Circular Operators

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Introduction. A familiar theorem, due essentially to Weyl and von Neumann, asserts that every normal operator is a small compact perturbation of a diagonal operator. In particular, a normal operator with continuous spectrum can be compactly perturbed so as to have discrete spectrum.

In this paper we are concerned with operators which are highly nonnormal, but which in some sense have "continuous" or "discrete" features. Specifically, we show that no weighted translation operator is unitarily equivalent to a weighted shift; but that every weighted translation operator is unitarily equivalent to a small compact perturbation of a weighted shift. The second result requires Voiculescu's theorem characterizing approximately equivalent representations of C^*-algebras [16], [4]. Some of the results below were announced in [8].

All Hilbert spaces considered in this paper are separable. $\mathcal{L}(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ (or, more briefly, $\mathcal{K}$) will denote the bounded operators, the unitary operators, and the compact operators on the Hilbert space $\mathcal{H}$. The inner product on $\mathcal{H}$ is denoted $\langle\xi,\eta\rangle$. If S is a set of operators, $C^*(S)$ denotes the C^*-algebra generated by S and the identity. If R is a set of vectors, $[R]$ denotes its norm closed linear span. Finally, $\boldsymbol{T}$ represents the compact abelian group of all complex numbers of modulus one and m is normalized Lebesgue measure on $\boldsymbol{T}$.

The authors want to thank the referee for pointing out the paper of Gellar [7] which overlaps somewhat with the material in Section 1.

1. Circular operators and weighted shifts. An operator $T \in \mathcal{L}(\mathcal{H})$ is called a *weighted bilateral shift* if there is an orthonormal basis $\{e_n : n \in \boldsymbol{Z}\}$ for $\mathcal{H}$ and a sequence w_n of complex numbers such that $Te_n = w_n e_{n+1}$. Note that some (or even all) of the weights may be zero, and necessarily we have $\sup_n |w_n| = \|T\| < +\infty$.

We introduce an invariant for single operators that will allow us to identify operators that are *not* weighted shifts. Fix $T \in \mathcal{L}(\mathcal{H})$; we denote by $G(T)$ the set of all complex numbers λ such that λT is unitarily equivalent to T.

Lemma 1.1. *If $T \neq 0$, then $G(T)$ is a subgroup of $\boldsymbol{T}$.*

Proof. Let $\lambda \in G(T)$, then $|\lambda|\|T\| = \|\lambda T\| = \|U_\lambda T U_\lambda^{-1}\| = \|T\|$ and hence $|\lambda| = 1$. It is clear that $G(T)$ is a group. □

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In general $G(T)$ is not closed in $\boldsymbol{T}$ (examples to follow). We say that T is *circular* if $G(T) = \boldsymbol{T}$. This simply asserts that T is geometrically indistinguishable from any rotation λT, $|\lambda| = 1$, of itself.

The following is a variation on the lore of weighted shifts, and follows from the argument of ([9], page 46).

Proposition 1.2. *All weighted shifts are circular.*

Let G be a second countable locally compact group. Recall that a *projective representation* of G is a mapping $g \to U_g$ of G into $\mathcal{U}(\mathcal{H})$ such that

(i) $U_e = 1$ *where e is the identity of* G,

(ii) $U_g U_h = c(g,h) U_{gh}$ *where* $c(g,h) \in \boldsymbol{T}$,

(iii) $g \to \langle U_g \xi, \eta \rangle$ *is a Borel function for each* $\xi, \eta \in \mathcal{H}$.

The function $c: G \times G \to \boldsymbol{T}$ is called the *multiplier* of U. It is uniquely determined by U and it is a Borel function on $G \times G$.

Let $f: G \times G \to \boldsymbol{T}$ be any Borel function such that $f(e) = 1$, and put

$$c(g,h) = f(gh) f(g)^{-1} f(h)^{-1}.$$

Then c is a multiplier. Such multipliers are called *exact*. It is known that every multiplier on $\boldsymbol{T}$ is exact ([15], Theorem 10.41, page 134).

The following result asserts that for an *irreducible* circular operator T, one can select a family $U_\lambda, \lambda \in \boldsymbol{T}$, of unitary operators which implements the circularity property $U_\lambda T U_\lambda^{-1} = \lambda T$, and which is multiplicative and appropriately continuous. The method of proof, while familiar in group representations, is not commonly seen in operator theory. For completeness, we include the details.

Recall first that $\mathcal{U}(\mathcal{H})$, in its strong operator topology, is a Polish group (i.e., it is metrizable, separable, and metrically complete). It follows that any strongly closed subgroup of $\mathcal{U}(\mathcal{H})$ is Polish in its relative topology.

Proposition 1.3. *An irreducible operator T is circular if and only if there is a strongly continuous unitary representation* $\lambda \to U_\lambda$ *of* $\boldsymbol{T}$ *such that*

$$U_\lambda T U_\lambda^{-1} = \lambda T.$$

Proof. We prove only the nontrivial assertion. We claim first that there is a Borel map $\lambda \in \boldsymbol{T} \to U_\lambda \in \mathcal{U}(\mathcal{H})$ satisfying $U_\lambda T U_\lambda^{-1} = \lambda T$ for all λ. For this, consider the group of unitaries

$$G = \{U \in \mathcal{U}(\mathcal{H}) : UTU^{-1} = \lambda T \text{ for some } \lambda \in \boldsymbol{T}\}.$$

Note first that G is closed in $\mathcal{U}(\mathcal{H})$. Indeed, if $U_n \in G$ converges strongly to $U \in \mathcal{U}(\mathcal{H})$, then we have $U_n T U_n^{-1} = \lambda_n T$ for appropriate $\lambda_n \in \boldsymbol{T}$. By passing to a suitable subsequence, we may assume that $\lambda_n \to \lambda$. Since U_n and U_n^{-1} converge weakly (and therefore strongly) to U and U^{-1} respectively, we conclude that

$$\lambda T = \lim_n \lambda_n T = \lim_n U_n T U_n^{-1} = UTU^{-1},$$

as asserted.

The map $U \in G \to \lambda_U \in \boldsymbol{T}$, defined by

$$UTU^{-1} = \lambda_U T$$

is a continuous homomorphism of the Polish group G onto the circle group; and in particular, the inverse image of a point in $\boldsymbol{T}$ under this mapping is a closed set in G.

Now we claim that $U \to \lambda_U$ maps open sets in G to Borel sets in $\boldsymbol{T}$. Since T is irreducible, the kernel K of this map is the compact group of all scalars $\mu \cdot 1$, $\mu \in \boldsymbol{T}$. The given map admits a unique factorization $\theta \circ \pi$ as follows

$$\begin{array}{ccc} & G & \\ {}^{\pi}\swarrow & & \searrow \\ G/K & \xrightarrow[\theta]{} & \boldsymbol{T} \end{array}.$$

G/K is Polish because K is compact, and the natural projection π is an open mapping of Polish spaces. Since θ is a continuous bijection of Polish spaces, it is a Borel isomorphism ([3], 3.2.3). Thus the composition $\theta \circ \pi$ maps open sets to Borel sets.

By the Cross-section Theorem ([3], 3.4.1) we may find the required map $\lambda \in \boldsymbol{T} \to U_\lambda$ satisfying $U_\lambda T U_\lambda^{-1} = \lambda T$.

One computes that

$$U_{\lambda\mu} T U_{\lambda\mu}^{-1} = U_\lambda U_\mu T U_\mu^{-1} U_\lambda^{-1},$$

and moreover $U_\mu^{-1} U_\lambda^{-1} U_{\lambda\mu}$ is a unitary operator which commutes with T. By irreducibility of T, this operator is a scalar $c(\lambda, \mu)1$, and hence

$$U_{\lambda\mu} = c(\lambda, \mu) U_\lambda U_\mu.$$

In particular, c is a multiplier. Since multipliers on $\boldsymbol{T}$ are exact, there is a Borel function $g : \boldsymbol{T} \to \boldsymbol{T}$ such that

$$c(\lambda, \mu) = g(\lambda, \mu) g(\lambda)^{-1} g(\mu)^{-1}.$$

Putting $V_\lambda = g(\lambda)^{-1} U_\lambda$, it follows that $V_{\lambda\mu} = V_\lambda V_\mu$ for all λ, μ; and clearly $V_\lambda T V_\lambda^{-1} = \lambda T$. Since a Borel measurable unitary representation of $\boldsymbol{T}$ is strongly continuous ([15], page 55), the proof is complete. □

Let (X, μ) be a finite nonatomic measure space, let $T : X \to X$ be an invertible transformation that leaves μ quasi-invariant and assume T is ergodic, that is, if E contained in X is a Borel set with $TE = E$, then $\mu(E) = 0$ or $\mu(X \backslash E) = 0$. We also assume that the underlying Borel structure on X is standard ([3], page 69).

Let $H = L^2(X, \mu)$ and let $\mathcal{M}$ be the algebra of multiplications by L^∞-functions. Then $\mathcal{M}$ is a maximal abelian von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ and since (X, μ) is nonatomic, $\mathcal{M}$ has no minimal projections.

Let $f : X \to \boldsymbol{C}$ be any measurable function such that the operator defined on $L^2(X, \mu)$ by

$$A\xi(x) = f(x)\xi(Tx) \tag{1.4}$$

is bounded. We may write A in the form $A = MU$ where

$$U\xi(x) = \left(\frac{d\mu \circ T}{d\mu}\right)^{1/2}(x)\xi(Tx)$$

is a unitary operator on H and where M is multiplication by

$$g(x) = f(x)\left(\frac{d\mu \circ T}{d\mu}\right)^{-1/2}(x); \tag{1.5}$$

notice that A is bounded iff g is in $L^\infty(X,\mu)$.

Definition 1.6. Any operator of the form (1.4) is called a *weighted translation operator*.

Notice that this definition is somewhat broader than the usual definition of weighted translation operators which requires that the measure should be invariant under the transformation T (Parrott, [12], in which these operators were first studied).

A *Bishop operator* is a weighted translation operator, acting on $L^2[0,1]$, of the particular form

$$(A\xi)(x) = x\xi(x \dotplus \alpha), \qquad x \in [0,1], \tag{*}$$

where α is a fixed irrational number and $x \dotplus \alpha$ denotes addition modulo one.

It is not hard to calculate $G(A)$ for these operators; in fact one has $G(A) = \{\lambda_0^n : n \in \mathbf{Z}\}$, where $\lambda_0 = \exp(2\pi i\alpha)$. To see this, let z denote the function

$$z(x) = \exp(2\pi i x),$$

and let W be the operator multiplication by z. We have $AW = \lambda_0 WA$, and hence λ_0^n belongs to $G(A)$ for every $n \in \mathbf{Z}$. Conversely, let $\lambda \in G(A)$. Then there is a unitary operator V such that $VAV^* = \lambda A$. It follows that V commutes with AA^*, and the polynomials in AA^* are weakly dense in the multiplication algebra, a maximal abelian von Neumann algebra. Hence V is multiplication by some function, say $V\xi(x) = f(x)\xi(x)$, where $|f(x)| = 1$ a.e. Using the polar decomposition on the relation $VAV^* = \lambda A$ we obtain $VT_\alpha V^* = \lambda T_\alpha$, where T_α is the translation operator

$$T_\alpha\xi(x) = \xi(x \dotplus \alpha).$$

It follows that $T_\alpha V^* T_\alpha^{-1} = \lambda V^*$, and so

$$f(x \dotplus \alpha) = \bar{\lambda} f(x)$$

a.e. This condition implies that f is a scalar multiple of z^n for some $n \in \mathbf{Z}$ (a fact easily verified by considering the Fourier expansion of f), from which we conclude that $\lambda_0^n = \bar{\lambda}$, as required.

In particular, $G(A) \neq \mathbf{T}$ for every Bishop operator A. The calculation of $G(A)$

for general weighted translation operators is more difficult, particularly in those cases where the transformation T preserves no σ-finite measure equivalent to μ. Nevertheless, we have

Theorem 1.7. *Let A be a weighted translation operator on $L^2(X,\mu)$ such that the function g, defined almost everywhere by (1.5), satisfies*
(i) $g > 0$ a.e., *and*
(ii) $\{g\circ T^n : n \in \mathbf{Z}\}$ *separates points of X.*
Then A is an irreducible operator which is not circular.

Remark. Hypothesis (ii) means that *some* version of g satisfies the stated condition. Note 1.7 implies that A *is not unitarily equivalent to a weighted shift.* There are other routes to that conclusion, but they do not involve circularity.

Proof of Theorem 1.7. We first show that A is irreducible. Indeed, because $g > 0$ and U is unitary, $A = MU$ is the polar decomposition of A and so any projection E that commutes with A must commute with both factors M and U. Thus E commutes with U^nMU^{-n} for all n, so we see that E commutes with all multiplications by functions $g\circ T^n$, $n \in \mathbf{Z}$. By condition 1.7(ii) and by ([3], Theorem 3.3.5, page 72), it follows that E commutes with the multiplication algebra $\mathcal{M}$; hence $E \in \mathcal{M}$. Since $UEU^{-1} = E$ we conclude from ergodicity that $E = 0$ or $E = 1$.

Next we show that A is not circular. If it were, 1.3 implies that we would have a strongly continuous representation $\lambda \to U_\lambda$ of $\mathbf{T}$ such that $U_\lambda A U_\lambda^{-1} = \lambda A$. It follows that if $M_{g\circ T^n}$ denotes the operator of multiplication by $g\circ T^n$, then
(1) $U_\lambda M_{g\circ T^n} U_\lambda^{-1} = M_{g\circ T^n}$ *for all n, and*
(2) $U_\lambda U U_\lambda^{-1} = \lambda U$.
By (1) and an argument similar to that of the preceding paragraph, we see that each U_λ belongs to $\mathcal{M}$. Note also that $\sigma(U_\lambda)$ is contained in $\{\lambda^n : n \in \mathbf{Z}\}$, since $\lambda \to U_\lambda$ is a representation. Fix an irrational $\lambda \in \mathbf{T}$. If $U_\lambda = \sum_{-\infty}^{\infty} \lambda^n E_n$ is the spectral decomposition of U_λ ([13], page 381), then of course $\sum_{-\infty}^{\infty} E_n = 1$. Moreover, since each E_n belongs to the von Neumann algebra generated by $\{U_\lambda : \lambda \in \mathbf{T}\}$, we have $E_n \in \mathcal{M}$.

Now (2) implies

$$UE_nU^{-1} = E_{n+1}.$$

Therefore the E_n's induce a partition of X into disjoint measurable sets X_n:

$$X = \bigcap_{n\in\mathbf{Z}} X_n$$

and we may assume $TX_n = X_{n+1}$ for each n. We claim that X_0 is an atom in the measure algebra of (X,μ). For if $X_0 = A \cup B$; $A \cap B = \varnothing$; $\mu(A) > 0$; and $\mu(B) > 0$, then $\bigcup_n T^nA$ and $\bigcup_n T^nB$ give a decomposition of X into T-invariant sets of positive measure, contradicting the ergodicity of T. Thus μ has an atom, contrary to hypothesis. □

2. Perturbation theory. The purpose of this section is to show that irreducible weighted translation operators are small compact perturbations of weighted shifts. For such an operator A, it is convenient to work with a (separable) C^*-algebra somewhat larger than $C^*(A)$. The general setting is as follows.

Let $\mathcal{M}$ be a nonatomic abelian von Neumann algebra acting on $\mathcal{H}$ and let U be a unitary operator satisfying

(2.1) *(i)* $U\mathcal{M}U^{-1} = \mathcal{M}$
(ii) (ergodicity) *for* $A \in \mathcal{M}$, $UAU^{-1} = A$ *implies* A *is a scalar operator.*

We require some information about the C^*-algebra $\mathcal{A} = C^*(\mathcal{M},U)$. Observe that the set $\mathcal{A}_0$ of all "polynomials" in U

$$P = \sum_{k=-n}^{n} D_k U^k, \qquad D_k \in \mathcal{M},\ n \geq 0,$$

is a norm dense *-subalgebra of $\mathcal{A}$, and the operator P uniquely determines the coefficients D_k ([1], page 96).

Although we do not require it, one can show [11] that $\mathcal{A}$ is *-isomorphic to the crossed product $\boldsymbol{Z} \times_\tau \mathcal{M}$, where the action of $\boldsymbol{Z}$ on the C^*-algebra $\mathcal{M}$ is given by

$$\tau_k(A) = U^k A U^{-k}, \qquad k \in \boldsymbol{Z}.$$

There is always a natural action α of the dual group $\boldsymbol{T} = \tilde{\boldsymbol{Z}}$ on any crossed product of the form $\boldsymbol{Z} \times_\tau \mathcal{M}$ ([11], 7.8.3). The following result essentially asserts the existence and certain properties of α in this concrete setting.

Proposition 2.2. *There is a unique action of the unit circle* $\boldsymbol{T}$, $\lambda \to \alpha_\lambda \in$ aut$(\mathcal{A})$, *satisfying*

(i) $\alpha_\lambda|_{\mathcal{M}} =$ identity, *and*

(ii) $\alpha_\lambda(U) = \lambda U$,

for all $\lambda \in \boldsymbol{T}$. α *is strongly continuous. Moreover,* $\mathcal{M}$ *is the fixed point algebra of this action and the map* Φ *defined by*

$$\Phi(A) = \int_{\boldsymbol{T}} \alpha_\lambda(A) dm(\lambda)$$

is a faithful positive linear projection of $\mathcal{A}$ *onto* $\mathcal{M}$.

Proof. Recalling that $\mathcal{A}_0$ is norm dense in $\mathcal{A}$, define $\Phi : \mathcal{A}_0 \to \mathcal{M}$ by

$$\Phi\left(\sum_{-n}^{n} D_k U^k\right) = D_0.$$

It follows from ([1], Lemma 1.1) that Φ is well defined on $\mathcal{A}_0$ and it is clear that Φ is a positive linear projection of $\mathcal{A}_0$ onto $\mathcal{M}$. From ([1], Proposition 1.4) Φ is bounded on $\mathcal{A}_0$ so it extends by continuity to a bounded linear map of $\mathcal{A}$ onto $\mathcal{M}$; by ([1], Theorem 1.5) the extended map, call it Φ, is faithful on $\mathcal{A}$.

Since $\mathcal{H}$ is separable, there is a faithful normal state ρ_0 of $\mathcal{M}$. Thus

$$\rho = \rho_0 \circ \Phi$$

is a faithful state of $\mathcal{A}$.

Now fix $\lambda \in \boldsymbol{T}$. We can define a *-automorphism α_λ of $\mathcal{A}_0$ by

$$\alpha_\lambda\left(\sum_{-n}^{n} D_k U^k\right) = \sum_{-n}^{n} D_k \lambda^k U^k.$$

Clearly, $\rho \circ \alpha_\lambda = \rho$ on $\mathcal{A}_0$, so ([5], Theorem 2.5) α_λ extends uniquely to a *-automorphism of $\mathcal{A}$, which we again denote by α_λ.

It is clear that $\alpha_\lambda \alpha_\mu = \alpha_{\lambda\mu}$; that $\alpha_1 = \mathrm{id}$; that $\alpha_\lambda|_{\mathcal{M}} = \mathrm{id}\ \mathcal{M}$; and that $\alpha_\lambda(U) = \lambda U$. Also, we clearly have

$$\lim_{\lambda \to 1} \|\alpha_\lambda(A) - A\| = 0$$

for all A in the dense subalgebra $\mathcal{A}_0$ and hence this persists for all A in $\mathcal{A}$. Finally, by checking on "polynomials" it is easy to see that the formula

$$\Phi(A) = \int_{\boldsymbol{T}} \alpha_\lambda(A) dm(\lambda)$$

holds on $\mathcal{A}_0$ and so this too persists for all elements of $\mathcal{A}$. □

Corollary 2.3. $\mathcal{A} \cap \mathcal{K} = \{0\}$.

Proof. By the spectral theorem for self-adjoint compact operators, it suffices to show that the only finite dimensional projection $E \in \mathcal{A}$ is $E = 0$.

Fix a finite dimensional projection $E \in \mathcal{A}$. For each $\lambda \in \boldsymbol{T}$, $\alpha_\lambda(E)$ is a projection in $\mathcal{A}$, and the function $\lambda \to \alpha_\lambda(E)$ moves continuously in the operator norm. Thus there is an $\varepsilon > 0$ such that

$$\|\alpha_\lambda(E) - \alpha_\mu(E)\| < 1$$

whenever $|\lambda - \mu| < \varepsilon$. Since two projections P,Q satisfying $\|P - Q\| < 1$ must have the same dimension, it follows that dim $\alpha_\lambda(E)$ is locally constant, hence constant in λ. In particular $\lambda \to \alpha_\lambda(E)$ is a norm-continuous function from $\boldsymbol{T}$ into the positive compact operators and so

$$\Phi(E) = \int_{\boldsymbol{T}} \alpha_\lambda(E) dm(\lambda)$$

is a positive compact operator in $\mathcal{M}$. Since $\mathcal{M}$ is nonatomic we have $\Phi(E) = 0$ and since Φ is faithful, $E = 0$. □

We are primarily concerned with separable C^*-subalgebras of $\mathcal{A}$ of the following type. Let $\mathfrak{D}$ be a norm-separable unital C^*-subalgebra of $\mathcal{M}$ which is weakly dense in $\mathcal{M}$ and which is invariant under the action of U:

$$U\mathfrak{D}U^{-1} = \mathfrak{D}.$$

Let $\mathcal{B} = C^*(\mathcal{D},U)$. The above action of $\boldsymbol{T}$ on $\mathcal{A}$ restricts to an action α_λ of $\boldsymbol{T}$ on $\mathcal{B}$ and thus we have a C^*-dynamical system $(\boldsymbol{T},\alpha,\mathcal{B})$. The subalgebra $\mathcal{D}$ may or may not contain projections, but we can say

Lemma 2.4. *$\mathcal{D}$ contains no nonzero minimal projections.*

Proof. Let E be a minimal projection in $\mathcal{D}$. Then $E\mathcal{D}E = \boldsymbol{C}\cdot E$. The set $\{T \in \mathcal{L}(\mathcal{H}) : ETE \in \boldsymbol{C}\cdot E\}$ is weakly closed, it contains $\mathcal{D}$, and therefore it contains $\mathcal{M}$. Hence $E\mathcal{M}E = \boldsymbol{C}\cdot E$, which implies E is a minimal projection in $\mathcal{M}$ and this is a contradiction unless $E = 0$. □

Proposition 2.5. *Let (π,W) be a covariant representation of the C^*-dynamical system $(\boldsymbol{T},\alpha,\mathcal{B})$ such that $\pi|_{\mathcal{D}}$ is faithful. Then π is fatthful and $\pi(\mathcal{B})$ contains no nonzero compact operators.*

Proof. π is a representation of $\mathcal{B}$ on a Hilbert space $\mathcal{H}'$ and W is a strongly continuous unitary representation of $\boldsymbol{T}$ on $\mathcal{H}'$ satisfying

$$W_\lambda \pi(X) W_\lambda^{-1} = \pi(\alpha_\lambda(X))$$

for all $\lambda \in \boldsymbol{T}$ and for all $X \in \mathcal{B}$.

To prove π is faithful, choose a positive operator $X \in \mathcal{B}$ with $\pi(X) = 0$. Then

$$\pi(\Phi(X)) = \int_{\boldsymbol{T}} \pi(\alpha_\lambda(X))dm(\lambda) = \int_{\boldsymbol{T}} W_\lambda \pi(X) W_\lambda^{-1} dm(\lambda) = 0.$$

Since $\Phi(X) \in \mathcal{D}$ and $\pi|_{\mathcal{D}}$ is faithful we have $\Phi(X) = 0$, therefore $X = 0$ because Φ is faithful.

Let K be a nonzero positive compact operator in $\pi(\mathcal{B})$. Then

$$K_0 = \int_{\boldsymbol{T}} W_\lambda K W_\lambda^{-1} dm(\lambda)$$

is a positive compact operator in $\pi(\mathcal{B})$ which commutes with W. If ξ is any vector in $\mathcal{H}'$ for which $\langle K\xi,\xi\rangle > 0$, then

$$\langle K_0\xi,\xi\rangle = \int_{\boldsymbol{T}} \langle KW_\lambda^{-1}\xi, W_\lambda^{-1}\xi\rangle\, dm(\lambda) > 0$$

since the integrand is a nonnegative continuous function which is strictly positive at $\lambda = 0$. Thus $K_0 \neq 0$.

By spectral theory we can find a finite dimensional nonzero spectral projection E for K_0. Since E is a continuous function of K_0 we have $E \in \pi(\mathcal{B})$ and E commutes with W. We claim that $E \in \pi(\mathcal{D})$. To see this let $X \in \mathcal{B}$ be such that $\pi(X) = E$: then for all $\lambda \in \boldsymbol{T}$,

$$\pi(X) = E = W_\lambda E W_\lambda^{-1} = W_\lambda \pi(X) W_\lambda^{-1} = \pi(\alpha_\lambda(X)).$$

Since π is faithful, $X = \alpha_\lambda(X)$ for all $\lambda \in \boldsymbol{T}$, so X is in the fixed point algebra $\mathcal{D}$.

This shows that $\pi(\mathcal{D})$ contains a finite dimensional projection. Hence $\pi(\mathcal{D})$ must contain a minimal projection and so $\mathcal{D}$ must have the same property. This contradicts 2.4. □

We will also make use of the following familiar result ([14], problem 7, page 47).

Proposition 2.6. *If* ρ *is an* α*-invariant state of* $\mathcal{B}$ *and* π *is the* GNS *representation obtained from* ρ *with unit cyclic vector* η *(that is,* $\rho(X) = \langle \pi(X)\eta,\eta\rangle$ *for all* $X \in \mathcal{B}$*), then there exists a unique strongly continuous unitary representation* W *of* $\boldsymbol{T}$ *such that*

(i) $W_\lambda \eta = \eta$

(ii) $W_\lambda \pi(X) W_\lambda^{-1} = \pi(\alpha_\lambda(X))$

for all $X \in \mathcal{B}$ *and for all* $\lambda \in \boldsymbol{T}$.

Since $\mathcal{D}$ is a unital abelian C^*-algebra the Gelfand theory provides a compact metric space Y such that $\mathcal{D} = C(Y)$. Since $U \cdot U^{-1}$ gives an automorphism of $\mathcal{D}$, there is an induced automorphism of $C(Y)$; let $\phi : Y \to Y$ be the corresponding homeomorphism of Y. Then we have the following.

Lemma 2.7. *There exists* $p \in Y$ *whose orbit* $\{\phi^n p : n \in \boldsymbol{Z}\}$ *is dense in* Y.

Proof. We claim that if G and V are open sets in Y, then $\phi^n(G) \cap V = \varnothing$ for all n implies either $G = \varnothing$ or $V = \varnothing$. To see this assume $G \neq \varnothing$ and $V \neq \varnothing$. Choose $f \neq 0$ and $g \neq 0$ in $C(Y)$ with $0 < f, g \leq 1$ such that f lives in G and g lives in V. Then

$$(f \circ \phi^n) g = 0$$

for all n. Hence there exist operators $A, B \in \mathcal{D}^+$ with $A \neq 0$ and $B \neq 0$ such that

$$U^n A U^{-n} B = 0$$

for all n. Since $\mathcal{D}$ is a subset of $\mathcal{M}$, by the spectral theorem we can find nonzero projections $E, F \in \mathcal{M}$ and $\varepsilon > 0$ such that

$$\varepsilon E \leq A \qquad \text{and} \qquad \varepsilon F \leq B.$$

The above implies that

$$U^n E U^{-n} F = 0$$

for all n. The last implies that

$$F \perp \bigvee_{n \in \boldsymbol{Z}} U^n E U^{-n}$$

which contradicts the ergodicity of the operator U. Hence either $G = \varnothing$ or $V = \varnothing$.

Since $C(Y)$ is separable, there is a countable base for the topology on Y, say $\{G_n : n \in \boldsymbol{Z}\}$. Let $V_n = \bigcup_{k \in \boldsymbol{Z}} \phi^k(G_n)$. From our claim $\bar{V}_n = Y$ for all n and hence, by the Baire category theorem, $\bigcap_{n\in Z} V_n$ is dense in Y. Any point $p \in \bigcap_n V_n$ is such that its orbit meets G_n for all n. □

Now let p be any point whose orbit is dense and let ρ_0 be the corresponding pure state of $\mathfrak{D}$. Put $\rho = \rho_0 \circ \Phi$. Since $\Phi \circ \alpha_\lambda = \Phi$, it follows that ρ is an α-invariant state on $\mathfrak{B}$. Let π be the GNS representation of $\mathfrak{B}$ obtained from ρ and let η be a unit vector such that

$$\rho(X) = \langle \pi(X)\eta, \eta \rangle$$

for all $X \in \mathfrak{B}$. Let $\mathcal{H}_\rho$ be the Hilbert space of the representation π.

Lemma 2.8. *The restriction of* ρ *to* $\mathfrak{D}$ *is faithful.*

Proof. Let $D \in \mathfrak{D}$ be such that $\pi(D) = 0$ and let g be the Gelfand transform of D. Since g is continuous and p has dense orbit, it suffices to show that $g(\phi^n p) = 0$ for all n. We have

$$\pi(U^n D U^{-n}) = \pi(U)^n \pi(D) \pi(U)^{-n} = 0$$

for all n. Therefore

$$0 = \langle \pi(U^n D U^{-n})\eta, \eta \rangle = \rho(U^n D U^{-n}) = \rho_0(\Phi(U^n D U^{-n}))$$
$$= \rho_0(U^n D U^{-n}) = g(\phi^n p)$$

for each n. Thus g is identically zero. Since the Gelfand transform is faithful, $D = 0$. □

It follows from 2.6 that there is a unique strongly continuous unitary representation W of $\boldsymbol{T}$ on $\mathcal{H}_\rho$ such that $W_\lambda \eta = \eta$ and $W_\lambda \pi(X) W_\lambda^{-1} = \pi(\alpha_\lambda(X))$ for all $X \in \mathfrak{B}$ and for all $\lambda \in \boldsymbol{T}$. For all $\lambda \in \boldsymbol{T}$, let $W_\lambda = \sum_{-\infty}^{\infty} \lambda^n E_n$ be the spectral decomposition of W_λ. Then $E_n = \int_T \bar{\lambda}^n W_\lambda dm(\lambda)$, and

Lemma 2.9. *We have*
(*a*) $\eta \in E_0 \mathcal{H}_\rho$
(*b*) $\pi(U)^n E_0 \mathcal{H}_\rho = E_n \mathcal{H}_\rho$, $n \in \boldsymbol{Z}$
(*c*) $\pi(D) E_0 \mathcal{H}_\rho$ *is contained in* $E_0 \mathcal{H}_\rho$, $D \in \mathfrak{D}$.

Proof. The subspaces $E_n \mathcal{H}_\rho$ of $\mathcal{H}_\rho$ are characterized by the condition

$$E_n \mathcal{H}_\rho = \{\xi \in \mathcal{H}_\rho : W_\lambda \xi = \lambda^n \xi, |\lambda| = 1\}.$$

By the definition of W, $W_\lambda \eta = \eta$, hence (a).

For (b), since $W_\lambda \pi(U) W_\lambda^{-1} = \lambda \pi(U)$ we have $W_\lambda \xi = \xi$ if $W_\lambda \pi(U)^n \xi = \lambda^n \pi(U)^n \xi$. (b) follows.

Part (c) follows by a similar argument using the fact that $W_\lambda \pi(D) W_\lambda^{-1} = \pi(D)$, $D \in \mathfrak{D}$. □

Proposition 2.10. *There exists an orthonormal base* $\{e_n : n \in \boldsymbol{Z}\}$ *for* $\mathcal{H}_\rho$ *such that for all* $D \in \mathfrak{D}$ *and all* $n \in \boldsymbol{Z}$:

(i) $\pi(U)e_n = e_{n+1}$
(ii) $\pi(D)e_n = \rho(U^{-n}DU^n)e_n$.

Proof. We claim first that $E_0\mathcal{H}_\rho$ is spanned by η. Note that, since the state

$$\rho_0(D) = \langle\pi(D)\eta,\eta\rangle$$

is multiplicative on $\mathcal{D}$, η must be an eigenvector for $\pi(\mathcal{D})$:

$$\pi(D)\eta = \rho_0(D)\eta.$$

Indeed, we have for $D \in \mathcal{D}$

$$\|\pi(D)\eta - \rho_0(D)\eta\|^2 = \|\pi(D)\eta\|^2 - 2\mathrm{Re}\langle\pi(D)\eta,\eta\rangle\,\overline{\rho_0(D)} + |\rho_0(D)|^2$$
$$= \rho_0(D^*D) - |\rho_0(D)|^2 = 0.$$

Now choose $\xi \in E_0\mathcal{H}_\rho$ with $\xi \perp \eta$. We will show that $\xi \perp [\pi(\mathcal{B})\eta]$ and hence $\xi = 0$. Since the polynomials $\sum_{-n}^{n} D_k U^k$ are dense in $\mathcal{B}$, it suffices to show that

$$\langle\pi(DU^n)\eta,\xi\rangle = 0$$

for all $n \in \mathbf{Z}$ and for all $D \in \mathcal{D}$.

If $n = 0$, then by the preceding paragraph

$$\langle\pi(D)\eta,\xi\rangle = \rho_0(D)\,\langle\eta,\xi\rangle = 0.$$

If $n \neq 0$, then by 2.9, $\pi(DU^n)\eta$ belongs to $E_n\mathcal{H}_\rho$, hence $\xi \perp \pi(DU^n)\eta$.

By 2.9(b), $E_n\mathcal{H}_\rho$ is spanned by $\pi(U)^n\eta$. Since $\sum E_n = 1$, we conclude that the $e_n = \pi(U)^n\eta$ form an orthonormal base for $\mathcal{H}_\rho$.

Assertion (i) is evident from the definition of e_n. For (ii) let $D \in \mathcal{D}$. Then

$$\pi(D)e_n = \pi(D)\pi(U)^n\eta = \pi(U^n)\pi(U^{-n}DU^n)\eta$$
$$= \rho(U^{-n}DU^n)\pi(U)^n\eta = \rho(U^{-n}DU^n)e_n,$$

since η is an eigenvector for $\pi(\mathcal{D})$. □

It follows that each operator on $\mathcal{H}_\rho$ of the form $A = \pi(DU)$, $D \in \mathcal{D}$, is a weighted bilateral shift.

In order to state the main result of this section, we require Voiculescu's notion of approximate equivalence [16], [4], suitably adapted for single operators. Two operators $A \in \mathcal{L}(\mathcal{H}_1)$, $B \in \mathcal{L}(\mathcal{H}_2)$ are *approximately equivalent* if, for every $\varepsilon > 0$, there is a unitary operator $W = W_\varepsilon : \mathcal{H}_1 \to \mathcal{H}_2$ and a compact operator $C = C_\varepsilon \in \mathcal{L}(\mathcal{H}_2)$ such that

(i) $WAW^{-1} = B + C$
(ii) $\|C\| \leq \varepsilon$.

In the proof of the following result, we make use of Voiculescu's theorem, as formulated in [4].

Theorem 2.11. *Every operator in $\mathcal{B}$ of the form DU, $D \in \mathcal{D}$, is approximately equivalent to a weighted bilateral shift.*

Proof. From 2.3 we have $\mathcal{B} \cap \mathcal{K} = \{0\}$ and from 2.5 we have $\pi(\mathcal{B}) \cap \mathcal{K} = \{0\}$, thus $\ker \mathring{\pi} = \ker \mathring{\mathrm{id}}$ (here $\mathring{\pi}$ is the composition of π with the canonical embedding of $\mathcal{L}(\mathcal{H}_\rho)$ into the Calkin algebra $\mathcal{L}(\mathcal{H}_\rho)/\mathcal{K}(\mathcal{H}_\rho)$ and id denotes the identity representation of $\mathcal{B}$). Also, since π is faithful, $\ker \pi = \ker \mathrm{id}$. Thus by ([4], Theorem 5), π is approximately equivalent to id and hence DU is approximately equivalent to $\pi(DU)$. From the preceding, $\pi(DU)$ is a weighted bilateral shift. □

We now state the main conclusion of this section.

Corollary. *Every irreducible weighted translation operator is approximately equivalent to a bilateral weighted shift.*

Proof. Let A, acting on $L^2(X,\mu)$, be an irreducible weighted translation operator as in 1.4. Consider the polar decomposition $A = MU$, where M belongs to the multiplication algebra $\mathcal{M}$ and U is a unitary operator satisfying $U\mathcal{M}U^{-1} = \mathcal{M}$.

Note that the action of U on $\mathcal{M}$ is ergodic. Indeed, if $E \in \mathcal{M}$ is a projection such that $UEU^{-1} = E$, then E commutes with both M and U, hence E commutes with $A = MU$, and hence $E = 0$ or 1 by irreducibility.

Now take $\mathcal{D}$ to be any norm-separable $U \cdot U^{-1}$-invariant C^*-subalgebra of $\mathcal{M}$ which is weakly dense in $\mathcal{M}$, which contains M and 1, and apply 2.11 □

The operators S and T are *algebraically equivalent* if there is a *-isomorphism of $C^*(S)$ onto $C^*(T)$ which carries S to T. The corollary implies that every irreducible weighted translation operator is algebraically equivalent to a weighted shift. This means that O'Donovan's classification of weighted shifts [10] can be applied to these operators as well, and thus one has available a rather concrete set of algebraic invariants for essentially all of these operators.

Added in Proof. It has been pointed out to us by Donal O'Donovan that Theorem 2.11 and its corollary can also be obtained by applying Voiculescu's theorem in an appropriate way to some of the results in O'Donovan [10].

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