commutative C^*-algebras. It is possible that this lemma is known; we have included a proof because we were unable to find a suitable reference in the literature.

LEMMA 1.1.6. *Let X and Y be compact Hausdorff spaces and let B be a complex-valued bilinear form on $C(X) \times C(Y)$ satisfying*

$$B(f, g) \geqslant 0$$

for all nonnegative functions $f \in C(X)$, $g \in C(Y)$. Then there is a unique positive Baire measure μ on $X \times Y$ such that

$$B(f, g) = \int_{X \times Y} f(x)\, g(y)\, d\mu(x, y),$$

for all $f \in C(X)$, $g \in C(Y)$.

Our proof of 1.1.6 is based on the following abstract version of Bochner's theorem. The latter is part of the lore of spectral theory and we omit the proof (a straightforward consequence of the GNS construction for positive linear functionals and standard spectral theory for commutative C^*-algebras of operators).

REPRESENTATION THEOREM. *Let C be a commutative Banach algebra with unit e ($\|e\| = 1$), having an isometric involution $x \mapsto x^*$. Let*

$$Z = \{\omega \in \mathcal{M}_c : \omega(x^*) = \overline{\omega(x)},\ x \in C\}$$

be the self-adjoint part of the maximal ideal space of C. Let β be a linear functional on C which is positive in the sense that $\beta(x^*x) \geqslant 0$, $x \in C$.

Then there is a unique positive Baire measure ν on Z such that

$$\beta(x) = \int_Z \omega(x)\, d\nu(\omega), \qquad x \in C.$$

Proof of 1.1.6. By a familiar argument, the bilinear form B is bounded in the sense that

$$|B(f, g)| \leqslant M\, \|f\| \cdot \|g\|,$$

and in fact M can be taken as $B(1, 1)$. There is a unique linear functional β on the *algebraic* tensor product $C(X) \otimes C(Y)$ satisfying

$$\beta(f \otimes g) = B(f, g).$$

We can make $C(X) \otimes C(Y)$ into a unital *-algebra by defining

$$(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) = f_1 f_2 \otimes g_1 g_2,$$
$$(f \otimes g)^* = \bar{f} \otimes \bar{g},$$
$$e = 1 \otimes 1.$$

The completion C of $C(X) \otimes C(Y)$ in the projective cross norm therefore becomes a unital commutative Banach *-algebra. Since the bilinear form B is bounded, β extends uniquely to a bounded linear functional on C.

We claim that β is positive. To prove that, if suffices to show that for every element $c \in C$ which is a finite sum of elementary tensors, say

$$c = \sum_{i=1}^{n} f_i \otimes g_i,$$

we have

$$\beta(c^*c) = \sum_{i,j=1}^{n} \beta(\bar{f}_i f_j \otimes \bar{g}_i g_j) = \sum_{i,j} B(\bar{f}_i f_j, \bar{g}_i g_j) \geqslant 0. \qquad (1.1.7)$$

To prove (1.1.7), we use the Riesz–Markov theorem to obtain complex Baire measures μ_{ij} on X such that

$$B(f, \bar{g}_i g_j) = \int_X f(x)\, d\mu_{ij}(x), \qquad f \in C(X),$$

for each $1 \leqslant i, j \leqslant n$. Let m be the finite positive measure

$$m = \sum_{i,j=1}^{n} |\mu_{ij}|.$$

Then by the Radon–Nikodym theorem we have

$$d\mu_{ij}(x) = w_{ij}(x)\, dm(x),$$

for certain functions $w_{ij} \in L^\infty(X, m)$. We will show first that the $n \times n$ matrix

$$W(x) = [w_{ij}(x)]$$

is positive semidefinite for almost every $x \in X$ (dm). Indeed, fixing $\lambda_1, ..., \lambda_n \in \mathbb{C}$, we have for each $f \in C(X)$,

$$\int |f(x)|^2 \sum_{i,j} \bar{\lambda}_i \lambda_j w_{ij}(x)\, dm = \sum_{i,j} \bar{\lambda}_i \lambda_j B(|f|^2, \bar{g}_i g_j)$$
$$= B\left(|f|^2, \left|\sum_j \lambda_j g_j\right|^2\right) \geqslant 0.$$

This implies that

$$\sum \bar{\lambda}_i \lambda_j w_{ij}(x) \geqslant 0 \qquad \text{a.e. } (dm). \tag{1.1.8}$$

The exceptional set in (1.1.8) depends on the n-tuple $(\lambda_1, ..., \lambda_n)$. However, we may assume that (1.1.8) holds on the complement of a single null set N simultaneously for all n-tuples $(\lambda_1, ..., \lambda_n)$ whose components belong to a countable subset D of $\mathbb{C}$; and now if D is dense in $\mathbb{C}$ then (1.1.8) clearly implies that the matrix $W(x)$ is positive semidefinite for every x in $X \setminus N$.

Now we have

$$\beta(c^*c) = \sum_{i,j} B(\bar{f}_i f_j, \bar{g}_i g_j) = \int_X \sum_{i,j} w_{ij}(x)\, f_j(x)\, \overline{f_i(x)}\, dm(x),$$

which must be nonnegative because the integrand is nonnegative almost everywhere.

By the general representation theorem cited above, there is a positive Baire measure ν on the space

$$Z = \{\omega \in \mathcal{M}_C \colon \omega^* = \omega\}$$

such that

$$\beta(c)=\int_Z \omega(c)\,d\nu(\omega). \tag{1.1.9}$$

It is a simple matter to identify Z with $X\times Y$ in such a way that a point (x, y) in $X\times Y$ corresponds to the complex homomorphism $\omega\in Z$ defined by

$$\omega(f\otimes g)=f(x)\,g(y).$$

Thus (1.1.9) provides the required representation

$$B(f, g)=\beta(f\otimes g)=\int_{X\times Y} f(x)\,g(y)\,d\nu(x, y). \quad \blacksquare$$

To prove 1.1.5(i), let $A: L^2(X_1, \mu_1)\to L^2(X_2, \mu_2)$ be a Markov operator. Let

$$\mathfrak{A}_i=L^\infty(X_i, \mu_i), \qquad i=1, 2.$$

$\mathfrak{A}_i$ is a commutative C^*-algebra with unit. Moreover, we can define a bounded bilinear form B on $\mathfrak{A}_1\times\mathfrak{A}_2$ by

$$B(f, g)=\langle Af, \bar{g}\rangle=\int_{X_2} (Af)(y)\,g(y)\,d\mu_2(y).$$

It is clear that $B(f, g)\geqslant 0$ when $f\geqslant 0$, $g\geqslant 0$ because A is order-preserving. Moreover,

$$B(1, 1)=\langle A1, 1\rangle=1.$$

It follows from Lemma 1.1.6 that there is a unique positive Baire measure ν on the compact Hausdorff space

$$Y=\hat{\mathfrak{A}}_1\times\hat{\mathfrak{A}}_2$$

such that

$$B(f, g)=\int_Y \omega_1(f)\,\omega_2(g)\,d\nu(\omega_1, \omega_2).$$

The condition $B(1, 1)=1$ implies that ν is a probability measure.

Define linear mappings $V_i: L^\infty(X_i, \mu_i)\to L^\infty(Y, \nu)$ by

$$(V_1 f)(\omega_1, \omega_2)=\omega_1(f),$$

$$(V_2 g)(\omega_1, \omega_2)=\omega_2(g),$$

for $f \in L^\infty(X_1, \mu_1)$, $g \in L^\infty(X_2, \mu_2)$. Note that V_1 and V_2 are isometries relative to the L^2 norms; e.g.,

$$\|V_1 f\|^2 = \int |\omega_1(f)|^2 \, d\nu(\omega_1, \omega_2) = \int \omega_1(|f|^2) \, d\nu(\omega_1, \omega_2)$$
$$= B(|f|^2, 1) = \langle A(|f|^2), 1 \rangle = \langle |f|^2, A^*(1) \rangle$$
$$= \langle |f|^2, 1 \rangle = \int_{X_1} |f|^2 \, d\mu_1,$$

$f \in L^\infty(X_1, \mu_1)$. We may therefore extend each V_i uniquely to an isometric embedding of $L^2(X_i, \mu_i)$ into $L^2(Y, \nu)$.

We have $V_i(1) = 1$ by definition, and since for each $f \in L^\infty(X_1, \mu_1)$ we have

$$\langle f, V_1^*(1) \rangle = \langle V_1(f), 1 \rangle = \int_Y \omega_1(f) \, d\nu(\omega_1, \omega_2)$$
$$= B(f, 1) = \langle Af, 1 \rangle = \langle f, A^*1 \rangle = \langle f, 1 \rangle,$$

it follows that $V_1^*(1) = 1$. Similarly, $V_2^*(1) = 1$.

Finally, V_1 and V_2 are clearly order preserving maps of their respective L^∞ spaces. Since L^∞ is dense in L^2, it follows that V_i is a Markov operator from $L^2(X_i, \mu_i)$ to $L^2(Y, \nu)$.

(V_1, V_2) is a minimal pair because the C^*-algebra generated by

$$V_1(L^\infty) \cup V_2(L^\infty)$$

consists of all continuous functions on $Y = \hat{\mathfrak{A}}_1 \times \hat{\mathfrak{A}}_2$ (Stone–Weierstrass theorem), and the latter are weak*-dense in $L^\infty(Y, \nu)$.

To establish the formula $A = V_2^* V_1$ choose $f \in L^\infty(X_1, \mu_1)$, $g \in L^\infty(X_2, \mu_2)$. Then we have

$$\langle V_2^* V_1 f, g \rangle = \langle V_1 f, V_2 g \rangle = \int_Y \omega_1(f) \overline{\omega_2(g)} \, d\nu(\omega_1, \omega_2)$$
$$= \int_Y \omega_1(f) \, \omega_2(\bar{g}) \, d\nu(\omega_1, \omega_2) = B(f, \bar{g}) = \langle Af, g \rangle.$$

We now prove the uniqueness assertion 1.1.5(ii). Let $(\tilde{V}_1, \tilde{V}_2)$ be another minimal pair of Markov isometries

$$\tilde{V}_i\colon \quad L^2(X_i, \mu_i) \to L^2(\tilde{Y}, \tilde{\nu})$$

satisfying $\tilde{V}_2^* \tilde{V}_1 = A = V_2^* V_1$.

Consider the linear span S in $L^2(Y, \nu)$ of all products

$$V_1(f_1)\, V_2(f_2), \qquad f_i \in L^\infty(X_i, \mu_i).$$

S is a self-adjoint subalgebra of $L^\infty(Y, \nu)$ which is weak*-dense in L^∞. Thus the L^2-closure of S is all of $L^2(Y, \nu)$.

We want to define an isometry $W: S \to L^2(Y, \nu)$ by the formula

$$W: \quad V_1(f_1)\, V_2(f_2) \mapsto \tilde{V}_1(f_1)\, \tilde{V}_2(f_2). \tag{1.1.10}$$

For that, it is enough to check inner products. Let $f_i, g_i \in L^\infty(X_i, \mu_i)$, $i = 1, 2$, and put

$$u = V_1(f_1)\, V_2(f_2), \qquad v = V_1(g_1)\, V_2(g_2),$$
$$\tilde{u} = \tilde{V}_1(f_1)\, \tilde{V}_2(f_2), \qquad \tilde{v} = \tilde{V}_1(g_1)\, \tilde{V}_2(g_2).$$

Then we have

$$\langle \tilde{u}, \tilde{v} \rangle = \int_{\tilde{Y}} \tilde{V}_1(f_1)\, \tilde{V}_2(f_2)\, \tilde{V}_1(\bar{g}_1)\, \tilde{V}_2(\bar{g}_2)\, d\tilde{\nu}$$
$$= \int_{\tilde{Y}} \tilde{V}_1(f_1 \bar{g}_1)\, \tilde{V}_2(f_2 \bar{g}_2)\, d\tilde{\nu} = \langle \tilde{V}_1(f_1 \bar{g}_1), \tilde{V}_2(\bar{f}_2 g_2) \rangle$$
$$= \langle V_1(f_1 \bar{g}_1), V_2(\bar{f}_2 g_2) \rangle = \langle u, v \rangle.$$

Thus there is a unique linear isometry W from $L^2(Y, \nu)$ to $L^2(\tilde{Y}, \tilde{\nu})$ satisfying (1.1.10).

Because of the minimality of the pair $(\tilde{V}_1, \tilde{V}_2)$, the argument given above to show that S is dense in $L^2(Y, \nu)$ can be repeated to show that $W(S)$ is dense in $L^2(\tilde{Y}, \tilde{\nu})$. Thus W is unitary.

Since V_1, V_2, $\tilde{V}_1$, $\tilde{V}_2$ all preserve multiplication in their respective L^∞ spaces, W must preserve multiplication in the sense that

$$W(uv) = W(u)\, W(v)$$

for all $u, v \in S$. Since the weak*-closed span of S is $L^\infty(Y, \nu)$, it follows that W is multiplicative on $L^\infty(Y, \nu)$.

Clearly $W(1) = 1$. To show that $W^*(1) = 1$ it suffices to show that $\langle u, W^*(1) \rangle = \langle u, 1 \rangle$ for every $u \in S$. But we have

$$\begin{aligned} \langle V_1(f)\, V_2(g), W^*(1) \rangle &= \langle W(V_1(f)\, V_2(g)), 1 \rangle \\ &= \langle \tilde{V}_1(f)\, \tilde{V}_2(g), 1 \rangle = \langle \tilde{V}_1(f), \tilde{V}_2(\bar{g}) \rangle \\ &= \langle V_1(f), V_2(\bar{g}) \rangle = \langle V_1(f)\, V_2(g), 1 \rangle, \end{aligned}$$

for all $f \in L^\infty(X_1, \mu_1)$, $g \in L^\infty(X_2, \mu_2)$.

Finally, the uniqueness of the unitary Markov operator W which satisfies

$$WV_1 = \tilde{V}_1, \qquad WV_2 = \tilde{V}_2$$

is apparent from the fact that $W|_{L^\infty}$ is multiplicative and S spans L^∞ in the weak*-topology. ▮

Let (X, μ) be a probability space. By a *reflection* of $L^2(X, \mu)$ we mean a unitary Markov operator

$$R: L^2(X, \mu) \to L^2(X, \mu)$$

satisfying $R^2 = 1$. If X is, say, a standard Borel space, then every reflection is induced by a measure-preserving automorphism $r: X \to X$ which satisfies $r \circ r = id_X$. Reflections play a control role in the theory of symmetric stochastic processes (see Sect. II). Here, we merely want to point out that they are associated with self-adjoint Markov operators in the following way.

PROPOSITION 1.1.11. *Let A be a Markov operator on $L^2(X, \mu)$ and let $A = V_2^* V_1$ be its canonical decomposition as in* 1.1.5, *where $V_i: L^2(X, \mu) \to L^2(Y, \nu)$. Then A is self-adjoint if and only if the pair (V_1, V_2) is symmetric in the sense that there is a reflection R of $L^2(Y, \nu)$ satisfying*

$$RV_1 = V_2, \qquad RV_2 = V_1.$$

Proof. First, assume that there is a reflection R satisfying $RV_1 = V_2$, $RV_2 = V_1$. Then we have

$$A^* = (V_2^* V_1)^* = V_1^* V_2 = (RV_2)^* RV_1 = V_2^* R^* R V_1 = V_2^* V_1 = A.$$

Conversely, assume $A^* = A$. Then we have

$$V_2^* V_1 = A = A^* = V_1^* V_2.$$

Since both (V_1, V_2) and (V_2, V_1) are minimal pairs and both represent A, 1.1.5(i) implies that there is a unitary Markov operator R on $L^2(Y, \nu)$ such that

$$RV_1 = V_2, \qquad RV_2 = V_1.$$

Clearly $R^2 V_i = V_i$, $i = 1, 2$, so that R^2 fixes the subalgebra of $L^\infty(Y, \nu)$ generated by $V_1(L^\infty) \cup V_2(L^\infty)$. By minimality, we conclude that $R^2 = 1$. ▮

1.2. *Factorization Theory*

Throughout this section, (X, μ) will denote a probability space.

DEFINITION 1.2.1. An operator A on $L^2(X, \mu)$ is said to be factorable if there is a (perhaps, infinite) measure space (Y, ν) and an order-preserving operator

$$B\colon \quad L^2(X, \mu) \to L^2(Y, \nu)$$

such that $A = B^*B$.

Since the adjoint of an order-preserving operator is order-preserving, and since the composition of two order-preserving operators is another such, it follows that any factorable operator is order-preserving. More significantly, a factorable operator A must be positive in the usual sense of operator theory:

$$\langle Af, f \rangle \geqslant 0, \qquad f \in L^2(X, \mu). \tag{1.2.2}$$

We shall be particularly concerned with determining which Markov operators are factorable. As we will see later (cf. Theorem 1.3.7), for such operators the necessary condition (1.2.2) is not sufficient. The purpose of this section and the next is to discuss this phenomenon.

First, consider the analogous finite-dimensional situation, in which A is a doubly stochastic $n \times n$ matrix $[a_{ij}]$. Assume that A is positive in the sense of (1.2.2). Then A has a positive semidefinite square root

$$A^{1/2} = [b_{ij}].$$

However, if $n \geqslant 3$, it is not very hard to give examples for which the entries b_{ij} of $A^{1/2}$ are not nonnegative.

If we replace $A^{1/2}$ by an $n \times n$ matrix of the form

$$B = UA^{1/2},$$

where U is a unitary $n \times n$ matrix, then we still have a factorization of A in the form $A = B^*B$, but it may not be possible to find a unitary matrix U such that the entries of $UA^{1/2}$ are nonnegative. More generally, there need not exist a rectangular $m \times n$ matrix B which has nonnegative entries and satisfies $A = B^*B$.

We digress for a moment to collect a somewhat unusual form of the Schwarz inequality, which will be needed at several points in the sequel.

LEMMA 1.2.3. *Let $\mathfrak{A}$ be a C^*-algebra, let (X, μ) be a positive measure space, and let $\rho\colon \mathfrak{A} \to L^2(X, \mu)$ be a linear mapping which is order-preserving in the sense that $\rho(a^*a) \geqslant 0$ a.e. $(d\mu)$, for each $a \in \mathfrak{A}$. Then*

$$\rho(a^*) = \overline{\rho(a)} \qquad \text{and} \qquad |\rho(b^*a)|^2 \leqslant \rho(a^*a)\,\rho(b^*b) \quad \text{a.e. } (d\mu)$$

for all $a, b \in \mathfrak{A}$.

Proof. Fix $a, b \in \mathfrak{A}$ and let D be a countable dense set in $\mathbb{C}$. For each pair $\lambda, \mu \in D$ the function

$$|\lambda|^2 \rho(a^*a) + \bar{\lambda}\mu\rho(a^*b) + \lambda\bar{\mu}\rho(b^*a) + |\mu|^2 \rho(b^*b)$$

is nonnegative almost everywhere because it has the form $\rho(c^*c)$ for $c = \lambda a + \mu b$. Since D is countable we can find a single Borel set $N \subseteq X$ of measure zero such that the above functions are simultaneously nonnegative on $X \backslash N$ for all $\lambda, \mu \in D$. Because D is dense in $\mathbb{C}$, this implies that the 2×2 matrices

$$\begin{bmatrix} \rho(a^*a)(x) & \rho(a^*b)(x) \\ \rho(b^*a)(x) & \rho(b^*b)(x) \end{bmatrix}, \qquad x \in X \backslash N,$$

are all positive semidefinite. Thus

$$\rho(ab^*)(x) = \overline{\rho(b^*a)}(x) \qquad \text{and} \qquad \rho(a^*a)(x)\,\rho(b^*b)(x) - |\rho(b^*a)(x)|^2 \geqslant 0$$

on the complement of N, as required. ▮

We begin with a result which implies that when an operator A is factorable, $A = B^*B$, then the order-preserving operator B can be chosen so as to have rather convenient properties.

PROPOSITION 1.2.4. *Let A be a factorable operator on $L^2(X, \mu)$.*

(i) *There is a finite measure space (Y, ν) and an order-preserving operator B: $L^2(X, \mu) \to L^2(Y, \nu)$ satisfying $A = B^*B$ and $B(1) = 1$.*

(ii) *If A is a Markov operator then there is a Markov operator B such that $A = B^*B$.*

Proof. (i) By hypothesis, we can write $A = B_0^*B_0$, where B_0 is an order-preserving operator from $L^2(X, \mu)$ to L^2 of some positive measure space (Y, ν_0).

The function $B_0(1)$ is a nonnegative element of $L^2(Y, \nu_0)$. Moreover, by the Schwarz inequality 1.2.3, we have

$$|B_0(f)|^2 \leqslant B_0(1)\, B_0(|f|^2) \qquad \text{a.e. } (d\nu_0).$$

for every $f \in L^\infty(X, \mu)$. From this inequality we conclude that each function $B_0(f)$, $f \in L^\infty(X \mu)$, vanishes almost everywhere $(d\nu_0)$ on the set

$$S = \{y \in Y: B_0(1)(y) = 0\}.$$

Consider the positive measure ν defined on Y by

$$d\nu = B_0(1)^2\, d\nu_0.$$

In fact, ν is finite because $B_0(1) \in L^2(Y, \nu_0)$, and we have

$$\nu(Y) = \langle B_0(1), B_0(1) \rangle$$
$$= \langle B_0^* B_0(1), 1 \rangle = \langle A1, 1 \rangle.$$

Note in particular that ν is a probability measure in the event that $A1 = 1$.

For each $f \in L^\infty(X, \mu)$, define a function $B(f)$ on Y by

$$B(f)(y) = \begin{cases} \dfrac{B_0(f)(y)}{B_0(1)(y)} & \text{if } y \notin S \\ 0 & \text{if } y \in S. \end{cases}$$

For $f \in L^\infty(X, \mu)$ we have

$$\int_Y |B(f)|^2 \, d\nu = \int_{Y \setminus S} |B_0(f)|^2 \, d\nu_0 = \int_Y |B_0(f)|^2 \, d\nu_0 < \infty;$$

thus B maps $L^\infty(X, \mu)$ into $L^2(Y, \nu)$. Moreover, in a similar way we have

$$\langle Bf, Bg \rangle_{L^2(Y,\nu)} = \langle B_0 f, B_0 g \rangle_{L^2(Y,\nu_0)} = \langle Af, g \rangle.$$

This implies that B extends uniquely to a bounded operator from $L^2(X, \mu)$ to $L^2(Y, \nu)$ which satisfies $B^*B = A$.

B is clearly order-preserving, and we have

$$B(1) = 1 - \chi_S = 1$$

since S is a set of ν-measure zero. That completes the proof of (i).

For (ii), assume in addition that A is a Markov operator. As we have already seen, $\nu(Y) = \langle A(1), 1 \rangle = 1$, so that (Y, ν) is a probability space. To see that $B^*(1) = 1$, note that

$$B^*(1) = B^*(B(1)) = A(1) = 1. \quad \blacksquare$$

Proposition 1.2.4 allows us to give the following description of factorable doubly stochastic matrices.

Corollary. *Let $A = [a_{ij}]$ be a self-adjoint $n \times n$ matrix with non-negative entries. Then A is factorable iff there is a finite measure space (Y, ν) and a set of measurable functions $f_1, \ldots, f_n$ on Y satisfying*

(i) $f_i \geq 0, f_1 + \cdots + f_n = 1$,

(ii) $a_{ij} = \int_Y f_i f_j \, d\nu$.

Proof. Coordinatize the vector space $\mathbb{C}^n$ so that a vector $(z_1,..., z_n)$ is positive iff $z_k \geqslant 0$ for all k, and so that

$$\|(z_1,..., z_n)\|^2 = \frac{1}{n}\,(|z_1|^2 + \cdots + |z_n|^2).$$

First, suppose that a_{ij} has the above form, for functions $f_1,..., f_n$ in $L^\infty(Y, \nu)$. Define $B: \mathbb{C}^n \to L^2(Y, \nu)$ by

$$B(z_1,..., z_n) = (1/\sqrt{n}) \sum_{k=1}^{n} z_k f_k.$$

B is clearly order-preserving and satisfies $B^*B = A$, hence A is factorable.

Conversely, if A is factorable then we can write $A = B^*B$, where $B: \mathbb{C}^n \to L^2(Y, \nu)$ satisfies the conclusion of 1.2.3(i). The functions

$$\begin{aligned} f_1 &= B(1, 0,..., 0),\\ f_2 &= B(0, 1, 0,..., 0),\\ &\vdots\\ f_n &= B(0,..., 0, 1), \end{aligned}$$

satisfy conditions (i) above, and we can obtain (ii) as well if we replace μ with the measure $n \cdot \mu$. ∎

Let $\mathscr{F}(X, \mu)$ denote the set of all factorable operators on $L^2(X, \mu)$. It is clear that $\mathscr{F}(X, \mu)$ is stable under multiplication by nonnegative scalars. Moreover, if A_1 and A_2 belong to $\mathscr{F}(X, \mu)$, say $A_i = B_i^* B_i$ where

$$B_i: \quad L^2(X, \mu) \to L^2(Y_i, \nu_i), \qquad i = 1, 2$$

is an order-preserving operator, then we can write

$$A_1 + A_2 = C^*C,$$

where C is the operator from $L^2(X, \mu)$ into the direct sum of Hilbert spaces $L^2(Y_1, \nu_1) \oplus L^2(Y_2, \nu_2)$ defined by

$$Cf = B_1 f \oplus B_2 f, \qquad f \in L^2(X, \mu).$$

There is an obvious way of coordinatizing the direct sum so as to make C an order-preserving operator, and thus $A_1 + A_2$ is a factorable operator. We conclude that *the factorable operators on $L^2(X, \mu)$ form a convex cone.*

Let f be a nonnegative function in $L^2(X, \mu)$. Then the rank-one operator

$$f \otimes f: \quad g \mapsto \langle g, f \rangle f$$

is factorable because it can be written $f\otimes f=B^*B$, where B is the order-preserving operator from $L^2(X,\mu)$ to $\mathbb{C}$ given by

$$Bg=\langle g,f\rangle.$$

The main result of this section asserts that the cone $\mathscr{F}(X,\mu)$ of factorable operators is closed in the σ-weak operator topology, and is in fact generated by the rank-one operators $f\otimes f$ of the above type.

To state this result, we require some preliminaries. Let E be a complex Banach space having an isometric involution $x\mapsto x^*$. For each $n\geqslant 1$, $\bigvee^n E$ will denote the symmetric tensor product of n copies of E, completed in the *projective* (i.e., greatest) cross norm. There is a natural projection S of the full projective tensor product

$$E\otimes E\otimes\cdots\otimes E \qquad (n\text{ times})$$

onto $\bigvee^n E$, determined on elementary tensors by the symmetrization operator

$$S(x_1\otimes\cdots\otimes x_n)=\frac{1}{n!}\sum_\pi x_{\pi(1)}\otimes\cdots\otimes x_{\pi(n)}, \tag{1.2.5}$$

the summation on the right extended over all permutations π of $\{1,2,\ldots,n\}$. For any n elements $x_1,\ldots,x_n$ of E, we will use the notation

$$x_1\vee x_2\vee\cdots\vee x_n$$

for the element of $\bigvee^n E$ defined by (1.2.5).

Recall that $\bigvee^n E$ has an important universal property which "linearizes" symmetric multilinear forms. More explicitly, if F is any Banach space and B is an n-variate multilinear mapping from E to F,

$$B\colon\quad E\times E\times\cdots\times E\to F$$

which is *bounded* and *symmetric* in all of its variables, then there is a unique bounded linear operator $\tilde{B}\colon\bigvee^n E\to F$ satisfying

$$\tilde{B}(x_1\vee\cdots\vee x_n)=B(x_1,\ldots,x_n).$$

Moreover, one has

$$\|\tilde{B}\|=\sup_{\|x_i\|<1}\|B(x_1,\ldots,x_n)\|.$$

A similar result is valid for n-variate symmetric forms which are antilinear in each variable. In particular, we may define an isometric involution $\xi\to\xi^*$ on $\bigvee^n E$ by requiring that

$$(x_1\vee\cdots\vee x_n)^*=x_1^*\vee\cdots\vee x_n^*.$$

Now put $\bigvee^0 E = \mathbb{C}$, and define a Banach space $\exp E$ to be the l^1-sum of Banach spaces

$$\exp E = \sum_{n=0}^{n} \bigvee^{n} E.$$

Every element of $\exp E$ is represented by a sequence $\xi = (\xi_0, \xi_1, \ldots)$, where $\xi_n \in \bigvee^n E$ and

$$\|\xi\| = \sum_{n=0}^{\infty} \|\xi_n\|.$$

The involutions in each summand extend uniquely so as to give a common involution $\xi \mapsto \xi^*$ of $\exp E$.

We make $\exp E$ into a commutative Banach algebra as follows. For each $m, n \geqslant 0$, there is a unique bounded bilinear map

$$(\xi, \eta) \in \bigvee^{m} E \times \bigvee^{n} E \mapsto \xi \vee \eta \in \bigvee^{m+n} E$$

defined by the requirement

$$(x_1 \vee \cdots \vee x_m) \vee (y_1 \vee \cdots \vee y_n) = x_1 \vee \cdots \vee x_m \vee y_1 \vee \cdots \vee y_n.$$

This operation extends uniquely to a bounded bilinear map of $\exp E$ into itself where, for $\xi = (\xi_n)$ and $\eta = (\eta_n)$, $\xi \vee \eta$ is the sequence $\zeta = (\zeta_n)$ defined by

$$\zeta_n = \sum_{p+q=n} \xi_p \vee \eta_q.$$

With this operation as multiplication, $\exp E$ becomes a commutative Banach *-algebra. The sequence $e = (1, 0, 0, \ldots)$ functions as a unit for this algebra structure, and we have

$$\|e\| = 1.$$

Now by its construction, $\exp E$ has the following universal property. If B is any unital Banach *-algebra and $L: E \to B$ is a linear mapping satisfying

(i) $\|L\| \leqslant 1$ and

(ii) $L(x^*) = L(x^*)$, $x \in E$,

then there is a unique unital *-homomorphism $\pi_L: \exp E \to B$ satisfying

$$\pi_L|_E = L.$$

One has, moreover, that $\|\pi_L\| = 1$.

We apply this construction to the factorization problem in the following way. Let A be a *self-adjoint* order-preserving operator on $L^2(X, \mu)$. Consider the involutive Banach space $E = L^\infty(X, \mu)$ (relative to the essential sup norm), and let $\mathscr{S}$ be the closed linear subspace of $\exp L^\infty(X, \mu)$ generated by the elements

$$\lambda e + f + g \vee h,$$

where $\lambda \in \mathbb{C}$, $f, g, h \in L^\infty(X, \mu)$, e denoting the unit of $\exp L^\infty(X, \mu)$.

LEMMA 1.2.6. *There is a unique bounded linear functional μ_A on $\mathscr{S}$ satisfying*

$$\mu_A\left(\lambda e + f + g \vee h\right) = \lambda\langle A1, 1\rangle + \langle Af, 1\rangle + \langle Ag, \bar{h}\rangle.$$

Proof. For the existence of μ_A, it suffices to define μ_A on each summand $\bigvee^0 L^\infty = \mathbb{C}\cdot e$, $\bigvee^1 L^\infty = L^\infty$, and $\bigvee^2 L^\infty$ of $\mathscr{S}$.

Clearly μ_A is well defined on $\mathbb{C}\cdot e$ and on L^∞. Moreover, since

$$|\langle Af, 1\rangle| \leqslant \|A\|\ \|f\|_2 \leqslant \|A\|\cdot\|f\|_\infty,$$

μ_A is bounded on L^∞.

Now we have

$$\langle Ah, \bar{g}\rangle = \langle g, \overline{Ah}\rangle = \langle g, A(\bar{h})\rangle = \langle A^*g, \bar{h}\rangle = \langle Ag, \bar{h}\rangle.$$

Thus $g, h \to \langle Ag, \bar{h}\rangle$ is a symmetric bilinear form on $L^\infty \times L^\infty$. We also have

$$|\langle Ag, \bar{h}\rangle| \leqslant \|A\|\cdot\|g\|_2\ \|h\|_2 \leqslant \|A\|\cdot\|g\|_\infty\ \|h\|_\infty.$$

Hence there is a unique bounded linear functional μ_A on $\bigvee^2 L^\infty$ satisfying

$$\mu_A\left(g \vee h\right) = \langle Ag, \bar{h}\rangle.$$

The bounded linear functional μ_A is obviously uniquely determined on $\mathscr{S}$ by these requirements. ∎

We can now state the main characterization of factorable operators.

THEOREM 1.2.7. *Let A be a self-adjoint order-preserving operator on $L^2(X, \mu)$. The following are equivalent*:

(i) *A is factorable.*

(ii) *A belongs to the σ-weakly closed cone generated by*

$$\{f \otimes f: f \in L^2(X, \mu), f \geqslant 0\}.$$

(iii) *There is a positive linear functional* ρ *on* $\exp L^{\infty}(X, \mu)$ *satisfying* $\rho|_{\mathscr{S}} = \mu_A$.

Proof. We will show that (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). First, we claim that the identity operator on $L^2(X, \mu)$ belongs to the σ-weakly closed cone generated by the rank-one operators $f \otimes f$, $f \geqslant 0$ in L^2.

For that, consider the family of all finite partitions

$$\mathscr{P} = \{E_1, ..., E_n\}$$

of X into disjoint Borel sets E_i of positive measure. The partial ordering in which $\mathscr{P}_1 \leqslant \mathscr{P}_2$ means that $\mathscr{P}_2$ is a refinement of $\mathscr{P}_1$ makes this family into an increasing directed system. For every such $\mathscr{P} = \{E_1, ..., E_n\}$, let $E_{\mathscr{P}}$ be the self-adjoint projection operator

$$E_{\mathscr{P}} = \sum_{k=1}^{n} \mu(E_k)^{-1} \chi_{E_k} \otimes \chi_{E_k}.$$

$E_{\mathscr{P}}$ is the projection of $L^2(X, \mu)$ onto the n-dimensional subspace of all linear combinations

$$\lambda_1 \chi_{E_1} + \cdots + \lambda_n \chi_{E_n},$$

and therefore $\{E_{\mathscr{P}}\}$ is an increasing directed system of finite-dimensional projections. The union of all their ranges is dense in $L^2(X, \mu)$ because every L^2 function can be approximated by simple functions.

We conclude that

$$\lim_{\mathscr{P}} E_{\mathscr{P}} = 1$$

in the σ-weak operator topology and, since each $E_{\mathscr{P}}$ belongs to the cone generated by $\{f \otimes f: f \geqslant 0\}$, the claim follows.

Now let ω be a σ-weakly continuous linear functional on $\mathscr{L}(L^2(X, \mu))$ for which

$$\omega(f \otimes f) \geqslant 0$$

for every positive function f in $L^2(X, \mu)$. We have to show that $\omega(A) \geqslant 0$ for every factorable operator A. Write

$$\omega(T) = \operatorname{trace}(\Omega T),$$

where Ω is a trace class operator on $L^2(X, \mu)$, and write $A = B^*B$, where

$$B: \quad L^2(X, \mu) \to L^2(Y, \nu)$$

is an order-preserving operator. By 1.2.4(i), we can assume that $\nu(Y) < \infty$.

Consider the trace class operator Ω_B on $L^2(Y, \nu)$ defined by

$$\Omega_B = B\Omega B^*.$$

If F is a positive function in $L^2(Y, \nu)$, then we have

$$\begin{aligned}\operatorname{trace}(\Omega_B F\otimes F) &= \langle \Omega_B F, F\rangle = \langle \Omega B^*F, B^*F\rangle \\ &= \omega(B^*F\otimes B^*F)\geqslant 0,\end{aligned}$$

since $f = B^*F$ is a positive function in $L^2(X, \mu)$. By the preceding paragraphs, the identity operator on $L^2(Y, \nu)$ belongs to the σ-weakly closed cone generated by such rank-one operators $F\otimes F$, and hence

$$\operatorname{trace}(\Omega A) = \operatorname{trace}(\Omega B^*B) = \operatorname{trace}(\Omega_B 1)\geqslant 0,$$

as required.

(ii) $\Rightarrow$ (iii). Let us call a linear functional λ on $\mathscr{S}$ *extendable* if there is a positive linear functional ρ on $\exp L^\infty$ such that $\rho|_{\mathscr{S}} = \lambda$.

The set $\mathscr{E}$ of extendable linear functionals is a cone in the dual $\mathscr{S}'$ of $\mathscr{S}$. We claim that $\mathscr{E}$ is closed in the weak*-topology of $\mathscr{S}'$. By the Krein–Smulyan theorem, this will follow if we show that the intersection of $\mathscr{E}$ with the unit ball of $\mathscr{S}'$ is weak*-closed. Now since the norm of a positive linear functional on $\exp L^\infty$ is achieved at e and since $e\in\mathscr{S}$, we have

$$\|\rho\| = |\rho(e)| = \|\rho|_{\mathscr{S}}\|$$

for all such ρ. Thus, $\rho\to\rho|_{\mathscr{S}}$ is a weak*-continuous map of the state space of $\exp L$ onto

$$\mathscr{E}\cap \operatorname{ball}(\mathscr{S}').$$

Since the state space of $\exp L^\infty$ is weak*-compact, we have the desired conclusion.

Next, note that for A of the form $A = f\otimes f$ with $f\geqslant 0$ in $L^2(X, \mu)$, the linear functional μ_A is extendable. To prove this, we may obviously assume that $\langle f, 1\rangle = 1$. Consider the linear functional F defined on $L^\infty(X, \mu)$ by

$$F(g) = \langle g, f\rangle.$$

F is self-adjoint because f is real, and we have $\|F\|\leqslant 1$ because

$$|F(g)|\leqslant \int_X |g|\, f\, d\mu\leqslant \|g\|_\infty\, \langle f, 1\rangle = \|g\|_\infty.$$

So by the universal properties of the Banach *-algebra $\exp L^\infty$, F extends uniquely to a complex *-homomorphism

$$\rho\colon\quad \exp L^\infty\to\mathbb{C}.$$

Note that $\rho|_{\mathscr{S}} = \mu_{f\otimes f}$. Indeed, we have

$$\rho(e) = 1 = \langle f, 1\rangle^2 = \langle (f\otimes f)\, 1, 1\rangle,$$
$$\rho(g) = \langle g, f\rangle = \langle (f\otimes f)\, g, 1\rangle,$$

and

$$\rho(g_1 \vee g_2) = \rho(g_1)\, \rho(g_2) = \langle (f\otimes f)\, g_1, \bar{g}_2\rangle.$$

Finally, let $\mathscr{C}$ be the cone of all self-adjoint order-preserving operators on $L^2(X, \mu)$. The mapping

$$A \in \mathscr{C} \mapsto \mu_A \in \mathscr{S}'$$

is additive, homogeneous with respect to multiplication by nonnegative scalars, and carries each operator of the form $f\otimes f$, $f \geqslant 0$, into $\mathscr{E}$. So to prove that (ii) $\Rightarrow$ (iii), it suffices to show that this map is continuous relative to the σ-weak topology on $\mathscr{C}$ and the weak*-topology on $\mathscr{S}'$.

Fix an element $\xi \in \mathscr{S}$. We will show that $A \in \mathscr{C} \mapsto \mu_A(\xi)$ is σ-weakly continuous. For that, we can decompose $\xi = \xi_0 + \xi_1 + \xi_2$, where $\xi_i \in \bigvee^i L^\infty$, and so it suffices to show that each of the three functions

$$A \in \mathscr{C} \mapsto \mu_A(\xi_i), \qquad i = 0, 1, 2$$

is σ-weakly continuous. This is obvious for $i = 0$, and for $i = 1$ it follows from the fact that ξ_i belongs to $L^\infty \subseteq L^2$ and therefore

$$\mu_A(\xi_1) = \langle A\xi_1, 1\rangle$$

is σ-weakly continuous in A.

Consider $\mu_A(\xi_2)$. Now every element ξ of the projective tensor product of two Banach spaces $E_1 \otimes E_2$ can be expressed as an absolutely convergent series of elementary tensors

$$\xi = \sum_{n=0}^{\infty} x_n \otimes y_n;$$

in fact, one can find such an expression for ξ in which

$$\sum_{n=1}^{\infty} \|x_n\| \cdot \|y_n\| \leqslant \|\xi\| + \varepsilon,$$

where ε is arbitrary small. In particular, we can write

$$\xi_2 = \sum_{n=1}^{\infty} f_n \vee g_n,$$

where $f_n, g_n \in L^\infty(X, \mu)$ satisfy

$$\sum_{n=1}^{\infty} \|f_n\|_\infty \|g_n\|_\infty < \infty.$$

Then $\mu_A(\xi_2)$ has the form

$$\mu_A(\xi_2) = \sum_{n=1}^{\infty} \langle Af_n, g_n \rangle.$$

Since

$$\sum_{n=1}^{\infty} \|f_n\|_2 \|g_n\|_2 \leqslant \sum_{n=1}^{\infty} \|f_n\|_\infty \|g_n\|_\infty,$$

the right side of this expression is clearly σ-weakly continuous in A.

(iii) $\Rightarrow$ (i). Assume that there exists a positive linear functional ρ on $\exp L^\infty(X, \mu)$ satisfying $\rho|_{\mathscr{S}} = \mu_A$. We have to show that A is factorable.

Let Y denote the spectrum of $\exp L^\infty$, i.e., the compact Hausdorff space of all complex homomorphisms ω of $\exp L^\infty$ satisfying

$$\omega(\xi^*) = \overline{\omega(\xi)}, \qquad \xi \in \exp L^\infty.$$

By the representation theorem cited in Section 1.1, there is a finite positive Baire measure ν on Y such that

$$\rho(\xi) = \int_Y \omega(\xi)\, d\nu(\omega),$$

for every $\xi \in \exp L^\infty$.

We claim first that if $u \in L^\infty(X, \mu)$ denotes the constant function $u(x) \equiv 1$, then

$$\omega(u) = 1 \qquad \text{almost everywhere } (d\nu).$$

Indeed, with e denoting the unit of $\exp L^\infty$ we have

$$\begin{aligned}
\int_Y |\omega(u) - 1|^2\, d\nu &= \int_Y |\omega(u - e)|^2\, d\nu \\
&= \rho((u - e) \vee (u^* - e)) \\
&= \mu_A(u \vee u^*) - 2 \operatorname{Re} \mu_A(u) + \mu_A(e) \\
&= \langle A1, 1 \rangle - 2 \operatorname{Re} \langle A1, 1 \rangle + \langle A1, 1 \rangle = 0,
\end{aligned}$$

as required.

Now consider the mapping B of $L^\infty(X, \mu)$ into $L^\infty(Y, \nu)$ defined by

$$(Bf)(\omega) = \omega(f).$$

B is an involution-preserving linear map of commutative C^*-algebras satisfying $\|B\| \leqslant 1$. Moreover, by the preceding paragraph, B carries the unit of $L^\infty(X, \mu)$ to the unit of $L^\infty(Y, \nu)$. It follows that B is a positive linear map (see [3, Corollary of 1.7.1]).

We claim that B is bounded relative to the respective L^2 norms. Indeed, we have

$$\int_Y |Bf|^2 \, d\nu = \int_Y \omega(f \vee f^*) \, d\nu(\omega) = \rho(f \vee f^*) = \langle Af, f \rangle$$

$$\leqslant \|A\| \int_X |f|^2 \, d\nu.$$

Hence, we can extend B uniquely to a bounded operator from $L^2(X, \mu)$ to $L^2(Y, \nu)$. The extended map is positive because of the preceding paragraph and the fact that the positive bounded functions are dense in the positive cone of $L^2(X, \nu)$.

Finally, to see that $A = B^*B$, we need only check that $\langle Af, g \rangle = \langle Bf, Bg \rangle$ for bounded functions $f, g \in L^\infty$. In this case we have

$$\langle Bf, Bg \rangle = \int_Y \omega(f) \overline{\omega(y)} \, d\nu(\omega) = \int_Y \omega(f \vee g^*) \, d\nu$$

$$= \rho(f \vee g^*) = \langle Af, g \rangle,$$

as required. ∎

COROLLARY. *The factorable Markov operators form a weakly compact convex set in* $\mathscr{L}(L^2(X, \mu))$.

Proof. Let $\mathscr{M}$ denote the set of all self-adjoint Markov operators on $L^2(X, \mu)$. $\mathscr{M}$ is a weakly closed convex subset of the unit ball of $\mathscr{L}(L^2(X, \mu))$. By the previous theorem, the set in question is the intersection of $\mathscr{M}$ with the σ-weakly closed cone generated by the operators $f \otimes f$, $f \geqslant 0$, $f \in L^2(X, \mu)$; hence it is a σ-weakly closed convex set in the unit ball of $\mathscr{L}(L^2(X, \mu))$. Since the weak and σ-weak topologies coincide on the unit ball of $\mathscr{L}(\mathscr{H})$ and since the latter is weakly compact, the assertion follows. ∎

We conclude this section with a few remarks about dilation theory, and we give a further characterization of factorability in those terms.

Every contraction T on a Hilbert space $\mathscr{H}$ has a unitary power dilation. More precisely, there is an isometric embedding

$$V\colon \ \mathscr{H} \to \tilde{\mathscr{H}}$$

of $\mathscr{H}$ in another Hilbert space $\tilde{\mathscr{H}}$ and a unitary operator U on $\tilde{\mathscr{H}}$ such that

$$T^n = V^* U^n V, \qquad n = 0, 1, 2....$$

The pair (U, V) is unique in an appropriate sense provided that the set of vectors

$$\{U^n V\xi : n \in \mathbb{Z}, \xi \in \mathscr{H}\}$$

spans $\tilde{\mathscr{H}}$.

Nelson has observed [12, pp. 681–82] that an analogous dilation theorem holds in the category whose objects are probability spaces (X, μ) and whose morphisms are Markov operators

$$A: \quad L^2(X_1, \mu_1) \to L^2(X_2, \mu_2).$$

His result is equivalent to the following assertion: if A is a Markov operator on $L^2(X, \mu)$ then there is a probability space $(\tilde{X}, \tilde{\mu})$, a unitary Markov operator U on $L^2(\tilde{X}, \tilde{\mu})$, and a Markov isometry $V: L^2(X, \mu) \to L^2(\tilde{X}, \tilde{\mu})$ satisfying

$$A^n = V^* U^n V, \qquad n = 0, 1, 2,.... \tag{1.2.8}$$

Again, the pair (V, U) is unique in the appropriate sense for this category if one imposes certain minimality conditions. The construction of U from A is simply an abstract form of the construction of a stationary Markov process out of its transition probabilities and initial probabilities. There is a corresponding dilation theorem for one-parameter semigroups A_t, $t \geqslant 0$, of Markov operators.

Another familiar theorem from dilation theory asserts that if A is an operator on a Hilbert space $\mathscr{H}$ such that $0 \leqslant A \leqslant 1$, then there is a Hilbert space $\tilde{\mathscr{H}}$, a projection E in $\mathscr{L}(\tilde{\mathscr{H}})$, and an isometric embedding $V: \mathscr{H} \to \tilde{\mathscr{H}}$ such that

$$A = V^* E V. \tag{1.2.9}$$

Here there is no uniqueness result. However, the proof of the existence of (E, V) is particularly simple and explicit: take

$$\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H},$$

$$V\xi = \xi \oplus 0,$$

$$E = \begin{bmatrix} A & (A - A^2)^{1/2} \\ (A - A^2)^{1/2} & 1 - A \end{bmatrix}.$$

Now suppose A is a Markov operator on $L^2(X, \mu)$. In the sequel, we will be led to seek a representation corresponding to (1.2.9), in which V and E belong to the class of Markov operators. More precisely, we ask if there is another probability space $(\tilde{X}, \tilde{\mu})$, a Markov isometry $V: L^2(X, \mu) \to L^2(\tilde{X}, \tilde{\mu})$, and a Markov projection E acting on $L^2(\tilde{X}, \tilde{\mu})$ such that

$$A = V^*EV. \tag{1.2.10}$$

The requirement on E is simply that E should be the conditional expectation operator corresponding to some sub σ-field of the given σ-field of subsets of $(\tilde{X}, \tilde{\mu})$.

For a representation like (1.2.10) to exist, it is necessary that A should be positive semidefinite; and since Markov operators are necessarily contractions, this is equivalent to the condition

$$0 \leqslant A \leqslant 1.$$

In this case, however, the necessary condition is no longer sufficient, for we have

THEOREM 1.2.11. *Let A be a Markov operator satisfying $A \geqslant 0$. Then A has a dilation of the form* 1.2.10 *if and only if A is factorable.*

Proof. If A has the form V^*EV as in (1.2.10), then we have $A = B^*B$, where B is the Markov operator

$$B = EV: \quad L^2(X, \mu) \to L^2(\tilde{X}, \tilde{\mu}).$$

To prove the converse, we apply the representation theorem 1.1.5 in the following way. Assume $A = B^*B$, where $B: L^2(X, \mu) \to L^2(Y, \nu)$ is order-preserving. By 1.2.3(ii), we may assume that (Y, ν) is a probability space and B is a Markov operator.

By Theorem 1.1.5, there is a probability space (Z, σ) and a (minimal) pair of Markov isometries (V, W),

$$V: \quad L^2(X, \mu) \to L^2(Z, \sigma),$$
$$W: \quad L^2(Y, \nu) \to L^2(Z, \sigma),$$

such that $B = W^*V$. It follows that

$$A = V^*WW^*V = V^*EV,$$

where E is the conditional expectation operator WW^* on $L^2(Z, \sigma)$. ∎

1.3. *Nonfactorable Operators and C*-Algebras*

In this section we describe a construction which produces a wide variety of nonfactorable Markov operators A on $L^2(X, \mu)$, all of which satisfy the operator-theoretic positivity condition

$$\langle Af, f \rangle \geqslant 0, \qquad f \in L^2(X, \mu).$$

The construction makes use of properties of noncommutative C^*-algebras in an essential way.

Let X be a compact Hausdorff space and let $\mathfrak{A}$, ϕ be a unital C^*-algebra $\mathfrak{A}$ together with a distinguished state ϕ on $\mathfrak{A}$. Suppose that, in addition, we are given a linear mapping

$$f \in C(X) \mapsto P_f \in \mathscr{L}(\mathfrak{A})$$

from the space of continuous functions on X to bounded linear operators on $\mathfrak{A}$, having the following properties:

$$\begin{aligned} &\text{(i)} \quad f \geqslant 0 \Rightarrow P_f \text{ is a positive linear map,} \\ &\text{(ii)} \quad P_1 \text{ fixes the unit of } \mathfrak{A}, \\ &\text{(iii)} \quad \phi(P_f(a)\, a^*) \geqslant 0 \text{ for } a \in \mathfrak{A},\ f \geqslant 0. \end{aligned} \tag{1.3.1}$$

A quadruple $(X, \mathfrak{A}, \phi, P)$ with these properties will be called a *Markov* system. A Markov system which possesses the additional property

(iv) $\{P_f(1) : f \in C(X), f \geqslant 0\}$ is dense in the positive cone of $\mathfrak{A}$ will be called *full*. (1.3.1)

Here is one way to obtain examples of Markov systems in which the space X is countable. Let $\mathfrak{A}$ be an arbitrary separable unital C^*-algebra which possesses a tracial state ϕ (i.e., $\phi(ab) = \phi(ba)$, $a, b \in \mathfrak{A}$). Choose a sequence $e_2, e_3, \ldots,$ of positive elements of $\mathfrak{A}$ which generates the positive cone of $\mathfrak{A}$. By scaling appropriately we can arrange that the sum $\sum \|e_n\|$ is smaller than 1, and by adjoining the element

$$e_1 = 1 - \sum_{n=2}^{\infty} e_n$$

to the sequence, we may assume that

$$\sum_{n=1}^{\infty} e_n = 1,$$

the series being absolutely convergent in the sense that $\|e_1\| + \|e_2\| + \cdots < \infty$. Let $C(X)$ be the unital extension $\tilde{c}_0$ of the commutative

C^*-algebra c_0, considered as the space of all convergent sequences $f=(f_n)_{n\geqslant 1}$ of complex numbers. For $f\in C(X)$, put

$$P_f(a)=\sum_{n=1}^{\infty} f_n e_n^{1/2} a e_n^{1/2}, \qquad a\in\mathfrak{A}.$$

The series on the right converges absolutely, and the properties 1.3.1(i) and (ii) are immediate. (1.3.1)(iii) follows from the observation that, since $\phi(xy)=\phi(yx)$ for all $x, y\in\mathfrak{A}$, we have

$$\begin{aligned}\phi(P_f(a)\,a^*) &= \sum_{n=1}^{\infty} f_n\phi(e_n^{1/2}ae_n^{1/2}a^*)\\ &= \sum_{n=1}^{\infty} f_n\phi(e_n^{1/4}ae_n^{1/2}a^*e_n^{1/4}),\end{aligned}$$

and of course the latter is nonnegative if $f=(f_n)$ is a nonnegative sequence. This Markov system is *full* because, taking $u_1=(1, 0\, 0,...)$, $u_2=(0, 1, 0,...,),...,$ in $C(X)^+$, we obtain the sequence

$$P_{u_n}(1)=e_n, \qquad n\geqslant 1,$$

which generates the positive cone of $\mathfrak{A}$ by the way we choose $e_1, e_2,...$.

Variations on this example give full Markov systems $(X, \mathfrak{A}, \phi, P)$ in which X is an arbitrary compact metric space.

Now let $(X, \mathfrak{A}, \phi, P)$ be a full Markov system, which will be fixed throughout the remainder of this section. We will indicate how one can construct a probability measure μ on X and a positive semidefinite Markov operator A on $L^2(X, \mu)$. By (1.3.1)(i) and (ii),

$$f\mapsto\phi(P_f(1))$$

is a state of $C(X)$, so by the Riesz-Markov theorem there is a unique Baire probability measure μ on X satisfying

$$\int f\,d\mu=\phi(P_f(1)), \qquad f\in C(X). \tag{1.3.2}$$

PROPOSITION 1.3.3. *There is a unique bounded operator A on $L^2(X, \mu)$ satisfying*

$$\langle Af, g\rangle=\phi(P_f(1)\,P_{\bar g}(1))$$

for $f, g\in C(X)$. A is a Markov operator and we have

$$\langle Af, f\rangle\geqslant 0, \qquad f\in L^2(X, \mu).$$

For the proof of 1.3.3, we require

LEMMA 1.3.4. *For every $f \in C(X)$ and $a, b \in \mathfrak{A}$, we have*

(i) $P_f(a) = P_f(a^*)^*$ *and*

(ii) $\phi(P_f(a)\, b) = \phi(a P_f(b))$.

Proof. (i) For fixed $a \in \mathfrak{A}$, both sides of (i) are antilinear in f; so it suffices to verify (i) for real-valued functions f. Since every real f in $C(X)$ is a difference of positive functions, we may assume $f \geqslant 0$. But then P_f is a positive linear map of $\mathfrak{A}$ into itself, and such mappings are self-adjoint in the sense that $P_f(a^*) = P_f(a)^*$.

(ii) Again, it suffices to prove the asserted identity for the case where $f \geqslant 0$. Fix such an f, and consider the sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$ defined by

$$[a, b] = \phi(P_f(a)\, b^*).$$

By (1.3.1)(iii), $[\cdot, \cdot]$ is positive semidefinite and in particular, $[\cdot, \cdot]$ must be self-adjoint. Thus we have

$$\begin{aligned} \phi(P_f(a)\, b^*) &= [a, b] = \overline{[b, a]} = \overline{\phi(P_f(b)\, a^*)} \\ &= \phi(a P_f(b)^*) = \phi(a P_f(b^*)). \quad \blacksquare \end{aligned}$$

To prove 1.3.3, consider the sesquilinear form $B\colon C(X) \times C(X) \to \mathbb{C}$ defined by

$$B(f, g) = \phi(P_f(1)\, P_{\bar{g}}(1)).$$

Note that B is positive semidefinite; for by Lemma 1.3.4(i) we have $P_{\bar{f}}(1) = P_f(1)^*$, and hence

$$B(f, f) = \phi(P_f(1)\, P_{\bar{f}}(1)) = \phi(P_f(1)\, P_f(1)^*) \geqslant 0.$$

We want to show that

$$|B(f, g)| \leqslant \|f\|_2\, \|g\|_2, \qquad f, g \in C(X), \tag{1.3.5}$$

$\|\cdot\|_2$ denoting the norm in $L^2(X, \mu)$. For this, note that if f, g are positive functions then $B(f, g) \geqslant 0$. Indeed, using 1.3.4(ii) we can write

$$B(f, g) = \phi(P_f(1)\, P_g(1)) = \phi(P_g(P_f(1))),$$

which is nonnegative because $P_g \circ P_f$ is a positive linear mapping and thus carries 1 to a positive element of $\mathfrak{A}$. By Lemma 1.1.6 there is a positive Baire measure ν on $X \times X$ such that

$$B(f, g) = \int_{X \times X} f(x)\, \overline{g(y)}\, d\nu(x, y). \tag{1.3.6}$$

ν is a probability measure because, by (1.3.1)(ii) we have

$$B(1, 1) = \phi(P_1(1)\, P_1(1)) = \phi(1) = 1.$$

Applying the Schwarz inequality to (1.3.6) we have

$$|B(f, g)|^2 \leqslant \int_{X \times X} |f(x)|^2 \, d\nu \int_{X \times X} |g(y)|^2 \, d\nu$$

$$= B(|f|^2, 1)\, B(1, |g|^2).$$

Now,

$$B(|f|^2, 1) = \phi(P_{|f|^2}(1)) = \int_X |f|^2 \, d\mu = \|f\|_2^2,$$

and moreover,

$$B(1, |g|^2) = \overline{B(|g|^2, 1)} = B(|g|^2, 1) = \|g\|_2^2.$$

That proves (1.3.5).

Since $C(X)$ is dense in $L^2(X, \mu)$ we may conclude from the Riesz lemma that there is a unique contraction A on $L^2(X, \mu)$ satisfying

$$\langle Af, g \rangle = B(f, g) = \phi(P_f(1)\, P_{\bar{g}}(1)).$$

We have $A \geqslant 0$ because $B(\cdot, \cdot)$ is a positive semidefinite sesquilinear form on $C(X) \times C(X)$.

Now if f and g are positive functions in $C(X)$, then as we have already seen,

$$\langle Af, g \rangle = B(f, g) \geqslant 0.$$

Since $C(X)^+$ is dense in the positive cone of $L^2(X, \mu)$ we have $\langle Af, g \rangle \geqslant 0$ for all positive $f, g \in L^2(X, \mu)$, and therefore A is order-preserving.

Finally, $A(1) = 1$ because for all $g \in C(X)$,

$$\langle A1, g \rangle = B(1, g) = \overline{B(g, 1)} = \overline{\phi(Pg(1))} = \phi(Pg(1)^*)$$

$$= \phi(P_{\bar{g}}(1)) = \int \bar{g} \, d\mu = \langle 1, g \rangle. \quad \blacksquare$$

Thus, every Markov system $(X, \mathfrak{A}, \phi, P)$ gives rise to a positive semidefinite Markov operator A. When the system $(X, \mathfrak{A}, \phi, P)$ is *full*, we have the following rather convenient characterization of the factorability of A. Let π_ϕ be the representation of $\mathfrak{A}$ obtained via the GNS construction,

$$\langle \pi_\phi(a)\, v, v \rangle = \phi(a),$$

where v is a unit vector in the Hilbert space $\mathscr{H}_\phi$ *which is cyclic for the operator algebra* $\pi_\phi(\mathfrak{A})$.

THEOREM 1.3.7. *Let A be the canonical Markov operator associated with a full Markov system $(X, \mathfrak{A}, \phi, P)$. Then A is factorable iff the C^*-algebra $\pi_\phi(\mathfrak{A})$ is abelian.*

The proof of Theorem 1.3.7 is based on the following result, which may be of some interest on its own. It provides a rather broad class of cones in Hilbert spaces which cannot be isometrically embedded in "commutative" cones, i.e., the cone of positive functions in a space of the form $L^2(X, \mu)$. In particular, one cannot introduce coordinates for $\mathscr{H}$ so as to make all vectors in the given cone nonnegative.

LEMMA 1.3.8. *Let $\mathfrak{A}$ be a C^*-algebra of operators on a Hilbert space $\mathscr{H}$, which contains the identity operator and has a cyclic vector v. Let $\mathscr{P}$ be the closure in $\mathscr{H}$ of the cone $\{av: a \in \mathfrak{A},\ a \geqslant 0\}$. Then the following are equivalent:*

(i) *There is a positive measure space (X, μ) and an isometry $U: \mathscr{H} \to L^2(X, \mu)$ such that $U\xi \geqslant 0$ for all $\xi \in \mathscr{P}$,*

(ii) *$\mathfrak{A}$ is commutative.*

Proof. (i) $\Rightarrow$ (ii). Let

$$U: \quad \mathscr{H} \to L^2(X, \mu)$$

be an isometry with the stated properties. We first show that one may modify (X, μ) and U so as to obtain a *finite* measure space and an isometry which carries vectors of the form av, $a \in \mathfrak{A}$, to bounded functions.

To see this, consider the linear mapping $\rho: \mathfrak{A} \to L^2(X, \mu)$ defined by

$$\rho(a) = U(av).$$

We have $\rho(a^*a) \geqslant 0$ for every $a \in \mathfrak{A}$, and so Lemma 1.2.3 implies that we have the Schwarz inequality

$$|\rho(b^*a)|^2 \leqslant \rho(b^*b)\, \rho(a^*a) \qquad \text{a.e. } (d\mu),$$

for every $a, b \in \mathfrak{A}$. Setting $b = 1$, we conclude that

$$|\rho(a)|^2 \leqslant \rho(1)\, \rho(a^*a) \qquad \text{a.e. } (d\mu) \tag{1.3.9}$$

for each $a \in \mathfrak{A}$. Let ν be the finite positive measure on X defined by

$$d\nu = \rho(1)^2\, d\mu.$$

Since ν is concentrated on the set

$$S = \{x \in X: \rho(1)(x) > 0\},$$

we can define a linear mapping $V: \mathfrak{A} \to L^2(X, \nu)$ by

$$V(\xi) = \rho(1)^{-1}\, U(\xi).$$

Indeed, V is a bounded operator because

$$\|V\xi\| = \int_X \rho(1)^{-2}\, |U(\xi)|^2\, d\nu = \int_S |U(\xi)|^2\, d\mu$$

$$\leqslant \int_X |U(\xi)|^2\, d\mu = \|\xi\|^2.$$

Actually, V is an isometry. To see that, let ξ have the form $\xi = av$, $a \in \mathfrak{A}$. Then by (1.3.9) we have

$$|U(\xi)|^2 = |\rho(a)|^2 \leqslant \rho(1)\, \rho(a^*a) \qquad \text{a.e. } (d\mu),$$

which shows that $U\xi$ lives essentially in S. Then,

$$\|V\xi\|^2 = \int_S |U\xi|^2\, d\mu = \int_X |U\xi|^2\, d\mu = \|\xi\|^2.$$

The assertion follows because $\mathfrak{A}v$ is dense in $\mathscr{H}$.

Finally, note that V carries v to the constant $1 \in L^2(X, \nu)$, and of course V carries $\mathscr{P}$ into the positive cone of $L^2(X, \nu)$. Note also that $V(av) \in L^\infty(X, \nu)$ for every $a \in \mathfrak{A}$. Indeed, if a is a self-adjoint element of $\mathfrak{A}$ with $\|a\| \leqslant 1$, then $-1 \leqslant a \leqslant 1$ and hence

$$-1 = V(v) \leqslant V(av) \leqslant V(v) = 1$$

a.e. $(d\mu)$, because $V(\mathscr{P}) \geqslant 0$. The assertion follows from this.

Thus we can define a linear mapping $\pi: \mathfrak{A} \to L^2(X, \nu)$ by

$$\pi(a) = V(av);$$

π is a positive linear map of C^*-algebras which carries the unit of $\mathfrak{A}$ to the unit of $L^\infty(X, \nu)$. We claim that π is multiplicative. By a standard polarization argument applied to the sesquilinear forms $\overline{\pi(x)}\,\pi(y)$ and $\pi(x^*y)$, it is enough to show that

$$|\pi(a)|^2 = \pi(a^*a) \qquad \text{a.e. } (d\nu),$$

for every $a \in \mathfrak{A}$. Corresponding to (1.3.9) we have

$$\pi(a^*a) - |\pi(a)|^2 \geqslant 0 \qquad \text{a.e. } (d\nu),$$

and so it is enough to show that the integral of the left side of the inequality is zero. But since V is an isometry we have

$$\int_X \pi(a^*a)\, d\nu = \int_X \pi(a^*a)\,\overline{\pi(1)}\, d\nu = \langle V(a^*av), Vv\rangle_{L^2(X,\nu)}$$
$$= \langle a^*av, v\rangle,$$

while

$$\int_X |\pi(a)|^2\, d\nu = \langle V(av), V(av)\rangle_{L^2(X,\nu)}$$
$$= \langle av, av\rangle = \langle a^*av, v\rangle.$$

The claim is follows.

To complete the proof of (i) $\Rightarrow$ (ii), it suffices to show that π has trivial kernel. But if $a \in \mathfrak{A}$ and $\pi(a)=0$, then for all $x, y \in \mathfrak{A}$ we have

$$\langle axv, yv\rangle = \langle V(axv), V(yv)\rangle_{L^2(X,\nu)}$$
$$= \int_X \pi(ax)\,\overline{\pi(y)}\, d\nu = \int_X \pi(a)\,\pi(x)\,\overline{\pi(y)}\, d\nu = 0.$$

We conclude that $a=0$ because v is a cyclic vector for $\mathfrak{A}$.

(ii) $\Rightarrow$ (i). This is a standard argument in spectral theory and we merely sketch the details. Let X be the spectrum of the commutative C^*-algebra $\mathfrak{A}$. By the Riesz–Markov theorem there is a positive Baire measure μ on X such that

$$\langle av, v\rangle = \int_X \hat{a}\, d\mu,$$

$\hat{a}$ denoting the Gelfand transform of $a \in \mathfrak{A}$. The map

$$U\colon av \to \hat{a}$$

extends to a unitary operator from $\mathscr{H}$ to $L^2(X, \mu)$ having the asserted properties. ∎

Turning now to the proof of 1.3.7, let $(X, \mathfrak{A}, \phi, P)$ be a full Markov system with associated Markov operator A defined as in Proposition 1.3.3. We will show that A is factorable iff there is an isometric embedding

$$U\colon \mathscr{H}_\phi \to L^2(Y, \nu)$$

of $\mathscr{H}_\phi$ into some L^2 space which carries the cone $\pi_\phi(\mathfrak{A})^+ v$ into the positive cone of $L^2(Y, \nu)$. An application of Lemma 1.3.8 will then complete the proof.

First, assume that there is a positive measure space (Y, ν) and an isometry $U: \mathscr{H}_\phi \to L^2(Y, \nu)$ having the stated properties. Define a linear map $B: C(X) \to L^2(Y, \nu)$ by

$$Bf = U(\pi_\phi(P_f(1))\, v).$$

B maps positive functions in $C(X)$ to positive functions in $L^2(Y, \nu)$ because of (1.3.1)(i).

Recall that A acts on $L^2(X, \mu)$, where μ is the probability measure

$$\int_X f\, d\mu = \phi(P_f(1)), \qquad f \in C(X),$$

and we have $\langle Af, g\rangle = \phi(P_f(1) P_{\bar{g}}(1))$. Let us check inner products. For $f, g \in C(X)$ we have

$$\begin{aligned}\langle Bf, Bg\rangle &= \langle U(\pi_\phi(P_f(1))\, v), U(\pi_\phi(P_g(1))\, v)\rangle \\ &= \langle \pi_\phi(P_f(1))\, v, \pi_\phi(P_g(1))\, v\rangle \\ &= \langle \pi_\phi(P_g(1)^* P_f(1))\, v, v\rangle \\ &= \phi(P_g(1)^* P_f(1)) = \phi(P_{\bar{g}}(1)\, P_f(1)) \\ &= \langle A\bar{g}, \bar{f}\rangle = \langle f, Ag\rangle = \langle Af, g\rangle.\end{aligned}$$

It follows that

$$\|Bf\|^2 \leqslant \|A\| \cdot \|f\|^2 \leqslant \|f\|^2,$$

$\|f\|$ denoting the norm of f in $L^2(X, \mu)$, and thus B extends uniquely to a bounded operator from $L^2(X, \mu)$ to $L^2(Y, \nu)$. The extended map is order-preserving because the positive cone of $C(X)$ is dense in that of $L^2(X, \mu)$. Finally, the preceding computation implies that $B^*B = A$, hence A is factorable.

Conversely, assume A can be factored in the form $A = B^*B$, where

$$B: \quad L^2(X, \mu) \to L^2(Y, \nu)$$

is an order-preserving operator. We claim that there is a unique isometry $U: \mathscr{H}_\phi \to L^2(Y, \nu)$ satisfying

$$U(\pi_\phi(P_f(1))\, v) = B(f), \qquad f \in C(X). \tag{1.3.10}$$

For the existence of U, fix $f, g \in C(X)$. Then as we have already seen above, we have

$$\begin{aligned}\langle \pi_\phi(P_f(1))\, v, \pi_\phi(P_g(1))\, v\rangle &= \phi(P_{\bar{g}}(1)\, P_f(1)) \\ &= \langle A\bar{g}, \bar{f}\rangle = \langle f, Ag\rangle = \langle Af, g\rangle \\ &= \langle B^*Bf, g\rangle = \langle Bf, Bg\rangle.\end{aligned}$$

Now $\{P_f(1): f \geqslant 0\}$ is dense in the positive cone of $\mathfrak{A}$ because the Markov system $(X, \mathfrak{A}, \phi, P)$ is full. Since π_ϕ maps the positive cone of $\mathfrak{A}$ onto that of $\pi_\phi(\mathfrak{A})$, it follows that the vectors of the form

$$\pi_\phi(P_f(1))\, v, \qquad f \in C(X)^+ \tag{1.3.11}$$

are dense in the cone $\{Tv: T \in \pi_\phi(\mathfrak{A})^+\}$. In particular, since v is cyclic for $\pi_\phi(\mathfrak{A})$ there is a unique linear isometry U satisfying (1.3.10).

The preceding observations imply that U maps the cone

$$\{Tv: T \in \pi_\phi(\mathfrak{A})^+\}$$

into the positive cone of $L^2(Y, \nu)$, and the proof is finished. ∎

EXAMPLE 1.3.12. We conclude this section with a discussion of some rather concrete examples of positive semidefinite Markov operators associated with nonatomic probability spaces, none of which are factorable. The proof of the latter makes essential use of Theorem 1.3.7.

Fix an integer $n \geqslant 2$, let M_n be the C^*-algebra of all complex $n \times n$ matrices, and let U_n be the unitary group in M_n. U_n is a compact group which admits a unique Haar measure m, normalized so that $m(U_n) = 1$. We may consider (U_n, m) as a nonatomic probability space.

If we cause M_n to act on $\mathbb{C}^n$ in the usual way, then for a fixed unit vector $\xi \in \mathbb{C}^n$ the function $\psi: U_n \to \mathbb{C}$ defined by

$$\psi(u) = |\langle u\xi, \xi\rangle|^2$$

is continuous, takes on nonnegative values, has integral 1, and satisfies the positive definiteness condition

$$\sum_{i,j=1}^{N} \lambda_i \bar{\lambda}_j \psi(u_j^* u_i) \geqslant 0$$

for all $\lambda_1, ..., \lambda_N \in \mathbb{C}$, $u_1, ..., u_N \in U_n$, $N \geqslant 1$. It follows that the convolution operator A, defined on $L^2(U_n, m)$ by

$$Af = f * \psi \tag{1.3.13}$$

is a positive semidefinite Markov operator.

We claim that A is not factorable. To see this, one applies Theorem 1.3.7 in the following way. Let e be the rank one projection whose range is spanned by the given unit vector ξ, and consider the quadruple $(X, \mathfrak{A}, \phi, P)$, where $X = U_n$, $\mathfrak{A} = M_n$, ϕ is the normalized trace on M_n, and P is defined by

$$P_f(a) = \int_{U_n} f(u)\, ueu^*aueu^*\, dm(u),$$

$f \in C(U_n)$, $a \in M_n$. We leave it for the reader to carry out the routine verification that $(X, \mathfrak{A}, \phi, P)$ is a full Markov system.

The canonical measure μ is defined on $X = U_n$ by 1.3.2 is clearly m, and the operator defined in the statement of proposition 1.3.3 turns out to be the operator A defined above in 1.3.13. Thus we are in position to apply Theorem 1.3.7. Since M_n is a simple noncommutative C^*-algebra, $\pi_\phi(\mathfrak{A})$ is noncommutative, and so we may conclude from 1.3.7 that A is not factorable.

II. OS-Positive Processes

2.1. *Connections with Quantum Theory*

The motivation for the work behind this paper arose from a mathematical problem connected with (Euclidean) quantum field theory. The purpose of this section is to discuss that problem for readers interested in constructive field theory. Those who are not may skip directly to Section 2.2 without loss of essential content.

From the mathematical point of view, a quantum system possessing a ground state consists of at least three elements: an abelian C^*-algebra of operators (the configuration observables at time zero), a one-parameter unitary group (which gives the time evolution of the system), and a distinguished unit vector (the vacuum). One is often presented with additional structure such as a Weyl system [15, 16] but here we wish to concentrate on these three basic constituents.

More precisely, we have a triple $(\mathfrak{A}, U, v)$ consisting of

(i) an abelian C^*-algebra $\mathfrak{A}$ of operators on a Hilbert space,

(ii) a one-parameter unitary group U_t having nonnegative spectrum,

and (2.1.1)

(iii) a distinguished unit vector v in $\mathscr{H}$ such that $U_t v = v$, $t \in \mathbb{R}$.

Nothing is lost if one assume (as we will) that v is a cyclic vector for the C^*-algebra generated by $\mathfrak{A} \cup \{U_t: t \in \mathbb{R}\}$.

The *Schwinger functions* associated with a quantum system $(\mathfrak{A}, U, v)$ are defined as follows. By Stone's theorem we can write

$$U_t = e^{itH},$$

where H is a uniquely determined self-adjoint (unbounded) operator. Property (2.1.1)(ii) asserts that the spectrum of H is nonnegative, and (iii) implies that v belongs to the domain of H and satisfies $Hv = 0$.

Since $H \geqslant 0$, we may utilize the standard functional calculus for normal operators to define a one-parameter semigroup P_t of positive contraction operators by

$$P_t = e^{-tH}, \qquad t \geqslant 0.$$

Formally, P_t is obtained by analytically continuing the unitary group $\{U_s: s \in \mathbb{R}\}$ to imaginary time: $P_t = U_{it}$. In any case, for each $n \geqslant 1$ we obtain a function

$$\phi_n: \quad [0, \infty)^n \times \mathfrak{A}^n \to \mathbb{C}$$

as follows

$$\phi_n(t_1, ..., t_n; a_1, ..., a_n) = \langle P_{t_1} a_1 P_{t_2} a_2 \cdots P_{t_n} a_n v, v \rangle.$$

For $t_1, ..., t_n$ fixed, $\phi_n(t_1, ..., t_n; a_1, ..., a_n)$ is a bounded multilinear form on $\mathfrak{A} \times \mathfrak{A} \times \cdots \times \mathfrak{A}$. The functions $\phi_1, \phi_2, ...$ are called the *Schwinger functions* of the system $(\mathfrak{A}, U, v)$.

It follows from work of Nelson [12, 13] that if one is given a symmetric stationary Markov process $\{X_t: t \in \mathbb{R}\}$ (taking values in an arbitrary state space Σ), then one can construct a quantum system. We will sketch this construction, utilizing a formulation similar to that proposed by Klein [10]. Let (Ω, p) be a probability space and let Σ be a standard Borel space (see [3]). By a *stochastic process* (with state space Σ) we mean a family $\{X_t: t \in \mathbb{R}\}$ of measurable functions

$$X_t: \quad \Omega \to \Sigma$$

which is *continuous* in the sense that for every $n \geqslant 1$ and every sequence $E_1, ..., E_n$ of Borel sets in Σ, the function

$$(t_1, ..., t_n) \in \mathbb{R}^n \to p\{X_{t_1} \in E_1, ..., X_{t_n} \in E_n\}$$

is continuous. A process $\{X_t\}$ is called *stationary* (resp. *symmetric*) if, for every $n \geqslant 1$, every $t_1,..., t_n \in \mathbb{R}$, and every sequence $E_1,..., E_n$ of Borel sets in Σ, one has

$$p\{X_{t_1+\tau} \in E_1,..., X_{t_n+\tau} \in E_n\} = p\{X_{t_1} \in E_1,..., X_{t_n} \in E_n\}$$

for every $\tau\mathbb{R}$, (and, resp.

$$p\{X_{-t_1} \in E_1,..., X_{-t_n} \in E_n\} = p\{X_{t_1} \in E_1,..., X_{t_n} \in E_n\}).$$

Finally, $\{X_t\}$ is called a *Markov process* if, for every $t_1 < t_2 < \cdots < t_n < t$ and every bounded Borel function $f\colon \Sigma \to \mathbb{C}$, we have

$$E(f(X_t) \mid X_{t_1},..., X_{t_n}) = E(f(X_t) \mid X_{t_n}) \tag{2.1.2}$$

Here, $E(\cdot, \cdot)$ denotes the usual conditional expectation [4]. In addition, we lose nothing if we always require that the σ-field generated by the sets $\{X_t \in E\}$, $t \in \mathbb{R}$, $E \subseteq \Sigma$, be essentially all of the Borel sets in Ω.

We now indicate how, starting with a symmetric stationary Markov process $\{X_t\}$, one can construct a quantum system $(\mathfrak{A}, U, v)$. Let $\mathscr{H}$ be the "time zero" subspace of $L^2(\Omega, p)$, consisting of all L^2 functions $F\colon \Omega \to \mathbb{C}$ which have the form

$$F(\omega) = f(X_0(\omega)) \qquad \text{a.e. } (dp) \tag{2.1.3}$$

for some Borel function $f\colon \Sigma \to \mathbb{C}$. Let v be the constant function 1 and let $\mathfrak{A}$ be the abelian von Neumann algebra of all operators

$$M_F|_{\mathscr{H}},$$

where F is a bounded function of the form (2.1.3) and M_F denotes the usual multiplication operator on $L^2(\Omega, p)$.

In order to define the unitary group $U_t \in \mathscr{L}(\mathscr{H})$, consider the one-parameter group W_t of unitary Markov operators on $L^2(\Omega, p)$ determined by translation by t; i.e.,

$$W_t(f_1(x_{t_1}) \cdots f_n(X_{t_n})) = f_1(X_{t_1+t}) \cdots f_n(X_{t_n+t})$$

for $n \geqslant 1$, $t_1,..., t_n \in \mathbb{R}$, and $f_1,..., f_n$ bounded Borel functions on Σ. Let E_0 be the projection of $L^2(\Omega, p)$ onto the subspace $\mathscr{H}$. E_0 is of course the conditional expectation operator $E_0(f) = E(f \mid X_0)$. In any case, the compression

$$P_t = E_0 W_t|_{\mathscr{H}}$$

of W_t to $\mathscr{H}$ is a contraction for every $t \geqslant 0$, and because of the Markov property of $\{X_t\}$ it is not hard to show that $\{P_t\}$ is a *semigroup*, i.e.,

$$P_s P_t = P_{s+t}, \qquad s, t \geqslant 0$$
$$P_0 = 1.$$

As it turns out, the symmetry hypothesis on the process $\{X_t\}$ implies that $P_t^* = P_t$ for each $t \geqslant 0$. It follows that P_t is a positive self-adjoint contraction for every $t \geqslant 0$.

Thus, there is a unique *positive* self-adjoint operator H on $\mathscr{H}$ such that

$$P_t = e^{-tH}, \qquad t \geqslant 0.$$

It is now a simple exercise with the functional calculus to construct the required unitary group

$$U_t = e^{itH}, \qquad t \in \mathbb{R}.$$

The quantum system $(\mathfrak{A}, U, v)$ constructed in this way has two important properties which are not shared by general quantum systems. First, the Schwinger functions ϕ_n must satisfy the following positivity condition

$$\phi_n(t_1,..., t_n; a_1,..., a_n) \geqslant 0 \tag{2.1.4}$$

whenever $a_1,..., a_n$ in $\mathfrak{A}$ satisfy $a_1 \geqslant 0,..., a_n \geqslant 0$. Second, and improving considerably on the general requirement that v should be cyclic for the C^*-algebra generated by $\mathfrak{A} \cup \{U_t\}$, here we actually have

$$\overline{\mathfrak{A}v} = \mathscr{H}. \tag{2.1.5}$$

Indeed, 2.1.5 implies that the weak closure of $\mathscr{H}$ is a *maximal abelian* von Neumann algebra in $\mathscr{L}(\mathscr{H})$.

First, let us consider the positivity condition (2.1.4). Osterwalder and Schrader [14] showed that one can relax significantly the Markov hypothesis on the process $\{X_t\}$, in such a way that something very close to Nelson's construction of $(\mathfrak{A}, U, v)$ still works, and moreover, it still yields a quantum system which obeys (2.1.4). Their replacement of the Markov hypothesis is now called OS-positivity, and is defined as follows. Let $\{X_t: t \in \mathbb{R}\}$ be a continuous stationary process on (Ω, p), $X_t: \Omega \to \Sigma$. $\{X_t\}$ is called OS-*positive* if, for every ordered n-tuple $0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ of non-negative real numbers and every bounded Borel function $f: \Sigma^n \to \mathbb{C}$, one has

$$\langle f(X_{t_1},..., X_{t_n}), f(X_{-t_1},..., X_{-t_n}) \rangle \geqslant 0, \tag{2.1.6}$$

$\langle \cdot, \cdot \rangle$ denoting the inner product in $L^2(\Omega, p)$. A stationary OS-positive process is necessarily symmetric. Now given such a process, there is a generalization of the above construction which yields a quantum system $(\mathfrak{A}, U, v)$ whose Schwinger functions satisfy (2.1.4) (see [8], for example).

Unfortunately, the cyclicity condition (2.1.5) is no longer true in general for the quantum systems constructed from OS-positive processes. This phenomenon was clarified by Klein, who also characterized the quantum systems which can be constructed from OS-positive processes [7, 8]. He showed first, that a quantum system $(\mathfrak{A}, U, v)$ arises from a stationary OS-positive process $\{X_t\}$ in this manner if and only if its associated Schwinger functions satisfy (2.1.4). Moreover, in this case the OS-positive process $\{X_t\}$ which gives rise to the quantum system is essentially unique. Second, one has

$$\overline{\mathfrak{A}v} = \mathscr{H}$$

if and only if the associated process $\{X_t\}$ is actually a symmetric stationary Markov process.

Let $(\tilde{\mathfrak{A}}, \tilde{U}, \tilde{v})$ be a quantum system on a Hilbert space $\tilde{\mathscr{H}}$ which satisfies the positivity condition (2.1.4). For every C^*-subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$, we can define a "smaller" quantum system $(\mathfrak{A}, U, v)$ as follows. Let $\mathscr{H}$ be the subspace of $\tilde{\mathscr{H}}$ generated by vectors of the form $b\tilde{v}$, where b belongs to the C^*-algebra generated by $\mathfrak{B} \cup \{\tilde{U}_t : t \in \mathbb{R}\}$. $\mathfrak{B}$ contains v, and is invariant under both the unitary group $\{\tilde{U}_t\}$ and the C^*-algebra $\mathfrak{B}$. Thus we can define $(\mathfrak{A}, U, v)$ by

$$v = \tilde{v},$$

$$U_t = \tilde{U}_t|_{\mathscr{H}},$$

$$\mathfrak{A} = \mathfrak{B}|_{\mathscr{H}}.$$

This new quantum system $(\mathfrak{A}, U, v)$ must also satisfy (2.1.4). In this situation we will say that the system $(\tilde{\mathfrak{A}}, \tilde{U}, \tilde{v})$ is an *extension* of the system $(\mathfrak{A}, U, v)$.

Now suppose that we start with an arbitrary quantum sustem $(\mathfrak{A}, U, v)$ on $\mathscr{H}$ which satisfies (2.1.4). It is natural to ask if it is possible to extend $(\mathfrak{A}, U, v)$ to a system $(\tilde{\mathfrak{A}}, \tilde{U}, \tilde{v})$ which acts on a larger Hilbert space $\tilde{\mathscr{H}}$, which satisfies (2.1.4), and which has the additional property that

$$[\tilde{\mathfrak{A}}v] = \tilde{\mathscr{H}}.$$

As a consequence of the results described above, Klein has observed that the answer is yes if, and only if, the OS-positive process $\{X_t\}$ associated with the original system $(\mathfrak{A}, U, v)$ can be extended to a *Markov* process, in a sense that we will now describe.

Let $\{\tilde{X}_t\}$ be a stationary Markov process on $(\tilde{\Omega}, \tilde{p})$, having state space $\tilde{\Sigma}$. Let Σ be another standard Borel space and let

$$\psi: \tilde{\Sigma} \to \Sigma$$

be a Borel function. Then we can define a new process $\{X_t\}$ on $(\tilde{\Omega}, \tilde{p})$, having state space Σ, by

$$X_t(\omega) = \psi(\tilde{X}_t(\omega)), \qquad \omega \in \tilde{\Omega}.$$

The process $\{X_t\}$ no longer generates the full Borel field on $(\tilde{\Omega}, \tilde{p})$, but we can replace it by an equivalent process on a new probability space (Ω, p) which does generate. A process X_t obtained in this way from a stationary Markov process $\{\tilde{X}_t\}$ is called a *function* of $\{\tilde{X}_t\}$, and similarly $\{\tilde{X}_t\}$ is called an *extension* of $\{X_t\}$.

Such a process $\{X_t\}$ must be stationary and symmetric, but it is not necessarily a Markov process. However, it is not hard to show that $\{X_t\}$ is an OS-positive process. More generally, any function of an OS-positive process is OS-positive [7].

Putting all of this together, we may draw the following conclusion. Let $(\mathfrak{A}, U, v)$ be a quantum system which satisfies (2.1.4) and let $\{X_t\}$ be its associated OS-positive process. Then $(\mathfrak{A}, U, v)$ can be extended to a quantum system $(\tilde{\mathfrak{A}}, \tilde{U}, \tilde{v})$ in which

$$[\tilde{\mathfrak{A}}v] = \tilde{\mathscr{H}}$$

if and only if $\{X_t\}$ can be extended to a stationary symmetric Markov process.

Klein then asked if it is always possible to find such an extension of any OS-positive process. He showed that the answer is yes for *Gaussian* processes $\{X_t\}$ [9], but the problem in general has remain unresolved. We remark, incidentally, that Klein and Landau have generalized these results so as to include the case of KMS states as well as ground states [11].

In this paper we consider the analogous problem for processes $\{X_n: n \in \mathbb{Z}\}$ in which the continuous time domain $\mathbb{R}$ is replaced by the discrete time domain $\mathbb{Z}$. It is a fact that, as in the case of continuous time, the answer is yes for Gaussian processes. We do not prove that here; however, we do show that the answer is no in general. This is accomplished in the following way. With every stationary OS-positive process $\{X_n: n \in \mathbb{Z}\}$ we associate a positive semidefinite Markov operator K, and we show that if $\{X_n\}$ is extendable to a Markov process then K is factorable in the sense of Section 1.2. We then construct examples of OS-positive processes whose associated operators are not factorable. This construction is a refinement of what was already done in Section 1.3.

Of course, our results give no new information about quantum systems because we only consider processes in discrete time. Nevertheless, we believe that these results strongly suggest that the answer to Klein's question is no. Whether or not our methods can be refined so as to give examples of nonextendable OS-positive processes in continuous time (and therefore nonextendable quantum systems) remains to be seen.

2.2. *Factorable Operators and Extensions of* OS-*Positive Processes*

Let (Ω, p) be a probability space and let Σ be a standard Borel space [3]. By a *stochastic process* on (Ω, p) with state space Σ we mean a sequence of Borel functions

$$X_n: \qquad \Omega \to \Sigma$$

such that the weak*-closed algebra generated by the $L^\infty(\Omega, p)$ functions

$$\omega \mapsto f(X_n(\omega)),$$

where $n \in \mathbb{Z}$ and $f: \Sigma \to \mathbb{C}$ is a bounded Borel function, is all of $L^\infty(\Omega, p)$. We will normally denote a process with the somewhat abbreviated notation $\{X_n\}$. On those occasions when it is necessary to keep track of all four entities associated with a process we will write $(\{X_n\}, \Sigma, \Omega, p)$.

Although it is not always necessary to do so, we will assume throughout the remainder of this paper that all processes are *stationary* in the sense that for every $n \geqslant 1$ and every sequence $E_{-n},..., E_n$ of Borel sets in Σ, one has

$$p\{X_{-n+1} \in E_{-n},..., X_{n+1} \in E_n\} = p\{X_{-n} \in E_{-n},..., X_n \in E_n\}.$$

For such a process there is a unique unitary Markov operator U on $L^2(\Omega, p)$ satisfying

$$Uf(X_{-n},..., X_n) = f(X_{-n+1},..., X_{n+1}),$$

for every $n \geqslant 1$ and every bounded Borel function $f: \Sigma^{2n+1} \to \mathbb{C}$. U is called the *shift operator* of the process $\{X_n\}$.

A process $\{X_n\}$ is called *symmetric* if, for every $n \geqslant 1$ and every sequence $E_{-n},..., E_n$ of Borel sets in Σ, we have

$$p\{X_{-n} \in E_{-n},..., X_n \in E_n\} = p\{X_n \in E_{-n},..., X_{-n} \in E_n\}.$$

For such a process there is a unique unitary Markov operator R on $L^2(\Omega, p)$ satisfying

$$Rf(X_{-n},..., X_n) = f(X_n,..., X_{-n}),$$

for all $n \geqslant 1$ and all $f: \Sigma^{2n+1} \to \mathbb{C}$ as above. R is in fact a *symmetry* in that $R^2 = 1$ and satisfies the following commutation relation with U:

$$RU = U^* R. \tag{2.2.1}$$

R is called the *reflection operator* of the processes $\{X_n\}$.

Now with any symmetric process $\{X_n\}$ one can associate a canonical Markov operator K, which will play a central role in the sequel. K is defined as follows. Let $\mathscr{E}_+$ be the "positive time" subspace of $L^2(\Omega, p)$, defined as the closed linear span of all random variables of the form

$$f_0(X_0) f_1(X_1) \cdots f_n(X_n),$$

where $n \geqslant 0$ and $f_0, ..., f_n$ are bounded complex-valued Borel functions on Σ. E_+ will denote the projection of $L^2(\Omega, p)$ onto $\mathscr{E}_+$. Note that $\mathscr{E}_+$ is the space of all L^2 functions on (Ω, p) which are measurable with respect to $\{X_0, X_1, ...\}$, and therefore E_+ is the conditional expectation

$$E_+ = E(\cdot \mid \mathscr{E}_+).$$

The operator K is defined by compressing R to $\mathscr{E}_+$:

$$K = E_+ R|_{\mathscr{E}_+}. \tag{2.2.2}$$

K is obviously self-adjoint because $R^* = R^{-1} = R$, and has the properties

(i) $F \in \mathscr{E}_+,\ F \geqslant 0 \Rightarrow KF \geqslant 0$

(ii) $K(1) = K^*(1) = 1.$

Thus one may consider K as a Markov operator on $\mathscr{E}_+$. We can, if we wish, introduce coordinates so as to realize $\mathscr{E}_+$ as the L^2 space $L^2(\Omega_+, p_+)$ associated with a probability space, but it will not be necessary to do so. Finally, define $\mathscr{E}_0$ and E_0 to be the "time zero" subspace of $L^2(\Omega, p)$ and its corresponding projection, i.e.,

$$\mathscr{E}_0 = \{f(X_0): f: \Sigma \to \mathbb{C}\}^{-L^2}.$$

E_0 is clearly a subprojection of E_+, and since R fixes each element of the space $\mathscr{E}_0$ we have $RE_0 = E_0$. Since R is self-adjoint we conclude that $RE_0 = E_0 = E_0 R$, and hence

$$KE_0 = E_0 K = E_0. \tag{2.2.3}$$

By a *Markov process* we mean a process $\{X_n\}$ such that, for every $n \geqslant 1$ and every bounded Borel function $f: \Sigma \to \mathbb{C}$ we have

$$E(f(X_0) \mid X_1, X_2, ..., X_n) = E(f(X_0) \mid X_1). \tag{2.2.4}$$

Because of our blanket hypothesis about processes being stationary, (2.2.4) implies that

$$E(f(X_k) \mid X_{k+1}, X_{k+2},..., X_l) = E(f(X_k) \mid X_{k+1})$$

for every $k < l$ in $\mathbb{Z}$, and is in turn equivalent to

$$E_+(f(X_{-1})) = E_0(f(X_{-1}))$$

for all bounded Borel functions $f: \Sigma \to \mathbb{C}$. A process $\{X_n\}$ is called OS-*positive* if, for every $n \geqslant 1$ and every bounded Borel function $f: \Sigma^{n+1} \to \mathbb{C}$ we have

$$\langle f(X_0, X_1,..., X_n), f(X_0, X_{-1},..., X_{-n}) \rangle \geqslant 0,$$

$\langle \cdot, \cdot \rangle$ denoting the inner product in $L^2(\Omega, p)$. We first collect a few elementary facts.

PROPOSITION 2.2.5. (i) *Let* $\{X_n\}$ *be a process satisfying*

$$\langle f(X_0)\, g(X_1,..., X_n), f(X_0)\, g(X_{-1},..., X_{-n}) \rangle \geqslant 0$$

for all bounded Borel functions $f: \Sigma \to \mathbb{C}$, $g: \Sigma^n \to \mathbb{C}$. *Then* $\{X_n\}$ *is* OS-*positive.*

(ii) *Every symmetric Markov process is* OS-*positive.*

(iii) *Every* OS-*positive process is symmetric.*

Proof. The proof of (i) is similar to the device used in the proof of Lemma 1.1.6. Let $g: \Sigma^{n+1} \to \mathbb{C}$ be a function of the form

$$g(s_0, s_1,..., s_n) = \sum_{j=1}^{N} f_j(s_0)\, h_j(s_1,..., s_n), \tag{2.2.6}$$

where the functions $f_j: \Sigma \to \mathbb{C}$ and $h_j: \Sigma^n \to \mathbb{C}$ are bounded and measurable. We will show that

$$\langle g(X_0, X_1,..., X_n), g(X_0, X_{-1},..., X_{-n}) \rangle \geqslant 0.$$

OS-positivity follows from this, together with routine applications of the bounded convergence theorem which we omit.

Fix a function g as in (2.2.6), and consider the bounded random variables

$$H_{ij} = E(h_i(X_1,..., X_n)\, \bar{h}_j(X_{-1},..., X_{-n}) \mid X_0).$$

We claim that the $N \times N$ matrices

$$H(\omega) = [H_{ij}(\omega)], \qquad \omega \in \Omega$$

are positive semidefinite almost everywhere (dp). To prove this, the argument used in the proof of Lemma 1.1.6 implies that it is enough to show that for each N-tuple $(\lambda_1, \ldots, \lambda_N)$ of complex numbers, we have

$$\sum_{i,j} \lambda_i \bar{\lambda}_j H_{ij}(\omega) \geqslant 0 \qquad \text{a.e. } (dp). \tag{2.2.7}$$

Since the H_{ij}'s are measurable relative to the time zero sigma field $\mathscr{E}_0$, it suffices to show that

$$\int_\Omega u(x_0) \sum_{i,j} \lambda_i \bar{\lambda}_j H_{ij}\, dp \geqslant 0 \tag{2.2.8}$$

for every nonnegative bounded Borel function $u: \Sigma \to [0, \infty)$. Setting $v = u^{1/2}$, the integral in (2.2.8) becomes

$$\begin{aligned} &\int_\Omega v(X_0)^2 \sum_{i,j} \lambda_i \bar{\lambda}_j H_{ij}\, dp \\ &= \sum_{i,j} \lambda_i \bar{\lambda}_j \int_\Omega v(X_0)^2\, E(h_i(X_1, \ldots, X_n)\, \bar{h}_j(X_{-1}, \ldots, X_{-n}))\, dp \\ &= \sum_{i,j} \lambda_i \bar{\lambda}_j \int_\Omega v(X_0)^2\, h_i(X_1, \ldots, X_n)\, \bar{h}_j(X_{-1}, \ldots, X_{-n})\, dp \\ &= \int_\Omega v(X_0)^2\, g(X_1, \ldots, X_n)\, \bar{g}(X_{-1}, \ldots, X_{-n})\, dp, \end{aligned} \tag{2.2.9}$$

where $g: \Sigma^n \to \mathbb{C}$ is the function

$$g(s_1, \ldots, s_n) = \sum_{j=1}^{N} \lambda_j\, g_j(s_1, \ldots, s).$$

The last term in (2.2.9) is nonnegative by hypothesis.

(ii) Let $f: \Sigma \to \mathbb{C}$, $g: \Sigma^n \to \mathbb{C}$ be bounded Borel functions. By part (i), it suffices to show that

$$\int_\Omega |f(X_0)|^2\, g(X_1, \ldots, X_n)\, \bar{g}(X_{-1}, \ldots, X_{-n})\, dp \geqslant 0. \tag{2.2.10}$$

Now since $\{X_n\}$ is a Markov process, we have, for $E_0 = E(\cdot \mid X_0)$,

$$\begin{aligned} &E_0(|f(X_0)|^2\, g(X_1, \ldots, X_n)\, \bar{g}(X_{-1}, \ldots, X_{-n})) \\ &\quad = |f(X_0)|^2\, E_0(g(X_1, \ldots, X_n))\, E_0(\bar{g}(X_{-1}, \ldots, X_{-n})) \end{aligned}$$

[4, p. 83]. Moreover, since the reflection R based on $\{X_n\}$ satisfies $E_0 = E_0 R$, we have

$$E_0(\bar{g}(X_{-1},..., X_{-n})) = E_0(\bar{g}(X_1,..., X_n)).$$

It follows that the integral in (2.2.10) is

$$\int_\Omega |f(X_0)|^2 \, |E_0((X_1,..., X_n))|^2 \, dp \geqslant 0.$$

(iii) Assume $\{X_n\}$ is OS-positive, and consider the space $\mathscr{S}$ of all bounded Borel functions

$$f\colon \Sigma^{n+1} \to \mathbb{C}.$$

To prove that $\{X_n\}$ is symmetric, it suffices to show that for every pair $f, g \in \mathscr{S}$, we have

$$\int_\Omega f(X_0, X_1,..., X_n)\, \bar{g}(X_0, X_{-1},..., X_{-n})\, dp$$

$$= \int_\Omega f(X_0, X_{-1},..., X_{-n})\, \bar{g}(X_0, X_1,..., X_n)\, dp. \qquad (2.2.11)$$

For that, consider the sesquilinear form $[\cdot, \cdot]$ defined on $\mathscr{S} \times \mathscr{S}$ by

$$[f, g] = \langle f(X_0,..., X_n), g(X_0, X_{-1},..., X_{-n})\rangle_{L^2(\Omega,p)}.$$

By hypothesis, $[\cdot, \cdot]$ is positive semidefinite. Therefore it is self-adjoint in the sense that $[f, g] = \overline{[g, f]}$, and the latter is just the assertion of (2.211). ∎

In particular, every OS-positive process admits a reflection R, and therefore has a canonical Markov operator

$$K = E_+ R|_{\mathscr{E}_+},$$

as defined in (2.2.2). In general, a symmetric process $\{X_n\}$ is OS-positive if and only if its canonical operator K is positive semidefinite in the sense that

$$\langle Kf, f\rangle \geqslant 0, \qquad f \in \mathscr{E}_+.$$

We now discuss extensions of processes and the corresponding dual concept. Let $(\{\tilde{X}_n\}, \tilde{\Sigma}, \tilde{\Omega}, \tilde{p})$ be a symmetric process, let Σ be a second standard Borel space, and let

$$\psi\colon \tilde{\Sigma} \to \Sigma$$

be a Borel function. Define a new sequence $X_n\colon \tilde{\Omega} \to \Sigma$ by

$$X_n(\omega) = \psi(\tilde{X}_n(\omega)).$$

The new sequence is a symmetric stationary process but it may not generate the full σ-field in $(\tilde{\Omega}, \tilde{p})$. We may, of course, replace $\{X_n\}$ with an equivalent process on a new probability space which does generate. The new process $(\{X_n\}, \Sigma, \Omega, p)$ is called a *function* of $(\{\tilde{X}_n\}, \tilde{\Sigma}. \tilde{\Omega}, \tilde{p})$, and similarly $(\{\tilde{X}_n\}, \tilde{\Sigma}, \tilde{\Omega}, \tilde{p})$ is called an *extension* of $(\{X_n\}, \Sigma, \Omega, p)$.

In more explicit terms, a process $(\{\tilde{X}_n\}, \tilde{\Sigma}, \tilde{\Omega}, \tilde{p})$ is an extension of a process $(\{X\}, \Sigma, \Omega, p)$ if there is a pair (ψ, V) consisting of a Borel function $\psi\colon \tilde{\Sigma} \to \Sigma$ and a Markov isometry

$$V\colon \quad L^2(\Omega, p) \to L^2(\tilde{\Omega}, \tilde{p})$$

such that for every $m \leqslant n$ in $\mathbb{Z}$ and every bounded Borel function $f\colon \Sigma^{n-m+1} \to \mathbb{C}$ we have

$$Vf(X_m, ..., X_n) = f(\psi(\tilde{X}_m), ..., \psi(\tilde{X}_n)). \tag{2.2.12}$$

A function of a symmetric Markov process is not necessarily a Markov process, but it is OS-positive. More generally, a function of an OS-positive process is OS-positive. These facts were pointed out by Klein [7] for processes in continuous time. The situation is the same here, and for completeness we present the following simple proof.

PROPOSITION 2.2.13. *If $\{X_n\}$ can be extended to an* OS-*positive process then $\{X_n\}$ is* OS-*positive.*

Proof. Let $(\{\tilde{X}_n\}, \tilde{\Sigma}, \tilde{\Omega}, \tilde{p})$ be an OS-positive process which extends the given process $(\{X_n\}, \Sigma, \Omega, p)$. Let

$$\psi\colon \quad \tilde{\Sigma} \to \Sigma,$$
$$V\colon \quad L^2(\Omega, p) \to L^2(\tilde{\Omega}, \tilde{p}),$$

satisfy (2.1.12). Then for every bounded Borel function $f\colon \Sigma^{n+1} \to \mathbb{C}$ we have

$$\begin{aligned}
&\langle f(X_0, X_1, ..., X_n), f(X_0, X_{-1}, ..., X_{-n})\rangle_{L^2(\Omega,p)} \\
&\quad = \langle Vf(X_0, X_1, ..., X_n), Vf(X_0, X_{-1}, ..., X_{-n})\rangle_{L^2(\tilde{\Omega},\tilde{p})} \\
&\quad = \langle f(\psi(\tilde{X}_0), \psi(\tilde{X}_1), ..., \psi(\tilde{X}_n)), f(\psi(\tilde{X}_0), \psi(\tilde{X}_{-1}), ..., \psi(\tilde{X}_{-n}))\rangle_{L^2(\tilde{\Omega},\tilde{p})}.
\end{aligned}$$

The latter term has the form

$$\langle g(\tilde{X}_0, \tilde{X}_1, ..., \tilde{X}_n), g(\tilde{X}_0, \tilde{X}_{-1}, ..., \tilde{X}_{-n})\rangle,$$

where $g\colon \tilde{\Sigma}^{n+1} \to \mathbb{C}$ is defined by

$$g(s_0, s_1, ..., s_n) = f(\psi(s_0), \psi(s_1), ..., \psi(s_n)),$$

and hence the latter is nonnegative because $\{\tilde{X}_n\}$ is OS-positive. ∎

In particular, if a given process $\{X_n\}$ can be extended to a symmetric Markov process, then $\{X_n\}$ is OS-positive. The main problem we want to consider in Section II is the extent to which the converse is true; i.e., is every OS-positive process extendable to a symmetric Markov process? The following gives a necessary condition for the existence of such an extension, and provides us with the starting point for the construction of counter examples in Section 2.3.

Proposition 2.2.14. *Let* $(\{X_n\}, \Sigma, \Omega, p)$ *be an* OS*-positive process and let*

$$K = E_+ R|_{\mathscr{E}_+}$$

be its canonical Markov operator, defined as in (2.2.2). *If* $\{X_n\}$ *can be extended to a symmetric Markov process, then* K *is factorable.*

Proof. Let $(\{\tilde{X}_n\}, \tilde{\Sigma}, \tilde{\Omega}, \tilde{p})$ be a symmetric Markov process, and let $\psi\colon \tilde{\Sigma} \to \Sigma$ and $V\colon L^2(\Omega, p) \to L^2(\tilde{\Omega}, \tilde{p})$ satisfy (2.1.12). Let $\tilde{E}_0$ be the conditional expectation, mapping $L^2(\tilde{\Omega}, \tilde{p})$ onto the time zero subspace:

$$\tilde{E}_0 = E(\cdot \mid \tilde{X}_0).$$

Then

$$B = \tilde{E}_0 V|_{\mathscr{E}_+}$$

is a Markov operator from $\mathscr{E}_+$ into $L^2(\tilde{\Omega}, \tilde{p})$; we will show that for every $n \geq 1$ and every pair of bounded Borel functions f, $g\colon \Sigma^{n+1} \to \mathbb{C}$, we have

$$\begin{aligned}\langle Rf(X_0, X_1, ..., X_n), g(X_0, X_1, ..., X_n)\rangle_{L^2(\Omega,p)} \\ = \langle \tilde{E}_0 Vf(X_0, ..., X_n), \tilde{E}_0 Vg(X_0, ..., X_n)\rangle_{L^2(\tilde{\Omega},\tilde{p})}. \qquad (2.2.15)\end{aligned}$$

Note that this gives the required factorization $K = B^*B$ of K.

The left side of (2.2.15) is

$$\begin{aligned}&\langle f(X_0, X_{-1},..., X_{-n}), g(X_0, X_1,..., X_n)\rangle\\ &\quad = \langle Vf(X_0, X_{-1},..., X_{-n}), Vg(X_0, X_1,..., X_n)\rangle\\ &\quad = \langle f(\psi(\tilde{X}_0), \psi(\tilde{X}_{-1}),..., \psi(\tilde{X}_{-n})), g(\psi(\tilde{X}_0), \psi(\tilde{X}_1),..., \psi(\tilde{X}_n))\rangle\\ &\quad = \int_\Omega f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_{-n}))\, \bar{g}(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n))\, d\tilde{p}.\end{aligned}$$

We may replace the last integral with its conditional expectation relative to $\tilde{\mathscr{E}}_+$ without changing the value of the integral. The new integrand is

$$\begin{aligned}&\tilde{E}_+[f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_{-n}))\, \bar{g}(\psi(\tilde{X}),..., \psi(\tilde{X}_n))]\\ &\quad = \bar{g}(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n))\, E_+[f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_{-n}))].\end{aligned}$$

Note now that

$$\tilde{E}_+[f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_{-n}))] = \tilde{E}_0[f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_{-n}))]. \quad (2.2.16)$$

To see this, we can assume that f is a finite linear combination of functions of the form

$$f(s_0, s_1,..., s_n) = u(s_0)\, v(s_1, s_2,..., s_n).$$

But since $u(\psi(\tilde{X}_0))$ is $\tilde{\mathscr{E}}_0$-measurable, we have

$$\begin{aligned}&\tilde{E}_+[u(\psi(\tilde{X}_0))\, v(\psi(\tilde{X}_{-1}),..., \psi(\tilde{X}_{-n}))]\\ &\quad = u(\psi(\tilde{X}_0))\, \tilde{E}_+[v(\psi(\tilde{X}_{-1}),..., \psi(\tilde{X}_{-n}))]. \qquad (2.2.17)\end{aligned}$$

Moreover, since $\{\tilde{X}_n\}$ is a Markov process and $v(\psi(\tilde{X}_{-1}),..., \psi(\tilde{X}_{-n}))$ is measurable with respect to the "past" $\{\tilde{X}_{-1}, \tilde{X}_{-2},...\}$, we have

$$\tilde{E}_+[v(\psi(\tilde{X}_{-1}),..., \psi(\tilde{X}_{-n}))] = E_0[v(\psi(\tilde{X}_{-1}),..., \psi(\tilde{X}_{-n}))]$$

(see [4, p. 83]). The right side of (2.2.17) becomes

$$\begin{aligned}&u(\psi(\tilde{X}_0))\, \tilde{E}_0[v(\psi(X_{-1}),..., \psi(X_{-n}))]\\ &\quad = \tilde{E}_0[u(\psi(X_0))\, v(\psi(X_{-1}),..., \psi(X_{-n}))],\end{aligned}$$

as asserted.

Now we have seen that $\tilde{E}_0\tilde{R} = \tilde{E}_0$; hence the right side of (2.2.16) is

$$\tilde{E}_0[\tilde{R}f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_{-n}))] = \tilde{E}_0[f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n))].$$

Thus the integrand in question becomes

$$g(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n))\, \tilde{E}_0[f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n))].$$

Again, we may replace this integrand by its conditional expectation with respect to $\tilde{\mathscr{E}}_0$ without altering the value of the integral, obtaining

$$\int_{\tilde{\Omega}} \bar{g}(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n))\, \tilde{E}_0[f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n))]\, d\tilde{p}$$

$$= \int_{\tilde{\Omega}} \tilde{E}_0[\bar{g}(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n))]\, \tilde{E}_0[f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n))]\, d\tilde{p}$$
$$= \langle \tilde{E}_0\, f(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n)), \tilde{E}_0\, g(\psi(\tilde{X}_0),..., \psi(\tilde{X}_n)))$$
$$= \langle \tilde{E}_0 Vf(X_0,..., X_n), \tilde{E}_0 Vg(X_0,..., X_n)\rangle,$$

which gives (2.2.15). ∎

2.3. *Construction of* OS-*Positive Processes*

In this section we show how, starting with a Markov system as defined in (1.3.1), one can construct a (stationary) OS-positive process. We then take up the question of whether or not the OS-positive processes constructed in this way can be extended to symmetric Markov processes. By exploiting the factorization criterion of Proposition 2.2.14 and the properties of noncommutative C^*-algebras, we are able to show that such extensions rarely exist.

If X is a compact Hausdorff space, then the doubly infinite product space

$$\Omega = X^{\mathbb{Z}}$$

is also a compact Hausdorff space. We will denote the nth component of a sequence ω in Ω by ω_n, $n \in \mathbb{Z}$. Every Baire probability measure μ on Ω gives rise to a stochastic process X_n on (Ω, μ), with values in X, by

$$X_n(\omega) = \omega_n, \qquad \omega \in \Omega,\ n \in \mathbb{Z}.$$

The process $\{X_n\}$ is stationary iff μ is invariant under the shift automorphism σ of Ω, defined by

$$(\sigma\omega)_n = \omega_{n+1}.$$

Similarly, $\{X_n\}$ is symmetric iff μ is invariant under the reflection $\rho: \Omega \to \Omega$ defined by $(\rho\omega)_n = \omega_{-n}$.

Now let $(X, \mathfrak{A}, \phi, P)$ be a Markov system (1.3.1).

THEOREM 2.3.1. *There is a unique Baire probability measure* μ *on* Ω *satisfying*

$$\int_\Omega f_m(\omega_m)\cdots f_n(\omega_n)\,d\mu(\omega)=\phi(P_{f_m}P_{f_{m+1}}\cdots P_{f_n}(1)),$$

for every pair of integers $m\leqslant n$ *in* $\mathbb{Z}$ *and every* $f_m,\ldots,f_n\in C(X)$.

The corresponding process $\{X_n\}$ *is stationary and* OS-*positive.*

The existence of μ can be based on the following extension of Lemma 1.1.6.

LEMMA 2.3.2. *Let* $X_1,\ldots,X_n$ *be compact Hausdorff spaces and let*

$$B\colon C(X_1)\times\cdots\times C(X_n)\to\mathbb{C}$$

be a multilinear form satisfying the condition $B(f_1,\ldots,f_n)\geqslant 0$ *whenever* $f_1\geqslant 0,\ldots,f_n\geqslant 0$. *Then there is a unique positive Baire measure* μ *on* $X_1\times\cdots\times X_n$ *satisfying*

$$B(f_1,\ldots f_n)=\int f_1(x_1)\cdots f_n(x_n)\,d\mu(x_1,\ldots,x_n).$$

Proof. The argument is a simple induction starting with the case $n=2$, which was established in (1.1.6). We sketch the argument for completeness.

Assume that the assertion is true for $n\geqslant 2$, and let $B(f_1,\ldots,f_{n+1})$ satisfy the above hypotheses. For each nonnegative function h in $C(X_{n+1})$, the induction hypothesis provides a unique positive Baire measure ν_h on $X_1\times\cdots\times X_n$ such that

$$B(f_1,\ldots,f_n,h)=\int f_1(x_1)\cdots f_n(x_n)\,d\nu_h$$

for all $f_i\in C(X_i)$, $1\leqslant i\leqslant n$. The mapping $h\mapsto\nu_h$ extends uniquely to a linear map of $C(X_{n+1})$ into the complex vector space of Baire measures on $X_1\times\cdots\times X_n$, and the above formula persists for the extended mapping. Applying (1.1.6) to the bilinear form

$$\tilde{B}\colon C(X_1\times\cdots\times X_n)\times C(X_{n+1})\to\mathbb{C}$$

defined by

$$\tilde{B}(u,h)=\int u(x_1,\ldots,x_n)\,d\nu_h(x_1,\ldots,x_n),$$

we obtain a measure μ on $X_1 \times \cdots \times X_{n+1}$ which represents $\tilde{B}$. One has

$$\int f_1(x_1) \cdots f_{n+1}(x_{n+1})\, d\mu = \tilde{B}(f_1 \otimes \cdots \otimes f_n, f_{n+1})$$

$$= B(f_1, ..., f_{n+1}),$$

as required. ∎

To prove the theorem, suppose that for each pair of integers $m \leqslant n$ we have a multilinear form $B_{m,n}(f_m, ..., f_n)$ in $n-m+1$ variables $f_m, ..., f_n \in C(X)$. Assume that the family $\{B_{m,n}\}$ satisfies

(i) (consistency)

$$B_{m-1,n}(1, f_m, ..., f_n) = B_{m,n}(f_m, ..., f_n),$$

$$B_{m,n+1}(f_m, ..., f_n, 1) = B_{m,n}(f_m, ..., f_n)$$

(ii) (positivity) (2.3.3)

$$f_m \geqslant 0, ..., f_n \geqslant 0 \Rightarrow B_{m,n}(f_m, ..., f_n) \geqslant 0,$$

(iii) (invariance)

$$B_{m+1,n+1}(f_m, ..., f_n) = B_{m,n}(f_m, ..., f_n),$$

together with the normalizing condition

(iv) $B_{0,0}(1) = 1$.

We claim first that there is a unique Baire probability measure μ on Ω satisfying

$$\int_\Omega f_m(\omega_m) \cdots f_n(\omega_n)\, d\mu = B_{m,n}(f_m, ..., f_n)$$

for all $m \leqslant n$, $f_m, ..., f_n \in C(X)$, and moreover that μ is translation invariant.

To see this, for each $m \leqslant n$ let $\mathfrak{A}_{m,n}$ be the C^*-subalgebra of $C(\Omega)$ generated by the functions

$$f_m \otimes \cdots \otimes f_n(\omega) = f_m(\omega_n) f_{m+1}(\omega_{n+1}) \cdots f_n(\omega_n),$$

$f_i \in C(X)$ (while the notation $f_m \otimes \cdots \otimes f_n$ is somewhat ambiguous, it will not cause problems). $\mathfrak{A}_{m,n}$ is a unital C^*-algebra which is in an obvious sense isomorphic to $C(X^{n-m+1})$. Because of the positivity condition (2.3.3)(ii), Lemma 2.3.2 implies that there is a unique positive linear functional $\rho_{m,n}$ on $\mathfrak{A}_{m,n}$ satisfying

$$\rho_{m,n}(f_m \otimes \cdots \otimes f_n) = B_{m,n}(f_m, ..., f_n),$$

for all $f_m, \ldots, f_n$ in $C(X)$. By (2.3.3)(i) we have

$$\rho_{m,n}(1) = B_{m,n}(1, \ldots, 1) = B_{n,n}(1).$$

Since $B_{n,n}(f)$ must agree with $B_{0,0}(f)$ (by (2.3.3)(i) again), we conclude that $\rho_{m,n}$ is a state: $\rho_{m,n}(1) = 1$.

Now the algebras $\mathfrak{A}_{m,n}$ obey the relations

$$\mathfrak{A}_{m,n} \subseteq \mathfrak{A}_{m-1,n},$$
$$\mathfrak{A}_{m,n} \subseteq \mathfrak{A}_{m,n+1},$$

and their union $\bigcup \mathfrak{A}_{m,n}$ is a dense unital *-subalgebra of $C(\Omega)$. Because of (2.3.3)(i) the family of states $\{\rho_{m,n}\}$ is coherent in the sense that

$$\rho_{m-1,n}|_{\mathfrak{A}_{m,n}} = \rho_{m,n}$$
$$\rho_{m,n+1}|_{\mathfrak{A}_{m,n}} = \rho_{m,n},$$

and therefore there is a unique state ρ on $C(\Omega)$ which satisfies

$$\rho|_{\mathfrak{A}_{m,n}} = \rho_{m,n}.$$

The existence of μ now follows from the Riesz–Markov theorem; μ is clearly unique, and translation invariance is an obvious consequence of (2.3.3)(iii) together with the fact that $\bigcup \mathfrak{A}_{m,n}$ is dense in $C(\Omega)$.

Applying this to the multilinear forms

$$B_{m,n}(f_m, \ldots, f_n) = \phi(P_{f_m} P_{f_{m+1}} \cdots P_{f_n}(1)) \tag{2.3.4}$$

we must check the four properties (2.3.3). For (i), we have

$$\begin{aligned} B_{m,n+1}(f_m, \ldots, f_n, 1) &= \phi(P_{f_m} \cdots P_{f_n} P_1(1)) \\ &= \phi(P_{f_m} \cdots P_{f_n}(1)) = B_{m,n}(f_m, \ldots, f_n), \end{aligned}$$

because P_1 fixes the unit of the C^*-algebra $\mathfrak{A}$. Also,

$$\begin{aligned} B_{m-1,n}(1, f_m, \ldots, f_n) &= \phi(P_1(P_{f_m} \cdots P_{f_n}(1))\, 1) \\ &= \phi(P_{f_m} \cdots P_{f_n}(1)\, P_1(1)) \\ &= \phi(P_{f_m} \cdots P_{f_n}(1)) = B_{m,n}(f_m, \ldots, f_n), \end{aligned}$$

because of Lemma 1.3.4(ii) and the fact that $P_1(1) = 1$.

Property (2.3.3)(ii) follows from (1.3.1)(i), and (iii) is immediate from the Definition 2.3.4. Finally, we have (iv) because

$$B_{0,0}(1) = \phi(P_1(1)) = \phi(1) = 1.$$

It remains to show that the corresponding process $\{X_n\}$ is OS-positive. For that, we note that by Proposition 2.2.5(i), it is enough to show that if $u \in C(X)$ and g is a function in $C(X^n)$ of the form

$$g(s_1, ..., s_n) = \sum_{k=1}^{m} f_{k1}(s_1) f_{k2}(s_2) \cdots f_{kn}(s_n)$$

then we have

$$\int_\Omega |u(\omega_0)|^2 \, g(\omega_1, ..., \omega_n) \, \bar{g}(\omega_{-1}, ..., \omega_{-n}) \, d\mu \geqslant 0.$$

The left side of this inequality can be expanded to obtain

$$\sum_{j,k=1}^{m} \int_\Omega \bar{f}_{jn}(\omega_{-n}) \cdots \bar{f}_{j1}(\omega_{-1}) \, |u(\omega_0)|^2 f_{k1}(\omega_1) \cdots f_{kn}(\omega_n) \, d\mu$$
$$= \sum_{j,k} \phi(P_{\bar{f}_{jn}} \cdots P_{\bar{f}_{j1}} P_{|u|^2} P_{f_{k1}} \cdots P_{f_{kn}}(1)).$$

Now if apply the identity $\phi(P_{\bar{f}}(a)\, b) = \phi(a P_f(b^*)^*)$ (cf. (1.3.4)(i) and (ii)) repeatedly to the summand, we may write

$$\begin{aligned}\phi(P_{\bar{f}_{jn}} \cdots P_{f_{kn}}(1)) &= \phi(P_{\bar{f}_{jn-1}} \cdots P_{f_{kn}}(1)\, P_{f_{jn}}(1)^*) \\ &= \phi(P_{\bar{f}_{jn-2}} \cdots P_{f_{kn}}(1) (P_{f_{jn-1}} P_{f_{jn}}(1))^*) \\ &= \\ &\;\;\vdots \\ &= \phi(P_{|u|^2} P_{f_{k1}} \cdots P_{f_{kn}}(1) (P_{f_{j1}} \cdots P_{f_{jn}}(1))^*).\end{aligned}$$

So if we define an element a in $\mathfrak{A}$ by

$$a = \sum_{k=1}^{m} P_{f_{k1}} P_{f_{k2}} \cdots P_{f_{kn}}(1),$$

then the right side of 2.3.5 becomes

$$\phi(P_{|u|^2}(a)\, a^*),$$

which is nonnegative by (1.3.1)(iii). ∎

Turning now to the main discussion, let us fix a Markov system $(X, \mathfrak{A}, \phi, P)$ throughout the remainder of this section. Let $\{X_n\}$ be the corresponding OS-positive process and let

$$K = E_+ R|_{\mathscr{E}_+} \tag{2.3.6}$$

be its canonical Markov operator as in (2.2.2). K is, of course, positive semidefinite. We will show that, in most cases, K is not factorable. The proof of this is somewhat more involved than the proof of the corresponding result (1.3.7) in Section I; for clarity, we have broken the proof into several lemmas.

The first result resembles a basic estimate from quantum field theory ([5], p. 93; [14]).

LEMMA 2.3.7. *Let Q be a bounded linear map of $\mathfrak{A}$ into itself satisfying $Q(a^*) = Q(a)^*$ and*

$$\phi(Q(a)\, b) = \phi(aQ(b)),$$

for all $a, b \in \mathfrak{A}$. Then we have

$$\phi(Q(a)^*\, Q(a)) \leqslant \|Q\|^2\, \phi(a^* a),$$

for every $a \in \mathfrak{A}$.

Proof. We have

$$\phi(Q(a)^*\, Q(a)) = \phi(Q(a^*)\, Q(a)) = \phi(Q^2(a^*)\, a) = \phi(Q^2(a)^*\, a);$$

so by the Schwarz inequality for positive linear functionals on $\mathfrak{A}$ we have

$$\phi(Q(a)^*\, Q(a)) \leqslant \phi(a^* a)^{1/2}\, \phi(Q^2(a)^*\, Q^2(a))^{1/2}.$$

Repeating the argument on the second term on the right side gives the estimate

$$\phi(Q^2(a)^*\, Q_2(a))^{1/2} \leqslant \phi(a^* a)^{1/4}\, \phi(Q^4(a)^*\, Q^4(a))^{1/2}.$$

Continuing in this way we obtain after n iterations

$$\phi(Q(a)^*\, Q(a)) \leqslant \phi(a^* a)^{(1/2) + (1/4) + \cdots + (1/2)^n}\, \phi(Q^{2^n}(a)^*\, Q^{2^n}(a))^{(1/2)^n}.$$

Since the second factor on the right is dominated by $\|Q\|^2\, \|a\|^{(1/2)^n}$, one may take the limit on n to obtain the required inequality. ∎

LEMMA 2.3.8. *Assume that the Markov system $(X, \mathfrak{A}, \phi, P)$ is full, and assume that the operator K of* (2.3.6) *is factorable.*

Then for every positive function $f \in C(X)$, there is a triple $(\mathfrak{B}, \rho, L)$ consisting of a unital commutative C^*-algebra $\mathfrak{B}$, a positive linear functional ρ on $\mathfrak{B}$, and a unital positive linear map $L\colon \mathfrak{A} \to \mathfrak{B}$ satisfying

$$\phi(P_f(a)\, b) = \rho(L(a)\, L(b)).$$

Proof. By the GNS construction, we find a Hilbert space $\mathscr{H}$, a representation π of $\mathfrak{A}$ on $\mathscr{H}$, and a unit cyclic vector v such that

$$\phi(a) = \langle \pi(a)\, v, v \rangle, \qquad a \in \mathfrak{A}.$$

Applying Lemma 2.3.7 to the operator $Q = P_f$, we may conclude that there is a (necessarily unique) bounded operator H on $\mathscr{H}$ satisfying

$$H(\pi(a)\, v) = \pi(P_f(a))\, v, \qquad a \in \mathfrak{A}.$$

Now suppose K factors into the form $K = B^*B$, where $B: \mathscr{E}_+ \to L^2(Y, \nu)$ is an order-preserving operator. By (1.2.4)(ii) we may assume that (Y, ν) is a probability space and B is a Markov operator. Let $f^{1/2}$ denote the positive square root of the function $f \in C(X)$, and let $\mathscr{H}_0$ be the closed subspace of $\mathscr{H}$ generated as follows,

$$\mathscr{H}_0 = [H^{1/2}\pi(P_g(1))\, v\colon g \in C(X)].$$

We claim that there is a unique isometry U from $\mathscr{H}_0$ into $L^2(Y, \nu)$ satisfying

$$UH^{1/2}\pi(P_g(1))\, v = B(f^{1/2}(X_0)\, g(X_1)),$$

$g \in C(X)$, where X_0, X_1 are the "time zero" and "time one" random variables from $\{X_n\}$. To prove this, it is enough to check inner products, and note that for $g, h \in C(X)$ we have

$$\begin{aligned}\langle H^{1/2}\pi(P_g(1))\, v, H^{1/2}\pi(P_h(1))\, v\rangle &= \langle H\pi(P_g(1))\, v, \pi(P_h(1))\, v\rangle \\ &= \langle \pi(P_f P_g(1))\, v, \pi(P_h(1))\, v\rangle \\ &= \phi(P_h(1)^*\, P_f P_g(1)) \\ &= \phi(P_{\bar{h}} P_f P_g(1)).\end{aligned}$$

In the last equality we have used the properties $P_h(a)^* = P_{\bar{h}}(a^*)$ and $\phi(P_{\bar{h}}(a)\, b) = \phi(aP_{\bar{h}}(b))$. Now the last term above is, by definition of the measure μ on Ω, identical with

$$\begin{aligned}\int_\Omega \bar{h}(X_{-1})\, f(X_0)\, g(X_1)\, d\mu &= \langle f(X_0)\, g(X_1), Rh(X_1)\rangle \\ &= \langle Rf(X_0)\, g(X_1), h(X_1)\rangle \\ &= \langle Rf^{1/2}(X_0)\, g(X_1), f^{1/2}(X_0)\, h(X_1)\rangle \\ &= \langle K(f^{1/2}(X_0)\, g(X_1)), f^{1/2}(X_0)\, h(X_1)\rangle \\ &= \langle B(f^{1/2}(X_0)\, g(X_1)), B(f^{1/2}(X_0)\, h(X_1))\rangle_{L^2(Y,\nu)}.\end{aligned}$$

The second equality follows from the fact that $R^* = R$, the third is because the automorphism of $L^\infty(\Omega, \mu)$ induced by R fixes all functions of the form $u(X_0)$, $u \in C(X)$, and the last one follows from the fact that $K = B^*B$.

Now since $(X, \mathfrak{A}, \phi, P)$ is *full*, the operators $P_g(1)$, $g \in C(X)$, are dense in $\mathfrak{A}$. Hence $\mathscr{H}_0$ is simply the closure $\overline{H\mathscr{H}}$. Thus we can define a bounded linear mapping $L\colon \mathfrak{A} \to L^2(Y, \nu)$ by

$$L(a) = UH^{1/2}\pi(a)\, v.$$

We claim next that L is order-preserving, i.e., $L(a) \geqslant 0$ a.e. $(d\nu)$ for every positive element a of $\mathfrak{A}$. Since the Markov system is full the elements $P_g(1)$, $g \in C(X)^+$, are dense in the positive cone of $\mathfrak{A}$, and so it suffices to show that $L(P_g(1)) \geqslant 0$ a.e. $(d\nu)$ for all such g. But by definition of L we have

$$L(P_g(1)) = UH^{1/2}\pi(P_g(1))\, v = B(f^{1/2}(X_0)\, g(X_1)),$$

which is nonnegative a.e. because $f^{1/2}(X_0)\, g(X_1) \geqslant 0$ and B is order-preserving.

Note that L carries the unit of $\mathfrak{A}$ to the nonnegative function $B(f^{1/2}(X_0))$ (set $g = 1$ in the formula displayed above). Since $B(f^{1/2}(X_0))$ is in $L^\infty(Y, \nu)$, it follows that $L(\mathfrak{A}) \subseteq L^\infty(Y, \nu)$. Indeed, for $a = a^*$ we have

$$-\|a\|\, 1 \leqslant a \leqslant \|a\|\, 1,$$

and so

$$-\|a\|\, B(f^{1/2}(X_0)) \leqslant L(a) \leqslant \|a\|\, B(f^{1/2}(X_0)),$$

from which the assertion is evident.

Finally, put $\mathfrak{B} = L^\infty(Y, \nu)$ and define a state ρ of $\mathfrak{B}$ by

$$\rho(F) = \int_Y F\, d\nu, \qquad F \in L^\infty(Y, \nu).$$

This triple $(\mathfrak{B}, \rho, L)$ has the asserted property, for if a, b are two elements of $\mathfrak{A}$ then we have

$$\begin{aligned}\rho(L(a)\, L(b)) &= \int_Y L(a)\, L(b)\, d\nu = \langle L(a), L(b^*)\rangle_{L^2(Y,\nu)} \\ &= \langle H^{1/2}\pi(a)\, v, H^{1/2}\pi(b^*)\, v\rangle \\ &= \langle H\pi(a)\, v, \pi(b^*)\, v\rangle = \langle \pi(P_f(a))\, v, \pi(b^*)\, v\rangle \\ &= \langle \pi(bP_f(a))\, v, v\rangle = \phi(bP(a)). \quad \blacksquare\end{aligned}$$

Before stating the next lemma, we recall the definition of the *opposite* C^*-algebra $\mathfrak{A}^0$ of a C^*-algebra $\mathfrak{A}$. $\mathfrak{A}^0$ is defined as the C^*-algebra consisting of the same elements as $\mathfrak{A}$, the same norm, and the same operations except that the multiplication in $\mathfrak{A}^0$ (written $a \circ b$) is reversed:

$$a \circ b = ba.$$

The natural map $\alpha: a \in \mathfrak{A}^0 \mapsto a \in \mathfrak{A}$ is an isometric $*$-anti-isomorphism of C^*-algebras.

LEMMA 2.3.9. *Let ϕ be a tracial state on a C^*-algebra $\mathfrak{A}$ such that the* GNS *representation π_ϕ associated with ϕ is faithful. Let $P: \mathfrak{A} \to \mathfrak{A}$ be a linear mapping.*

Assume that there is a triple $(\mathfrak{B}, \rho, L)$ as in Lemma 2.3.8, *with*

$$\phi(P(a)\, b) = \rho(L(a)\, L(b)), \qquad a, b \in \mathfrak{A}.$$

Then the composition $P\alpha$ of P with the anti-isomorphism α: $\mathfrak{A}^0 \to \mathfrak{A}$ is a completely positive linear map of $\mathfrak{A}^0$ into $\mathfrak{A}$.

Proof. We may identify $\mathfrak{A}$ with a C^*-subalgebra of $\mathscr{L}(\mathscr{H}_\phi)$ and ϕ with a vector state

$$\phi(a) = \langle av, v\rangle,$$

when v is a unit cyclic vector for $\mathfrak{A}$. It suffices to show that if $\xi_1, ..., \xi_n \in \mathscr{H}_\phi$ and $a_1, ..., a_n \in \mathfrak{A}^0$, then

$$\sum_{i,j=1}^{n} \langle P\alpha(a_i^* \circ a_j)\, \xi_j, \xi_i\rangle \geq 0.$$

Since v is cyclic we may assume that ξ_j has the form $\xi_j = b_j v$ for $b_1, ..., b_n \in \mathfrak{A}$. The left side of the above inequality becomes

$$\begin{aligned}\sum_{i,j} \langle P(a_j a_i^*)\, b_j v, b_i v\rangle &= \sum_{i,j} \phi(b_i^* P(a_j a_i^*)\, b_j) \\ &= \sum_{i,j} \phi(P(a_j a_i^*)\, b_j b_i^*) \\ &= \sum_{i,j} \rho(L(a_j a_i^*)\, L(b_j b_i^*)). \qquad (2.3.10)\end{aligned}$$

Now, since $L: \mathfrak{A} \to \mathfrak{B}$ is a positive linear map and $\mathfrak{B}$ is commutative, L must be completely positive [1, 17]. It follows that each of the two $n \times n$ matrices of functions

$$[L(a_j a_i^*)] \qquad \text{and} \qquad [L(b_j b_i^*)]$$

is a positive element of $M_n \otimes \mathfrak{B}$. Since the Hadamard product of two positive semidefinite scalar matrices is positive semidefinite, it follows that

$$[L(a_j a_i^*)\, L(b_j b_i^*)]$$

is a positive element of $M_n \otimes \mathfrak{B}$. Thus,

$$\sum_{i,j} L(a_j a_i^*)\, L(b_j b_i^*)$$

is a positive element of $\mathfrak{B}$ and therefore the last term of (2.3.10) is non-negative. ∎

Finally, we require the following characterization of abelian C^*-algebras.

LEMMA 2.3.11. *If $\mathfrak{A}$ is a unital C^*-algebra for which the natural anti-isomorphism $\alpha\colon \mathfrak{A}^0 \to \mathfrak{A}$ is completely positive, then $\mathfrak{A}$ is abelian.*

Proof. We may assume that $\mathfrak{A} \subseteq \mathscr{L}(\mathscr{H})$ acts on a Hilbert space. Then α is a unital completely positive linear map of $\mathfrak{A}^0$ into $\mathscr{L}(\mathscr{H})$. By Stinespring's theorem ([1] or [17]), there is a Hilbert space $\mathscr{H}'$, an isometry $V\colon \mathscr{H} \to \mathscr{H}'$, and a representation π of $\mathfrak{A}^0$ on $\mathscr{H}'$ such that

$$\alpha(a) = V^*\pi(a)\, V, \qquad a \in \mathfrak{A}^0.$$

We claim that $V\mathscr{H}$ is an invariant subspace for $\pi(\mathscr{A}^0)$. To see that, let u be a unitary element of $\mathfrak{A}^0$. Then for every $\xi \in \mathscr{H}$ we have

$$\begin{aligned} \|V\alpha(u)\,\xi - \pi(u)\, V\xi\|^2 &= \|\xi\|^2 - 2Re\langle V\alpha(u)\,\xi, \pi(u)\, V\xi\rangle + \|\xi\|^2 \\ &= 2(\|\xi\|^2 - Re\langle \alpha(u)\,\xi, V^*\pi(u)\, V\xi\rangle) \\ &= 2(\|\xi\|^2 - \|\alpha(u)\,\xi\|^2) = 0, \end{aligned}$$

because $\alpha(u)$ is a unitary operator in $\mathfrak{A} \subseteq \mathscr{L}(\mathscr{H})$. Thus

$$V\alpha(u) = \pi(u)\, V$$

for every unitary element of $\mathfrak{A}^0$ and, since $\mathfrak{A}^0$ is spanned by its unitary elements, we see that $V\alpha(a) = \pi(a)\, V$ holds for all $a \in \mathfrak{A}^0$. In particular, this implies that $\pi(\mathfrak{A}^0)$ leaves $V\mathscr{H}$ invariant.

The above also implies that the map $\alpha\colon \mathfrak{A}^0 \to \mathscr{L}(\mathscr{H})$ is unitarily equivalent to the *-homomorphism $\beta\colon \mathfrak{A}^0 \to \mathscr{L}(V\mathscr{H})$ given by

$$\beta(a) = \pi(a)|_{V\mathscr{H}}.$$

Thus α is both a homomorphism and an anti-homomorphism. So for each $x, y \in \mathfrak{A}$ we have

$$xy - yx = \alpha(x)\,\alpha(y) - \alpha(y)\,\alpha(x) = \alpha(xy) - \alpha(xy) = 0,$$

proving that $\mathfrak{A}$ is abelian. ∎

The main result below applies to processes constructed from full Markov systems $(X, \mathfrak{A}, \phi, P)$ which have the following additional property:

> For every $n \geqslant 1$ and every self-adjoint $n \times n$ matrix $[a_{ij}]$ in $M_n \otimes \mathfrak{A}$ for which $[P_f(a_{ij})] \geqslant 0$ for every $f \in C(X)^+$, one has $[a_{ij}] \geqslant 0$. (2.3.12)

It is very easy to find full Markov systems having the property (2.3.12). For instance, if, in the examples constructed at the beginning of Section 1.3, some element e_j of the sequence $e_1, e_2, \ldots$, is a positive *invertible* element of $\mathfrak{A}$, then (2.3.12) follows. The reason is that if e is a positive invertible element of a C^*-algebra $\mathfrak{B}$ and $a \in \mathfrak{B}$ satisfies $eae \geqslant 0$, then $a \geqslant 0$ because we can multiply the given operator inequality on the left and right by e^{-1}.

THEOREM 2.3.13. *Let $(X, \mathfrak{A}, \phi, P)$ be a full Markov system. Assume further that*

(i) *$\mathfrak{A}$ is not commutative,*

(ii) *ϕ is a trace,*

(iii) *the associated representation π_ϕ is faithful, and*

(iv) *condition* (2.3.12) *is satisfied.*

Then the associated OS-*positive process cannot be extended to a Markov process.*

Proof. By Proposition 2.2.14, it suffices to show that the canonical Markov operator

$$K = E_+ R|_{\mathscr{E}_+}$$

is not factorable. Assume, to the contrary, that K is factorable. We will show that $\mathfrak{A}$ is abelian, contradicting the hypothesis (i). To prove this, Lemma 2.3.11 implies that it is enough to show that the anti-isomorphism $\alpha: \mathfrak{A}^0 \to \mathfrak{A}$ is completely positive. To see that, let $[a_{ij}]$ be a positive element of $M_n \otimes \mathfrak{A}^0$. Then $[\alpha(a_{ij})]$ is a self-adjoint element of $M_n \otimes \mathfrak{A}$. So by (2.3.12), it suffices to show that for every positive function $f \in C(X)$, one has

$$[P_f(\alpha(a_{ij}))] \geqslant 0. \tag{2.3.14}$$

Fix such an $f \in C(X)^+$. Lemma 2.3.8 implies that there is a triple $(\mathfrak{B}, \rho, L)$ where $\mathfrak{B}$ is an abelian C^*-algebra, ρ is a state of $\mathfrak{B}$, and $L: \mathfrak{A} \to \mathfrak{B}$ is a unital positive linear map satisfying

$$\phi(P_f(a)\, b) = \rho(L(a)\, L(b)), \qquad a, b \in \mathfrak{A}.$$

By Lemma 2.3.9, $P_f\alpha$ is a completely positive linear map of $\mathfrak{A}^0$ into $\mathfrak{A}$, and clearly 2.3.14 follows from this. ▮

Concluding Remarks

It follows from Theorem 2.3.13 that the examples of Markov systems $(X, \mathfrak{A}, \phi, P)$ in Section 1.3 give rise to a wide variety of nonextendable OS-positive processes. For example, one may take $\mathfrak{A}$ to be the $n \times n$ complex matrices ($n \geqslant 2$), a UHF algebra, an irrational rotation C^*-algebra, or the reduced C^*-algebra of the free group on two generators. All of these are separable simple C^*-algebras having faithful tracial states ϕ and, as we have already remarked, it is easy to arrange the remaining hypothesis of (2.3.13) in all of these examples. Clearly, the OS-processes so obtained will vary widely in their stochastic properties, and very little is known as yet about the classification of these processes.

Another problem concerns the characterization of those processes $\{X_n\}$ which *can* be extended to (stationary, symmetric) Markov processes. Such processes must be OS-positive of course, and moreover, (2.2.14) implies that the canonical Markoc operator K associated with $\{X_n\}$ must be factorable. It seems unlikely that these necessary conditions could also be sufficient, but we are in possession of neither a proof nor a counter example.

However, in the case of a *Gaussian* process $\{X_n\}$ (with state space an arbitrary Banach space, say), the situation is clear: every Gaussian OS-positive process can be extended to a symmetric Gaussian Markov process. The proof of this is parallel to Klein's proof of the corresponding result for continuous parameter processes [9], and will be taken up elsewhere.

Finally, there remains the corresponding problem for continuous parameter processes that was originally posed by Klein: does every OS-positive process extend to a Markov process? The above results seem to suggest rather strongly that the answer must be no. However, at this point it is not clear how one might adapt Theorem 2.3.13 to the case of continuous time; in particular, the problem of constructing a Euclidean field with these properties remains open.

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