CONTINUOUS NESTS AND THE ABSORPTION PRINCIPLE

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1. INTRODUCTION

In [2], a general absorption principle is established which provides a unification of theorems of Dan Voiculescu and Niels Toft Andersen (to be described presently). Andersen's theorem was subsequently generalized to a rather broad class of commutative subspace lattices. Since a substantial amount of work is required to set up this generalization, it is not made very clear in [2] that one can proceed in a simple way from the absorption principle to Andersen's theorem. The purpose of this note is to show how this can be done. We will discuss the absorption principle (without proof) and we will indicate (with proof) how one goes about deducing Andersen's theorem from it.

Throughout this paper, all Hilbert spaces will be separable, and the generic symbol K will denote the C*-algebra of compact operators on the appropriate Hilbert space.

Voiculescu's theorem [6] asserts that if $A \subseteq \mathcal{L}(\mathcal{H})$ is a *separable* C*-algebra of operators which contains the identity and σ is a non-degenerate representation of A which annihilates $A \cap K$, then

$$id \oplus \sigma \underset{a}{\sim} id \quad ,$$

where id denotes the identity representation of A. Here, $\underset{a}{\sim}$ is Voiculescu's notion of approximate equivalence: for two representations π_1, π_2 of A on spaces $\mathcal{H}_1, \mathcal{H}_2$, $\pi_1 \underset{a}{\sim} \pi_2$ means that there is a sequence W_n of unitary operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that for each $A \in A$

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(i) $\quad W_n \pi_1(A) W_n^* - \pi_2(A) \in K \quad ,$

and

(ii) $\quad \lim_{n \to \infty} \| W_n \pi_1(A) W_n^* - \pi_2(A) \| = 0 \quad .$

Using this theorem, one can easily deduce

<u>Corollary</u>. *Let* $A_i \subseteq \mathcal{L}(\mathcal{H})$ *be two separable* C^**-algebras of operators which contain 1. Assume that*

i) A_1 *and* A_2 *are* **-isomorphic*, *and*

ii) $A_i \cap K = \{0\}$, $i = 1,2$.

Then $A_1 + K$ *and* $A_2 + K$ *are unitarily equivalent.*

The corollary has a classical predecessor, due to Weyl and von Neumann. *Let* A_1, A_2 *be self-adjoint operators such that*

i) $sp(A_1) = sp(A_2)$,

ii) *neither* A_1 *nor* A_2 *has any isolated eigenvalue of finite multiplicity.*

Then A_1 *is unitarily equivalent to a compact perturbation* $A_2 + K$ *of* A_2; *moreover,* K *can be chosen so that its norm is arbitrarily small.* Actually, K can be chosen to be a small Hilbert-Schmidt operator, but that is not relevant to our purpose here (the essential step can be found on p.525 of [5]).

We want to point out that the corollary fails if one drops the separability hypothesis. Indeed, if A_1 is a nonatomic maximal abelian von Neumann algebra in $\mathcal{L}(\mathcal{H})$ and A_2 is the abelian von Neumann algebra on $\mathcal{H} \oplus \mathcal{H}$ defined by

$$A_2 = \{A \oplus A : A \in A_1\} \quad ,$$

then A_1 and A_2 are *-isomorphic, $A_i \cap K = \{0\}$ for i = 1,2, but $A_1 + K$ and $A_2 + K$ are not unitarily equivalent. The argument can be found in the introduction of [2].

Let us recall Andersen's theorem ([1], 3.5.5) about continuous nests. By a continuous nest we mean here a projection-valued function $t \in [0,1] \mapsto P_t \in \mathcal{L}(\mathcal{H})$ satisfying

i) $P_0 = 0, \quad P_1 = 1$

ii) $s < t \Rightarrow P_s < P_t$

iii) $t \mapsto \langle P_t \xi, \eta \rangle$ *is continuous for every* ξ, η *in* $\mathcal{H}$.

Andersen's theorem asserts that if $\{P_t\}$ and $\{Q_t\}$ are two continuous nests, then there is a sequence W_n of unitary operators such that

i) $W_n P_t W_n^* - Q_t$ *is compact for all* t.

(1.1) ii) $\sup_{0 \leq t \leq 1} \|W_n P_t W_n^* - Q_t\|$ *tends to zero as* n *tends to* ∞.

iii) $t \mapsto W_n P_t W_n^* - Q_t$ *is a norm-continuous operator-valued function, for each* $n \geq 1$.

Notice that the assertions of (1.1) resemble the definitions of approximate equivalence of representations to some extent, but there are some essential differences. First, the commutative C*-algebras generated by $\{P_t\}$ and $\{Q_t\}$ are invariably inseparable. Second, the condition (1.1)ii) asserts that, for large n, the infinite set of norms

$$\{\|W_n P_t W_n^* - Q_t\| : \quad 0 \leq t \leq 1\}$$

are simultaneously small: the assertion of "approximate equivalence" would require only a finite number of these norms to be small. Finally, there is no counterpart whatsoever to the third property (1.1)iii) in the definition of approximate equivalence.

In the next section we will introduce a context appropriate for obtaining conclusions of this nature, and in section 3 we will derive Andersen's theorem from the main result of section 2.

2. THE ABSORPTION PRINCIPLE

By a *-semigroup we mean a second countable locally compact Hansdorff space X, on which there is defined a jointly continuous associative multiplication

$$(x,y) \in X \times X \longmapsto x \cdot y \in X \quad ,$$

and a continuous involution $x \longmapsto x^*$, i.e., a mapping of X satisfying $x^{**} = x$, $(xy)^* = y^*x^*$. For convenience, we also assume X has a unit e: $e \cdot x = x \cdot e = x$, $x \in X$.

To recover the context of Voiculescu's theorem, one chooses X to be a countable norm-dense subgroup of the unitary group of a separable C^*-algebra A, endowed with its discrete topology. The multiplication and involution in X are the obvious operations inherited from A. To recover the context of Andersen's theorem, take X to be the closed unit interval with its natural topology, and the operations

$$x \cdot y = \min(x,y) \quad ,$$

$$x^* = x \quad ,$$

$x,y \in [0,1]$. Other applications are described in [2].

A *representation* of a *-semigroup X is a mapping $x \mapsto U(x)$ from X to the operators on some Hilbert space $\mathcal{H}$, which is

i) *strongly continuous*

(2.1) ii) *a homomorphism of unital *-semigroups*

iii) *bounded:* $\sup_{x \in X} \|U(x)\| < \infty$.

It is easy to see that, in fact, we must have

$$\|U(x)\| \leq 1$$

for all $x \in X$. We will also write $\mathcal{H}_U$ for the Hilbert space on which a given representation U acts.

Definition 2.2 (Norm equivalence). *If* U,V *are representations of* X, *we will write* $U \sim V$ *if for every compact subset* K *of* X *and* $\varepsilon > 0$, *there is a unitary operator* W *from* $\mathcal{H}_U$ *to* $\mathcal{H}_V$ *such that*

$$\sup_{x \in K} \|WU(x)W^* - V(x)\| \leq \varepsilon \quad .$$

This is clearly an equivalence relation in the collection of all representations of X. This relation has a simple definition and is easy to work with. But what we are really interested in is the following much stronger relation.

Definition 2.3 (Approximate equivalence). *For two representations* U,V *of* X, $U \underset{a}{\sim} V$ *means that for every compact subset* K *of* X *and* $\varepsilon > 0$, *there is a unitary operator* W *from* $\mathcal{H}_U$ *to* $\mathcal{H}_V$ *satisfying*

i) $\sup_{x \in K} \|WU(x)W^* - V(x)\| \leq \varepsilon$, *and*

ii) $x \longrightarrow WU(x)W^* - V(x)$ *is a norm-continuous function from* X *to the compact operators.*

Let U,V be two representations of X. We require some criteria for determining when V is "absorbed" by U in the following sense,

$$U \oplus V \sim U \quad .$$

These criteria should involve the action of U and V on their respective spaces, and should involve properties that can be checked in specific examples. We will see that such criteria exist, but that they involve not only U and V but a sequence of representations associated with U and V.

This sequence is defined as follows. Let X be a *-semigroup. For each positive integer n, let G_n be a finite subgroup of the unitary group of the C^*-algebra M_n of all $n \times n$ matrices, such that

$$M_n = \text{span } G_n \quad .$$

For instance, one may take G_n to be the group of all $n \times n$ matrices having exactly one nonzero entry, consisting of ±1, in each row and each column. G_n is considered to be fixed throughout the remainder of the discussion.

G_n is a *-semigroup in its discrete topology. So for each $n > 1$ we may form the Cartesian product of *-semigroups $G_n \times X$. Finally, if U is a representation of X on $\mathcal{H}$ then we can form a sequence of representations $U_n \colon G_n \times X \longrightarrow \mathcal{L}(\mathbb{C}^n \otimes \mathcal{H})$ by

$$U_n(u,x) = u \otimes U(x) ,$$

$u \in G_n$, $x \in X$. The process whereby one considers the sequence of representations $U_1, U_2, \ldots$ along with U is somewhat analogous to the process of considering, along with a completely positive linear map of C^*-algebras

$$\phi: A \to \mathcal{L}(\mathcal{H}) ,$$

its associated sequence of completely positive maps

$$\mathrm{id} \otimes \phi: M_n \otimes A \to \mathcal{L}(\mathbb{C}^n \otimes \mathcal{H}) ,$$

$n = 1, 2, \ldots$.

Finally, we will say that a representation V is *subordinate* to a representation U if, for every normal state ρ of $\mathcal{L}(\mathcal{H}_V)$, there is a sequence ξ_n of unit vectors in $\mathcal{H}_u$ such that

(2.4)

i) $\xi_n \to 0$ *weakly in* $\mathcal{H}_u$, *and*

ii) $\rho(V(x)) = \lim_{n\to\infty} \langle U(x)\xi_n, \xi_n \rangle$ *uniformly on compact subsets of* X.

Roughly speaking, (2.4) says that normal states of V can be approximated by vector states of U, where the approximating vectors are "near infinity".

We can now state the main result.

Theorem 2.5. *Let* U,V *be representations of* X. *The following are equivalent:*

i) $U \oplus V \sim U$.

ii) $U \oplus V \underset{a}{\sim} U$.

iii) V_n *is subordinate to* U_n, *for every* $n = 1,2,\ldots$.

3. CONTINUOUS NESTS

We now prove the following theorem from ([1], 3.5.5).

Theorem 3.1. *Let* $\{P_t : 0 \le t \le 1\}$, $\{Q_t : 0 \le t \le 1\}$ *be continuous nests. Then there is a unitary operator* W *such that the properties* 1.1 *are satisfied.*

We require the following variation of the continuity theorem of probability theory.

Lemma. *Let* μ_n *be a sequence of positive finite measures on* [0,1] *which converges to a nonatomic measure* μ *in the weak* topology of* C[0,1]. *Then*

$$\sup_{0 \le t \le 1} |\mu_n([0,t]) - \mu([0,t])|$$

tends to zero as n *tends to* ∞.

Proof of Lemma. We may clearly assume that $\mu_n([0,1]) = \mu([0,1]) = 1$ for every n . Let

$$F_n(t) = \mu_n([0,t])$$

$$F(t) = \mu([0,t]) \quad , \qquad 0 \le t \le 1 \quad .$$

By ([4], theorem 1, p.249), the sequence F_n converges *pointwise* to F. We have to show that this convergence is actually uniform.

For that, fix $\varepsilon > 0$. Since μ is nonatomic, F is continuous and therefore uniformly continuous. So we may find points

$$0 = t_0 < t_1 < \ldots < t_N = 1$$

in $[0,1]$ such that $|F(t_j) - F(t_{j-1})| \le \varepsilon$ for all $1 \le j \le N$. Choose n large enough so that

$$\max_{0 \le j \le N} |F_k(t_j) - F(t_j)| \le \varepsilon \quad ,$$

for all $k \ge n$. Then for every such k and every $s \in [0,1]$, say $t_{j-1} \le s \le t_j$, we can write

$$F_k(s) \le F_k(t_j) \le F(t_j) + \varepsilon$$

$$\le F(t_{j-1}) + 2\varepsilon \le F(s) + 2\varepsilon \quad .$$

Hence, $F_k(s) \le F(s) + 2\varepsilon$. Similarly, $F_k(s) \ge F(s) - 2\varepsilon$, and so $\|F_k - F\|_\infty \le \varepsilon$. □

To prove theorem 3.1, we consider the *-semigroup structure

$$X = [0,1] \qquad \textit{(usual topology)}$$

$$x \cdot y = \min(x,y)$$

$$x^* = x \quad .$$

For $t \in X = [0,1]$, put

$$U(t) = P_t , \qquad V(t) = Q_t .$$

U and V are representations of X, and we have to show that $U \underset{a}{\sim} V$. It is enough to prove $U \oplus V \underset{a}{\sim} U$ and $V \oplus U \underset{a}{\sim} V$.

We will prove that $U \oplus V \underset{a}{\sim} U$; the rest will follow by symmetry. By theorem 2.5, we need only prove that V_n is subordinate to U_n for every $n = 1,2,\ldots$.

For that, fix $n \geq 1$ and let ρ be a normal state of $\mathcal{L}(\mathbb{C}^n \otimes \mathcal{H}_v)$. We have to find a sequence of unit vectors $\xi_1, \xi_2, \ldots$ in $\mathbb{C}^n \otimes \mathcal{H}_u$ such that $\xi_p \to 0$ weakly, as $p \to \infty$, and

$$\sup_{0 \leq t \leq 1} |\rho(u \otimes Q_t) - \langle u \otimes P_t \xi_p, \xi_p \rangle| \longrightarrow 0 \tag{3.2}$$

as $p \to \infty$, for every u in G_n. We will actually prove (3.2) for every u in M_n and, for that, it is enough to prove it for $u \geq 0$.

So fix $u \geq 0$ in M_n, and consider the representations π, σ of $C[0,1]$ defined by

$$\pi(f) = \int_0^1 f(t)\, dP_t ,$$

$$\sigma(f) = \int_0^1 f(t)\, dQ_t .$$

Now the range of π contains no nonzero compact operators because the spectral measure defined by $\{P_t : 0 \leq t \leq 1\}$ is nonatomic. It follows that the C^*-algebra of operators

$$\mathcal{A} = \mathrm{id}_n \otimes \pi(M_n \otimes C[0,1]) ,$$

being essentially the $n \times n$ operator matrices over $\pi(C[0,1])$, contains no compact operators either.

Note, too, that π and σ are both faithful representations of $C[0,1]$, because of the conditions $s < t \Rightarrow P_s < P_t$ and $Q_s < Q_t$. It follows that

$$\| id_n \otimes \pi(z) \| = \| id_n \otimes \sigma(z) \| ,$$

for every z in $M_n \otimes C[0,1]$.

It follows that the linear functional $\lambda: \mathfrak{A} \to \mathbb{C}$ defined by

$$\lambda(id_n \otimes \pi(a \otimes f)) = \rho(id_n \otimes \sigma(a \otimes f)) ,$$

is a well-defined state of $\mathfrak{A}$. Since $\mathfrak{A} \cap K = \{0\}$, we may extend λ to a state $\tilde{\lambda}$ of the perturbed algebra $\mathfrak{A} + K$ in the obvious way,

$$\tilde{\lambda}(A + K) = \lambda(A) , \qquad A \in \mathfrak{A} , \quad K \in K$$

and we now have a state of $\mathfrak{A} + K$ which annihilates all compact operators.

Now Glimm's lemma ([3], 11.2.1) implies that $\tilde{\lambda}$ is a weak* limit on $\mathfrak{A} + K$ of vector states. Since $\mathfrak{A} + K$ is separable, we obtain a *sequence* of unit vectors $\xi_p \in \mathbb{C}^n \otimes \mathcal{H}_u$ such that

(3.3) $$\lambda(A) = \tilde{\lambda}(A + K) = \lim_{p \to \infty} \langle (A + K)\xi_p, \xi_p \rangle ,$$

for every $A \in \mathfrak{A}$, $K \in K$. Take $A = 0$ to obtain $\langle K\xi_p, \xi_p \rangle \to 0$ for every compact K, and hence

(3.4) $\xi_p \to 0$ *is the weak topology of* $\mathbb{C}^n \otimes \mathcal{H}_u$.

Taking $K=0$ and $A = u \otimes \pi(f)$ for $f \in C[0,1]$, we obtain from (3.3):

$$(3.5) \qquad \rho(u \otimes \sigma(f)) = \lim_{p \to \infty} \langle u \otimes \pi(f)\, \xi_p, \xi_p \rangle \quad .$$

Now we can define measures $\mu, \mu_1, \mu_2, \ldots$ on $[0,1]$ by

$$\int_0^1 f(t)\, d\mu(t) = \rho(u \otimes \sigma(f)) \quad ,$$

$$\int_0^1 f(t)\, d\mu_p(t) = \langle u \otimes \pi(f)\, \xi_p, \xi_p \rangle$$

for all $f \in C[0,1]$, by the Riesz-Markov theorem. All of these are finite positive measures, and $\mu_p \to \mu$ is the weak* topology of $C[0,1]$ because of (3.5). Finally, since μ is a nonatomic measure (because the spectral measure defined by $\{Q_t\}$ is nonatomic), we may conclude from the lemma that

$$(3.6) \qquad \sup_{0 \le t \le 1} |\mu([0,t]) - \mu_p([0,t])|$$

tends to zero as $p \to \infty$. Since we clearly have

$$\mu([0,t]) = \rho(u \otimes Q_t)$$

$$\mu_p([0,t]) = \langle u \otimes P_t\, \xi_p, \xi_p \rangle$$

by definition of μ and μ_p, we have established (3.2). □

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