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Perturbation Theory for Groups and Lattices

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A generalization of a theorem of Dan Voiculescu on perturbations of separable C^*-algebras is proved. This is applied to solve two problems relating to the perturbation theory of unitary group representations, and of commutative subspace lattices. The latter generalizes a theorem of Niels Toft Andersen on compact perturbations of nests.

Contents.

1. INTRODUCTION

Voiculescu's theorem [18] has the following consequence. If $\mathscr{A}_1$ and $\mathscr{A}_2$ are two (separable, separably acting, nondegenerate) isomorphic C^*-algebras of operators which contain no nonzero compact operators, then the perturbed C^*-algebras $\mathscr{A}_1 + \mathscr{K}$ and $\mathscr{A}_2 + \mathscr{K}$ are unitarily equivalent; here $\mathscr{K}$ stands for the algebra of all compact operators on the appropriate Hilbert space. The assertion is a direct consequence of Corollary 1(i) on p. 343 of [7]. This conclusion may certainly fail if the separability assumption on the two algebras is dropped. As a simple example, let $\mathscr{A}_1$ be a nonatomic maximal abelian von Neumann algebra and let $\mathscr{A}_2$ be an abelian von Neumann algebra which is isomorphic to $\mathscr{A}$ but has uniform multiplicity $n \geqslant 2$ (e.g., if $\mathscr{A}_1$ acts on $\mathscr{H}$ we may take $\mathscr{A}_2$ to be all operators on an n-fold direct sum of copies of $\mathscr{H}$ having the form $A \oplus A \oplus \cdots \oplus A$, $A \in \mathscr{A}_1$). To show that $\mathscr{A}_1 + \mathscr{K}$ and $\mathscr{A}_2 + \mathscr{K}$ are not unitarily equivalent, we appeal to a theorem of Johnson and Parrott which implies that the essential commutant of an abelian von Neumann algebra $\mathscr{B}$ (i.e., the set of all operators T such that $TB - BT$ is compact for all $B \in \mathscr{B}$) is $\mathscr{B}' + \mathscr{K}$ [12, Theorem 2.1]. It

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follows that the essential commutants of $\mathscr{A}_1$ and $\mathscr{A}_2$ are $\mathscr{A}_1' + \mathscr{K}$ and $\mathscr{A}_2' + \mathscr{K}$; therefore the images of these essential commutants in the respective Calkin algbras fail to be isomorphic (one is abelian and isomorphic to $\mathscr{A}_1' \cong \mathscr{A}_1$, the other is nonabelian and isomorphic to $\mathscr{A}_2' \cong M_n \otimes \mathscr{A}_1$). It follows that $\mathscr{A}_1 + \mathscr{K}$ and $\mathscr{A}_2 + \mathscr{K}$ cannot be unitarily equivalent.

Nevertheless, there are several good reasons for seeking something like Voiculescu's theorem for *certain* inseparable C^*-algebras. For example, given two unitary representations U, V of a (second countable) locally compact group G, then Voiculescu's theorem provides criteria for comparing the "smeared" operators

$$U_f = \int_G f(x)\, U_x\, dx \qquad \text{and} \qquad V_f = \int_G f(x)\, V_x\, dx,$$

for $f \in L^1(G)$, but one has no basis for comparing the unitary operators U_x, V_x themselves, simply because the C^*-algebras generated by these two sets of operators are inspearable. For instance, if one is interested in relating the behaviour of the two automorphisms groups

$$\alpha_x(A) = U_x A U_x^* \qquad \text{and} \qquad \beta_x(A) = V_x A V_x^*$$

"modulo compacts," then the information one has about the smeared operators U_f, V_f is not directly applicable. Similar problems occur if one seeks to compare two covariant representation of a C^*-dynamical system.

A second class of problems arises in connection with nest algebras. A theorem of Andersen [3, Theorem 3.5.5] implies that if $\mathscr{P}$ and $\mathscr{Q}$ are two separably acting continuous nests (considered as families of self-adjoint projection) and $\theta: \mathscr{P} \to \mathscr{Q}$ is an order isomorphism, then there is a unitary operator W such that

$$\{WPW^* - \theta(P) : P \in \mathscr{P}\} \tag{1.1}$$

is a norm compact set of compact operators. Using [11, Proposition 2.2], it follows easily that the unitary equivalence

$$A \mapsto WAW^*$$

carries the quasitriangular algebra $\operatorname{alg} \mathscr{P} + \mathscr{K}$ onto $\operatorname{alg} \mathscr{Q} + \mathscr{K}$, and one obtains Andersen's result that all quasitriangular algebras based on *continuous* nests are unitarily equivalent. That result has led to additional progress. For example, using this result, Larson [14, 15] has recently solved a central and long-standing problem of Ringrose (the latter has recently been simplified by Andersen [4], using a perturbation theoretic result of Lance [13]). Ringrose's problem assumes that $\mathscr{P}$ and $\mathscr{Q}$ are two maximal linearly ordered families of subspaces of a separable Hilbert space which are *similar* in the sense that there is an invertible operator T which transforms the subspaces of $\mathscr{P}$ onto the subspaces of $\mathscr{Q}$, and it asks if $\mathscr{P}$ and $\mathscr{Q}$ are unitarily equivalent in the obvious sense. The answer turns out to be no, and this fact has significant consequences about the invariant subspaces structure of single operators. These are discussed at length in [15]. It follows from the above that nests transform quite differently under invertible operators than they do under unitary operators, and that one must take care in making analogies with the finite-dimensional situation. An appropriate classification of arbitrary nests has been carried out by Kenneth Davidson [22].

Andersen's proof of (1.1) is difficult and does not generalize. We were intrigued by the fact that (1.1) resembles the conclusion of Voiculescu's theorem to some extent, but there are two critical differences. First, the C^*-algebra generated by a continuous nest is invariably inseparable, and second, the applicaton to Ringrose's problem requires that the norms

$$\| WPW^* - \theta(P)\|$$

be small for all $P \in \mathscr{P}$ simultaneously: the conclusion of "Voiculescu's theorem" here would make only a finite number of these quantities small.

This paper is the result of our attempt to find a generalization of Voiculescu's result which is applicable to such problems and, especially, would lead to a "general principles" proof of (1.1) which could be applied to other commutative subspace lattices. Section 4 concerns a general absorption principle for representations of topological $*$-semigroups. This applies to certain inseparable C^*-algebras and does in fact generalize Voiculescu's theorem. Indeed, the development in Sections 2–4 follows the broad pattern of our paper [7], but there are essential differences at several points. In Section 5 we apply this to unitary group representations and obtain a new result (Theorem 4). Sections 6 and 7 contain the generalization of Andersen's theorem to certain commutative subspace lattices, together with some new general results which are required for the application of the results of Section 4. In Sections 2 and 3 we make repeated reference to our paper [7]. Most of these remarks simply compare the methods and results here to those of [7], and are not essential to the development of this paper. We do make use of one lemma from [7] in the proof of Theorem 2 below, and the reader unfamiliar with quasicentral approximate units is referred back to [7] for a discussion of concrete examples.

We want to point out that Donald Hadwin has found an interesting reformulation of Voiculescu's theorem and has shown that this reformation can be extended to inseparable C^*-algebras [21]. Unfortunately, we do not see how Hadwin's results can be brought to bear on the problems considered in Sections 5–7, and in the reformulation presented below it has been necessary for us to restructure the arguments of [7] from the beginning.

2. Approximate Units

Let J be an ideal in a C^*-algebra A. Recall that a *quasicentral approximate unit* for J is an increasing directed set Λ in the positive part of the unit ball of J such that

$$\lim_{e \in \Lambda} \|ke - k\| = 0, \qquad k \in J,$$

and

$$\lim_{e \in \Lambda} \|ae - ea\| = 0, \qquad a \in A.$$

The approximate unit Λ is called *convex* if Λ is a convex set, and *countable* if Λ has the order structure of the positive integers $1, 2, \ldots$. The existence of quasicentral approximate units was established in [7] and independently in [1].

Let $\Lambda = \{e_1, e_2, \ldots\}$ be a countable quasicentral approximate unit. It is very easy to see that if K is any *norm*-compact subset of A, then

$$\sup_{a \in K} \|ae_n - e_n a\| \to 0 \tag{2.1}$$

as $n \to \infty$. The purpose of this section is to show that under certain circumstances, one can arrange to have uniform convergence to zero in (2.1) over sets K which are not norm-compact, but which are compact in a weaker topology. When A is a C^*-algebra of operators and J is the compact operators, this weaker topology turns out to be the $*$-strong operator topology. We note that, while the development of this section and the next runs parallel to Section 1 of [7], there are significant differences which require some care.

Let $J \subseteq A$ be as above. There is a natural $*$-homomorphism of A into the multiplier algebra $M(\bar{J})$ of the closure $\bar{J}$ of J; this map associates to $a \in A$ the multiplier (L_a, R_a), where

$$L_a(k) = ak, \qquad R_a(k) = ka, \qquad k \in \bar{J}.$$

Recall that the *strict* topology on $M(\bar{J})$ is the topology for which convergence of a net (L_α, R_α) of multipliers to zero means that $L_\alpha \to 0$ and $R_\alpha \to 0$ in the strong operator topology of $\mathscr{L}(\bar{J})$:

$$\lim_\alpha \|L_\alpha(k)\| = \lim_\alpha \|R_\alpha(k)\| = 0,$$

for all $k \in \bar{J}$ [2, 8]. Finally, a subset $K \subseteq A$ will be called *J-compact* if its image in $M(\bar{J})$ is strictly compact.

LEMMA. *Let Λ be a convex approximate unit for J and let K be a J-compact set. For every $\varepsilon > 0$ there is an $e \in \Lambda$ such that*

$$\sup_{a \in K} \| ae - ea \| < \varepsilon.$$

Proof. Let X be the image of K in $M(\bar{J})$. X is a compact Hausdorff space in its relative strict topology. Let B be the space of all continuous functions from X to $M(\bar{J})$, where $M(\bar{J})$ is topologized with its *strict* topology. B is clearly a complex vector space. Moreover, we have

$$\|F\| = \sup_{x \in X} \|F(x)\| < \infty \tag{2.2}$$

for each $F \in B$ because $F(X)$ is a strictly compact (and therefore bounded) subset of $M(\bar{J})$. B is closed under the adjoint operation

$$F^*(x) = F(x)^*, \qquad x \in X,$$

and it is closed under pointwise multiplication because multiplication in $M(\bar{J})$ is jointly strictly continuous on norm-bounded sets. Finally, B is complete in the norm defined by (2.2) simply because a uniformly convergent sequence of continuous functions into the space $M(\bar{J})$ has a continuous limit function.

Therefore B is a C^*-algebra. It contains the tensor product

$$C(X) \otimes M(\bar{J})$$

in a natural way (the latter is identified with all functions $F: X \to M(\bar{J})$ which are continuous relative to the *norm*-topology on $M(\bar{J})$), but of course it is much larger than the tensor product.

Let L be the C^*-algebra of all *norm-continuous* functions

$$G: X \to M(\bar{J})$$

whose range lies in $\bar{J}$ (or, more properly, in the natural image of $\bar{J}$ in its multiplier algebra). L is a C^*-subalgebra of B which, by the preceding remarks, is isomorphic to $C(X) \otimes \bar{J}$.

Notice that L is an ideal in B. This is equivalent to the fact that if $G: X \to \bar{J}$ is a norm continuous function and

$$F: x \in X \mapsto (L_x, R_x) \in M(\bar{J})$$

is a strictly continuous function, then

$$\lim_{x \to x_0} \|L_x(G(x)) - L_{x_0}(G(x_0))\| = 0,$$

and

$$\lim_{x \to x_0} \|R_x(G(x)) - R_{x_0}(G(x_0))\| = 0;$$

i.e., if a net vectors x_α in a Banach space converges in norm to a vector x_0, and if a uniformly bounded net T_α of operators converges strongly to an operator T_0, then $\|T_\alpha(x_\alpha) - T_0(x_0)\| \to 0$.

Finally, for each $e \in \Lambda$ we can define an element $\tilde{e}$ in the positive part of the unit ball of L by

$$\tilde{e}(x) = e \qquad \text{for all} \quad x \in X.$$

Let $\tilde{\Lambda} = (\tilde{e}: e \in \Lambda\}$. $\tilde{\Lambda}$ is a convex set which is directed increasing. We claim that $\tilde{\Lambda}$ is an approximate unit for L. To see this, fix $G \in L$. Considering G as a norm-continuous function form X to $\bar{J}$, we may cover the norm compact set $G(X)$ with a finite number of ε-balls

$$G(X) \subseteq \sum_{k=1}^{n} B_\varepsilon(y_k),$$

where $y_1, ..., y_n \in \bar{J}$. Because Λ is an approximate unit for $\bar{J}$ we can find $e_0 \in \Lambda$ for which

$$\|ey_k - y_k\| \leqslant \varepsilon, \qquad 1 \leqslant k \leqslant n,$$

for every $\tilde{e} \in \tilde{\Lambda}$ satisfying $\tilde{e} \geqslant \tilde{e}_0$ we have $e \geqslant e_0$ (because $e \longmapsto \tilde{e}$ is an order isomorphism), and for every $x \in X$ we can find y_k so that $\|G(x) - y_k\| \leqslant \varepsilon$. Hence

$$\begin{aligned} \|\tilde{e}(x)\, G(x) - G(x)\| &= \|eG(x) - G(x)\| \\ &\leqslant 2\varepsilon + \|ey_k - y_k\| \leqslant 3\varepsilon. \end{aligned}$$

Thus $\|\tilde{e}G - G\| \leqslant 3\varepsilon$, and the assertion follows.

Now define $F_0 \in B$ by

$$F_0(x) = x;$$

i.e., F_0 carries each multiplier in X to itself. Since $\tilde{\Lambda}$ is a convex approximate unit for $L \subseteq B$, we may apply the lemma on pp. 330–331 of [7] to infer the existence of an $e \in \Lambda$ for which

$$\|F_0\tilde{e} - \tilde{e}F_0\| \leqslant \varepsilon,$$

i.e., $\|F_0(x)e - eF_0(x)\| \leqslant \varepsilon$ for every x in X. Finally, for each $a \in K$ we can choose $x \in X$ to be the multiplier (L_a, R_a) and use the preceding inequality to obtain the desired conclusion

$$\|ae - ea\| \leqslant \varepsilon, \qquad a \in K. \quad \blacksquare$$

THEOREM 1. *Let J be an ideal in a C^*-algebra A. Assume that J has a countable approximate unit, and let $K_1 \subseteq K_2 \subseteq \cdots$ be a sequence of J-compact sets in A. Then J has an approximate unit $e_1, e_2, \ldots$ such that for each n,*

$$\sup_{a \in K_n} \|e_k a - a e_k\|$$

tends to zero as $k \to \infty$.

If A is generated by J and $\bigcup_n K_n$, then e_n is a quasicentral approximate unit.

Proof. Let $u_1 \leqslant u_2 \leqslant \cdots$ be the given approximate unit for J. To prove the principal assertion, it is enough to construct an increasing sequence $e_1 \leqslant e_2 \leqslant \cdots$ satisfying

(i) $u_k \leqslant e_k$, $\|e_k\| \leqslant 1$

(ii) $\sup_{a \in K_n} \|e_k a - a e_k\| \leqslant 1/K$ for all $k \geqslant n$.

Note that property (i) implies that $\{e_k\}$ is an approximate unit for J.

Let Λ be the convex hull of $\{u_1, u_2, \ldots\}$. Λ is a convex approximate unit for J (see [7, p. 330]) which contains each u_k. $\Lambda_1 = \{e \in \Lambda : e \geqslant u_1\}$ is a confinal convex subset of Λ and thus it too is a convex approximate unit for J. By the lemma, Λ_1 contains an element e_1 such that

$$\sup_{a \in K_1} \|e_1 a - a e_1\| \leqslant 1.$$

Assuming that $e_1 \leqslant e_2 \leqslant \cdots \leqslant e_n$ have been found in Λ so that properties (i) and (ii) are satisfied, we repeat the above argument on

$$\Lambda_{n+1} = \{e \in \Lambda : e \geqslant u_n, e \geqslant e_n\}$$

to obtain $e_{n+1} \in \Lambda_{n+1}$ satisfying

$$\sup_{a \in K_{n+1}} \|e_{n+1} a - a e_{n+1}\| \leqslant \frac{1}{n+1}.$$

That completes the induction.

For the last assertion of the theorem, we simply note that if e_n is the constructed approximate unit for J, then the set of elements $a \in A$ satisfying

$$\lim_{n \to \infty} \|e_n a - a e_n\| = 0$$

is a C^*-subalgebra of A which contains J and $\bigcup_n K_n$, hence it contains A. ∎

Remarks. In order to compare Theorem 1 with its counterpart from [7], consider the following example: Let U be a strongly continuous unitary representation of a locally compact group G on a separable Hilbert space $\mathscr{H}$, and let $\mathscr{A}$ be the C^*-algebra generated by $\{U_x : x \in G\}$ and the compact operators $\mathscr{K}$. $\mathscr{A}$ is inseparable in the typical cases. By the result of [7], one can assert that $\mathscr{A}$ has a quasicentral approximate unit E_α consisting of positive finite rank operators, but because of the inseparability of $\mathscr{A}$ the E_α's constructed in [7] will not be countable.

However, if G is σ-compact, then $\{U_x : x \in G\}$ is a countable union of subsets of $\mathscr{A}$ which are compact in the $*$-strong topology of $\mathscr{L}(\mathscr{H})$. Considering $\mathscr{L}(\mathscr{H})$ as the multiplier algebra of $\mathscr{K}$ and noting that the $*$-strong topology on $\mathscr{L}(\mathscr{H})$ coincides with the strict topology of $\mathscr{M}(\mathscr{K})$, we may conclude from Theorem 1 that there is an increasing sequence E_n, $0 \leqslant E_n \leqslant 1$, of finite rank operators which is a quasicentral approximate unit for $\mathscr{A}$. Moreover, if we express G as a countable union of compact subsets $K_n \subseteq K_{n+1}$, then we can arrange that

$$\sup_{x \in K_n} \|E_p U_x - U_x E_p\|$$

tends to zero as $p \to \infty$, for each $n \geqslant 1$.

It seems unlikely that countable quasicentral approximate units should exist "in general." In more concrete terms, it appears that there is no sequential approximate unit E_n for $\mathscr{K}$ which satisfies

$$\lim_{n \to \infty} \|E_n T - T E_n\| = 0$$

for every operator $T \in \mathscr{L}(\mathscr{H})$, but we have not checked this carefully.

3. Localization for *-Semigroups

The results of this paper concern perturbation theory for two classes of objects: group representations (i.e., strongly continuous unitary representations of separable locally compact groups) and commutative subspace lattices (i.e., strongly closed lattices of mutually commuting projections containing 0 and 1). In order to include both applications, we formulate the general results of this section and the next in terms of representations of *-semigroups.

By a *-semigroup we mean a second countable locally compact Hausdorff space X, endowed with a jointly continuous associative multiplication

$$(x, y) \in X \times X \mapsto xy \in X$$

and a continuous self-mapping $x \mapsto x^*$ satisfying

$$x^{**} = x, \qquad (xy)^* = y^*x^*.$$

We also require that X should contain a multiplicative unit e, which is necessarily self-adjoint in the sense that $e^* = e$. For our purposes here, there are three examples

(a) a locally compact group X in which $x^* = x^{-1}$,

(b) a commutative subspace lattice in which multiplication is operator multiplication, the involution is trivial ($P^* = P$ for all P), and the topology is the relative strong operator topology,

(c) a countable dense self-adjoint subset of a unital separable C^*-algebra which is closed under multiplication, contains 1, and is topologized discretely.

Regarding example (b), it is significant that the subspace lattices of interest to us are actually *compact* (see Theorem 5 and Proposition 7.1).

By a *representation* of a *-semigroup X we mean a *-homomorphism $x \mapsto U(x)$ of X into the *-semigroup of all bounded operators on a *separable* Hilbert space $\mathscr{H}$ such that

$$\sup_x \| U(x) \| < \infty$$

$$x \to \langle U(x)\xi, \eta \rangle$$

is continuous for all $\xi, \eta \in \mathscr{H}$, and which is *nondegenerate* in the sense that the only vector annihilated by all operators $U(x)$, $x \in X$, is $\xi = 0$. This is equivalent to the condition $U(e) = 1$.

It is easy to see that a representation U of X is *-strongly continuous (i.e., if $x_n \to x$, then $U(x_n) \to U(x)$ and $U(x_n)^* \to U(x)^*$ in the strong operator topology), and that we have $\| U(x) \| \leqslant 1$ for all x. The latter follows from the fact that if ϕ is a bounded complex-valued function on X satisfying

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j}\, \phi(x_j^* x_i) \geqslant 0 \tag{3.1}$$

for all $n \geqslant 1$, $x_1, \ldots, x_n \in X$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, then $|\phi(x)| \leqslant \phi(e)$. This is seen as follows. For each $x \in X$, the above condition for $n=2$ implies that the 2×2 matrix

$$\begin{vmatrix} \phi(e) & \phi(x^*) \\ \phi(x) & \phi(x^*x) \end{vmatrix}$$

is positive self-adjoint, hence its determinant is nonnegative and so

$$\begin{aligned} |\phi(x)| &\leqslant \phi(x^*x)^{1/2}\,\phi(e)^{1/2} \\ &\leqslant \phi((x^*x)^2)^{1/4}\,\phi(e)^{1/2+1/4} \\ &\;\vdots \\ &\leqslant \phi((x^*x)^{2^{n-1}})^{1/2^n}\,\phi(e)^{1/2+\cdots+1/2^n}. \end{aligned}$$

Since $|\phi(y)| \leqslant M < \infty$ for all $y \in X$, the above implies

$$|\phi(x)| \leqslant M^{1/2^n}\phi(e)^{1/2+\cdots+1/2^n}$$

for every $n \geqslant 1$, and we may take the limit on n to conclude that $|\phi(x)| \leqslant \phi(e)$.

Let $\mathscr{A}$ be a C^*-algebra of operators on a separable Hilbert space $\mathscr{H}$. Recall [7] that a *localizing map* for $\mathscr{A}$ is a completely positive linear map $\lambda: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ of the form

$$\lambda(A) = \sum_{n=1}^{\infty} E_n A E_n,$$

where $E_1, E_2, \ldots,$ is a sequence of positive finite rank operators satisfying

$$\sum_{n=1}^{\infty} E_n^2 = 1,$$

and which has the further property that $A - \lambda(A)$ is compact for every $A \in \mathscr{A}$.

THEOREM 2. *Let U be a representation of the $*$-semigroup X and let $\mathscr{A}$ be the C^*-algebra generated by $\{U(x): x \in X\}$.*

Then there is a sequence $\lambda_1, \lambda_2, \ldots,$ of localizing maps for $\mathscr{A}$ such that the functions $F_n: X \to K$ defined by

$$F_n(x) = U(x) - \lambda_n(U(x))$$

are norm-continuous and tend to zero uniformly on compact subsets of X as $n \to \infty$.

Proof. Let $\varepsilon > 0$ and let K be a compact subset of X. We will construct a localizing map λ for $\mathscr{A}$ such that

$$x \mapsto U(x) - \lambda(U(x)) \text{ is norm-continuous} \tag{3.2}$$

and

$$\sup_{x \in K} \| U(x) - \lambda(U(x)) \| \leqslant \varepsilon. \tag{3.3}$$

Note that the theorem follows from (3.2) and (3.3). Indeed, since X is locally compact and second countable, we may find an increasing sequence G_n of open sets in X whose closures are compact, such that $X = \bigcup_n G_n$. Letting $K_n = \overline{G_n}$, it follows that every compact subset of X is contained in some K_n, and hence if we choose λ_n so as to satisfy (3.2) and (3.3) for $K = K_n$ and $\varepsilon = 1/n$, then $\lambda_1, \lambda_2, \ldots,$ is the required localizing sequence.

So fix $\varepsilon > 0$ and K. Choose increasing compact sets $K_n = \overline{G_n}$ as in the preceding paragraph, so that K_1 contains K. By the lemma on p. 322 of [7] we can find a decreasing sequence of positive numbers $\delta_1 \geqslant \delta_2 \geqslant \cdots$ with the property that if A, F are two operators in the unit ball of $\mathscr{L}(\mathscr{H})$ which satisfy $F \geqslant 0$ and $\|FA - AF\| \leqslant \delta_n$, then

$$\|F^{1/2}A - AF^{1/2}\| \leqslant \varepsilon/2^{n+1}.$$

Applying Theorem 1 to the ideal of all finite rank operators in the C^*-algebra $\mathscr{A} + \mathscr{K}$, we may obtain a sequence $F_1 \leqslant F_2 \leqslant \cdots$ of positive finite rank operators such that $F_j \to 1$ strongly and

$$\lim_{j \to \infty} \sup_{x \in K_n} \|F_j U(x) - U(x) F_j\| = 0;$$

for every $n = 1, 2, \ldots$. By passing to a subsequence if necessary we can assume that

$$\sup_{x \in K_{n+1}} \|F_j U(x) - U(x) F_j\| \leqslant \delta_n/2$$

for all $j \leqslant n$. It follows that

$$\|F_1 U(x) - U(x) F_1\| \leqslant \delta_1$$

for all $x \in K_1$ and, for $n \geqslant 2$, we have

$$\begin{aligned} \sup_{x \in K_n} \|(F_n - F_{n-1}) U(x) - U(x)(F_n - F_{n-1})\| &\leqslant \tfrac{1}{2}\delta_n + \tfrac{1}{2}\delta_{n-1} \\ &\leqslant \delta_{n-1}. \end{aligned}$$

Define $E_1 = F_1^{1/2}$ and $E_n = (F_n - F_{n-1})^{1/2}$ for $n \geqslant 2$. By the choice of δ_n we have

$$\sup_{x \in K_n} \|E_n U(x) - U(x) E_n\| \leqslant \varepsilon/2^n. \tag{3.4}$$

Clearly E_n is a positive finite rank operator and we have

$$\sum_{n=1}^{\infty} E_n^2 = F_1 + \sum_{n=2}^{\infty} (F_n - F_{n-1}) = 1,$$

the sums converging in the strong operator topology. Thus we can define a completely positive linear map λ of $\mathscr{L}(\mathscr{H})$ into itself by

$$\lambda(A) = \sum_{n=1}^{\infty} E_n A E_n.$$

We can write $A - \lambda(A)$ as follows:

$$A - \lambda(A) = \sum_{n=1}^{\infty} (AE_n - E_n A) E_n,$$

where the right side is interpreted as a strongly convergent infinite series. Thus for each n we have

$$U(x) - \lambda(U(x)) = \sum_{k=1}^{n} (U(x) E_k - E_k U(x)) E_k$$
$$+ \sum_{k=n+1}^{\infty} (U(x) E_k - E_k U(x)) E_k.$$

Note that when x belongs to K_n, the second term is a small compact operator; indeed, each operator $(U(x) E_k - E_k U(x)) E_k$ is of finite rank and we have

$$\sum_{k=n+1}^{\infty} \sup_{x \in K_n} \| U(x) E_k - E_k U(x)\| \leqslant \sum_{k=n+1}^{\infty} \varepsilon/2^k \leqslant \varepsilon/2^n.$$

Moreover, since $x \longmapsto U(x)$ is *-strongly continuous, the function

$$f_n(x) = \sum_{k=1}^{n} (U(x) E_k - E_k U(x)) E_k$$

is a norm-continuous function from X into the Banach spaces $\mathscr{K}$ of all compact operators. The above estimate implies that

$$\sup_{x \in K_n} \| U(x) - \lambda(U(x)) - f_n(x)\| \leqslant \varepsilon/2^m$$

for all $m \leqslant n$, and hence $f_m(x)$ converges to $U(x) - \lambda(U(x))$ uniformly on compact subsets of X. This proves that $x \mapsto U(x) - \lambda(U(x))$ is a continuous function from X to $\mathscr{K}$.

Finally, since $K \subseteq K_n$ for every n, we may use (3.4) to estimate $U(x) - \lambda(U(x))$ directly for $x \in K$ to obtain

$$\begin{aligned}\sup_{x \in K} \|U(x) - \lambda(U(x))\| &\leqslant \sum_{n=1}^{\infty} \sup_{x \in K} \|U(x)\, E_k - E_n\, U(x)\| \\ &\leqslant \sum_{1}^{\infty} \varepsilon/2^n = \varepsilon,\end{aligned}$$

and the proof is complete. ∎

Here, as in [7], the significance of localizing maps is that they can be dilated to block diagonal maps with convenient properties relative to the representation U. A *block diagonal map* of a unital C^*-algebra A is a unital completely positive linear map $\delta: A \to \mathscr{L}(\mathscr{M})$ which is (unitarily equivalent to) a countable direct sum of maps $\delta_j: A \to \mathscr{L}(\mathscr{M}_j)$ where each $\mathscr{M}_j$ is finite dimensional. Let U be a representation of a $*$-semigroup X, let $\mathscr{A}$ be the C^*-algebra generated by the range of U, and let

$$\rho: \mathscr{A} \to \mathscr{L}(\mathscr{M})$$

be a unital completely positive map. A projection $P \in \mathscr{L}(\mathscr{M})$ will be called *essentially reducing* for $\rho(\mathscr{A})$ if $P\rho(U(x)) - \rho(U(x))P$ is compact for every $x \in X$ and

$$x \mapsto P\rho(U(x)) - \rho(U(x))P$$

is a norm-continuous function. We remark that this is a considerably stronger notion than the corresponding one in [7, p. 334].

The following result asserts that every representation of X is approximately a direct summand of an appropriate block diagonal operator-valued function on X.

COROLLARY. *Let U and $\mathscr{A}$ be as in Theorem 2. Then there is a normal block diagonal map $\delta: \mathscr{L}(\mathscr{H}_U) \to \mathscr{L}(\mathscr{M})$ and an isometry $W: \mathscr{H}_U \to \mathscr{M}$ such that WW^* is an essentially reducing projection for $\delta(\mathscr{A})$ and*

$$x \mapsto WU(x) - \delta(U(x))\, W$$

is a norm continuous function from X to the Banach space $\mathscr{K}(\mathscr{H}_u, \mathscr{M})$ of all compact operators from $\mathscr{H}_U$ to $\mathscr{M}$.

Moreover, if K is a compact set in X and $\varepsilon > 0$, we can arrange that

$$\sup_{x \in K} \| WU(x) - \delta(U(x))W \| \leqslant \varepsilon.$$

Proof. Let L be a compact subset of X which contains K, K^*, and $K^*K = \{y^*x : x, y \in K\}$. We can find a localizing map λ for $\mathscr{A}$ satisfying the condition of Theorem 2, and

$$\sup_{x \in L} \| U(x) - \lambda(U(x)) \| \leqslant \varepsilon^2/3.$$

Write

$$\lambda(A) = \sum_{n=1}^{\infty} E_n A E_n,$$

where the E_n are positive finite rank operators with $\sum E_n^2 = 1$. Let $\mathscr{M}_n$ be the range of E_n and let

$$\delta_n(A) = P_{\mathscr{M}_n} A \,|_{\mathscr{M}_n}$$

be the compression map of $\mathscr{L}(\mathscr{H}_U)$ onto $\mathscr{L}(\mathscr{M})$. Define

$$\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2 \oplus \cdots,$$
$$\delta = \delta_1 \oplus \delta_2 \oplus \cdots,$$

and define an isometry $W : \mathscr{K} \to \mathscr{M}$ by

$$W\xi = (E_1\xi, E_2\xi, \ldots).$$

Since $E_n \delta_n(A) E_n = E_n A E_n$ for every operator A, we have

$$W^*\delta(A)W = \sum_n E_n \delta_n(A) E_n = \lambda(A). \tag{3.5}$$

To prove that $P = WW^*$ is essentially reducing for $\delta(\mathscr{A})$, it is enough to show that the function

$$f(x) = (1 - P)\,\delta(U(x))P \tag{3.6}$$

is a norm-continuous function from X to the compact operators on $\mathscr{M}$ (indeed, $\delta(U(x))P - P\delta(U(x)) = f(x) - f(x^*)^*$). In order to do this we make use of the following elementary fact: if J is a closed ideal in a C^*-algebra B and $f \in B$ is an element such that f^*f is dominated by a positive element of

J, then $f \in J$ [17, 1.5.2, p. 15]. Let B be the C^*-algebra of all bounded $*$-strongly continuous functions $g: X \to \mathscr{L}(\mathscr{M})$ relative to the norm

$$\|g\| = \sup_{x \in X} \|g(x)\|.$$

Let J denote the subalgebra of B consisting of all functions g which take on compact operator values and which are norm-continuous,

$$\lim_{x \to x_0} \|g(x) - g(x_0)\| = 0.$$

J is a closed two-sided ideal in B (see the proof of the lemma of Section 2). The function f defined in (3.6) clearly belongs to B. Thus it suffices to show that f^*f is dominated by a positive element of J. We have

$$\begin{aligned} f(x)^* f(x) &= P\delta(U(x))^* (1-P)\,\delta(U(x))P \\ &= P\delta(U(x))^*\,\delta(U(x))P - P\delta(U(x^*))\,P\delta(U(x))P. \end{aligned}$$

By the Schwarz inequality for completely positive maps, the first term on the right is dominated by $P\delta(U(x)^*\,U(x))P = P\delta(U(x^*x))P$. Since $P\delta(U(y))P = W\lambda(U(y))\,W^*$ by (3.5), we have

$$f(x)^* f(x) \leqslant W(\lambda(U(x^*x)) - \lambda(U(x^*))\,\lambda(U(x)))\,W^*,$$

and the term on the right is clearly in J because

$$y \mapsto \lambda(U(y)) - U(y)$$

is a norm-continuous function from X to the compacts and U is a $*$-homomorphism of X into $\mathscr{L}(\mathscr{H}_U)$.

Now from (3.5) we have $W\lambda(U(x)) = P\delta(U(x))W$, and hence

$$\begin{aligned} WU(x) - \delta(U(x))W &= W\lambda(U(x)) - \delta(U(x))W + W(U(x) - \lambda(U(x))) \\ &= P\delta(U(x))W - \delta(U(x))W + W(U(x)) - \lambda(U(x))) \\ &= (P\delta(U(x)) - \delta(U(x))P)W + W(U(x)) - \lambda(U(x))), \end{aligned}$$

and the right side is a sum of two norm-continuous functions from X to $\mathscr{K}(\mathscr{H}_U, \mathscr{M})$.

Finally, the inequality established above implies

$$\begin{aligned}\|f(x)\|^2 &= \|f(x)^* f(x)\| \leqslant \|\lambda(U(x^*x)) - \lambda(U(x))^* \lambda(U(x))\| \\ &\leqslant \|\lambda(U(x^*x)) - U(x^*x)\| + \|U(x)^* U(x) - \lambda(U(x))^* \lambda(U(x))\| \\ &\leqslant \|\lambda(U(x^*x)) - U(x^*x)\| + 2\,\|U(x) - \lambda(U(x))\| \\ &\leqslant \varepsilon^2\end{aligned}$$

for every $x \in K$. ∎

4. The Absorption Principle

Let X be a *-semigroup in the sense of the preceding section and let U, V be two representations of X, acting perhaps on different separable Hilbert spaces $\mathscr{H}_U$ and $\mathscr{H}_V$. We will say that U and V are *approximately equivalent* (written $U \sim V$) if, for every compact subset $K \subseteq X$ and every positive real number ε, there is a unitary operator $W: \mathscr{H}_U \to \mathscr{H}_V$ such that

$$\sup_{x \in K} \|WU(x) - V(x)W\| \leqslant \varepsilon.$$

We will see presently that this relation actually implies a stronger version of itself (at least in the cases of interest to us), and the latter is precisely a generalization of Voiculescu's notion of approximate equivalence [7, 18].

We will also say that a representation V is *absorbed* by a representation U if $U \oplus V \sim U$, where the direct sum $U \oplus V$ of two representations has its traditional meaning. This is clearly related to the concept of a neutral element in the theory of C^*-algebraic extensions. But here there is no single underlying C^*-algebra in evidence, and the various C^*-algebras that do appear (for example, the C^*-algebra generated by the operators in the range of a given representation) are inseparable. Another difference from extension theory is that this relation makes no reference to compact operators.

The purpose of this section is to give a characterization of the relation $U \oplus V \sim U$ in terms of criteria that can be checked; these criteria will be used to solve specific problems in later sections.

We begin by collecting a few immediate consequences of the definitions. Since $\sim$ is obviously an equivalence relation, we see by a simple induction that $U \oplus V \sim U$ implies $U \oplus n \cdot V \sim U$ for every $n \geqslant 1$, where $n \cdot V$ denotes the direct sum of n copies of V.

Let M_n denote the C^*-algebra of all $n \times n$ complex matrices, which we consider to be the operator algebra $\mathscr{L}(\mathbb{C}^n)$. For each n, we choose a finite subgroup G_n of the unitary group of M_n which spans M_n linearly. *G_n will be fixed throughout the remainder of this paper.* We may regard G_n as a (discrete) *-semigroup, and so for any *-semigroup X we can form the Cartesian product of *-semigroups

$$X_n = G_n \times X, \qquad n = 1, 2, \ldots.$$

Every representation U of X on a Hilbert space $\mathscr{H}$ gives rise to a representation U_n of $G_n \times X$ on the Hilbert space $\mathbb{C}^n \otimes \mathscr{H}$ in a natural way

$$U_n(u, x) = u \otimes U(x),$$

$u \in G_n$, $x \in X$. It follows that for any two representations U, V of X, we have

$$U \sim W \text{ implies } U_n \sim W_n \text{ for every } n \geqslant 1, \tag{4.1}$$

and therefore

$$U \oplus V \sim U \text{ implies } U_n \oplus V_n \sim U_n \text{ for every } n \geqslant 1, \tag{4.2}$$

because $U_n \oplus V_n$ is unitarily equivalent to $(U \oplus V)_n$,

We now want to derive a consequence of the relation $U \oplus V \sim U$ which is somewhat less obvious. It is convenient to introduce some terminology.

DEFINITION 4.3. V is said to be subordinate to U if, for every normal state ρ of $\mathscr{L}(\mathscr{H}_V)$, there is a sequence ξ_n of unit vectors in $\mathscr{H}_U$ such that

(i) $\xi_n \to 0$ in the weak topology of $\mathscr{H}_U$, and

(ii) for every compact set $K \subseteq X$,

$$\sup_{x \in K} |\langle U(x)\, \xi_n, \xi_n \rangle - \rho(V(x))|$$

tends to zero as $n \to \infty$.

Remarks. Condition (i) asserts that the vectors ξ_n are going to infinity, while of course (ii) asserts that the sequence of functions

$$f_n(x) = \langle U(x)\, \xi_n, \xi_n \rangle$$

converges to $f(x) = \rho(V(x))$ uniformly on compact subsets of X.

It is significant that if V is subordinate to U, then so is the direct sum $\tilde{V} = V \oplus V \oplus \cdots$ of a *countably infinite* number of copies of V. Indeed, if ρ is a normal state of $\mathscr{L}(\mathscr{H}_{\tilde{v}})$, then we may define a normal state ρ_0 of $\mathscr{L}(\mathscr{H}_v)$ by

$$\rho_0(A) = \rho(A \oplus A \oplus \cdots);$$

since we can approximate the function $x \mapsto \rho_0(V(x))$ in the sense of (i) and (ii), the same is true of $\rho(\tilde{V}(x)) = \rho_0(V(x))$.

PROPOSITION 4.4. *Let U and V be representations of X. If $U \oplus V \sim U$, then V_n is subordinate to U_n for every $n = 1, 2, \ldots$.*

Proof. By (4.2), it suffices to prove this proposition for the special case $n = 1$. Let ρ be a normal state of $\mathscr{L}(\mathscr{H}_V)$, choose $\varepsilon > 0$ and a compact set $K \subseteq X$. It suffices to show that for every finite-dimensional projection P in $\mathscr{L}(\mathscr{H}_U)$, there is a unit vector $\xi \in \mathscr{H}_U$ satisfying

$$\|P\xi\| \leqslant \varepsilon \tag{4.5i}$$

and

$$\sup_{x \in K} |\langle U(x)\xi, \xi\rangle - \rho(V(x))| \leqslant 2\varepsilon. \tag{4.5ii}$$

Choose N large enough that $N\varepsilon^2$ exceeds the dimension of P. We claim that it is enough to exhibit N mutually orthogonal unit vectors $\xi_1, \ldots, \xi_N$ in $\mathscr{H}_U$ such that

$$\sup_{x \in K} |\langle U(x)\, \xi_j, \xi_j\rangle - \rho(V(x))| \leqslant 2\varepsilon,$$

for every $j = 1, 2, \ldots, N$. Indeed, assuming that such vectors $\xi_1, \ldots, \xi_N$ exist, notice that some one of them must satisfy $\|P\xi_j\| \leqslant \varepsilon$. For if $\|P\xi_j\| > \varepsilon$ for every j, then

$$\dim P \geqslant \sum_{j=1}^{n} \langle P\xi_j, \xi_j\rangle = \sum_{1}^{N} \|P\xi_j\|^2 > N\varepsilon^2,$$

contradicting the choice of N. Any ξ_j with $\|P\xi_j\| \leqslant \varepsilon$ has the required properties.

$\xi_1, \ldots, \xi_N$ are constructed as follows: We can express ρ in the form

$$\rho(A) = \sum_{k=1}^{\infty} \langle Au_k, u_k\rangle, \qquad A \in \mathscr{L}(\mathscr{H}_V),$$

where the vector u_k satisfy $\sum \|u_k\|^2 = 1$ and where $u_1 \neq 0$. So if we put

$$\rho_n(A) = \left(\sum_{k=1}^{n} \|u_k\|^2\right)^{-1} \sum_{k=1}^{n} \langle Au_k, u_k\rangle,$$

then we can choose n large enough that $\|\rho - \rho_n\| \leqslant \varepsilon$. Let $n \cdot \mathscr{H}_V$ be the direct sum of n copies of $\mathscr{H}_V$ and, for every $A \in \mathscr{L}(\mathscr{H}_V)$, write

$$n \cdot A = A \oplus \cdots \oplus A \in \mathscr{L}(n \cdot \mathscr{H}_V).$$

Then we can express ρ_n as a vector state as follows:

$$\rho_n(A) = \langle n \cdot Av, v \rangle,$$

where v is the unit vector in $n \cdot \mathscr{H}_V$ defined by

$$v = \left(\sum_1^n \|u_k\|^2 \right)^{-1/2} u_1 \oplus u_2 \oplus \cdots \oplus u_n.$$

We consider $\tilde{\mathscr{H}} = \mathscr{H}_U \oplus N \cdot n \cdot \mathscr{H}_v$ to be the direct sum of $\mathscr{H}_U$ with N copies of $n \cdot \mathscr{H}_V$,

$$\tilde{\mathscr{H}} = \mathscr{H}_U \oplus n \cdot \mathscr{H}_V \oplus n \cdot \mathscr{H}_V \oplus \cdots \oplus n \cdot \mathscr{H}_V.$$

Define N mutually orthogonal unit vectors $\zeta_1, ..., \zeta_N$ in $\tilde{\mathscr{H}}$ by

$$\begin{aligned}
\zeta_1 &= 0 \oplus v \oplus 0 \oplus \cdots \oplus 0, \\
\zeta_2 &= 0 \oplus 0 \oplus v \oplus \cdots \oplus 0, \\
&\vdots \\
\zeta_N &= 0 \oplus 0 \oplus 0 \oplus \cdots \oplus v.
\end{aligned}$$

Now since $U \oplus V \sim U$ implies $U \oplus N \cdot n \cdot V \sim U$, there is a unitary operator $W: \tilde{\mathscr{H}} \to \mathscr{H}_U$ such that

$$\sup_{x \in K} \|(U(x) \oplus n \cdot V(x) \oplus \cdots \oplus n \cdot V(x)) - W^* U(x) W\| \leqslant \varepsilon.$$

Let $\xi_j = W\zeta_j$, $1 \leqslant j \leqslant N$. The ξ_j's are mutually orthogonal unit vectors in $\mathscr{H}_U$, and for each j and each $x \in K$ we have

$$\begin{aligned}
&|\langle U(x)\, \xi_j, \xi_j \rangle - \rho(V(x))| \\
&\quad = |\langle W^* U(x)\, W\zeta_j, \zeta_j \rangle - \rho(V(x))| \\
&\quad \leqslant \varepsilon + |\langle (U(x) \oplus n \cdot V(x) \oplus \cdots \oplus n \cdot V(x))\, \zeta_j, \zeta_j \rangle - \rho(V(x))| \\
&\quad = \varepsilon + |\rho_n(V(x)) - \rho(V(x))| \leqslant 2\varepsilon. \quad \blacksquare
\end{aligned}$$

The principal result of this section asserts that the necessary condition of Proposition 4.4. is also sufficient, and in fact we have

THEOREM 3. *For any two representations U, V of X, the following three conditions are equivalent*:

(i) $U \oplus V \sim U$,

(ii) V_n *is subordinate to* U_n *for every* $n \geqslant 1$,

(iii) *for every compact set* $K \subseteq X$ *and* $\varepsilon > 0$, *there is a unitary operator*

$W: \mathscr{H}_U \to \mathscr{H}_U \oplus \mathscr{H}_V$ such that the function $x \mapsto WU(x) - (U(x) \oplus V(x))W$ is compact operator valued, is operator norm continuous, and satisfies

$$\sup_{x \in K} \| WU(x) - (U(x) \oplus V(x))W \| \leqslant \varepsilon.$$

Proof. The assertion (iii) $\Rightarrow$ (i) is trivial, and (i) $\Rightarrow$ (ii) is the content of Proposition 4.4. We preface the proof of (ii) $\Rightarrow$ (iii) with some remarks.

Notice that the hypothesis (ii) is also valid for the inflated representation $\tilde{V} = V \oplus V \oplus \cdots$. Indeed, for each n, the inflation $(V_n)^{\sim}$ of V_n is unitarily equivalent to $(\tilde{V})_n$, the remarks following Definition 4.3 imply that $(V_n)^{\sim}$ is subordinate to U_n for each n, and hence $(\tilde{V})_n$ is subordinate to U_n for every n.

Note also that, in order to prove that (ii) implies (iii), it suffices to show that (ii) implies the following assertion. Let $K \subseteq X$ be compact and let $\varepsilon > 0$. Then there is an operator valued function

$$F: X \to \mathscr{L}(\mathscr{N})$$

and a unitary operator $W: \mathscr{H}_U \to \mathscr{N} \oplus \mathscr{H}_V$ such that the function

$$x \in X \mapsto WU(x) - (F(x) \oplus V(x))W \in \mathscr{L}(\mathscr{H}_U, \mathscr{N} \oplus \mathscr{H}_V) \tag{4.6}$$

is compact operator-valued, norm-continuous, and has operator norm at most ε for x in K. This implies (iii), because if we apply the above to the inflation $\tilde{V}$ of V (by the preceding remarks) and observe that the operator valued function

$$x \mapsto F(x) \oplus \tilde{V}(x)$$

is *unitarily equivalent* to its direct sum with V,

$$x \mapsto F(x) \oplus \tilde{V}(x) \oplus V(x).$$

we obtain (iii).

Assuming now that V_n is subordinate to U_n, for each n, we have

LEMMA 1. *Let M be a finite-dimensional Hilbert space and let $\rho: \mathscr{L}(\mathscr{H}_V) \to \mathscr{L}(\mathscr{M})$ be a normal completely positive unital map. Then there is a sequence of isometries $W_j: \mathscr{M} \to \mathscr{H}_U$ such that*

(i) *$W_j \to 0$ weakly as $j \to \infty$, and*

(ii) $\sup_{x \in K} \| W_j^* U(x) W_j - \rho(V(x)) \| \to 0$

as $j \to \infty$, for every compact set $K \subseteq X$.

Proof. Let n be the dimension of $\mathscr{M}$ and let $e_1, e_2, \ldots, e_n$ be an orthonormal base for $\mathscr{M}$. Define a linear functional σ on the C^*-algebra $M_n \otimes \mathscr{L}(\mathscr{H}_V)$ by

$$\sigma(A) = \sum_{i,j=1}^{n} \langle \rho(A_{ij})\, e_j, e_i \rangle,$$

where A denotes the $n \times n$ matrix (A_{ij}) of operators $A_{ij} \in \mathscr{L}(\mathscr{H}_V)$. σ is a positive linear functional because

$$\mathrm{id} \otimes \rho \colon M_n \otimes \mathscr{L}(\mathscr{H}_V) \to M_n \otimes \mathscr{L}(\mathscr{M})$$

is a positive linear map. Moreover, for every $n \times n$ matrix of complex numbers $a = (a_{ij})$, $a \otimes V(x)$ is the $n \times n$ operator matrix $(a_{ij} V(x))$, and hence

$$\sigma(a \otimes V(x)) = \sum_{i,j=1} a_{ij} \langle \rho(V(x))\, e_j, e_i \rangle.$$

Since V_n is subordinate to U_n, there is a sequence $\xi_1, \xi_2, \ldots$ of vectors in $\mathbb{C}^n \otimes \mathscr{H}_U$, satisfying $\|\xi_j\|^2 = \sigma(1) = n$, such that $\xi_j \to 0$ weakly and such that, for every unitary $a \in G_n$, the sequence of functions

$$\psi_k(x) = \langle a \otimes U(x)\, \xi_k, \xi_k \rangle$$

tends to $\psi(x) = \sigma(a \otimes V(x))$ uniformly on compact subsets of X. If we express $\mathbb{C}^n \otimes \mathscr{H}_U$ as a direct sum of n copies of $\mathscr{H}_U$ and write out each vector ξ_k in components

$$\xi_k = \xi_k(1) \oplus \xi_k(2) \oplus \cdots \oplus \xi_k(n),$$

then $\langle a \otimes U(x)\, \xi_k, \xi_k \rangle$ becomes

$$\sum_{i,j} a_{ij} \langle U(x)\, \xi_k(j), \xi_k(i) \rangle.$$

We now define a sequence $W_1, W_2, \ldots,$ of linear operators from $\mathscr{M}$ to $\mathscr{H}_U$ by

$$W_k e_j = \xi_k(j), \qquad 1 \leqslant j \leqslant n.$$

The W_i's are not necessarily isometries, but since the vectors ξ_k tend weakly to zero in $\mathbb{C}^n \otimes \mathscr{H}_U$, the sequence of operators W_k tends to zero in the weak operator topology of $\mathscr{L}(\mathscr{M}, \mathscr{H}_U)$. Now since G_n spans M_n linearly, the preceding assertions persist if we replace $a \in G_n$ by an arbitrary element $a \in M_n$. Taking a to be a matrix unit, we conclude that for every i, j between 1 and n,

$$\langle W_k^* U(x)\, W_k e_j, e_i \rangle = \langle U(x)\, \xi_k(i), \xi_k(j) \rangle$$

tends uniformly on compact subsets of X to the limit function

$$\langle \rho(V(x))\, e_j, e_i \rangle.$$

Since $e_1, \ldots, e_n$ spans $\mathcal{M}$ and since the weak operator topology agrees with the norm topology of $\mathcal{L}(\mathcal{M})$, we may assert that

$$\lim_{x \to \infty} \| W_k^* U(x)\, W_k - \rho(V(x)) \| = 0$$

uniformly on compact subsets of X.

The W_k's may be made into isometries with the same properties. Indeed, taking $x = e$ (the unit of X) in the preceding statement, we see that $\| W_k^* W_k - 1 \| \to 0$. So for k sufficiently large we can define isometries $\tilde{W}_k : \mathcal{M} \to \mathcal{H}_U$ by

$$\tilde{W}_k = W_k (W_k^* W_k)^{-1/2}.$$

The sequence $\tilde{W}_k$ has both required properties (i) and (ii). ∎

We remark that if $\mathcal{N}$ is a given finite-dimensional subspace of $\mathcal{H}_U$, then the sequence of isometries $W_k : \mathcal{M} \to \mathcal{H}_U$ can all be chosen so as to have range orthogonal to $\mathcal{N}$. Indeed, letting P be the orthogonal projection of $\mathcal{H}_U$ onto $\mathcal{N}$, we have (for the sequence W_k constructed in Lemma 1)

$$P W_k \to 0$$

in the weak operator topology of $\mathcal{L}(\mathcal{M}, \mathcal{N})$. Since $\mathcal{M}$ and $\mathcal{N}$ are both finite dimensional this entails

$$\| P W_k \| \to 0,$$

and hence for k sufficiently large we can define new isometries $\tilde{W}_k$ from $\mathcal{M}$ to $\mathcal{N}^{\perp}$ by

$$\tilde{W}_k = (1 - P)\, W_k (1 - W_k^* P W_k)^{-1/2}.$$

This new sequence has all of the properties of the original sequence W_k because

$$\lim_{k \to \infty} \| \tilde{W}_k - W_k \| = 0.$$

We can now improve Lemma 1 to cover the case of block diagonal maps (see Section 3), under the same hypothesis on U and V.

LEMMA 2. *Let ρ be a normal unital completely positive map of $\mathcal{L}(\mathcal{H}_V)$ into $\mathcal{L}(\mathcal{M})$, which is block diagonal.*

Then there is an isometry $W: \mathcal{M} \to \mathcal{H}_U$ such that $W^*U(x)W - \rho(V(x))$ is compact for all $x \in X$ and is a norm-continuous operator valued function on X.

Moreover, if K is a given compact subset of X and $\varepsilon > 0$, we can arrange that

$$\| W^*U(x)W - \rho(V(x))\| \leqslant \varepsilon$$

for each $x \in K$.

Proof. This argument runs parallel to the proof of Theorem 4 in [7]. By hypothesis, we have a decomposition

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots$$

of $\mathcal{M}$ into finite-dimensional subspaces $\mathcal{M}_i$ which induces a decomposition

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots$$

of ρ into finite-dimensional normal unital maps ρ_i.

Let $K_1 \subseteq K_2 \subseteq \cdots$ be compact sets such that $K_i^* = K_i$, K_1 contains the given set K, and the interiors of the sets K_i cover X. We will construct a sequence of isometries

$$W_k: \mathcal{M}_k \to \mathcal{H}_U$$

such that

(a) $\operatorname{ran} W_m \perp \operatorname{ran} W_n$ for all $1 \leqslant m < n$

(b) $\| W_n^* U(x)\, W_m\| \leqslant \varepsilon/2^{m+n}$ for all $x \in K_n$

and all $1 \leqslant m < n$

(c) $\|\rho_n(V(x)) - W_n^* U(x)\, W_n\| \leqslant \varepsilon/2^n$ for all $n \geqslant 1$.

The argument is by induction. By Lemma 1, we obtain an isometry $W: \mathcal{M}_1 \to \mathcal{H}_U$ such that (c) is satisfied. Assume that $W_1, \ldots, W_{n-1}$ have been defined, and consider the set of vectors $\mathcal{N}_j \subseteq \mathcal{H}_U$ defined by

$$\mathcal{N}_j = \{U(x)\, W_j \xi : x \in K_n, \xi \in \mathcal{M}_j, \|\xi\| \leqslant 1\},$$

$1 \leqslant j \leqslant n-1$. For each j, the image of the unit ball of $\mathcal{M}_j$ under W_j is a (norm) compact subset of $\mathcal{H}_U$; and since by strong continuity of $x \mapsto U(x)$, the set of operators $\{U(x): x \in K_n\}$ is compact in the strong operator topology, $\mathcal{N}_j$ is a norm-compact subset of $\mathcal{H}_U$.

Now if $A_1, A_2, \ldots,$ is any sequence of operators from $\mathscr{M}_n$ to $\mathscr{H}_U$ which converges weakly to zero, then A_k^* converges strongly to zero and moreover the strong convergence to zero is uniform over norm-compact subsets of $\mathscr{H}_U$. Thus we may conclude that

$$\sup_{x \in K_n} \|A_k^* U(x) W_j\| = \sup_{\zeta \in \mathscr{N}_j} \|A_k^* \zeta\| \to 0$$

as $k \to \infty$, for every $j = 1, 2, \ldots, n-1$. Lemma 1, together with the preceding remarks, implies that there is an isometry $W_n: \mathscr{M}_n \to \mathscr{H}_U$ satisfying

$$\sup_{x \in K_n} \| W_k^* U(x) W_k - \rho_k(V(x))\| \leqslant \varepsilon/2^n,$$

and such that

$$\sup_{x \in K_n} \| W_n^* U(x) W_j\| \leqslant \varepsilon/2^{n+j},$$

for every $j = 1, 2, \ldots, n-1$. By the remarks following Lemma 1, we may also assume that the range of W_n is orthogonal to the finite-dimensional subspace

$$W_1 \mathscr{M}_1 + W_2 \mathscr{M}_2 + \cdots + W_{n-1} \mathscr{M}_{n-1},$$

and the induction is complete.

Thus we can define an isometry W from $\mathscr{M}$ into $\mathscr{H}_U$ by requiring that W should agree with W_n on $\mathscr{M}_n$ for every $n = 1, 2, \ldots$.

Now for each $x \in X$ we have a formal decomposition of the operator $\rho(V(x)) - W^* U(x) W$:

$$\begin{aligned}\rho(V(x)) - W^*U(x)W = &\sum_k (\rho_k(V(x)) - W_k^* U(x) W_k) \\ &- \sum_{k<l} W_k^* U(x) W_l - \sum_{k>l} W_k^* U(x) W_l.\end{aligned}$$

But if $x \in K_n$ then

$$\sum_{k=n}^{\infty} \|\rho_k(V(x)) - W_k^* U(x) W_k\| \leqslant \sum_n^{\infty} \varepsilon/2^k \leqslant \varepsilon/2^{n-1},$$

$$\sum_{k>l\geqslant n} \| W_k^* U(x) W_l\| \leqslant \sum_{k>l\geqslant n} \varepsilon/2^{k+l} \leqslant \varepsilon/4^{n-1},$$

and

$$\sum_{n\leqslant k<l} \| W_k^* U(x) W_l\| = \sum_{n\leqslant k<l} \| W_l^* U(x^*) W_k\| \leqslant \varepsilon/4^{n-1}.$$

These estimates show that the function $x \in X \mapsto \rho(V(x)) - W^*U(x)W$ has the form

$$\sum_{p=1}^{\infty} F_p(x),$$

where, for each p, F_p is a norm-continuous function into the finite rank operators on $\mathscr{M}$, for which the series of norms

$$\sum_{p=1}^{\infty} \|F_p(x)\|$$

is uniformly convergent on compact subsets of X. Thus $x \mapsto \rho(V(x)) - W^*U(x)W$ is a norm-continuous function from X to the compact operators on $\mathscr{M}$. Moreover, the same estimates imply that

$$\|\rho(V(x)) - W^*U(x)W\| \leqslant \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,$$

when x belongs to $K \subseteq K_1$. ∎

We turn now to the proof of the required relation (4.6). Let $K \subseteq X$ be a compact set and choose $\varepsilon > 0$. Let L be a compact set which contains K, K^*, and K^*K. By the corollary of Theorem 2 there is a normal block diagonal map

$$\delta\colon \mathscr{L}(\mathscr{H}_V) \to \mathscr{L}(\mathscr{M})$$

and an isometry $W_1\colon \mathscr{M} \to \mathscr{H}_V$ such that the function

$$x \in X \mapsto W_1 U(x) - \delta(V(x))\, W_1$$

is norm-continuous and compact operator valued, which satisfies

$$\| W_1 V(x) - \delta(V(x))\, W_1\| \leqslant \varepsilon^2/6$$

for $x \in L$. The same assertions are valid for the function

$$x \in X \mapsto V(x) - W^*\delta(V(x))W = W^*(WV(x) - \delta(V(x))W).$$

By Lemma 2 above, there is an isometry $W_2\colon \mathscr{M} \to \mathscr{H}_U$ such that

$$x \mapsto W_2^* U(x)\, W_2 - \delta(V(x))$$

is a norm-continuous compact operator-valued function, which satisfies

$$\| W_2^* U(x)\, W_2 - \delta(V(x))\| \leqslant \varepsilon^2/6,$$

for $x \in L$. Put $W = W_2 W_1$. Then W is an isometry from $\mathscr{H}_V$ into $\mathscr{H}_U$, and the function

$$V(x) - W^*U(x)W = V(x) - W_1^*\delta(V(x))\, W_1 \\ + W_1^*(\delta(V(x)) - W_2^*U(x)\, W_2)\, W_1$$

is norm-continuous, compact operator valued, and satisfies

$$\| V(x) - W^*U(x)W\| \leqslant \varepsilon^2/3$$

for $x \in L$. Let $P = WW^*$ be the range projection of W. We claim that P is essentially reducing for U in the sense that $PU(x) - U(x)P$ is compact for all $x \in X$ and moves continuously in the operator norm. Indeed, putting

$$f(x) = (1 - P)\, U(x)P,$$

we see that the function

$$\begin{aligned} f(x)^* f(x) &= PU(x^*x)P - PU(x)^*\, PU(x)P \\ &= W(W^*U(x^*x)W - V(x^*x))\, W^* \\ &\quad + WV(x)^*\, (V(x) - W^*U(x)W)\, W^* \\ &\quad + W(V(x) - W^*U(x)W)^*\, W^*U(x)P \end{aligned}$$

is compact operator valued and norm-continuous, so we can employ the ideal-theoretic device used in the proof of the corollary of Theorem 2 to conclude that $f(X) \subseteq \mathscr{K}(\mathscr{H}_U)$ and f is norm-continuous. If $x \in K$, we see from the preceding formula that

$$\|f(x)\|^2 = \|f(x)^* f(x)\| \leqslant 3 \sup_{y \in L} \| W^*U(y)W - V(y)\| \leqslant \varepsilon^2,$$

and hence $\| U(x)P - PU(x)\| \leqslant \varepsilon$ for $x \in K$.

Let $\mathscr{N} = (1 - P)\mathscr{H}_U$, and define an operator function $F\colon X \to \mathscr{L}(\mathscr{N})$ by

$$F(x) = (1 - P)\, U(x)\,|_{\mathscr{N}}.$$

We have to show that $F \oplus V$ is "approximately" unitarily equivalent to U. But the direct sum of operators

$$1 \oplus W\colon \mathscr{N} \oplus \mathscr{H}_V \to \mathscr{N} \oplus P\mathscr{H}_U = \mathscr{H}_U$$

is unitary, and we have

$$\begin{aligned} &(1 \oplus W)(F(x) \oplus V(x)) - U(x)(1 \oplus W) \\ &\quad = ((1 - P)\, U(x)(1 - P) - U(x)(1 - P)) + WV(x) - U(x)W \\ &\quad = P(U(x)P - PU(x)) - (WV(x) - U(x)W). \end{aligned}$$

The right side is clearly a norm-continuous map of X into compact operators whose norm over K is at most 2ε, and the proof is complete. ∎

In order to illustrate how one applies Theorem 3, we use it to deduce the result from which Theorem 3 has itself evolved.

COROLLARY (Voiculescu's theorem [18]). *Let $\mathscr{A}$ be a unital separable C^*-algebra of operators and let π be a nondegenerate representation of $\mathscr{A}$ which annihilates all compact operators in $\mathscr{A}$.*

Then there is a sequence of unitary operators U_n such that $U_n(A \oplus \pi(A)) - AU_n$ is compact for every n and

$$\lim_{n\to\infty} \| U_n(A \oplus \pi(A)) - AU_n \| = 0,$$

for every $A \in \mathscr{A}$.

Proof. By replacing $\mathscr{A}$ with $\mathscr{A} + \mathscr{K}$ and noting the isomorphism of $\mathscr{A}/\mathscr{A} \cap \mathscr{K}$ with $(\mathscr{A} + \mathscr{K})/\mathscr{K}$, we may assume that $\mathscr{A}$ contains all compact operators. Let X be a countable norm-dense subgroup of the unitary group of $\mathscr{A}$, considered as a discrete $*$-semigroup. For $x \in X$, define

$$U(x) = x, \qquad V(x) = \pi(x).$$

We have to show that $U \oplus V \sim U$. By Theorem 3, it is enough to show that V_n is subordinate to U_n for every $n \geqslant 1$.

Fix $n \geqslant 1$ and let ρ be a normal state of $\mathscr{L}(\mathbb{C}^n \otimes \mathscr{H}_\pi)$. Considering $M_n \otimes \mathscr{A}$ as a C^*-algebra of operators on $\mathbb{C}^n \otimes \mathscr{H}_{\mathscr{A}}$, we have a nondegenerate C^*-algebraic representation

$$\mathrm{id}_n \otimes \pi: \mathscr{M}_n \otimes \mathscr{A} \to \mathscr{L}(\mathbb{C}^n \otimes \mathscr{H}_\pi),$$

such that

$$V_n(a \otimes x) = \mathrm{id}_n \otimes \pi(a \otimes x)$$

whenever $a \in G_n$ and $x \in X$. $\mathrm{id}_n \otimes \pi$ annihilates all compact operators in $\mathscr{L}(\mathbb{C}^n \otimes \mathscr{H}_{\mathscr{A}})$, and therefore

$$\sigma(b) = \rho(\mathrm{id}_n \otimes \pi(b))$$

is a state of $M_n \otimes \mathscr{A}$ which is null on compact operators. A theorem of Glimm [10, 11.2.1] (plus separability) implies that there is a sequence ξ_n of unit vectors in $\mathbb{C}^n \otimes \mathscr{H}_{\mathscr{A}}$ such that the associated vector states ω_{ξ_n} converge to σ in the weak $*$-topology of $(M_n \otimes \mathscr{A})'$. The ξ_n's must necessarily converge weakly to zero because σ annihilates every one-dimensional projection. In particular,

$$\lim_{n\to\infty} \langle a\otimes x\xi_n, \xi_n\rangle = \rho(a\otimes \pi(x))$$

for every $(a, x)\in G_n\times X$, and this proves that V_n is subordinate to U_n. ∎

5. Perturbations of Group Representations

A familiar theorem of Weyl and von Neumann implies that if A and B are separably acting bounded self-adjoint operators having the same spectrum and no eigenvalues of finite multiplicity, then there is a sequence of unitary operators W_n such that $W_nA - BW_n$ is compact for all n and

$$\lim_{n\to\infty} \|W_nA - BW_n\| = 0.$$

If A and B are unbounded self-adjoint operators, then one can make sense out of these considerations by regarding a self-adjoint operator A as the generator of a one-parameter unitary group

$$U_t = e^{itA}, \qquad t\in\mathbb{R}.$$

The spectrum of A can then be defined as the spectrum of the group U [5, Definition 2.2] that is, $\mathrm{sp}(U)$ is the hull of the ideal of all functions $f\in L^1(\mathbb{R})$ such that

$$U_f = \int_{-\infty}^{\infty} f(t)\, U_t\, dt = 0.$$

Similarly, the essential spectrum $\mathrm{sp}_e(U)$ of U is the hull of the ideal of all $f\in L^1(\mathbb{R})$ for which U_f is compact. To say that A has no eigenvalues of finite multiplicity is equivalent to the condition $\mathrm{sp}(U) = \mathrm{sp}_e(U)$. If V is another one-parameter group satisfying $\mathrm{sp}(V) = \mathrm{sp}_e(V) = \mathrm{sp}(U)$, then by a consequence of Voiculescu's theorem (see [7, Theorem 5]) there is a sequence of unitary operators W_n such that $W_nU_f - V_fW_n$ is compact for all f in $L^1(\mathbb{R})$ and

$$\lim_{n\to\infty} \|W_nU_f - V_fW_n\| = 0$$

for all f. These conditions do not imply that the operators $W_nU_x - V_xW_n$ are compact or small in norm for $x\in\mathbb{R}$. Indeed, if we choose a sequence f_k of integrable functions which approximates the delta function at the point

$x \in \mathbb{R}$, then the operators $W_n U_{f_k} - V_{f_k} W_n$ converge to $W_n U_x - V_x W_n$ in the strong operator topology as $k \to \infty$, but they do not converge in norm and there is no reason to expect $\| W_n U_x - V_x W_n \|$ to be small.

Nevertheless, natural situations do occur in which one requires information about the groups U_x, V_x and not the smeared operators U_f, V_f. For example, one might want to relate the *-automorphism group $\alpha_t(S) = U_t S U_t^*$ of $\mathscr{L}(\mathscr{H}_U)$ to the corresponding *-automorphism group $\beta_t(S) = V_t S V_t^*$ of $\mathscr{L}(\mathscr{H}_V)$. The preceding assertions about the smeared operators give no information that is useful in relating α to β.

Suppose, however, that we know that there is a sequence W_n of unitaries such that

$$\sup_{|x| \leqslant M} \| W_n U_x - V_x W_n \| \to 0$$

as $n \to \infty$, for every $M > 0$. Then it is easy to deduce that the sequence of *-isomorphisms

$$\theta_n \colon T \in \mathscr{L}(\mathscr{H}_U) \to W_n T W_n^* \in \mathscr{L}(\mathscr{H}_V)$$

is an "approximate" conjugacy of two groups α and β in the following rather strong sense:

$$\lim_{n \to \infty} \| \beta_t \circ \theta_n - \theta_n \circ \alpha_t \| = 0$$

uniformly on compact t-subsets of $\mathbb{R}$.

The purpose of this section is to present a consequence of Theorem 3 which provides a basis for approaching problems like the above concerning perturbations of automorphism groups.

Let G be a second countable locally compact group and let U be a strongly continuous unitary representation of G. Let $\operatorname{Prim}(G)$ denote the primitive ideal space of $C^*(G)$. For every closed ideal J in $C^*(G)$, the hull of J is a closed subset of $\operatorname{Prim}(G)$. Let U be a strongly continuous unitary representation of G (always on a separable Hilbert space). We define the *spectrum* $\operatorname{sp}(U)$ of U and the *essential spectrum* $\operatorname{sp}_e(U)$ of U to be the respective hulls of the ideals

$$\{a \in C^*(G) \colon \pi(a) = 0\},$$

and

$$\{a \in C^*(G) \colon \pi(a) \text{ is compact}\},$$

where π is the unique extension to $C^*(G)$ of the representation $f \in L^1(G) \mapsto U_f$ of $L^1(G)$ associated with U.

THEOREM 4. *Let U and V be two unitary representations of G such that*

$$\operatorname{sp}(U) = \operatorname{sp}_e(U) = \operatorname{sp}_e(V) = \operatorname{sp}(V).$$

Then U and V are approximately equivalent in the sense of Section 4.

We require the following bit of lore from the theory of functions of positive type [10, Théorème 13.5.2].

LEMMA. *Let ψ_0, ψ_1,..., be a sequence of continuous functions of positive type on G such that*

$$\lim_{n \to \infty} \psi_n(e) = \psi_0(e)$$

and

$$\lim_{n \to \infty} \int_G f(x)\, \psi_n(x)\, dx = \int_G f(x)\, \psi_0(x)\, dx,$$

for every $f \in L^1(G)$. Then ψ_n converges to ψ_0 uniformly on compact subsets of G.

Proof of Theorem 4. We need only prove that $U \oplus V \sim U$, for by symmetry we will also have $V \oplus U \sim V$, and hence $U \sim U \oplus V \sim V \oplus U \sim V$. By Theorem 3, it is enough to show that V_n is subordinate to U_n for every $n \geqslant 1$.

Fix n, and let ρ be a normal state of $\mathscr{L}(\mathbb{C}^n \otimes \mathscr{H}_V)$. Let $\mathscr{A}_0$ be the C^*-algebra generated by the set of operators

$$U_f = \int_G f(x)\, U_x\, dx, \qquad f \in L^1(G).$$

The hypothesis $\operatorname{sp}(U) = \operatorname{sp}_e(U)$ is equivalent to the asseration that $\mathscr{A}_0$ should contain no compact operators. Similarly, $\operatorname{sp}(U) = \operatorname{sp}(V)$ is the condition that the representations of $C^*(G)$ determined by U and V should have the same kernel. Therefore we must have

$$\|U_f\| = \|V_f\| \qquad \text{for all} \quad f \in L^1(G).$$

In general, $\mathscr{A}_0$ will not contain an identity. However, since $\mathscr{A}_0$ has trivial null space its unital extension

$$\mathscr{A} = \mathscr{A}_0 + \mathbb{C} \cdot 1$$

also contains no nontrivial compact operators.

We consider $M_n \otimes \mathscr{A}$ to be an operator algebra on $\mathbb{C}^n \otimes \mathscr{H}_U$ in the obvious way, and this C^*-algebra contains no nontrivial compact operators. Thus we can define a *-representation

$$\pi: M_n \otimes \mathscr{A} + \mathscr{K} \to \mathscr{L}(\mathbb{C}^n \otimes \mathscr{H}_V)$$

by

$$\pi(u \otimes U_f + K) = u \otimes V_f,$$

for $f \in L^1(G)$, $u \in M_n$, $K \in \mathscr{K}$. Composing π with ρ, we obtain a state $\sigma = \rho \circ \pi$ of the separable C^*-algebra $M_n \otimes \mathscr{A} + \mathscr{K}$ which annihilates the compact operators. By Glimm's lemma [10, 11.2.1] there is a sequence of unit vectors $\xi_k \in \mathbb{C}^n \otimes \mathscr{H}_U$ such that

$$\sigma(\pi(B)) = \lim_{k \to \infty} \langle B\xi_k, \xi_k \rangle$$

for every $B \in M_n \otimes \mathscr{A} + \mathscr{K}$.

Since σ annihilates every rank one operator, we see that the sequence ξ_k must converge weakly to 0. Moreover, we claim that for every $a \in M_n$ we have

$$\lim_{k \to \infty} \langle a \otimes U_x \xi_k, \xi_k \rangle = \rho(a \otimes V_x) \tag{5.1}$$

uniformly on compact subsets of G. Note that this implies that V_n is subordinate to U_n. Since M_n is spanned by its positive elements, it suffices to prove (5.1) for $a \geqslant 0$. Put

$$\psi_0 = \rho(a \otimes V_x),$$
$$\psi_k(x) = \langle a \otimes U_x \xi_k, \xi_k \rangle, \qquad k = 1, 2, \ldots.$$

$\psi_0, \psi_1, \ldots$ is a sequence of continuous functions of positive type on G (ψ_0 is continuous because ρ is normal), and since $\mathscr{A}$ contains the identity we have

$$\psi_k(e) = \langle a \otimes 1\xi_k, \xi_k \rangle \to \rho(a \otimes 1) = \psi_0(e),$$

as $k \to \infty$. Moreover, for each $f \in L^1(G)$

$$\int_G f(x)\, \psi_k(x)\, dx = \langle a \otimes U_f \xi_k, \xi_k \rangle,$$

which converges to

$$\rho(\pi(a \otimes U_f)) = \rho(a \otimes V_f) = \int_G f(x)\, \psi_0(x)\, dx$$

as $k \to \infty$. The required conclusion (5.1) now follows from the lemma. ∎

6. Continuous Measures and Compact Lattices

We turn now to perturbation theory for commutative subspace lattices. In the last section we were able to make use of a lemma on functions of positive type in showing that a certain representation was subordinate to a second one. In the context of Theorem 7, there was nothing known that was analogous to that lemma, and it has been necessary to develop some new material about these lattices which will allow us to apply Theorem 3. We feel that this material has some interest on its own.

The purpose of this section is to show that many commutative subspace lattices are *compact* in their relative strong (or weak) operator topology. In the next section we will generalize Andersen's theorem [3, 3.5.5] to these lattices.

Throughout this section and the next, (G, Σ) will denote an ordered abelian group. That is, G is a second countable locally compact abelian group and Σ is a *cone* in G, that is, a subset of G satisfying

(i) $\Sigma \cap -\Sigma = \{0\}$,

(ii) $\Sigma + \Sigma \subseteq \Sigma$,

(iii) Σ *is the closure of its interior.*

Two significant examples are

(a) $G = \mathbb{R}^n$,

$\Sigma = \{x \in \mathbb{R}^n : x_1 \geqslant 0, \ldots, x_n \geqslant 0\}$

and the "light cone" in $\mathbb{R}^{n+1}$

(b) $G = \mathbb{R}^n \times \mathbb{R}$,

$\Sigma = \{(x, t) : |x| \leqslant t\}$,

$|x|$ denoting the Euclidean norm of a vector $x \in \mathbb{R}^n$.

We may define a partial order $\leqslant$ in G by $x \leqslant y$ iff $y - x \in \Sigma$. A Borel set $E \subseteq G$ is called *increasing* if, for every $x \in E$ and $y \in G$, $y \geqslant x$ implies $y \in E$. $L(\Sigma)$ will denote the σ-lattice of all increasing Borel sets. Equivalently, $L(\Sigma)$ consists of all Borel sets in G which are invariant under translations by elements of Σ.

Let m be a σ-finite measure on G. Each Borel set $E \subseteq G$ gives rise to a projection operator P_E on the Hilbert space $L^2(G, M)$, namely, multiplication by the characteristic function of E, and we define

$$\mathscr{L}(\Sigma, m) = \{P_E : E \in L(\Sigma)\}.$$

$\mathscr{L}(\Sigma, m)$ is clearly a lattice of mutually commuting projections which contains 0 and 1, and it was shown in [19, 1.2.1] that $\mathscr{L}(\Sigma, m)$ is closed in the strong operator topology.

The weak and strong operator topologies coincide on $\mathscr{L}(\Sigma, m)$, and make $\mathscr{L}(\Sigma, m)$ into a *topological lattice* in the sense that both lattice operations $x \vee y$ and $x \wedge y$ are jointly continuous. We are interested in determining when $\mathscr{L}(\Sigma, m)$ is compact. Of course this depends on the behavior of the measure m relative to Σ. For example, take $G = \mathbb{R}^2$ and Σ to be the positive quadrant. Let m be a nonatomic measure concentrated on any straight line of slope -1. Then $\mathscr{L}(\Sigma, m)$ turns out to be a nonatomic Boolean algebra and such lattices are never compact. On the other hand, if m is a nonatomic measure concentrated on a straight line of slope $+1$ (or, for that matter, is two-dimensional Lebesgue measure), then Theorem 5 implies that $\mathscr{L}(\Sigma, m)$ is compact. We now introduce the relevant class of measures.

DEFINITION 6.1. A finite positive measure m on G is said to be Σ-continuous if, for every increasing Borel set E,

$$x \in G \mapsto \mu(E + x)$$

is continuous.

Remarks. Since every translate of an increasing set is increasing, it is enough to have continuity of $x \mapsto \mu(E + x)$ at $x = 0$ (for every $E \in L(\Sigma)$).

By the following result, the continuity of a measure depends only on its equivalence class under mutual absolute continuity. Thus we can define Σ-continuity for infinite measures in terms of the finite measures mutually absolutely continuous with them.

We also remark that much (if not all) of the sequel can be generalized to the case of cones Σ in noncommutative locally compact groups G. In order to avoid irrelevant technicalities, we have limited our discussion to the abelian case.

We will use the usual symbol ∂E to denote the topological boundary of a subset E of G. The following result gives a useful criterion for Σ-continuity. For example, when $G = \mathbb{R}$ and $\Sigma = [0, +\infty)$, it tells us that a measure is Σ-continuous iff it is nonatomic.

PROPOSITION 6.2. *A finite measure m is Σ-continuous if and only if*

$$m(\partial E) = 0$$

for every closed increasing set E.

Proof. Assume first that m is continuous and let E be an increasing Borel set. We will show that $m(\partial E) = 0$.

Indeed, if x belongs to the interior of Σ, then $E+x$ is a subset of the open increasing set $E+\operatorname{int}\Sigma$ which, in turn, is contained in E. Hence

$$\partial E=\bar{E}\backslash\operatorname{int}E\subseteq E\backslash(E+x),$$

and so

$$m(\partial E)\leqslant m(E\backslash(E+x))=m(E)-m(E+x).$$

Since 0 belongs to the closure of the interior of Σ, we can let x tend to 0 in the right side of the above inequality to obtain $m(\partial E)\leqslant 0$.

Conversely, assume $m(\partial E)=0$ for every closed increasing set E. Let E be an arbitrary increasing Borel set. Note that the closure $\bar{E}$ of E is increasing, simply because the closure of a set which is invariant under all translations $x\mapsto x+\sigma$, $\sigma\in\Sigma$, has the same property. Let x_n be a sequence in G converging to 0. We will show that for every $E\in L(\Sigma)$,

$$\limsup_{n\to\infty} m(E+x_n)\leqslant m(\bar{E}) \tag{6.3}$$

and

$$\liminf_{n\to\infty} m(E+x_n)\geqslant m(\operatorname{int}\bar{E}). \tag{6.4}$$

Since $m(\bar{E})-m(\operatorname{int}\bar{E})=m(\partial\bar{E})=0$, we may conclude that

$$\lim_{n\to\infty} m(E+x_n)=m(\bar{E})=m(\operatorname{int}\bar{E}).$$

Note that this also proves that

$$m(E)=m(\bar{E})$$

for every increasing Borel set E.

Fix $E\in L(\Sigma)$. To prove (6.3), let U_n be a sequence of open neighborhoods of 0 such that $x_n\in U_n$, $U_n\supseteq U_{n+1}$, and $\bigcap_n U_n=\{0\}$. Then the sequence of open sets $E+U_n$ is a decreasing sequence of sets and their intersection is $\bar{E}$. Thus

$$\limsup_{n\to\infty} m(E+x_n)\leqslant \lim_{n\to\infty} m(E+U_n)=m(\bar{E}).$$

To prove (6.4), we claim first that there is a sequence $\sigma_1,\sigma_2,\ldots$, in the interior of Σ such that $\sigma_1\geqslant\sigma_2\geqslant\cdots$, and $\lim_{n\to\infty}\sigma_n=0$. Indeed let $U_1\supseteq U_2\supseteq\cdots$ be open neighborhoods of 0 such that $\bigcap_n U_n=\{0\}$. Choose σ_1

arbitrarily in $(\operatorname{int} \Sigma) \cap U_1$. Assuming that $\sigma_1, \ldots, \sigma_n$ have been defined so that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n$ and $\sigma_j \in U_j \cap \operatorname{int} \Sigma$, note that

$$U_{n+1} \cap (\sigma_1 - \operatorname{int} \Sigma) \cap \cdots \cap (\sigma_n - \operatorname{int} \Sigma)$$

is an open neighborhood of 0 which must therefore intersect $\operatorname{int} \Sigma$, and so we can choose σ_{n+1} to be any point in the common intersection.

Now the sets $\operatorname{int} \bar{E} + \sigma_n$ are open, they satisfy

$$\operatorname{int} \bar{E} + \sigma_n \subseteq \operatorname{int} \bar{E} + \sigma_{n+1}$$

because $\operatorname{int} \bar{E}$ is invariant under translation by $\sigma_n - \sigma_{n+1} \in \Sigma$, and their union is the interior of $\bar{E}$. Hence

$$m(\operatorname{int} \bar{E}) = \lim_{n \to \infty} m(\operatorname{int} \bar{E} + \sigma_n).$$

So it suffices to show that

$$m(\operatorname{int} \bar{E} + \sigma_n) \leqslant \liminf_{k \to \infty} m(E + x_k)$$

for every $n = 1, 2, \ldots$. Fix n. Then $\sigma_n - \operatorname{int} \Sigma$ is an open neighborhood of 0 and so it contains $-x_k$ for large enough k, say $k \geqslant k_n$. Then $\sigma - x_k - \operatorname{int} \Sigma$ contains 0 for $k \geqslant k_n$ and so for each $x \in \bar{E}$,

$$x + \sigma_n - x_k - \operatorname{int} \Sigma$$

contains x for $k \geqslant k_n$. Since the latter set is open and since x belongs to the closure of E, it must contain a point y_k in E. Hence

$$x + \sigma_n - x_k \geqslant y_k \in E$$

for all $k \geqslant k_n$, which implies that $x + \sigma_n - x_k \in E$ for $k \geqslant k_n$ because E is an increasing set. Thus

$$\bar{E} + \sigma_n \subseteq E + x_k$$

for large enough k, and the latter clearly implies (6.5). ∎

Remarks. A Σ-continuous measure is necessarily nonatomic, since every singleton $\{x\}$ is contained in the boundary of a closed increasing set (namely, $x + \Sigma$; note that here we have used the condition $\Sigma \cap -\Sigma = \{0\}$).

The proof shows that if m is a Σ-continuous measure, then $m(E) = m(\bar{E})$ for every increasing Borel set.

Any finite measure on G which is absolutely continuous with respect to Haar measure is Σ-continuous. Indeed, if

$$m(E)=\int_E f(x)\,dx,$$

where $f\in L^1(G)$, then for every Borel set E we have

$$|m(E+x)-m(E)|\leqslant \|f(\cdot+x)-f(\cdot)\|_{L^1(G)}$$

which tends to zero as $x\to 0$. There are, however, many singular measures which are Σ-continuous. For example, let $G=\mathbb{R}^2$ and let Σ be the positive quadrant. Let $L_1, L_2,\ldots,$ be a sequence of disjoint straight lines in $\mathbb{R}^2$ all having slope $+1$. For every $j\geqslant 1$ let m_j be a nonatomic probability measure which is concentrated on L_j, and let

$$m=\sum_{j=1}^{\infty} 2^{-j}m_j.$$

A few moments' thought shows that if E is a closed increasing set in $\mathbb{R}^2$, then ∂E meets each line L_j in at most one point. Thus $m_j(\partial E)=0$ for all j and so $m(\partial E)=0$.

We come now to the main result of this section.

THEOREM 5. *Let m be a Σ-continuous measure on Σ. Then the lattice $\mathscr{L}(\Sigma, m)$ is compact.*

Proof. $\mathscr{L}(\Sigma, m)$ is a bounded family of self-adjoint operators which is closed in the strong operator topology ([19], 1.2.1). Thus it is a Polish space in its relative strong topology. So it suffices to establish the following assertion: if $E_1, E_2,\ldots,$ is a sequence of increasing Borel sets then there is a subsequence $E_{n_1}, E_{n_2},\ldots,$ such that the limit

$$f(x)=\lim_{k\to\infty}\chi_{E_{n_k}}(x)$$

exists almost everywhere (dm). For this we require

LEMMA. *Let $E_1, E_2,\ldots,$ be sets in $L(\Sigma)$ such that the characteristic functions $\chi_{E_1}(x)$, $\chi_{E_2}(x),\ldots,$ converge for all x in a dense subgroup D of G. Then there is a closed increasing set E such that*

$$\lim_{n\to\infty}\chi_{E_n}(x)=\chi_E(x),$$

for all $x\notin \partial E$.

Proof of the Lemma. Let f be the limit function

$$f(x) = \lim_{n\to\infty} \chi_{E_n}(x), \qquad x \in D.$$

Clearly $f(x)$ is 0 or 1 for every $x \in D$ so there is a subset $E_0 \subseteq D$ such that $f = \chi_{E_0}$. Let E be the closure of E_0 in G.

We claim first that in E is increasing. Because D is dense in G, $D \cap \operatorname{int} \Sigma$ is dense in Σ. So to prove that the *closed* set E is invariant under translations by Σ, it suffices to show that E is invariant under translations by element of $D \cap \operatorname{int} \Sigma$. Finally, since E_0 is dense in E and translations are continuous, it suffices to show that

$$E_0 + D \cap \operatorname{int} \Sigma \subseteq E_0.$$

For that, choose $x \in E_0$, $\sigma \in D \cap \operatorname{int} \Sigma$. Since

$$\lim_{n\to\infty} \chi_{E_n}(x) = \chi_{E_0}(x) = 1,$$

we must have $\chi_{E_n}(x) = 1$ for all $n \geqslant n_x$. Thus $x \in E_n$ for large n and since E_n is increasing we have $x + \sigma \in E_n$ for large n. Since $x + \sigma \in D$ (D is a group) we have

$$\chi_{E_0}(x+\sigma) = \lim_{n\to\infty} \chi_{E_n}(x+\sigma) = 1,$$

and thus $x + \sigma \in E_0$, proving the assertion.

Next we show that

$$\lim_{n\to\infty} \chi_{E_n}(x) = 1$$

for every x in the interior of E. Choose such an x. Note that $x - \operatorname{int} \Sigma$ must intersect E_0; for $x - \operatorname{int} \Sigma$ is an open set and therefore $(x - \operatorname{int} \Sigma) \cap E_0$ is dense in $(x - \operatorname{int} \Sigma) \cap \bar{E}_0 = (x - \operatorname{int} \Sigma) \cap E$, and the latter set cannot be void since it contains a sequence converging to x (for example, $x - \sigma_n$ will do, where σ_n is any sequence converging to x). Thus we can find $\sigma \in \operatorname{int} \Sigma$ such that $x - \sigma \in E$. Therefore $\chi_{E_n}(x - \sigma) \to 1$ as $n \to \infty$ and, since E_n is an increasing set in G and $x \geqslant x - \sigma$, we must have $\chi_{E_n}(x) \to 1$ as $n \to \infty$, as asserted.

It remains to show that

$$\lim_{n\to\infty} \chi_{E_n}(x) = 0,$$

for all x in the complement of E. Note that $x + \operatorname{int} \Sigma$ must intersect $D \backslash E_0$.

Indeed, since D is dense in G and $(x+\operatorname{int}\Sigma)\cap(G\backslash E)$ is an open set containing x in its closure, x must be in the closure of

$$(x+\operatorname{int}\Sigma)\cap(G\backslash E)\cap D\subseteq(x+\operatorname{int}\Sigma)\cap(D\backslash E_0).$$

So choose $\sigma\in\operatorname{int}\Sigma$ such that $x+\sigma\in D\backslash E_0$. Then

$$\lim_{n\to\infty}\chi_{E_n}(x+\sigma)=0,$$

and hence $x+\sigma\notin E_n$ for all large values of n. Since $x\leqslant x+\sigma$ and each E_n is an increasing set, $x\notin E_n$ for all large n, and hence

$$\lim_{n\to\infty}\chi_{E_n}(x)=0. \quad \blacksquare$$

To prove Theorem 5, let D be any countable dense subgroup of G, and let $E_1, E_2,\ldots,$ be a sequence of increasing Borel sets. By the Cantor diagonalization procedure, we can find a subsequence $E_{n_1}, E_{n_2},\ldots,$ whose sequence of characteristic functions converges at every point of D. The lemma implies that there is a closed set $E\in L(\Sigma)$ such that

$$\chi_{E_{n_k}}\to\chi_E$$

on the complement of ∂E. Because m is Σ-continuous, ∂E is a set of measure zero, and we are done. $\blacksquare$

Note that Theorem 5 (and its proof) are closely related to and in some sense generalize the Helley compactness theorem for monotonic functions of a real variable.

7. Perturbation Theory for Lattices

Let (G,Σ) be as in the preceding section. In this section we will classify perturbations of lattices of the form $\mathscr{L}(\Sigma, m)$, where m is a finite Σ-continuous measure on G. We will make essential use of the following result, which plays a role analogous so that of the lemma on functions of positive type used in Section 5. Theorem 6 is related to (and improves substantially on) certain results about weak* convergence that are quite useful in probability theory [20, pp. 247–249]. In the simplest case, the latter asserts that if $\mu,\mu_1,\mu_2,\ldots,$ are probability measures on the real line such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}f(x)\,d\mu_n(x)=\int_{-\infty}^{\infty}f(x)\,d\mu(x)$$

for every continuous function f vanishing at infinity, then the sequence of distribution functions

$$F_n(x) = \mu_n((-\infty, x])$$

converges pointwise and uniformly to the distribution function $F(x) = \mu((-\infty, x])$, whenever F is continuous.

$C_0(G)$ will denote the Banach space of all real-valued continuous functions on G which vanish at infinity.

THEOREM 6. *Let $\mu, \mu_1, \mu_2, \ldots$ be a sequence of finite positive measures on G such that*

(i) $\mu_n(G) \to \mu(G)$, *and*

(ii) $\int_G f\, d\mu_n \to \int_G f\, d\mu$ *for every* $f \in C_0(G)$.

If μ is Σ-continuous, then

$$\sup_E |\mu_n(E) - \mu(E)| \to 0 \qquad as \quad n \to \infty,$$

where the supremum is taken over all increasing Borel sets E.

We require some preliminaries.

LEMMA 1. *Let μ be a finite positive Σ-continuous measure on G. Then for every $\varepsilon > 0$, there is a neighborhood U of 0 such that*

$$|\mu(E + x) - \mu(E)| \leqslant \varepsilon$$

for every $x \in U$ and every increasing Borel set E.

Proof. As in the proof of Proposition 6.2, we may find a sequence of elements σ_n in the interior of Σ such that

$$\sigma_n \geqslant \sigma_{n+1} \qquad \text{for all } n$$

and

$$\lim_{n \to \infty} \sigma_n = 0.$$

Let

$$U_n = (\sigma_n - \operatorname{int} \Sigma) \cap (\operatorname{int} \Sigma - \sigma_n).$$

U_n is a neighborhood of 0. We claim that for each $x \in U_n$ and each increasing Borel set E,

$$|\mu(E+x)-\mu(E)| \leqslant \sup_F \mu(F)-\mu(F+\sigma_n), \tag{7.1}$$

where the supremum is taken over all increasing Borel sets F. Indeed, if $x \in U_n$, then $-\sigma_n \leqslant x \leqslant \sigma_n$ and so for each increasing set E we have

$$E+\sigma_n \subseteq E+x \subseteq E-\sigma_n.$$

Thus

$$-(\mu(E)-\mu(E+\sigma_n)) \leqslant \mu(E+x)-\mu(E) \leqslant \mu(E-\sigma_n)-\mu(E),$$

from which the assertion is evident. Thus it suffices to show that the right side of (7.1) can be made small by choosing n large enough. This we will do by an application of Dini's theorem.

Consider the finite measure

$$m(S)=\mu(S)+\sum_{n=1}^{\infty} 2^{-n}\mu(S+\sigma_n).$$

Note that m is Σ-continuous. For if E is a closed increasing set, then $E+\sigma_n$ is a closed increasing set whose boundary is $\partial E+\sigma_n$, hence

$$m(\partial E)=\mu(\partial E)+\sum_{n=1}^{\infty} 2^{-n}\mu(\partial(E+\sigma_n))=0,$$

and the claim follows from (6.2).

By Theorem 5, the subspace lattice $\mathscr{L}(\Sigma, m)$ is a compact Hausdorff space in its strong operator topology. For each $n \geqslant 1$, define a function $\phi_n: \mathscr{L}(\Sigma, m) \to \mathbb{R}$ by

$$\phi_n(P_E)=\mu(E)-\mu(E+\sigma_n),$$

where E is an increasing Borel set and P_E is its corresponding projection in $\mathscr{L}(\Sigma, m)$. We claim that ϕ_n is continuous (and well defined). For that, note that since the two finite measures μ and $S \to \mu(S+\sigma_n)$ are absolutely continuous with respect to m, there are nonnegative functions ξ, ξ_n in $L^2(G, m)$ such that

$$\mu(S)=\int_S \xi(x)^2\, dm(x), \qquad \mu(S+\sigma_n)=\int_S \xi_n(x)^2\, dm(x).$$

These formulas show that ϕ_n has the form

$$\phi_n(P) = \langle P\xi, \xi\rangle - \langle P\xi_n, \xi_n\rangle$$

for every projection $P \in \mathscr{L}(\Sigma, m)$ (where $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(G, m)$), and hence ϕ_n is strongly continuous.

We have to show that the sequence ϕ_n tends uniformly to zero on $\mathscr{L}(\Sigma, m)$. Now since $\sigma_{n+1} \leqslant \sigma_n$, we have $E + \sigma_n \subseteq E + \sigma_{n+1}$ for every increasing Borel set E, and therefore $\phi_{n+1}(P_E) \leqslant \phi_n(P_E)$. So by Dini's theorem it suffices to show that

$$\lim_{n\to\infty} \phi_n(P_E) = \lim_{n\to\infty} (\mu(E) - \mu(E + \sigma_n)) = 0,$$

for every fixed increasing set E. By the remark following Proposition 6.2, we know that $\mu(E) = \mu(\bar{E})$. So it suffices to show that for every *closed* increasing set E,

$$\lim_{n\to\infty} \mu(E + \sigma_n) = \mu\left(\bigcup_{n=1} (E + \sigma_n)\right) = \mu(E).$$

Now $\bigcup_n (E + \sigma_n)$ contains every interior point of E; for if $x \in \operatorname{int} E$, then $x - \sigma_n \in \operatorname{int} E \subseteq E$ for large n, and hence $x \in E + \sigma_n$. Thus

$$\operatorname{int} E \subseteq \bigcup_n (E + \sigma_n) \subseteq E.$$

Since $\mu(\partial E) = 0$ by Σ-continuity, we have the desired conclusion. ∎

LEMMA 2. *Let μ be a finite positive Σ-continuous measure. For every compact subset $K \subseteq G$ and every $\varepsilon > 0$, there is a norm-compact set of functions $\mathscr{F} \subseteq C_0(G)$ with the following property: for every increasing Borel set E there is a pair of functions $f, g \in \mathscr{F}$ satisfying*

(i) $0 \leqslant f \leqslant g \leqslant 1$,

(ii) $f \leqslant \chi_E \leqslant g$ *on* K,

(iii) $\int_G (g - f)\, d\mu \leqslant \varepsilon$.

Proof. Let V be an open neighborhood of 0, to be specified later. Let u be a nonnegative continuous function having support in $V \cap \operatorname{int} \Sigma$ and such that

$$\int u \, d\mu = 1.$$

For every increasing Borel set E, define

$$f_E(x) = \int_G u(y)\,\chi_E(x-y)\,dy,$$

$$g_E(x) = \int_G u(y)\,\chi_E(x+y)\,dy,$$

dy denoting Haar measure on G. f_E and g_E are nonnegative continuous functions because the convolution of an L^1 function with an L^∞ function is continuous. Indeed, we claim that both families of functions

$$\{f_E: E \text{ increasing}\} \qquad \text{and} \qquad \{g_E: E \text{ increasing}\}$$

are equicontinuous. This follows from the fact that the larger set of functions

$$\{u * h: h \in L^\infty, \|h\|_\infty \leqslant 1\}$$

is equicontinuous, by the estimate

$$\begin{aligned} |u * h(y) - u * h(x)| &\leqslant \int_G |u(y-t) - u(x-t)| \cdot |h(t)|\,dt \\ &\leqslant \int_G |u(-t+y-x) - u(-t)|\,dt, \end{aligned}$$

and the fact that the last term tends to zero as $y - x \to 0$.

We claim next that

$$f_E \leqslant \chi_E \leqslant g_E$$

for every increasing Borel set E. For if y belongs to Σ, then since E is increasing we have

$$\chi_E(x-y) \leqslant \chi_E(x) \leqslant \chi_E(x+y)$$

for all x, and the assertion follows by multiplying this string of inequalities by $u(y)$ and integrating y over the support of u (a subset of Σ).

Now choose a continuous compactly supported function w such that $0 \leqslant w \leqslant 1$ and $w \equiv 1$ on K. Then

$$\{w \cdot f_E: E \text{ increasing}\} \cup \{w \cdot g_E: E \text{ increasing}\}$$

is a bounded equicontinuous subset of $C_0(G)$, so by Ascoli's theorem its norm-closure $\mathscr{F}$ is compact.

It remains only to show that, for an appropriate choice of V and u, we will have

$$\int_G w(x)(g_E(x) - f_E(x))\, d\mu(x) \leqslant \varepsilon$$

for every E.

This is done as follows: Define a measure ν on G by

$$\nu(S) = \int_S w(x)\, d\mu(x).$$

Since ν is absolutely continuous with respect to μ, ν is also a Σ-continuous measure By Lemma 1, we can find a neighborhood U of 0 such that

$$|\nu(E) - \nu(E + x)| \leqslant \varepsilon$$

for every $x \in U$ and every increasing set E. Let V be an open neighborhood of 0 such that $V + V \subseteq U$, and let u be as stipulated above. Then for every increasing set E we have

$$\begin{aligned}
&\int_G w(x)(g_E(x) - f_E(x))\, d\mu(x) \\
&\quad = \int_\Sigma u(y) \left(\int_G w(x)\, \chi_E(x+y)\, d\mu(x) - \int_G w(x)\, \chi_E(x-y)\, d\mu(x)\right) dy \\
&\quad = \int_\Sigma u(y)(\nu(E-y) - \nu(E+y))\, dy \\
&\quad \leqslant \sup_{y \in V} (\nu(E-y) - \nu(E+y)) \leqslant \varepsilon. \quad \blacksquare
\end{aligned}$$

We now prove Theorem 6. Let $\mu_n \to \mu$ weak*, with $\mu_n(G) \to \mu(G)$. Let U be an open set with compact closure such that

$$\mu(U) > 1 - \varepsilon.$$

Note that $\mu_n(U) > 1 - \varepsilon$ for large n. For if we find $f \in C_0(G)$ such that f lives in U, $0 \leqslant f \leqslant 1$, and

$$\int_G f\, d\mu > 1 - \varepsilon,$$

then for large n we will have

$$\int_G f\, d\mu_n > 1 - \varepsilon,$$

and hence

$$\mu_n(U) \geqslant \int f\, d\mu_n > 1 - \varepsilon.$$

Let $K = \bar{U}$. Then by throwing away a finite number of terms we can arrange that

$$\mu(G \backslash K) < \varepsilon \qquad \text{and} \qquad \mu_n(G \backslash K) < \varepsilon$$

for every $n = 1, 2, \ldots$.

By Lemma 2, we can find a norm compact subset $\mathscr{F} \subseteq C_0(G)$ having the properties (i)–(iii) for K and ε. Because $\mathscr{F}$ is *norm*-compact and μ_n is a uniformly bounded subset of the dual of $C_0(G)$ which converges weak* to μ, we must have

$$\sup_{f \in \mathscr{F}} \left| \int f\, d\mu_n - \int f\, d\mu \right| < \varepsilon$$

for all n larger than some given integer $N = N_\varepsilon$. For each increasing Borel set E, find $f, g \in \mathscr{F}$ such that

$$0 \leqslant f \leqslant g \leqslant 1,$$
$$f \leqslant \chi_E \leqslant g \qquad \text{on} \quad K,$$
$$\int (g - f)\, d\mu \leqslant \varepsilon.$$

Then for each $n \geqslant N$ we have

$$\begin{aligned} \mu_n(E) &\leqslant \mu_n(K \cap E) + \varepsilon \leqslant \int_K g\, d\mu_n + \varepsilon \\ &\leqslant \int_G g\, d\mu_n + \varepsilon \leqslant \int_G g\, d\mu + 2\varepsilon \\ &\leqslant \int_G f\, d\mu + 3\varepsilon \leqslant \int_K f\, d\mu + 4\varepsilon \\ &\leqslant \mu(K \cap E) + 4\varepsilon \leqslant \mu(E) + 5\varepsilon. \end{aligned}$$

The first inequality is because $\mu_n(G \backslash K) \leqslant \varepsilon$, the second is because $\chi_E \leqslant g$ on K, the fourth is because

$$\sup_{h \in \mathscr{F}} \left| \int h\, d\mu_n - \int h\, d\mu \right| \leqslant \varepsilon,$$

the fifth is because

$$\int (g-f)\,d\mu \leqslant \varepsilon,$$

the sixth is because $f \leqslant 1$ and $\mu(G\backslash K) \leqslant \varepsilon$, the seventh is because $f \leqslant \chi_E$ on K, and the last is because $\mu(G\backslash K) \leqslant \varepsilon$. Similarly we have

$$\begin{aligned}
\mu(E) &\leqslant \mu(K \cap E) + \varepsilon \leqslant \int_K g\,d\mu + \varepsilon \\
&\leqslant \int_G g\,d\mu + \varepsilon \leqslant \int_G f\,d\mu + 2\varepsilon \\
&\leqslant \int_G f\,d\mu + 3\varepsilon \leqslant \int_K f\,d\mu_n + 4\varepsilon \\
&\leqslant \mu_n(K \cap E) + 4\varepsilon \leqslant \mu_n(E) + 5\varepsilon.
\end{aligned}$$

Thus,

$$\sup_E |\mu_n(E) - \mu(E)| \leqslant 5\varepsilon$$

whenever $n \geqslant N$. This is, of course, good enough. ∎

It is convenient to formulate the main result in terms of projection valued measures. By a projection valued measure on G we mean a countably additive function P from the Borel sets of G to the projections on a separable Hilbert space satisfying

$$P(\varnothing) = 0 \qquad \text{and} \qquad P(G) = 1.$$

Such a P is said to be Σ-continuous if, for each vector $\xi \in \mathscr{H}_p$, the measure

$$m_\xi(S) = \langle P(S)\xi, \xi\rangle$$

is Σ-continuous. This is equivalent to the assertion that, for every increasing Borel set E,

$$x \longmapsto P(E + x)$$

is a strongly continuous map from G to the projections in $\mathscr{L}(\mathscr{H}_p)$. One readily deduces from Proposition 6.2 that P is Σ-continuous iff

$$P(\partial E) = 0$$

for every closed increasing set E.

We will write $L(\Sigma)$ for the σ-lattice of all increasing Borel set in G.

PROPOSITION 7.2. *Let P be a Σ-continuous projection valued measure. Then*

$$\mathscr{P} = \{P(E): E \in L(\Sigma)\}$$

is a compact commutative subspace lattice.

Proof. Let ρ be a faithful normal state of $\mathscr{L}(\mathscr{H}_p)$, and let m be the probability measure on G defined by

$$m(S) = \rho(P(S));$$

m is clearly Σ-continuous.

For each Borel set $S \subseteq G$, let $Q(S)$ be the corresponding projection operator in $L^2(G, m)$:

$$Q(S)\,\xi(x) = \chi_S(x)\,\xi(x),$$

$\xi \in L^2(G, m)$. By a familiar result on abelian von Neumann algebras [9, Chap I, Sect. 7], there is a unique *-isomorphism θ of the multiplication algebra of $L^2(G, m)$ onto the von Neumann algebra generated by $\{P(S): S$ Borel$\}$ satisfying

$$\theta(Q(S)) = P(S)$$

for every Borel set S. The restriction of θ to the lattice

$$\mathscr{L}(\Sigma, m) = \{Q(E): E \in L(\Sigma)\}$$

is a strongly continuous lattice isomorphism with range $\mathscr{P}$. By Theorem 5, we conclude that $\mathscr{P}$ is compact. ∎

We will say that two projection valued measures P, Q are *Σ-equivalent* (written $P \sim Q$) if, for every $\varepsilon > 0$, there is a unitary operator $W: \mathscr{H}_P \to \mathscr{H}_Q$ such that

(i) $\{WP(E) - Q(E)W: E \in L(\Sigma)\}$ is a norm-compact set of compact operators, and

(ii) $\sup_{E \in L(\Sigma)} \| WP(E) - Q(E)W \| \leqslant \varepsilon$.

Let $\mathscr{P}$ and $\mathscr{Q}$ be the subspace lattices determined by $\mathscr{P}$ and $\mathscr{Q}$

$$\mathscr{P} = \{P(E): E \in L(\Sigma)\}, \qquad \mathscr{Q} = \{Q(E): E \in L(\Sigma)\}.$$

It is not hard to see that (i) and (ii) (with $\varepsilon < \frac{1}{2}$) imply that there is a topological isomorphism of commutative subspace lattices $\theta: \mathscr{P} \to \mathscr{Q}$ such that

$$\theta: P(E) \mapsto Q(E)$$

for every $E \in L(\Sigma)$. Moreover, the function

$$P \mapsto WP - \theta(P)W$$

is a continuous map of the topological space $\mathscr{P}$ into the Banach space of compact operators $\mathscr{K}(\mathscr{H}_P, \mathscr{H}_Q)$. We omit these arguments since we do not require the results.

The *direct sum* of P and Q is the projection valued measure $P \oplus Q$ on $\mathscr{H}_P \oplus \mathscr{H}_Q$ given by

$$P \oplus Q(S) = P(S) \oplus Q(S).$$

Finally, the *support* of a projection valued measure P is the (necessarily closed) set of all points $x \in G$ such that $P(U) \neq 0$ for every open set U containing x.

THEOREM 7. *Let P and Q be two Σ-continuous projection valued measures having the same support. Then P and Q are Σ-equivalent.*

Proof. By symmetry, it suffices to show that $P \oplus Q \sim P$.

Let $\mathscr{P} = \{P(E) : E \in L(\Sigma)\}$. $\mathscr{P}$ is a $*$-semigroup relative to operator multiplication and the trivial involution $x^* = x$. Since multiplication is strongly continuous on the unit ball of $\mathscr{L}(\mathscr{H}_P)$, $\mathscr{P}$ is a topological $*$-semigroup. By Proposition 7.1, it is compact.

We will show first, that there is a (continuous) representation θ of $\mathscr{P}$ defined by

$$\theta(P(E)) = Q(E), \qquad E \in L(\Sigma),$$

and second, that θ_n is subordinate to id_n for every $n \geqslant 1$, where id is the identity representation of $\mathscr{P}$. We may then conclude from Theorem 3 that $\mathrm{id} \oplus \theta$ is approximately equivalent to id in the sense of Section 4, and this clearly implies $P \oplus Q \sim P$.

For the existence of θ, it suffices to show that if E_n, E are increasing Borel sets for which

$$P(E_n) \to P(E) \qquad \text{strongly},$$

then $\rho(Q(E_n)) \to \rho(Q(E))$ for every ultraweakly continuous linear functional ρ on $\mathscr{L}(\mathscr{H}_Q)$. Clearly we may assume ρ is a normal state. For such a ρ, we will show that there is a continuous function $\phi : \mathscr{P} \to \mathbb{R}$ such that

$$\phi(P(E)) = \rho(Q(E)), \qquad E \in L(\Sigma).$$

The assertion follows from this.

In order to get ϕ, consider the commutative C^*-algebra

$$A = C_0(G) + \mathbb{C} \cdot 1,$$

and let π, σ be the representations of A defined by

$$\pi(f) = \int_G f(x)\, dP(x), \qquad \sigma(f) = \int_G f(x)\, dQ(x).$$

Note that $\pi(A)$ contains no nonzero compact operators. For if it did, there would be a nonzero finite-dimensional minimal projection E in $\pi(A)$, and hence there would be a point $x_0 \in G \cup \{\infty\}$ such that

$$E\pi(f)E = f(x_0)E$$

for every $f \in A$. This implies that

$$EP(S)E = E \neq 0$$

for every Borel set $S \subseteq G \cup \{\infty\}$ which contains x_0. Since $P(\{\infty\}) = 0$, we must have $x_0 \in G$, and the preceding implies that $P(\{x_0\}) \neq 0$, contradicting the fact that a Σ-continuous measure must be nonatomic.

Since P and Q have the same support $S \subseteq G$, we have

$$\|\pi(f)\| = \|\sigma(f)\| = \sup_{x \in S} |f(x)|$$

for every $f \in A$. Hence there is a unital $*$-isomorphism $\alpha: \pi(A) \to \sigma(A)$ satisfying $\alpha \circ \pi = \sigma$. Now the composition $\rho \circ \alpha$ defines a state of $\pi(A)$. Since $\pi(A) \cap \mathscr{K} = \{0\}$, we have a natural isomorphism

$$\pi(A) \cong (\pi(A) + \mathscr{K})/\mathscr{K}$$

and thus $\pi \circ \alpha$ can be regarded as a state of $\pi(A) + \mathscr{K}$ which annihilates $\mathscr{K}$. Glimm's lemma [9, 11.2.1] plus separability of $\pi(A) + \mathscr{K}$ provides a sequence of unit vectors ξ_n in $\mathscr{H}_P$ such that

$$\lim_{n \to \infty} \langle K\xi_n, \xi_n \rangle = 0, K \in \mathscr{K} \qquad \text{and} \qquad \lim_{n \to \infty} \langle \pi(f)\, \xi_n, \xi_n \rangle = \rho(\sigma(f)), \quad (7.3)$$

for every $f \in A$. The first expression implies $\xi_n \to 0$ weakly. Consider the probability measures μ, μ_n defined on G by

$$\mu(S) = \rho(Q(S)), \qquad \mu_n(S) = \langle P(S)\, \xi_n, \xi_n \rangle.$$

By (7.3), we have

$$\int_G f\, d\mu = \lim_{n \to \infty} \int_G f\, d\mu_n$$

for every $f \in A$. Since μ is Σ-continuous, Theorem 6 implies that

$$\sup_{E \in L(\Sigma)} |\langle P(E)\,\xi_n, \xi_n\rangle - \rho(Q(S))|$$

tends to zero as $n \to \infty$. This shows that the sequence of continuous functions $\phi_n \in C(\mathscr{P})$ defined by

$$\phi_n(P) = \langle P\xi_n, \xi_n\rangle$$

is uniformly convergent. The limit function $\phi \in C(\mathscr{P})$ is of course continuous and satisfies the required condition

$$\phi(P(E)) = \rho(Q(E)), \qquad E \in L(\Sigma).$$

Thus θ is a continuous $*$-representation of $\mathscr{P}$ in $\mathscr{L}(\mathscr{H}_Q)$. We now show that θ_n is subordinate to id_n for every $n \geqslant 1$. Fix n, and let ρ be a normal state of $\mathscr{L}(\mathbb{C}^n \otimes \mathscr{H}_Q)$. We will find a sequence ξ_n of unit vectors in $\mathbb{C}^n \otimes \mathscr{H}_P$ such that $\xi_n \to 0$ weakly and

$$\sup_{P \in \mathscr{P}} |\langle a \otimes P\xi_n, \xi_n\rangle - \rho(a \otimes \theta(P))| \tag{7.4}$$

tends to zero as $n \to \infty$, for every $\mathscr{A} \in M_n$. Let $\alpha: \pi(A) \to \pi(A)$ be the representation described above. Considering $M_n \otimes \pi(A)$ as a subalgebra of $\mathscr{L}(\mathbb{C}^n \otimes \mathscr{H}_P)$, we have

$$(M_n \otimes \pi(A)) \cap \mathscr{K} = \{0\}$$

because $\pi(A) \cap \mathscr{K} = \{0\}$. Arguing exactly as before, we may apply Glimm's lemma to the state

$$a \otimes \pi(f) \mapsto \rho(a \otimes \alpha \circ \pi(f)) = \rho(a \otimes \sigma(f))$$

of $M_n \otimes \pi(A)$ to obtain a sequence of unit vectors $\xi_k \in \mathbb{C}^n \otimes \mathscr{H}_P$ tending weakly to zero such that

$$\rho(a \otimes \sigma(f)) = \lim_{k \to \infty} \langle a \otimes \pi(f)\,\xi_k, \xi_k\rangle.$$

Choose a positive $a \in M_n$, and define measures μ, μ_k on G by

$$\mu(S) = \rho(a \otimes Q(S)), \qquad \mu_k(S) = \langle a \otimes P(S)\,\xi_k, \xi_k\rangle.$$

All these measures are positive because $a \geqslant 0$, and by the preceding we have

$$\int_G f\,d\mu = \lim_{k \to \infty} \int_G f\,d\mu_k$$

for every $f \in A$. μ is Σ-continuous because Q is Σ-continuous and so Theorem 6 implies that

$$\sup_{E \in L(\Sigma)} |\langle a \oplus P(E)\, \xi_k, \xi_k \rangle - \rho(a \otimes Q(E))|$$

tends to zero as $k \to \infty$. This implies the required condition (7.4) when $a \geqslant 0$, and the condition for an arbitrary $n \times n$ matrix a follows by taking finite linear combinations. ∎

We indicate briefly how one deduces the theorem of Andersen [3, 3.5.5] from Theorem 7. By a *continuous nest* we will mean here a strongly continuous mapping $t \mapsto P_t$ of the closed unit interval $[0, 1]$ into the projections on a separable Hilbert space such that

(i) $s < t \Rightarrow P_s < P_t$,
(ii) $P_0 = 0$, $P_1 = 1$.

Andersen's theorem asserts that *if P_t, Q_t are two continuous nests, then there is a unitary operator U such that*

$$t \mapsto UP_tU^* - Q_t \tag{7.5}$$

is a norm-continuous function from $[0, 1]$ *to the compact operators.* He has also shown that one can choose U so that

$$\sup_{0 \leqslant t \leqslant 1} \| UP_tU^* - Q_t \|$$

is arbitrarily small. In order to derive these results, consider the case

$$G = \mathbb{R}, \qquad \Sigma = (-\infty, 0].$$

Extend the functions P_t, Q_t to $\mathbb{R}$ by requiring them to be zero if $t < 0$ and 1 if $t > 1$. Then there are unique projection valued measures $\tilde{P}$, $\tilde{Q}$ on $\mathbb{R}$ such that

$$\tilde{P}((-\infty, t]) = P_t, \qquad \tilde{Q}((-\infty, t]) = Q_t,$$

for $-\infty < t < +\infty$. $\tilde{P}$ and $\tilde{Q}$ are nonatomic, and therefore they are Σ-continuous because the boundary of any closed increasing set is, in this case, a single point. Moreover, the supports of $\tilde{P}$ and $\tilde{Q}$ are both the closed unit interval. Thus (7.5) and (7.6) follow Theorem 7. We remark, by the way, that the proofs of the ancillary results to Theorem 7 (6.2, Theorems 5 and 6) are greatly simplified in the case $G = \mathbb{R}$ and $\Sigma = (-\infty, 0]$, and so the path from Theorem 3 to Andersen's theorem is relatively short.

By [1, Proposition 2.2), (7.5) implies that the quasitriangular algebras

$$\operatorname{alg}\{UP_tU^*\} + \mathscr{K} \qquad \text{and} \qquad \operatorname{alg}\{Q_t\} + \mathscr{K}$$

are identical, and therefore the quasitriangular algebras

$$\operatorname{alg}\{P_t\} + \mathscr{K} \qquad \text{and} \qquad \operatorname{alg}\{Q_t\} + \mathscr{K}$$

are unitarily equivalent. However, the results of [11] depend essentially on the distance formula for nest algebras [6, Theorem 1.1] and nothing like that is known for more general operator algebras. This raises a significant problem in connection with the above results. Let G, Σ be as above and let P, Q be two Σ-continuous projection valued measures on G having the same support. Are the operator algebras

$$\operatorname{alg} \mathscr{P} + \mathscr{K} \qquad \text{and} \qquad \operatorname{alg} \mathscr{Q} + \mathscr{K}$$

unitarily equivalent? We remark, that these two operator algebras are norm-closed (essentially by [11, pp. 138–139], or by [16, 5.2 and 7.1]).

A second problem of interest concern *norm* perturbations. If P, Q are as in the preceding paragraph and

$$\sup_{E \in L(\Sigma)} \|P(E) - Q(E)\|$$

is small, then are $\operatorname{alg} \mathscr{P}$ and $\operatorname{alg} \mathscr{Q}$ similar via an operator close to the identity? This is also known to be the case for nest algebras [4, 13, 15].

We conclude with an application of Theorem 7 to order automorphisms. An *order automorphism* of (G, Σ) is a homeomorphism $\psi: G \to G$ such that $x \leqslant y$ iff $\psi(x) \leqslant \psi(y)$. If P is a projection valued measure, then we can define a new projection valued measure P_ψ by

$$P_\psi(S) = P(\psi(S)),$$

$S \subseteq G$. Order automorphisms can be quite singular measure-theoretically, a familar phenomenon in the case $G = \mathbb{R}$, $\Sigma = (-\infty, 0]$. Nevertheless, the following result implies that for many Σ-continuous projection valued measures P, order automorphism induce bicontinuous lattice automorphisms of the subspace lattice

$$\mathscr{P} = \{P(E): E \in L(\Sigma)\}$$

which are approximately unitarily implemented.

COROLLARY. *Let P be a Σ-continuous projection valued measure which is supported everywhere on G. Then P and P_ψ are Σ-equivalent for every order automorphism ψ of G.*

Proof. The support of P_ψ is clearly G. If E is a closed increasing set, then $\psi(E)$ is a closed increasing set whose boundary is $\psi(\partial E)$. Hence

$$P_\psi(\partial E) = P(\partial\psi(E)) = 0,$$

which implies that $P_\psi \sim P$ by Theorem 7 ▮

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