Proceedings of Symposia in Pure Mathematics
Volume 38 (1982), Part 1

THE HARMONIC ANALYSIS OF AUTOMORPHISM GROUPS

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Contents

INTRODUCTION. This paper is an expanded version of a lecture given by the author at the AMS summer institute on operator algebras in Kingston, 1980. The basic objects of study are derivations and the one-parameter automorphism groups they generate. But we have made no attempt to cover these two subjects in an even-handed way. We concentrate on the "harmonic analysis" of automorphism groups, at the cost of neglecting basic aspects of the subject which are essential for a balanced perspective. For instance, we have not introduced the cohomology theory of Johnson and of Kadison and Ringrose, and we have not addressed the significant problem of "integrating" an unbounded derivation to obtain a flow (for example, see Sakai's forthcoming book). What we have attempted is to delineate the role of harmonic analysis in the subject and to indicate what we regard as milestones in the development of these ideas. The latter, especially, involves my own judgment and personal taste.

Section 1 contains an exposition of harmonic analysis in $L^\infty(\mathbb{R})$. We have organized the development so that §1 can be omitted without loss of technical content. §§2-5 are partly expository and partly new. We have given complete proofs of all significant results.

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[1]This material is based upon work supported by the National Science Foundation under Grant MCS 78-07740.

0082-0717/81/0000-0498/$18.75

In preparing the manuscript I have had the benefit of valuable conversations with Henry Helson about harmonic analysis, with Wolfgang Arendt about mapping properties of $sp(U)$, and with Jean de Cannière and Jonathan Block about reconstruction problems associated with spectral subspaces. To these people, and to the members of a fall quarter 1980 seminar at Berkeley who allowed me to inflict some of this material on them, I am pleased to record my thanks.

1. THE HARMONIC ANALYSIS OF BOUNDED FUNCTIONS. Let $\{\alpha_t : t \in \mathbb{R}\}$ be an ultraweakly continuous one-parameter group of $*$-automorphisms of a von Neumann algebra M . Each pair of elements $x \in M$, $\rho \in M_*$ gives rise to a bounded continuous function $f_{x,\rho} : \mathbb{R} \to \mathbb{C}$, defined by

$$f_{x,\rho}(t) = \rho(\alpha_t(x)) .$$

The harmonic analysis of the automorphism group $\{\alpha_t\}$ can be reduced to an analysis of the frequency distribution of the family of functions $\{f_{x,\rho}\}$. The purpose of this section is to review the classical spectral theory of such families of functions in a form convenient for our purposes. We have taken some care to formulate the basic concepts and to give complete proofs (modulo technical results on the existence of "sufficiently many" Fourier transforms of integrable fucntions). In some ways our approach differs from that in, say, Rudin's book [23]; these differences are pointed out in context.

Finally, though we will speak only of the additive group $\mathbb{R}$ of real numbers and its dual (which we distinguish from $\mathbb{R}$ by writing $\hat{\mathbb{R}}$) , everything in this section is valid with essentially no change for the general case of a locally compact abelian group in place of $\mathbb{R}$.

The most natural domain for harmonic analysis on the real line is the space $L^1 = L^1(\mathbb{R})$, not $L^\infty = L^\infty(\mathbb{R})$. That is simply because integrable functions have Fourier transforms while bounded functions do not. For instance, if f is an integrable function then one can define the <u>spectrum</u> of f to be the closed support of the Fourier transform $\hat{f}$:

1.1. $$\hat{f}(\lambda) = \int_{-\infty}^{+\infty} e^{-i\lambda t} f(t)\,dt .$$

If the transform $\hat{f}$ happens to be integrable over $\hat{\mathbb{R}}$ then the Fourier inversion theorem ([23], p. 22) asserts that

1.2. $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda t}\hat{f}(\lambda)d\lambda \ ,$$

hence f is expressed as a continuous superposition of the pure frequencies occurring in the spectrum of f. If $\hat{f}$ is not integrable then of course 1.2. is meaningless, but as we will see presently it is still appropriate to regard the spectrum of f as the set of pure frequencies "occurring" in f.

If f is merely a bounded function then the first integral 1.1 is meaningless as it stands. It is possible to make sense out of 1.1 if one considers $\hat{f}$ to be a tempered distribution, and in this case one may define the spectrum of f to be the closed support of the distribution $\hat{f}$. But this procedure does not work well for more general groups, and it is necessary to proceed along different lines.

Suppose we are given a bounded function f which has the particular form

1.3 $$f(t) = \int_{-\infty}^{+\infty} e^{i\lambda t}d\mu(\lambda) \ ,$$

where μ is a complex-valued measure on $\hat{\mathbb{R}}$ having finite total variation. The formula 1.3 itself implies that frequencies outside the closed support of μ are clearly absent from the function f. More generally, the variation of μ on a set $E \subseteq \hat{\mathbb{R}}$ measures the relative abundance of the frequencies of E which are present in the behavior of f. While there is no effective way of calculating $\mu(E)$ (or $|\mu|(E)$) from f, one can determine the closed support of μ from f in the following way.

Let $U \subseteq \hat{\mathbb{R}}$ be an open set. Then every continuous complex-valued function of $\hat{\mathbb{R}}$, which vanishes at ∞ and on $\hat{\mathbb{R}} \setminus U$, can be uniformly approximated by a sequence of Fourier transforms $\hat{\phi}_n$, where $\phi_n \in L^1$ and $\hat{\phi}_n$ has compact support in U (this is a consequence of the Stone-Weierstrass theorem and lemma B below). We conclude that $|\mu|(U) = 0$ if, and only if,

$$\int_{-\infty}^{+\infty} \hat{\phi}(\lambda)d\mu(\lambda) = 0$$

for every $\phi \in L^1$ such that $\hat{\phi}$ lives in U. By the Fubini theorem, this integral can be written

$$\iint e^{-i\lambda t}\phi(t)\,d\mu(\lambda)\,dt = \int_{-\infty}^{+\infty} f(-t)\phi(t)\,dt .$$

It will be convenient to consider the following pairing between functions $f \in L^\infty$ and $\phi \in L^1$:

$$\langle f, \phi\rangle = \int_{-\infty}^{+\infty} f(-t)\phi(t)\,dt .$$

The minus sign in the integrand here will allow us to avoid more unpleasant minus signs later. This pairing identifies L^∞ with the dual space of L^1, and, denoting the function $t \mapsto e^{i\lambda t}$ by $e^{i\lambda\cdot}$, we have

$$\hat{\phi}(\lambda) = \langle e^{i\lambda\cdot}, \phi\rangle .$$

In any event, the above assertion is that $|\mu|(U) = 0$ iff $\langle f, \phi\rangle = 0$ for every $\phi \in L^1$ such that $\hat{\phi}$ lives in U. Equivalently, we have the following description of the closed support of μ (defined as the set of all points $\lambda \in \hat{\mathbb{R}}$ for which $|\mu|(U) > 0$ for every neighborhood U of λ):

λ _belongs to support_(μ) _iff for every neighborhood_ U _of_ λ, _there exists_ $\phi \in L^1$ _such that_ $\hat{\phi}$ _lives in_ U _and_ $\langle f, \phi\rangle \neq 0$.

This shows how the closed set support(μ) can be determined from the function f. In order to see how this set enters into the harmonic analysis of f, let us consider the weak*-closed linear subspace S of L^∞ spanned by the translates of f. We have

PROPOSITION 1.4. (i) _For every_ $\lambda \in \hat{\mathbb{R}}$, _the character_ $e^{i\lambda\cdot}$ _belongs to_ S _iff_ λ _belongs to_ support(μ).

(ii) S _is spanned by_ $\{e^{i\lambda\cdot} : \lambda \in \text{support}(\mu)\}$.

Statement (i) identifies the pure harmonic constituents of f precisely with the points of support(μ), while (ii) asserts that S contains enough of these pure frequencies to reconstitute f.

SKETCH OF PROOF. We will show that for every $\phi \in L^1$, one has $\phi \perp S$ iff

$$\int_{-\infty}^{+\infty} e^{-it\lambda}\phi(t)\,dt = 0$$

for every λ in support(μ) . Notice that both statement (i) and (ii) follow directly from this assertion and an elementary separation theorem.

Indeed, assume $\phi \in L^1$ satisfies $\phi \perp S$. Then for every $s \in \mathbb{R}$ we have

$$\int_{-\infty}^{+\infty} f(s - t)\phi(t)dt = 0$$

and, using 1.3, we may rewrite this as

$$\int_{-\infty}^{+\infty} e^{is\lambda}\hat{\phi}(\lambda)d\mu(\lambda) = 0 , \quad s \in \mathbb{R} .$$

Since $\hat{\phi}d\mu$ is a finite measure on $\hat{\mathbb{R}}$ and such a measure is determined by its Fourier transform ([23], p. 29), we conclude that $\hat{\phi}d\mu = 0$. Thus $\hat{\phi}$ vanishes on the closed support of μ , as asserted. This argument is clearly reversible ∎

We turn now to the general case. Let S be any nonvoid subset of L^∞ .

DEFINITION 1.5. <u>A point</u> $\lambda \in \hat{\mathbb{R}}$ <u>is said to be an essential point of</u> S <u>if for every neighborhood</u> U <u>of</u> λ <u>there is a function</u> $\phi \in L^1$ <u>such that</u> $\hat{\phi}$ <u>lives in</u> U <u>and</u>

$$\langle S, \phi\rangle \neq \{0\} .$$

The set of all essential points of S is called the <u>spectrum</u> of S and is written $sp(S)$. For a singleton $S = \{f\}$ we will simply write $sp(f)$.

Let $\phi \in L^1$ and let ϕ_x denote the translate of ϕ by x :

$$\phi_x(t) = \phi(t - x) .$$

Then $\hat{\phi}_x$ is the product of $\hat{\phi}$ with an appropriate character of $\hat{\mathbb{R}}$, and in particular the support sets of $\hat{\phi}$ and $\hat{\phi}_x$ are identical. It follows that for any subset $S \subseteq L^\infty$, the spectrum of S is identical with the spectrum of the weak*-closed translation invariant subspace of L^∞ generated by S .

It is also easy to see that $sp(S)$ is a <u>closed</u> set in $\hat{\mathbb{R}}$. Indeed, if $\lambda \notin sp(S)$ then there is an open set U containing λ such that $\langle S, \phi\rangle = \{0\}$ for any $\phi \in L^1$ whose transform lives in U . Clearly no point of U can be an essential point of S , and hence the complement of $sp(S)$ is open.

This definition of $sp(S)$ (for the case where S is a weak*-

closed translation invariant subspace of L^∞) differs from the definition one frequently sees in harmonic analysis. For example, in [23], sp(S) is defined as *the set of all characters contained in* S . Assuming $S \neq \{0\}$, it is not obvious that $sp(S) \neq \emptyset$; and it is even less obvious that S must contain a character (indeed the latter assertion is more or less equivalent to Wiener's Tauberian theorem, see corollary 2 of theorem 1.7). Nevertheless, the definition we have given has turned out to be better for our purposes here; and since it does differ from that of [23] we have endeavored to give complete proofs of the main general assertions of the subject.

It is convenient to divide the harmonic analysis of a set S of bounded functions into two parts. The first, the problem of *spectral analysis*, asks roughly if the functions S can be decomposed into their pure frequencies by means of linear combinations of translation operators. More precisely:

Does each character in sp(S) *belong to the weak*-closed translation invariant subspace generated by* S ?

The second, the problem of *spectral synthesis*, asks if it is possible in principle to reconstruct each function in S as some kind of infinite linear combination (or integral) of these pure frequencies. That is

Is every function of S *contained in the weak*-closed linear span of the characters in* sp(S) ?

We will see that analysis is always possible but that synthesis may fail. We also want to point out that these properties are dual to significant assertions about closed ideals in the convolution algebra L^1 ; this dual formulation was first formulated and exploited by Beurling [3].

In order to proceed further we need two basic results about Fourier transforms of integrable functions, both of which are nontrivial and are valid for arbitrary locally compact abelian groups. Proofs can be found in ([23], pp. 49-51).

LEMMA A. *Let* $\phi \in L^1$, $\varepsilon > 0$. *Then there is a function* $\psi \in L^1$ *such that* $\hat{\psi}$ *has compact support and*

$$\|\phi - \phi_* \psi\|_1 < \varepsilon .$$

LEMMA B. *Let* $K \subset \hat{\mathbb{R}}$ *be compact and let* U *be an open set in* $\hat{\mathbb{R}}$ *containing* K . *Then there is a function* $\phi \in L^1$ *such that* $0 \leqslant \hat{\phi} \leqslant 1$, $\hat{\phi} = 1$ *on* K , *and* $\hat{\phi} = 0$ *off* U .

In addition, we will make repeated use of the fact that the Banach algebra L^1 is semisimple:

$$\phi \in L^1 , \quad \hat{\phi} \equiv 0 \Rightarrow \phi \equiv 0 .$$

(See [23], p. 17 and p. 29).

We will say that a property is satisfied near a set E in a topological space if it is true for all points in some open set containing E.

THEOREM 1.6. Let S be a nonvoid set in L^∞ and let $\phi \in L^1$ be such that $\hat{\phi}$ vanishes near $sp(S)$. Then $\phi \perp S$.

REMARK. In the degenerate case where $sp(S) = \emptyset$, we understand the assertion of the theorem to be that every fucntion in L^1 is orthogonal to S: i.e., $S = \{0\}$.

PROOF. By the above remarks, we may assume that S is a weak* closed translation invariant linear subspace of L^∞. Consider first the case where

$$E = \overline{\{\lambda \in \hat{\mathbb{R}}: \hat{\phi}(\lambda) \neq 0\}}$$

is compact. We will construct a function $\psi \in L^1$ such that $\langle S, \psi\rangle = \{0\}$ and $\phi = \phi*\psi$. The conclusion follows easily from this; for the space

$$S = \left\{u \in L^1: \langle S, u\rangle = \{0\}\right\}$$

is a translation invariant linear subspace of L^1, hence it is an ideal relative to convolution ([23], p. 157), hence $\phi = \phi*\psi$ belongs to $S_\perp$.

ψ is constructed as follows. By hypothesis, E is disjoint from $sp(S)$. Thus for every point λ in E there is an open set V_λ such that $\langle S, u\rangle = \{0\}$ for every $u \in L^1$ such that $\hat{u}$ lives in V_λ. Lemma B allows us to find, for each $\lambda \in E$, a function $u_\lambda \in L^1$ such that

$$\hat{u}_\lambda = 1 \text{ near } \lambda , \text{ and}$$

$$\hat{u}_\lambda \text{ lives in } V_\lambda .$$

Since E is compact we may find $\lambda_1, \ldots, \lambda_n$ in E so that the sets $V_{\lambda_1}, \ldots, V_{\lambda_n}$ cover E. Notice that

$$E \subseteq \bigcup_{j=1}^{n} \text{interior}\{\xi: \hat{u}_{\lambda_j}(\xi) = 1\} .$$

Define a polynomial p in the n complex variables $z_1, \ldots, z_n$ by

$$p(z_1, \ldots, z_n) = 1 - \prod_{j=1}^{n} (1 - z_j) .$$

Since p vanishes at the origin, it operates on n-tuples drawn from any commutative ring, and thus we may define a function ψ in the convolution algebra L^1 by

$$\psi = P(u_{\lambda_1}, \ldots, u_{\lambda_n}) .$$

Since each u_{λ_j} annihilates S and $S_\perp$ is an algebra (in fact, an ideal) under convolution, we have $\psi \perp S$. Moreover,

$$\psi(\xi) = 1 - \prod_{j=1}^{n} (1 - \hat{u}_{\lambda_j}(\xi)) , \quad \xi \in \mathbb{R} ,$$

and thus $\hat{\psi} = 1$ whenever some one of the u_{λ_j}'s equals 1. Thus $\hat{\psi} = 1$ near E and hence $\hat{\phi}\hat{\psi} = \hat{\phi}$. Since L^1 is semisimple we conclude that $\phi * \psi = \phi$, as required.

If E is not compact one may argue as follows. Choose a sequence $u_n \in L^1$ such that $\hat{u}_n$ has compact support and $\phi * u_n$ converges to ϕ in the L^1 norm (Lemma A). The preceding paragraph implies that $\phi * u_n$ S for every n, and hence $\phi \perp S$ because S is norm-closed ∎

COROLLARY. (i) *If* $S \subseteq L^\infty$ *contains a nonzero function then* $sp(S) \neq \emptyset$.

(ii) $sp(S)$ *is the smallest closed set* $F \subseteq \hat{\mathbb{R}}$ *having the property*:

$$\phi \in L^1 , \ \hat{\phi} = 0 \ \text{near} \ F \Rightarrow \phi \perp S .$$

PROOF. (i) is immediate. For (ii), let F be any closed set with the stated property and let λ be an essential point of S. We can show that $\lambda \in F$ as follows. If $\lambda \notin F$, choose a neighborhood V of λ whose closure is disjoint from F. Since λ is essential there is a function $\phi \in L^1$ for which $\hat{\phi}$ lives in V and $\langle S, \phi \rangle \neq \{0\}$. But since $\hat{\phi}$ vanishes on the open set $\mathbb{R} \setminus V$ containing F, we also have $\langle S, \phi \rangle = \{0\}$, a contradiction ∎

We now take up the question of the validity of spectral analysis. The critical information is contained in the following result, which identifies $sp(S)$ with the set of common zeros of a certain set of Fourier transforms.

THEOREM 1.7. *Let* S *be any nonvoid subset of* L^∞, *and let* $\tilde{S}$ *denote the weak***-closed translation invariant subspace generated* by S . Then

$$sp(S) = \{\lambda \in \hat{\mathbb{R}}: \hat{\phi}(\lambda) = 0 \text{ for every } \phi \in L^1, \phi \perp \tilde{S}\} .$$

Before giving the proof, we require another result about Fourier transforms. Wiener's Tauberian theorem for the additive group $\mathbb{Z}$ is equivalent to the assertion that if f is a complex-valued continuous function on the unit circle whose Fourier series is absolutely convergent, and which has no zeros on the unit circle, then the reciprocal of f has an absolutely convergent Fourier series. Wiener's proof of this was classical and difficult [27]. One of the earliest achievements of Gelfand's theory of commutative Banach algebras was a strikingly elegant conceptual approach to the proof of this assertion, and that proof now appears near the beginning of most elementary courses on commutative Banach algebras. The following result (for general LCA groups in place of $\mathbb{R}$) can be regarded as one generalization of Wiener's theorem to general groups; we will encounter another more familiar generalization below (Corollary 2 of theorem 1.7).

LEMMA. *Let* $K \subseteq \hat{\mathbb{R}}$ *be a compact set and let* $\phi \in L^1$ *be such that* $\hat{\phi}$ *has no zeros on* K . *Then there is a function* $\psi \in L^1$ *such that* $\hat{\phi}\hat{\psi} = 1$ *on* K .

PROOF. Let

$$J = \{\psi \in L^1: \hat{\psi} = 0 \text{ on } K\} .$$

J is a closed ideal in L^1 (relative to convolution) and we may form the commutative Banach algebra

$$A = L^1/J .$$

Let $u \in L^1 \mapsto \dot{u} \in A$ be the natural projection of L^1 on A .

Note first that A has a unit. Indeed, by lemma B there is a function $e \in L^1$ such that $\hat{e} = 1$ on K . So if $u \in L^1$ is arbitrary then the Fourier trnasform of $u_* e - u$ vanishes on K , hence $\dot{u}\dot{e} = \dot{u}$.

Now let $\phi \in L^1$ be as stated in the lemma. We want to show that there is a function $\psi \in L^1$ such that $\phi_*\psi - e \equiv 0 \pmod J$; i.e., $\dot{\phi}$ is invertible in A . By the standard facts about commutative Banach algebras *with unit*, it suffices to show that for every complex homomorphism $\omega \neq 0$ of A , we have $\omega(\dot{\phi}) \neq 0$.

Fix ω . Then $u \mapsto \omega(\dot{u})$ is a nontrivial continuous homomorphism of L^1 into the complex numbers. Hence ([23], p. 7) there is a point $\lambda \in \mathbb{R}$ such that

$$\omega(\dot{u}) = \hat{u}(\lambda) \ , \quad u \in L^1 \ .$$

If $\hat{u}(K) = \{0\}$ then $u \in J$ and hence $\hat{u}(\lambda) = \omega(\dot{u}) = \omega(0) = 0$. It follows that $\lambda \in K$ (by the separation property expressed in lemma B). Thus $\omega(\dot{\phi}) = \hat{\phi}(\lambda) \neq 0$, as required ■

PROOF OF 1.7. We first prove the more elementary inclusion $\supseteq$. Let λ be a common zero of $\{\hat{\phi}: \phi \in S\perp\}$. Then for each $\phi \perp S$ we have

$$\langle e^{i\lambda\cdot}, \phi\rangle = \hat{\phi}(\lambda) = 0 \ ,$$

and hence by a standard separation theorem (and the fact that $\tilde{S}$ is a weak*-closed linear subspace of L^∞), we have $e^{i\lambda\cdot} \in \tilde{S}$.

Assume $\lambda \notin sp(S)$. We have observed that $sp(S) = sp(\tilde{S})$, so there is a neighborhood V of λ having the property that $\langle \tilde{S}, \psi\rangle = \{0\}$ for every $\psi \in L^1$ such that $\hat{\psi}$ lives in V. By lemma B there is a $\psi \in L^1$ such that $\hat{\psi}$ lives in V and $\hat{\psi}(\lambda) \neq 0$. Hence

$$\langle e^{i\lambda\cdot}, \psi\rangle = \hat{\psi}(\lambda) \neq 0 \ ,$$

and since $e^{i\lambda\cdot} \in \tilde{S}$, this is a contradiction.

For the other inclusion choose λ in $sp(S)$ and $\phi \in L^1$, $\phi \perp S$. We claim: $\hat{\phi}(\lambda) = 0$. But if $\hat{\phi}(\lambda) \neq 0$ then $\hat{\phi} \neq 0$ on some compact neighborhood of λ and the preceding lemma provides a $\psi_0 \in L^1$ such that $\hat{\phi}\hat{\psi}_0 = 1$ on an open set V containing λ. Now choose any $\psi \in L^1$ such that $\hat{\psi}$ lives in V. Then

$$\hat{\phi}\hat{\psi}_0\hat{\psi} = \hat{\psi} \quad \underline{\text{everywhere}}\ \underline{\text{in}}\ \hat{\mathbb{R}} \ ,$$

and hence $\phi * \psi_0 * \psi = \psi$. Since the annihilator of $\tilde{S}$ is a closed translation invariant subspace of L^1, it is an ideal relative to convolution ([23], p. 157), hence $\psi \perp S$. This show that no point of V can be an essential point of $\tilde{S}$ (or of S), and we have arrived at a contradiction ■

COROLLARY 1. *Every weak*-closed translation invariant subspace* S *of* L^∞ *admits analysis: indeed,*

$$e^{i\lambda\cdot} \in S \quad \underline{\text{iff}} \quad \lambda \in sp(S) \ .$$

PROOF. 1.7 identifies $sp(S)$ with the set of all $\lambda \in \hat{\mathbb{R}}$ such that

$$\langle e^{i\lambda\cdot}, \phi\rangle = \hat{\phi}(\lambda) = 0$$

for every $\phi \in L^1$ which is orthogonal to S. Since S is a weak*-closed linear subspace of the dual of L^1, this assertion follows from an application of a separation theorem ■

COROLLARY 2 (WIENER'S TAUBERIAN THEOREM [27]). *Let* ϕ *be an integrable function whose Fourier transform never vanishes. Then the translates of* ϕ *span* L^1 .

PROOF. Let Φ be the closed linear span of the translates of ϕ , and put

$$S = \Phi^{\perp} = \{f \in L^{\infty}: \langle f, \Phi \rangle = \{0\}\} .$$

By the Hahn-Banach theorem it is enough to show that $S = \{0\}$.

If $S \neq \{0\}$ then we can find a point $\lambda \in sp(S)$, by the corollary of 1.6. By Corollary 1 above we have $e^{i\lambda \cdot} \in S$, and hence

$$\hat{\phi}(\lambda) = \langle e^{i\lambda \cdot}, \phi \rangle = 0$$

because $\phi \in \Phi$, a contradiction ■

REMARK. The same proof shows that if Φ is a closed ideal in L^1 whose set of Fourier transforms has *no common zero*, then necessarily $\Phi = L^1$. This somewhat more general statement is also called the Wiener Tauberian theorem.

We turn now to the problem of spectral synthesis, and for this it is convenient to shift attention from a given weak*-closed translation invariant subspace $S \subseteq L$ to its spectrum.

Let F be a closed subset of $\hat{\mathbb{R}}$. We associate two weak*-closed translation invariant subspaces with F as follows:

$$S_{min}(F) = \overline{\text{span}}^{w^*} \{e^{i\lambda \cdot}: \lambda \in F\} ,$$

$$S_{max}(F) = \{f \in L^{\infty}: sp(f) \subseteq F\} .$$

It is clear that $S_{min}(F) \subseteq S_{max}(f)$, and we leave it to the reader to verify that both spaces have spectrum F .

DEFINITION 1.8. F *is called a set of spectral synthesis if*

$$S_{min}(F) = S_{max}(F) .$$

We will see presently that not every closed set in $\hat{\mathbb{R}}$ admits synthesis. The relevance of this property to the problems we have been discussing is apparent. For if S is any (closed translation invariant) linear subspace of L^{∞} having spectrum F , then clearly $S \subseteq S_{max}(F)$. Moreover, since S must admit analysis we have in fact

$$S_{min}(F) \subseteq S \subseteq S_{max}(F) .$$

So if $F = sp(S)$ is a *set* of synthesis, then necessarily the linear space $S = S_{min}(F)$ admits systhesis.

Conversely, if F is not a set of systhesis then there is such a linear space S having spectrum F for which systhesis fails ($S = S_{max}(F)$ provides one example; in general there are infinitely many such spaces lying between $S_{min}(F)$ and $S_{max}(F)$ [33]).

The first and simplest example of a closed set failing spectral synthesis was given by L. Schwartz [26]. This example is for the group $\mathbb{R}^3$ rather than $\mathbb{R}$, and will be described presently. Subsequently, Malliavan [21] constructed examples in the dual of any noncompact LCA group.

The following result gives a criterion for synthesis that is somewhat more tractable than the definition itself.

PROPOSITION 1.9. _A closed set_ $F \subseteq \mathbb{R}$ _is a set of sepctral systhesis if, and only if, for every function_ $\phi \in L^1$ _whose Fourier transform vanishes on_ F , _there is a sequence_ $\phi_n \in L^1$ _satisfying_

(i) $\hat{\phi}_n$ _vanishes near_ F , _and_

(ii) $\|\phi - \phi_n\|_1 \to 0$ _as_ $n \to \infty$.

PROOF. Define two subspaces Φ_0 , Φ_1 , of L^1 as follows:

$$\Phi_0 = \{\phi \in L^1 : \hat{\phi} \text{ vanishes on } F\}$$

$$\Phi_1 = \overline{\{\phi \in L^1 : \hat{\phi} \text{ vanishes near } F\}} ,$$

the bar denoting norm closure in L^1 . Clearly Φ_0 contains Φ_1 and, by the Hahn-Banach theorem, it suffices to show that

$$S_{min}(F) = \Phi_0^{\perp}$$

$$S_{max}(F) = \Phi_1^{\perp} .$$

The first assertion is immediate; for if $\phi \in L^1$ then

$$\hat{\phi}(\lambda) = \langle e^{i\lambda\cdot}, \phi\rangle ,$$

and hence $\hat{\phi} = 0$ on F iff $\phi \perp S_{min}(F)$.

For the second assertion, let f be a bounded function orthogonal to Φ_1 . Then for every $\phi \in L^1$ satisfying $\hat{\phi} = 0$ near F we have $\phi \in \Phi_1$ by definition of Φ_1 and hence $\langle f, \phi\rangle = 0$. By statement (ii) of the corollary of theorem 1.6, we have $sp(f) \subseteq F$.

Conversely, suppose $f \in S_{max}(F)$. Then for every $\phi \in L^1$ such that $\hat{\phi} = 0$ near F , we have $\hat{\phi} = 0$ near $sp(f) \subseteq F$, hence theorem 1.6 implies $\langle f, \phi\rangle = 0$. Since such functions ϕ are dense in Φ_1 we conclude that $f \perp \Phi_1$ ■

Now let Φ be a closed ideal of L^1 and let S be its annihilator in L^{∞} :

$$S = \Phi^{\perp} .$$

By theorem 1.7, $sp(S)$ is the set of all common zeros of the Fourier transforms of the functions in Φ . Thus, one can restate Wiener's Tauberian theorem in an irritatingly vacuous way: <u>the empty set is a set of spectral synthesis</u>.

The assertion that <u>finite</u> sets admit synthesis has the following interpretation. Let S be a weak*-closed translation invariant subspace of L^{∞} having spectrum $\{\lambda_1, \ldots, \lambda_n\}$. Then every element f of S is a linear combination of the form

$$f(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t} ,$$

where $a_1, \ldots, a_n$ are scalars. Using Lemma B, it is easy to reduce the proof of this assertion about finite sets to the case of singletons, and the latter follows from 1.9 and the following nontrivial fact about Fourier transforms ([23], p. 51).

LEMMA C. <u>Let</u> $\phi \in L^1$, $\varepsilon > 0$, <u>and suppose</u> ϕ <u>vanishes at</u> $\lambda \in \hat{\mathbb{R}}$. <u>Then there is a function</u> $u \in L^1$ <u>such that</u> $\|u\|_1 < 3$, $\hat{u}$ <u>vanishes near</u> λ , <u>and</u> $\|\phi - \phi * u\|_1 < \varepsilon$.

Notice the connection between Lemmas A and C: Lemma A is the analogue of Lemma C for the point at infinity.

We conclude this section with a brief description of Schwartz's example [26] of a compact set in $\mathbb{R}^3$ which fails spectral synthesis. The set is simply the 2-sphere

$$\Sigma = \{\lambda \in \hat{\mathbb{R}}^3 : |\lambda| = 1\} ,$$

$|\cdot|$ denoting the Euclidean norm in $\hat{\mathbb{R}}^3$. Let Φ_0 and Φ_1 be the two associated subspaces in $L^1(\mathbb{R}^3)$;

$$\Phi_0 = \{\phi \in L^1(\mathbb{R}^3) : \hat{\phi} \text{ vanishes on } \Sigma\} ,$$

$$\Phi_1 = \overline{\{\phi \in L^1(\mathbb{R}^3) : \hat{\phi} \text{ vanishes near } \Sigma\}} .$$

To prove that synthesis fails for Σ it is equivalent to prove that $\Phi_0 \neq \Phi_1$; and for that it is sufficient to exhibit a bounded function f on $\mathbb{R}^3$ whose associated linear functional

$$\rho(\phi) = \langle f, \phi \rangle , \quad \phi \in L^1(\mathbb{R}^3) ,$$

annihilates Φ_1 but not Φ_0 .

Initially, ρ is defined as a linear functional on the space $C_0^{\infty}(\mathbb{R}^3)$ of all compactly supported infinitely differentiable functions as follows:

$$\rho(\phi) = \int_{\Sigma} \frac{\partial \hat{\phi}}{\partial x}\, d\sigma$$

where σ is normalized area measure on Σ and $\frac{\partial}{\partial x}$ denotes partial differentiation relative to the first coordinate variable in $\hat{\mathbb{R}}^3$. Notice that the derivative in the integrand exists and is continuous for functions $\phi \in C_0^\infty(\mathbb{R}^3)$.

It is obvious that if $\phi \in C_0^\infty(\mathbb{R}^3)$ is such that $\hat{\phi} = 0$ near Σ, then $\rho(\phi) = 0$. And it is almost equally obvious that there exist functions $\phi \in C_0^\infty(\mathbb{R})^3$ for which $\hat{\phi}$ vanishes on Σ but $\rho(\phi) \neq 0$.

It is less obvious that ρ is bounded relative to the L^1 norm on $C_0^\infty(\mathbb{R}^3)$, but in fact we have

$$|\rho(\phi)| \leq (2\pi)^{-3}\|\phi\|_1 .$$

Indeed, one can exhibit a bounded function f (explicitly in terms of the inverse Fourier transform of the measure σ) such that

$$\rho(\phi) = \langle f, \phi \rangle ,$$

and from this it follows quite easily that $\Phi_0 \neq \Phi_1$. For the details see (23 , pp. 165-166) ■

In the next section we will discuss the application of these ideas to automorphism groups of von Neumann algebras. We also want to point out that there are significant and very close parallels in the theory of reflexive (non self-adjoint) operator algebras. These are discussed in [29], and we will have no more to say about that subject here.

2. THE SPECTRUM OF A GROUP OF ISOMETRIES.

Let G be a locally compact abelian group, fixed throughout this section and the next. Let $t \mapsto U_t$ be a representation of G as isometries of a complex Banach space X. Thus, each U_t is an invertible linear isometry on X and we have $U_{s+t} = U_s U_t$, $s, t \in G$. For the moment, we will assume the strong continuity condition

2.0 $$\lim_{t \to 0} \|U_t x - x\| = 0 , \quad \text{for every } x \in X .$$

Let X_* denote the dual space of X (the star appears as a subscript rather than a superscript in order to accommodate a more general situation we will encounter presently). In order to define the spectrum of U we make use of the associated representation of the convolution group algebra $L^1 = L^1(G)$, defined as follows. For each $\phi \in L^1$ there is a unique bounded operator U_ϕ on X

satisfying

$$\rho(U_\phi x) = \int_G \phi(t)\rho(U_t^{-1}x)dt ,$$

for every $x \in X$, $\rho \in X_*$. This will also be written as an operator-valued integral

$$U_\phi = \int_G \phi(t)U_t^{-1}dt .$$

We have $\|U_\phi\| \leq \|\phi\|$, $U_{\phi*\psi} = U_\phi U_\psi$, and for any bounded approximate identity ϕ_n for L^1 , $U_{\phi_n} \to 1$ in the strong operator topology.

A character $\lambda \in \hat{G}$ is said to be an <u>essential point</u> of U if for every neighborhood V of λ , there is a ϕ in L^1 such that

2.1 (i) $\hat{\phi}$ lives in V

(ii) $U_\phi \neq 0$.

DEFINITION 2.2. The <u>spectrum</u> of U is the set of all essential points of U . This subset of $\hat{G}$ is denoted $sp(U)$. It is clear that $sp(U)$ is closed; for if $\lambda \in \hat{G}$ is not essential, then there is a neighborhood V of λ such that for every $\phi \in L^1$ whose Fourier transform lives in V , we have $U_\phi = 0$. Thus no point of V can be in $sp(U)$, and so the complement of $sp(U)$ is open.

For our purposes, the main examples are where X is a Hilbert space and $\{U_t\}$ is a group of unitary operators, or where X is a C^*-algebra and U is a group of $*$-automorphisms. In order to discuss weakly continuous $*$-automorphism groups of von Neumann algebras, one has to broaden the above setting somewhat.

Specifically, one is given along with a Banach space X , a norm closed linear subspace X_* of the dual of X , satisfying

$$\sup\{|\rho(x)|: \rho \in X_*, \|\rho\| \leq 1\} = \|x\|$$

for every $x \in X$, and which satisfies the following "completeness" axioms:

the X_*-closed convex hull of any X_*-compact subset of X is X_*-compact,

and the dual condition obtained by exchanging X and X_* . We refer somewhat abusively to the X_*-topology on X as the <u>weak</u> topology. In the case where X is a Hilbert space or a C^*-algebra,

X_* will normally be the full dual of X ; if X is a von Neumann algebra then X_* will be the predual of X .

For a group $\{U_t\}$ of isometries on X to be admissible, it is necessary that each U_t be <u>weakly</u> continuous, and that <u>one</u> of the following two continuity conditions should be satisfied:

(i) $\lim_{t\to 0}\|U_t x - x\| = 0$, for every $x \in X$,

or its dual

(ii) $\lim_{t\to 0}\|\rho \circ U_t - \rho\| = 0$, for every $\rho \in X$.

When U is a weakly continuous group of $*$-automorphisms of a von Neumann algebra X , it is not hard to show that (ii) is satisfied even though (i) normally fails. For the proof of this, as well as for technical details about vector integrals omitted from the discussion to follow, we refer the reader to [1].

The point is that in this more general setting one can smear the group representation U to a representation of L^1 exactly as if $\{U_t\}$ were strongly continuous,

$$U_\phi = \int_G \phi(t)U_t^{-1}dt , \quad \phi \in L^1 .$$

More generally, for a complex measure μ in the convolution algebra $M(G)$ of all regular Borel measures of G having finite total variation, we define an operator U_μ as follows. For each $x \in X$ and $\rho \in X_*$, we can integrate the function $t \mapsto \rho(U_t^{-1}x)$ against μ to obtain a bounded bilinear form in ρ and x . Then (see [1], section 1) there is a unique operator U_μ on X satisfying

$$\rho(U_\mu x) = \int_G \rho(U_t^{-1}x)\,d\mu(t) ,$$

for all $\rho \in X_*$, $x \in X$. U_μ is not only bounded but <u>weakly</u> continuous on X . This integration procedure will be expressed in the usual notation

$$U_\mu = \int_G U_t^{-1}d\mu(t) ,$$

for every μ in $M(G)$. $\mu \mapsto U_\mu$ is a unital homomorphism of Banach algebras satisfying $\|U_\mu\| \leqslant \|\mu\|$ for every $\mu \in M(G)$.

Now $\phi \mapsto U$ is, in particular, a <u>bona fide</u> representation of L^1 . Thus we may define essential points and $sp(U)$ exactly as we did in 2.1 and 2.2, and the argument given above shows that $sp(U)$ is closed. We will make frequent use of the following assertion, which is a straightforward application of a separation theorem:

2.4 X is the weakly closed linear span of the ranges of all operators $\{U_\phi : \phi \in L^1\}$.

The kernel of the map $\mu \mapsto U_\mu$ is a closed ideal in $M(G)$. The following result shows that this ideal is nested between two natural ideals associated with the set $sp(U)$, and has several significant consequences.

THEOREM 2.5. For every $\mu \in M(G)$ we have:

(i) If $\hat{\mu} = 0$ near $sp(U)$, then $U_\mu = 0$.

(ii) If $U_\mu = 0$, then $\hat{\mu} = 0$ on $sp(U)$.

We remark that if $sp(U) = \emptyset$, (i) should be interpreted as the assertion $U_\mu = 0$ for every $\mu \in M(G)$.

PROOF OF 2.5. (i) is a small variation of theorem 1.6. Indeed, assume that $\hat{\mu} = 0$ near $sp(U)$. Since the ranges of the operators U_f , $f \in L^1(G)$, span X , it suffices to show that $U_\mu U_f = U_{\mu * f} = 0$ for every $f \in L^1(G)$.

But $\phi = \mu * f$ belongs to $L^1(G)$ and its transform vanishes near $sp(U)$. The proof can now be completed exactly as in 1.6. That is, one first assumes that $\hat{\phi}$ has compact support, and constructs a function $\psi \in L'(G)$ such that $U_\psi = 0$ and $\phi = \phi * \psi$. The general case can be reduced to this by an application of lemma A as in the end of the proof of 1.6.

For (ii), let $\mu \in M(G)$ be such that $U_\mu = 0$. Notice that we can assume $\mu \in L^1(G)$. Indeed, to prove that $\hat{\mu} = 0$ on $sp(U)$ it clearly suffices to prove that $\hat{\mu}\hat{f} = 0$ on $sp(U)$ for each $f \in L^1(G)$; and since $\mu * f \in L^1(G)$ satisfies $U_{\mu * f} = U_\mu U_f = 0$, the reduction is apparent.

We claim that if $\phi \in L^1$, $U_\phi = 0$, and $\lambda \in \hat{G}$ is such that $\phi(\lambda) \neq 0$, then λ is not essential for U . For by the lemma preceding the proof of 1.7, we may find a function $\psi \in L^1$ satisfying $\hat{\phi}\hat{\psi} = 1$ near λ . So any functions $u \in L^1$ whose transform lives in the λ-neighborhood

$$\{\gamma \in \hat{G} : \hat{\phi}(\gamma)\psi(\gamma) = 1\}$$

must staisfy $\hat{u} = \hat{u}\hat{\phi}\hat{\psi}$, hence $u = u * \phi * \psi$, hence $U_u = U_u U_\phi U_\psi = 0$. This clearly implies that λ is inessential ∎

The special case of 2.5(i) in which $sp(U) = \emptyset$ is especially significant, for we have

COROLLARY 1. $sp(U) \neq \emptyset$ whenever $X \neq \{0\}$.

PROOF. If $sp(U) = \emptyset$, then (i) implies that $U_\phi = 0$ for every $\phi \in L^1$. Since X is spanned by vectors of the form

$U_\phi x$, $\phi \in L^1$, $x \in X$, we conclude that $X = \{0\}$

The following result shows that the definition of sp(U) given in [1], in terms of hulls and kernels of ideals, is equivalent to the one we have given here. Putting $\ker U = \{\phi \in L^1: U_\phi = 0\}$, we have

COROLLARY 2.

$$sp(U) = \{\lambda \in \hat{G}: \hat{\phi}(\lambda) = 0 \text{ for every } \phi \in \ker U\} .$$

PROOF. If $\lambda \in sp(U)$, then for every $\phi \in \ker U$ we have $\hat{\phi}(\lambda) = 0$ by 2.5(ii).

Conversely, if λ is not an essential point of U then there is a neighborhood V of λ such that $U_\phi = 0$ for every $\phi \in L^1$ such that $\hat{\phi}$ lives in V . By lemma B of sections 1 we may find such a ϕ satisfying $\hat{\phi}(\lambda) = 1$. This shows that λ is not a common zero for the set of Fourier transforms of ker U ∎

COROLLARY 3. If F is any closed set in $\hat{G}$ having the property (i) in the sense that for every $\phi \in L^1$, $\hat{\phi} = 0$ near $F \Rightarrow U_\phi = 0$, then F contains sp(U) .

PROOF. Suppose $\lambda \in sp(U)$ and $\lambda \notin F$. By lemma B, we can find $\phi \in L^1$ such that $\hat{\phi}(\lambda) = 1$ and $\hat{\phi} = 0$ outside a closed neighborhood of λ which misses F . Then $\hat{\phi}$ vanishes near F , hence $U_\phi = 0$ by hypothesis, and by 2.5(ii) we obtain a contradiction $\hat{\phi}(\lambda) = 0$ ∎

Corollary 3 allows us to get rough estimates on the spectra of subrepresentations and quotient representations of U . Specifically, let M be a closed subspace of X invariant under $\{U_t: t \in G\}$. Then we may define a subrepresentation V of G on M by

$$V_t = U_t|_M , \quad t \in G ,$$

and a quotient representation W of G on X/M by

$$W_t(x + M) = W_t x + M , \quad t \in G , \quad x \in X .$$

For each ϕ in L^1 such that $U_\phi = 0$, we clearly have $V_\phi = 0$ and $W_\phi = 0$. Thus, if $\hat{\phi}$ vanishes near sp(U) , we conclude from 2.5(ii) that $V_\phi = W_\phi = 0$. So Corollary 3 implies

2.6
$$sp(V) \subseteq sp(U) , \text{ and}$$
$$sp(W) \subseteq sp(U) .$$

It is known that if the Fourier transform of a function ϕ in $L^1(G)$ never vanishes, then the translates of ϕ span $L^1(G)$ For the case $G = \mathbb{R}$ this is equivalent to Wiener's Tauberian theorem. Now if we take U to be the regular representation of G on L^1, i.e.,

$$U_t f(x) = f(x+t) ,$$

then the closure of the range of the operator U_ϕ is easily seen to be identical with the closed linear span of the translates of ϕ. Thus the following is a generalization of Wiener's theorem to the context of arbitrary representations of G :

COROLLARY 4 (TAUBERIAN THEOREM). Let $\mu \in M(G)$. If $\hat{\mu}$ has no zeros on $sp(U)$, then U_μ has dense range and trivial null-space.

PROOF. Let M be the kernel of U_μ. M is invariant under the action of U and so it determines a subrepresentation V. Clearly $V_\mu = U_\mu|_M = 0$, and so by 2.5(ii), $\hat{\mu}$ vanishes on $sp(V)$. By 2.6, $sp(V) \subseteq sp(U)$ and thus $sp(V)$ must be void. By corollary 1 above, we conclude that $M = \{0\}$.

Similarly, let W be the quotient representation of U on $X/\overline{U_\mu X}$. Again, we have $W_\mu = 0$ and hence $\hat{\mu} = 0$ on $sp(W)$; since $sp(W) \subseteq sp(U)$ we can argue in the same way to conclude that $X/\overline{U_\mu X} = \{0\}$, and hence $U_\mu X$ is dense in X ■

We turn now to a discussion of the mapping properties of $sp(U)$. For each $\phi \in L^1$, U_ϕ is a bounded linear operator on the Banach space X and therefore has an operator-theoretic spectrum $\sigma(U_\phi)$. The basic assertion connecting these two spectra is the following:

2.7
$$\sigma(U_\phi) = \hat{\phi}(sp(U)) , \quad \text{or}$$
$$\sigma(U_\phi) = \hat{\phi}(sp(U)) \cup \{0\} ,$$

according as $sp(U)$ is compact or noncompact. So in general $\sigma(U_\phi)$ is the <u>closure</u> of the range of $\hat{\phi}$ on $sp(U)$.

We will prove 2.7 presently. Note first that 2.7 suggests a more general conjecture, namely that for every measure μ in $M(G)$,

2.8
$$\sigma(U_\mu) = \overline{\{\hat{\mu}(\lambda) : \lambda \in sp(U)\}} .$$

We will prove the inclusion $\supseteq$ below; but in general $\hat{\mu}(sp(U))$ is not dense in $\sigma(U_\mu)$. The precise nature of $\sigma(U_\mu)$ is not very well understood, and the mystery is connected with the fact that $\hat{G}$ fails to be dense in the maximal ideal space of $M(G)$ whenever G is nondiscrete. For example, taking U_x to be translation by x in $L^1(\mathbb{R})$, we have

PROPOSITION 2.9. For every $\mu \in M(\mathbb{R})$, $\sigma(U_\mu)$ is the range of the Gelfand transform of μ on the maximal ideal space of $M(\mathbb{R})$.

PROOF. The map

$$\mu \in M(\mathbb{R}) \mapsto U_\mu \in \mathcal{L}(L^1)$$

is an isometric isomorphism of $M(\mathbb{R})$ onto a maximal abelian algebra $\mathfrak{A}$ of operators on L^1 (23 , pp. 74-75). Therefore the operator theoretic spectrum of U_μ is the same as the spectrum of U_μ relative to the Banach algebra $\mathfrak{A}$ which contains it, and the assertion follows ■

Since the real line is not dense in the maximal ideal space of $M(\mathbb{R})$ ([23], pp. 107-108), it follows that there are measures μ such that $\hat{\mu}(\hat{\mathbb{R}})$ is not dense in $\sigma(U_\mu)$. The failure of 2.7 for measures μ was first pointed out in this context by D'Antoni, Longo, and Zsido [31].

The best thing one can say in general is the following

PROPOSITION 2.10. For every $\mu \in M(G)$,

$$\hat{\mu}(\mathrm{sp}(U)) \subseteq \sigma(U_\mu) .$$

For the proof, we require

LEMMA. Let μ belong to $M(G)$, take $\lambda \in \hat{G}$ and $\varepsilon > 0$. Then there is a function $\phi \in L^1$ such that $\hat{\phi} = 1$ near λ and $\|\mu * \phi - \hat{\mu}(\lambda)\phi\|_1 < \varepsilon$.

PROOF. Choose any u in L^1 such that $\hat{u} = 1$ near λ (lemma B), and define $v = \mu * u - \hat{\mu}(\lambda)u$. v belongs to L^1 and $\hat{v}$ vanishes at λ . By lemma C there is a function $\psi \in L^1$ such that $\hat{\psi}$ vanishes near λ and $\|v - v*\psi\|_1 < \varepsilon$. Put $\phi = u - u*\psi$ ■

PROOF OF 2.10. Fix $\lambda \in \hat{G}$. It is enough to show that the operator $U_\mu - \hat{\mu}(\lambda)1$ is a topological divisor of zero (for no such operator can be invertible). For this, fix $\varepsilon > 0$. We will construct a nonzero operator $T = T_\varepsilon$ so that

$$\|U_\mu T - \hat{\mu}(\lambda)T\| \leqslant \varepsilon\|T\| .$$

Choose $\phi \in L^1$ as in the lemma. If ψ is any integrable function such that $\hat{\psi}(\lambda) = 1$ and $\hat{\psi}$ lives in $\{\gamma \in \hat{G}: \hat{\phi}(\gamma) = 1\}$, then $U_\psi \neq 0$ (by 2.5(ii)) and $\psi = \phi * \psi$ (because $\hat{\psi} = \hat{\phi}\hat{\psi}$ on $\hat{G}$). Thus $U_\psi = U_\phi U_\psi$ and hence

$$\begin{aligned}\|(U_\mu - \hat{\mu}(\lambda))U_\psi\| &= \|(U_\mu - \hat{\mu}(\lambda))U_\phi U_\psi\| \\ &= \|(U_{\mu*\phi} - \hat{\mu}(\lambda)U_\phi)U_\psi\| \\ &\leqslant \|\mu*\phi - \hat{\mu}(\lambda)\phi\|_1 \|U_\psi\| \leqslant \varepsilon\|U_\psi\| \ .\end{aligned}$$

Thus we can put $T = U_\psi$ ■

Turning now to the opposite inclusion of 2.8, let G and $U: G \to L(X)$ be fixed throughout the remainder of this section. For any commutative Banach algebra A (with or without unit), we write $\sigma(A)$ for the space of all nontrivial complex homomorphisms of A. A is said to regular if the Gelfand transforms of the elements of A separate points from closed sets in $\sigma(A)$; that is, for any closed set $F \subseteq \sigma(A)$ and any point $\omega \notin F$, there should exist an element $a \in A$ satisfying $\hat{a} = 0$ on F and $\omega(a) = 1$. The most familiar examples are the group algebras L^1 of a locally compact abelian group. The full measure algebra $M(G)$ is never regular (when G is nondiscrete). Finally, for $\phi \in L^1$ and $\sigma \in M(G)$, we will write $\phi + \sigma$ for the measure

$$d\mu(x) = \phi(x)dx + d\sigma(x) \ ,$$

and similarly $L^1 + S$ will denote the set of all L^1-perturbations of elements of S, for S any subset of $M(G)$. Our objective is to show that if A is any regular Banach subalgebra of $M(G)$ containing 1, then one has a mapping theorem

$$\sigma(U_\mu) = \overline{\hat{\mu}(sp(U))}$$

for every measure μ in the closure of $L^1 + A$.

Now with any subalgebra A of $M(G)$ we have an associated Banach algebra of operators on X:

$$U(A) = \overline{\{U_\mu : \mu \in A\}}^{\|\cdot\|} \ .$$

Since L^1 is an ideal in $M(G)$, $U(A) \cap U(L^1)$ is a closed ideal in $U(A)$. An element $\omega \in \sigma(U(A))$ is said to be singular if it vanishes on $U(A) \cap U(L^1)$; the remaining elements of $\sigma(U(A))$ are called nonsingular. Of course $U(A) \cap U(L^1)$ may be zero, in which case every element of the spectrum of $U(A)$ is singular.

Notice that we have a natural mapping of $sp(U)$ into $\sigma(U(A))$. Indeed, if $\lambda \in sp(U)$ then by 2.10 we have $\hat{\mu}(\lambda) \in \sigma(U_\mu)$, and hence $|\hat{\mu}(\lambda)| \leqslant U_\mu$. It follows that there is a unique ω_λ in $\sigma(U(A))$ satisfying

$$\omega_\lambda(U_\mu) = \hat{\mu}(\lambda) \ , \quad \mu \in A \ .$$

The map $\lambda \mapsto \omega_\lambda$ is continuous but not necessarily one-to-one. We will require some information about singular elements.

PROPOSITION 2.11. Assume A is a unital regular subalgebra of $M(G)$ and let ω be a singular element of $\sigma(U(A))$. Then for every compact set $K \subseteq \hat{G}$, ω belongs to the closure in $\sigma(U(A))$ of $\{\omega_\lambda : \lambda \in sp(U) \setminus K\}$.

PROOF: Contrapositively, let $K \subseteq \hat{G}$ be compact and assume ω is not in the closure Ω of $\{\omega_\lambda : \lambda \in sp(U) \setminus K\}$. We will exhibit a measure $\sigma \in A$ for which $U_\sigma \in U(L^1)$ and $\omega(U_\sigma) = 1$.

Now the map $\mu \in A \mapsto U_\mu \in U(A)$ induces, by composition, a <u>one-to-one</u> continuous map ϕ of $\sigma(U(A))$ into $\sigma(A)$. Thus $\phi(\Omega)$ is compact in $\sigma(A)$ and $\phi(\omega) \notin \phi(\Omega)$. Let W be an open set in $\sigma(A)$ such that

$$\phi(\Omega) \subseteq W,$$

$$\phi(\omega) \notin \bar{W}.$$

By regularity of A, there is a $\sigma \in A$ whose Gelfand transform vanishes on $\bar{W}$ for which $\omega(U_\sigma) = \phi(\omega)(\sigma) = 1$.

We claim now that the <u>Fourier</u> transform of σ vanishes near $sp(U) \setminus K$. Indeed, letting $\gamma : \hat{G} \to \sigma(A)$ be the natural map (defined by $\gamma_\lambda(\mu) = \hat{\mu}(\lambda)$, $\mu \in A$), we see that γ is continuous but again, not necessarily injective. In any case,

$$W_0 = \{\lambda \in \hat{G} : \gamma_\lambda \in W\}$$

is certainly an open set in $\hat{G}$ and, since $\gamma_\lambda = \phi(\omega_\lambda)$ for every $\lambda \in sp(U)$ and W contains $\phi(\omega_\lambda)$ for every $\lambda \in sp(U) \setminus K$, W_0 must contain $sp(U) \setminus K$ as a subset of $\hat{G}$. Thus $\hat{\sigma}$ vanishes on W_0, as asserted.

To complete the proof we show that U_σ belongs to $U(L^1)$. For that, let ϕ be any integrable function such that $\hat{\phi} = 1$ near K (lemma B). Then $\hat{\phi}\hat{\sigma} = \hat{\sigma}$ near K and of course $\hat{\phi}\hat{\sigma} = \hat{\sigma} = 0$ near $sp(U) \setminus K$. Thus $\hat{\phi}\hat{\sigma} = \hat{\sigma}$ near $sp(U)$. By Theorem 2.5 we conclude that $U_{\phi*\sigma-\sigma} = 0$, and hence $U_\sigma = U_{\phi*\sigma}$. Since $\phi*\sigma$ belongs to L^1, we are done ■

THEOREM 2.12. Let A be a unital regular subalgebra of $M(G)$. Then $\{\omega_\lambda : \lambda \in sp(U)\}$ is dense in $\sigma(U(L^1 + A))$, and consists precisely of the nonsingular elements of $\sigma(U(L^1 + A))$.

PROOF. Let $\lambda \in sp(U)$. Then there is a function $\phi \in L^1$ such that $\hat{\phi}(\lambda) \neq 0$, hence $\omega_\lambda(U_\phi) \neq 0$ and ω_λ is nonsingular.

Conversely, let $\omega \in \sigma(U(L^1 + A))$ be nonsingular. Then $\phi \in L^1 \mapsto \omega(U_\phi)$ is a nontrivial complex homomorphism of L^1 so that there is a $\lambda \in \hat{G}$ such that $\omega(U_\phi) = \hat{\phi}(\lambda)$ ([23], p. 7). Since U vanishes when $\hat{\phi} = 0$ near $sp(U)$, we have $\hat{\phi}(\lambda) = \omega(U_\phi) = 0$ for all such ϕ's, and hence $\lambda \in sp(U)$. We have $\omega = \omega_\lambda \neq 0$ on $U(L^1)$, and since $U(L^1)$ is an ideal in $U(L^1 + A)$, it follows that $\omega = \omega_\lambda$.

It remains to prove that every singular ω in $\sigma(U(L^1 + A))$ can be weak*-approximated by elements ω_λ, $\lambda \in sp(U)$. Fix such an ω. Since $U(L^1 + A)$ is spanned by elements $\{U_\phi: \phi \in L^1\}$ and $\{U_\sigma: \sigma \in A\}$, and since ω annihilates each U_ϕ, it suffices to show that for every $\varepsilon > 0$, $\phi_1, \ldots, \phi_n \in L^1$, $\sigma_1, \ldots, \sigma_n \in A$, there is a point λ in $sp(U)$ such that

$$\max_j |\hat{\phi}_j(\lambda)| \leqslant \varepsilon, \quad \underline{\text{and}}$$

$$\max_j |\omega(U_{\sigma_j}) - \sigma_j(\lambda)| \leqslant \varepsilon .$$

By the Riemann-Lebesgue lemma the functions $\hat{\phi}_j$ all vanish at infinity, so there is a compact set K in $\hat{G}$ such that $\max_j |\phi_j(\lambda)| < \varepsilon$ for all λ not in K. Now

$$\{\gamma \in \sigma(U(L \quad + A)): \max_j |\gamma(\sigma_j) - \omega(\delta_j)| < \varepsilon\}$$

is a neighborhood of ω in $\sigma(U(L \quad + A))$ which, by 2.8, must intersect $\{\omega_\lambda: \lambda \in sp(U) \quad K\}$. Any λ whose ω_λ is in this intersection will suffice ∎

Thus we have

COROLLARY 1. If A is a regular unital subalgebra of $M(G)$, then

$$\sigma(U_\mu) = \overline{\hat{\mu}(sp(U))},$$

for every measure μ in the closure of $L^1 + A$.

PROOF. Fix μ and choose a point Z in $\sigma(U_\mu)$. Then $U_\mu - z1$ fails to be invertible in $U(L^1 + A)$ so there is a complex homomorphism ω of $U(L' + A)$ such that $\omega(U_\mu) - z = 0$. By the preceding result, ω is a weak*-limit of a net ω_{λ_j}, $\lambda_j \in sp(U)$, thus

$$z = \omega(U_\mu) = \lim_j \omega_{\lambda_j}(U_\mu) = \lim_j \hat{\mu}(\lambda_j),$$

as required ∎

REMARKS. For μ in L^1, this result was discovered and rediscovered by several people (it appears explicitly in [8], 2.3.7, for automorphism groups of von Neumann algebras). For point masses μ it is due to Connes ([8], 2.3.8). These results were generalized significantly by D'Antoni, Longo, and Zsido [31] to the case where μ admits a decomposition

$$\mu = \phi + \sigma$$

where $\phi \in L^1$ and σ is a discrete measure on G. To deduce their result from the above, take A to be the algebra of all discrete measures on G and notice that regularity of A follows from the fact that A is isomorphic to the group algebra $L^1(G_d)$, where G_d is the "discretized" group obtained from G.

One can obtain other examples by taking a closed subgroup $H < G$. The set of all measures which are concentrated on H and absolutely continuous with resepct to the Haar measure on H generates, together with the identity $\delta_0 \in M(G)$, another such regular subalgebra A. If G is not discrete and $H \neq G$, these measures are all singular relative to Haar measure on G.

It seems appropriate to point out that the first theorems of this general type are due to Beurling [3], [4] and, independently, to Wiener and Pitt [28]. These papers were published in 1938. In modern terms, Beurling's theorem deals with the measure algebra $L^1 + M_d$ on the real line (here M_d stands for the regular algebra of all discrete measures on $\mathbb{R}$), and implies that if $\mu \in L^1 + M_d$ satisfies $|\hat{\mu}(\lambda)| \geqslant \delta > 0$ for all $\lambda \in \hat{\mathbb{R}}$, then the reciprocal of $\hat{\mu}$ is the Fourier transform of a measure in $L^1 + M_d$. The result of Wiener and Pitt is that if $\mu \in M(\mathbb{R})$ has a decomposition

$$\mu = \phi + \delta + \sigma$$

where $\phi \in L^1$, $\delta \in M_d$, and σ is a nonatomic singular measure for which

$$\|\sigma\| < \inf_{\lambda \in \mathbb{R}} |\hat{\delta}(\lambda)| ,$$

then $|\mu| \geqslant \delta > 0$ on $\hat{\mathbb{R}}$ implies that the reciprocal of $\hat{\mu}$ is the Fourier transform of something in $M(\mathbb{R})$. They also constructed a measure $\mu \in M(\mathbb{R})$ for which $|\hat{\mu}| \geqslant \delta > 0$ on $\hat{\mathbb{R}}$ but the reciprocal of $\hat{\mu}$ is not the Fourier transform of a measure in $M(\mathbb{R})$. Notice that this already implies that the real line $\hat{\mathbb{R}}$ is not dense in the maximal ideal space of $M(\mathbb{R})$.

Notice that one can obtain Beurling's theorem from the corollary 2.10 by by considering the group U of all translation operators acting on $L^1(\mathbb{R})$. In this case we have $sp(U) = \hat{\mathbb{R}}$ and $U(L^1 + M_d)$ is appropriately isomorphic to $L^1 + M_d$.

The corollary of 2.10 implies the following Tauberian-like generalization of Beurling's theorem. Let A be a unital regular algebra in $M(G)$, let $U: G \to L(X)$ be a representation, and let $\mu \in L^1 + A$ satisfy $|\hat{\mu}(\lambda)| \geqslant \delta > 0$ for every λ in $sp(U)$. Then there is a sequence $\nu_n \in L^1 + A$ such that

$$\lim_{n\to\infty} \|U_\mu U_{\nu_n} - 1\| = 0 .$$

Of course, the sequence U_{ν_n} converges in the operator norm to the inverse of U_μ .

It would clearly be of interest to have additional classes of measures in $M(G)$ for which the mapping property 2.8 holds. In particular, if $\mu \in M(G)$ is such that $\hat{\mu}$ vanishes at infinity, is $\hat{\mu}(sp(U))$ dense in $\sigma(U_\mu)$? The algebra of all such measures is an ideal in $M(G)$ which has an apparently more tractable maximal ideal space than does $M(G)$ itself; but the answer is unknown to us. An elementary example of such a measure on $G = \mathbb{R}^3$ is the surface area measure concentrated on the 2-sphere $S^2 \subseteq \mathbb{R}^3$ ([10], pp. 201-205).

COROLLARY 2. The map $\lambda \mapsto \omega_\lambda$ establishes a homeomorphism of $sp(U)$ onto $\sigma(U(L^1))$.

PROOF. Let A be the one-dimensional subalgebra of $M(G)$ consisting of all scalar multiples of the unit point mass δ_0 . A is of course regular.

By 2.12, $\lambda \mapsto \omega_\lambda$ is a map of $sp(U)$ onto the set of nonsingular elements of $U(L^1 + A)$, and the latter is clearly identifiable with $\sigma(U(L^1))$.

Thus it suffices to show that if $\lambda, \lambda_\alpha \in sp(U)$ are such that $\omega_{\lambda_\alpha} \to \omega_\lambda$ in $\sigma(U(L^1))$, then $\lambda_\alpha \to \lambda$ in the topology of $\hat{G}$. But if $\omega_{\lambda_\sigma} \to \omega_\lambda$, then for every $\phi \in L^1$ we have

$$\hat{\phi}(\lambda_\alpha) = \omega_{\lambda_\alpha}(U_\phi) \to \omega_\lambda(U_\phi) = \hat{\phi}(\lambda) ,$$

and the conclusion follows from the fact that $\hat{G}$ is homeomorphically identified with $\sigma(L^1)$ ∎

This result has various formulations for the case $G = \mathbb{R}$. It can be shown, for example, that Corollary 2 above implies that the spectrum of a one-parameter group of isometries having infinitesimal generator D , i.e.,

$$U_t = e^{itD}$$

is identical with the operator-theoretic spectrum of the generator D . The latter result is due to D.E. Evans [34]. A version of this (for bounded generators) is proved in 4.7 below.

The following characterization of norm-continuous representations was discovered by Olesen [22].

THEOREM 2.13. $sp(U)$ is compact if, and only if, $\lim_{x\to 0}\|U_x - 1\| = 0$.

PROOF. Assume first that $sp(U)$ is compact. Then we may find a ϕ in L^1 for which $\hat{\phi} = 1$ near $sp(U)$ (lemma B). Notice that $U_\phi = 1$. For if $\psi \in L^1$ is arbitrary, then $\hat{\phi}\hat{\psi} = \hat{\psi}$ near $sp(U)$ so that $U_\phi U_\psi = U_{\phi*\psi} = U_\psi$. Hence $U_\phi x = x$ for every x in the range of any operator U_ψ , $\psi \in L^1$, and these x's weakly span X .

Thus we have

$$\|U_x - 1\| = \|U_x U_\phi - U_\phi\| = \|U_{\phi_x - \phi}\| \leq \|\phi_x - \phi\|_1 ,$$

which tends to zero as $x \to 0$.

Conversely, assume U is norm-continuous. Then if ϕ_n is an approximate identity for L^1 it follows that $\|U_{\phi_n} - 1\| \to 0$ as $n \to \infty$. Thus $1 \in U(L^1)$ and hence $\sigma(U(L^1))$ is compact. Moreover, every element of $\sigma(U(L^1))$ is nonsingular. By Corollary 2 of 2.12 we conclude that $sp(U)$ is compact ■

We want to present two applications of spectral theory which give useful information about the structure of representations of G . The first asserts that every representation is uniformly continuous on large invariant subspaces in X ; the second describes the harmonic structure of uniformly continuous representations. Taken together, these provide a partial substitute for Stone's theorem for unitary representations in Hilbert space.

U will be a fixed representation of G on X as above.

THEOREM 2.14. There is an increasing directed family M_α of hyperinvariant subspaces in X such that $\cup_\alpha M_\alpha$ is weakly dense in X and, for each α , $U|_{M_\alpha}$ has compact spectrum (and is therefore uniformly continuous).

REMARK. The term <u>hyperinvariant</u> means that each M_α is weakly closed and invariant under each weakly continuous operator on X which commutes with $\{U_t : t \in G\}$.

PROOF OF 2.14. Let K be the family of all compact subsets of $\hat{G}$, directed upward by inclusion. For each K in $\mathcal{K}$, let M_K denote the weakly closed span of the ranges of all operators U_ϕ, where ϕ runs over all L^1 functions whose Fourier transforms live in K. We know that X is weakly spanned by vectors of the form $U_\phi x$, $\phi \in L^1$, $x \in X$; and functions $\phi \in L^1$ whose transforms have compact support are dense in L^1 (Lemma A). Thus $\cup_K M_K$ is weakly dense in X.

Each M_K is clearly hyperinvariant, and notice that the spectrum of $U|_{M_K}$ is contained in K. For if $\phi \in L^1$ is such that $\hat{\phi} = 0$ near K, then for every $\psi \in L^1$ such that $\hat{\psi}$ lives in K we have $\hat{\phi}\hat{\psi} = 0$, hence $\phi * \psi = 0$, hence $U_\phi U_\psi x = 0$ for every x in X. Thus, $U_\phi|_{M_K} = 0$ as required ■

Stone's theorem implies that every unitary representation U of G on a Hilbert space is the Fourier transform of a projection valued measure P which is supported on $sp(U)$:

$$U_t = \int_{sp(U)} \lambda(t) dP(\lambda), \quad t \in G.$$

This expresses the representation U as an appropriate superposition of the pure frequencies in $sp(U)$. The following result gives a sense in which something like this is approximately true in general, at least when $sp(U)$ is compact. We remark that 2.14 and 2.15 together imply an "approximate" version of Stone's theorem even when $sp(U)$ is not compact; but of couse the convergence must be taken relative to a weaker topology. We leave the formulation of this result for the reader.

Let $\mathcal{L}_w(X)$ denote the Banach space of all weakly continuous linear operators on X. If $A: \hat{G} \to \mathcal{L}_w(X)$ is a (norm) continuous function on $\hat{G}$ which vanishes off some compact set $K \subseteq \hat{G}$, then we may form the inverse Fourier transform of A as a conventional Bochner integral:

$$F(t) = \int_{\hat{G}} \lambda(t) A(\lambda) d\lambda,$$

$d\lambda$ denoting Haar measure on $\hat{G}$ (we choose the version of $d\lambda$ so that the Fourier inversion theorem

$$\phi(t) = \int_{\hat{G}} \lambda(t) \hat{\phi}(\lambda) d\lambda$$

is valid for scalar functions $\phi \in L^1$ whose transforms are integrable [23], p. 22). $F: G \to \mathcal{L}_w(X)$ is norm-continuous, and we have

$$\sup_{t\in G}\|F(t)\| \leq \sup_{\lambda\in K}\|A(\lambda)\|\cdot\lambda(K) < \infty .$$

THEOREM 2.15. Assume $sp(U)$ is compact and let V be a compact set in $\hat{G}$ which contains $sp(U)$ in its interior. For every compact set $K \subseteq G$ and $\varepsilon > 0$, there is a norm-continuous function $A: \hat{G} \to \mathcal{L}_w(X)$ which vanishes off V, such that

$$\text{(i)} \quad \left\|\int_V \lambda(t)A(\lambda)d\lambda\right\| \leq 1 , \quad t \in G , \quad \text{and}$$

$$\text{(ii)} \quad \sup_{t\in K}\left\|U_t - \int_V \lambda(t)A(\lambda)d\lambda\right\| \leq \varepsilon .$$

For the proof, we require the following result for scalar functions in L^∞.

LEMMA. Let $f \in L^\infty$, $\phi \in L^1$. Then $\widehat{\phi f}$ vanishes off $sp(f) + \{\lambda: \hat{\phi}(\lambda) \neq 0\}$.

PROOF. Let $E = \{\lambda \in \hat{G}: \hat{\phi}(\lambda) \neq 0\}$, and fix $\lambda \notin sp(f) + E$. By Lemma A, there is a sequence $\phi_n \in L^1$ such that $\hat{\phi}_n$ has compact support in E and $\|\phi - \phi_n\|_1 \to 0$. Since the mapping $\phi \to \widehat{\phi f}(\lambda)$ is continuous in the L^1 norm, it suffices to show that $\widehat{\phi_n f}(\lambda) = 0$ for each n.

So fix n, and let K be a compact subset of E such that $\widehat{\phi_n} = 0$ off K. Because $sp(f) + K$ is closed and fails to contain λ, we can find an open set $W \supseteq sp(U)$ so that $\lambda \notin W + K$ (i.e., choose a neighborhood W_0 of 0 in $\hat{G}$ so that $\lambda - W_0$ misses $sp(f) + K$, and put $W = sp(f) + W_0$). Now $\lambda - W$ misses K and therefore $\widehat{\phi_n}(\lambda - \omega) = 0$ for all $\omega \in W$. So if we put

$$S = \{g \in L^\infty: \widehat{\phi_n g}(\lambda) = 0\} ,$$

then since $\widehat{\phi_n \omega}(\lambda) = \hat{\phi}_n(\lambda - \omega)$, we may conclude that $\omega \in S$ for every $\omega \in W$. Since S is a weak*-closed linear subspace of L^∞, it suffices to show that f is contained in the weak*-closed span of $\{\omega: \omega \in W\}$. For that, choose $\psi \in L^1$, $\psi \perp \{\omega: \omega \in W\}$. Then $\hat{\psi}(\omega) = \langle \omega, \psi\rangle = 0$ for each $\omega \in W$ and thus $\hat{\psi} = 0$ near $sp(f)$. By Theorem 1.6, we have the required conclusion $\langle f, \psi\rangle = 0$ ∎

PROOF OF 2.15. Choose any compact neighborhood W of 0 in $\hat{G}$ so that $sp(U) + W \subseteq V$. Now choose $\phi \in L^1$ such that $\hat{\phi}$ is nonnegative and lives in W (Lemma B). By scaling ϕ if necessary, we can assume that

$$\int_{\hat{G}} \hat{\phi}(\lambda)\,d\lambda = 1 \ .$$

Notice that if W shrinks to $\{0\}$ in $\hat{G}$, the corresponding $\hat{\phi}$'s form an approximate identity for the group algebra $L^1(\hat{G})$.

Holding W and ϕ fixed, define $A\colon \hat{G} \to \mathcal{L}_w(X)$ by

$$A(\lambda) = \int_G \overline{\lambda(t)}\phi(t)U_t\,dt \ .$$

We have $\|A(\lambda) - A(\mu)\| \leq \int |\lambda(t) - \mu(t)|\cdot|\phi(t)|\,dt$, and therefore A is norm-continuous.

We claim that A vanishes off V. Indeed, if $x \in X$ and $\rho \in X_*$, then

$$\rho(A(\lambda)x) = \int_G \overline{\lambda(t)}\phi(t)\rho(U_t x)\,dt \ .$$

Now the function $f_{\rho,x}(t) = \rho(U_t x)$ belongs to $L^\infty(G)$ and its spectrum is contained in $\mathrm{sp}(U)$. So by the lemma, we have $\rho(A(\lambda)x) = 0$ off $\mathrm{sp}(U) + W \subseteq V$. The conclusion follows since ρ and x are arbitrary. Thus we can form the inverse Fourier transform of A. Notice that

$$\int_{\hat{G}} \lambda(t)A(\lambda)\,d\lambda = \phi(t)U_t \quad \text{for all} \quad t \in G \ .$$

Indeed, for x, ρ as above, we have

$$\rho\Big(\Big(\int_{\hat{G}} \lambda(t)A(\lambda)\,d\lambda\Big)x\Big) = \int_{\hat{G}} \lambda(t)\rho(A(\lambda)x)\,d\lambda$$

and, by the Fourier inversion theorem ([23], p. 22) the second term is simply $\phi(t)\rho(U_t x)$. The asserted formula follows.

Now as W shrinks to $\{0\}$, the $\hat{\phi}_W$'s are an approximate unit for $L^1(\hat{G})$, and hence

$$|\phi_W(t)| \leq 1 \quad \text{for all} \quad t \text{, and}$$

$$\sup_{t\in K} |\phi_W(t) - 1| \to 0 \quad \text{as} \quad W \downarrow \{0\} \ .$$

It follows that for an appropriate ϕ we will have

$$\Big\|\int_{\hat{G}} \lambda(t)A(\lambda)\,d\lambda\Big\| = \|\phi(t)U_t\| \leq 1 \ , \quad t \in G \ ,$$

and

$$\sup_{t\in K}\|U_t - \int_{\hat{G}} \lambda(t)A(\lambda)d\lambda\| = \sup_{t\in K}|1 - \phi(t)| \leqslant \varepsilon \quad ,$$

as required ■

To conclude this section, we want to reformulate the material on analysis and synthesis described in section 1 so that it comes to bear on certain problems concerning the Banach algebras $U(L^1)$. Let S be a weak*-closed translation invariant subspace of L^∞ and, as in section 1, we put

$$S_\perp = \{\phi \in L^1 : \phi \perp S\} .$$

$S_\perp$ is a closed ideal in L^1 and hence the group of trnaslation operators acts naturally on the quotient space

$$X = L^1/S \quad .$$

Let U be this representation of G on X . It is not hard to see that the set $sp(S)$ as defined in section 1 is exactly $sp(U)$.

Now L^1/S is also a commutative Banach algebra and it has a bounded approximate identity (because L^1 does). From this it follows that the mapping

$$\alpha : \phi + S_\perp \in L^1/S \mapsto U_\phi \in \mathcal{L}(L^1/S)$$

defines an isomorphism of the Banach algebra L^1/S onto $U(L^1)$. If we identify the dual of L^1/S with S in the natural way, then the transpose α' of α gives an isomorphism of vector spaces

$$\alpha' : U(L^1)' \to S$$

which is a homeomorphism relative to the respective weak* topologies. Thus we can identify S with the dual space of $U(L^1)$.

Now by the results of section 1 we know that S admits <u>analysis</u>. In terms of $U(L^1)$, this means that each point $\lambda \in sp(U)$ gives rise to a complex homomorphism $\omega_\lambda \in \sigma(U(L^1))$ such that

$$\omega_\lambda(U_\phi) = \hat{\phi}(\lambda) \quad ,$$

for every $\phi \in L^1$, a fact we have already established in general.

To say that S admits <u>synthesis</u> in this context is to say that the complex homomorphisms $\{\omega_\lambda : \lambda \in sp(U)\}$ of $U(L^1)$ have all of the dual of $U(L^1)$ as their weak*-closed linear span.

Generalizing this, we will say that an arbitrary representation $U: G \to \mathcal{L}(X)$ admits <u>synthesis</u> if the weak*-closed linear span of $\{\omega_\lambda : \lambda \in sp(U)\}$ exhausts the dual space of $U(L^1)$. We have the following convenient characterization .

PROPOSITION 2.16. U admits synthesis if, and only if, the Banach aglebra $U(L^1)$ is semisimple.

REMARK. For our purposes, the assertion that $U(L^1)$ is semisimple means that the only quasinilpotent operator is $U(L^1)$ in 0 .

PROOF. Assume U admits synthesis and let $T \in U(L^1)$ be quasinilpotent. Then $\omega_\lambda(T) = 0$ for every $\lambda \in sp(U)$ because $|\omega_\lambda(T)|$ is at most the spectral radius of T . Since the ω_λ's weak*-span $U(L^1)'$, it follows that $f(T) = 0$ for every $f \in U(L^1)'$, and hence $T = 0$ by the Hahn-Banach theorem.

If U fails synthesis then by an elementary separation theotem there is a $T \neq 0$ in $U(L^1)$ such that $\omega_\lambda(T) = 0$ for every λ in $sp(U)$. By 2.10, $\omega(T) = 0$ for every $\omega \in \sigma(U(L^1))$. By elementary Gelfand theory, the spectral radius of T is zero and hence T is quasinilpotent. Thus $U(L^1)$ is not semisimple ■

From section 1 we know that if S is a translation invariant subspace of L^∞ such that $sp(S)$ admits synthesis, then S admits systhesis. This leads us to conjecture that if U is a representation of G on a Banach space X whose spectrum $sp(U)$ admits synthesis, then the Banach algebra $U(L^1)$ is semisimple. One can see using 1.9 that no element in $U(L^1)$ of the form U_ϕ , $\phi \in L^1$, can be quasinilpotent and nonzero in this case; but since these elements are only dense in $U(L^1)$ in general, this line of argument is inconclusive.

On the other hand, one can say that if F is any closed set in $\hat{G}$ which fails sythesis, then there is a group U acting on an appropriate quotient algebra of L^1 such that $sp(U) = F$ and $U(L^1)$ is not semisimple. Again, this situation is described by the results of section 1.

Finally, let us consider the case where U is a strongly continuous unitary representation of G on a Hilbert space H . By Stone's theorem, there is a (regular Borel) projection valued measure P on $\hat{G}$ such that

$$U_t - \int_{\hat{G}} \lambda(t)\,dP(t) \ , \quad t \in G \ .$$

If $\phi \in L^1$, then a routine application of Fubini's theorem shows that

$$U_\phi = \int_{\hat{G}} \hat{\phi}(\lambda)\,dP(\lambda) \quad .$$

By the elementary properties of spectral measures, we see that $U_\phi = 0$ if, and only if, $\hat{\phi}$ vanishes on the closed support of P. Therefore, $sp(U)$ <u>is the closed support of</u> P. In this case $U(L^1)$ is a commutative C^*-algebra, necessarily semisimple. Thus, every <u>unitary</u> representation of G admits synthesis.

For additional results relating to semisimplicity, we refer the reader to ([32], [35]).

3. SPECTRAL SUBSPACES. Let $\{U_t : t \in G\}$ be a group of isometries of X as in the preceding section. Each vector $x \in X$ gives rise to a translation invariant linear subspace of L^∞:

$$\{f_{\rho,x} : \rho \in X_*\} \; ,$$

where $f_{\rho,x}(t) = \rho(U_t x)$. We define the <u>spectrum</u> of x (written $sp_U(x)$) to be the spectrum of this family of functions.

Equivalently, let us say that a character $\lambda \in \hat{G}$ is a U-<u>essential point</u> for x if, for every neighborhood V of λ there is a $\phi \in L^1$ for which

3.1. (i) $\hat{\phi}$ lives in V, and
(ii) $U_\phi x \neq 0$.

Then the spectrum of x is just the set of all U-essential points of x. Notice that (since each operator U_ϕ is weakly continuous) $sp_U(x)$ is simply the spectrum of the subrepresentation of U obtained by restricting each U_t to the weakly closed U-invariant subspace of X generated by x.

Consider the case where H is a Hilbert space and U is a unitary group on H. By Stone's theorem we have

$$U_t = \int_{\hat{G}} \lambda(t)\,dP(\lambda) \quad , \quad t \in G \; ,$$

for an appropriate projection valued measure P on $\hat{G}$. Thus each vector x in H gives rise to a vector valued measure P_x, defined on the Borel sets in $\hat{G}$ by $P_x(S) = P(S)x$, and taking values in H.

One can easily see that $sp_U(x)$ is the closed support of this vector measure P_x. It follows that for every closed set $E \subseteq \hat{G}$ and every x in H, we have $P(E)x = x$ iff the measure P_x is supported in E . Thus we have the following description of the range of these projections:

$$\text{range } P(E) = \{x \in X: sp_U(x) \subseteq E\} .$$

If X is not a Hilbert space then there is no analogue of Stone's theorem and the left side of this expression is meaningless. But the right side makes sense in general, defines a linear subspace of X , and thus we may define the <u>spectral subspaces</u> of X as follows.

DEFINITION 3.2. For every closed set E in $\hat{G}$, $M^U(E)$ will denote the linear space of all $x \in X$ satisfying $sp_U(x) \subseteq E$.

Corresponding to 2.3 (iii and (iv), we have the characterization:

3.3 $x \in M^U(E)$ <u>if, and only if,</u> $U_\phi x = 0$ <u>for every</u> $\phi \in L^1$ <u>whose transform vanishes near</u> E .

Since each operator U_ϕ is weakly continuous, it follows that <u>spectral subspaces are weakly closed</u>. When there is no chance of confusion, we will write the spectral subspace $M^U(E)$ more concisely as $M(E)$, and similarly $sp_U(x)$ will be written $sp(x)$.

Notice first that $M(E) = M(E \cap sp(U))$, for every closed set E in $\hat{G}$. Indeed, the inclusion $\supseteq$ is clear from the fact that E contains $E \cap sp(U)$; and the opposite inclusion follows from the fact that if $sp(x) \subseteq E$ then also $sp(x) \subseteq sp(U)$ (by 3.3 and the fact that $U_\phi = 0$ if $\hat{\phi}$ vanishes near $sp(U)$), hence every U-essential point of x must belong to $E \cap sp(U)$. This observation shows that there is no loss if we think of the mapping $E \mapsto M(E)$ as being defined on the <u>relatively</u> closed subsets of $sp(U)$, and it is occasionally useful to take this point of view (see prop. 3.4 below). But we will normally consider closed set E as subsets of $\hat{G}$.

The following result shows that the spectral subspace structure associated with a representation acts as much like a spectral measure as one could expect in this generality.

THEOREM 3.3. Let U be fixed as above, and let all E's be closed sets in G . Then

(i) $M(\emptyset) = \{0\}$, $M(\hat{G}) = X$

(ii) $M(E)$ is a hyperinvariant subspace for $\{U_t: t \in G\}$

(iii) $M(\bigcap_\alpha E_\alpha) = \bigcap_\alpha M(E_\alpha)$ for arbitrary families $\{E_\alpha\}$

(iv) If F is compact and $E \cap F = \emptyset$, then $M(E \cup F) = M(E) + M(F)$.

REMARKS. Assertion (ii) means that $M(E)$ is weakly closed and invariant under every weakly continuous operator on X which commutes with $\{U_t: t \in G\}$. In (iv), the sum on the right represents a direct sum of Banach spaces.

PROOF. The proofs of (i) through (iii) are left for the reader. For (iv), it is apparent that both $M(E)$ and $M(F)$ are contained in $M(E \cup F)$. By lemma B, choose $\phi \in L^1$ such that $\hat{\phi} = 1$ near F and $\hat{\phi} = 0$ near E, and let P be the restriction of U_ϕ to $M(E \cup F)$. We show that P is an idempotent having range $M(F)$ and kernel $M(E)$.

Indeed, since $\hat{\phi}^2 - \hat{\phi} = 0$ near $E \cup F$ we have $U_\phi^2 - U_\phi = 0$ on $M(E \cup F)$, and hence $P^2 = P$.

If $x \in M(F)$, then since $\hat{\phi} = 1$ near F we have $U_\phi x = x$, hence $M(F) \subseteq$ range P. If $\psi \in L^1$ is such that $\hat{\psi} = 0$ near F, then for each $x \in M(E \cup F)$ we have

$$U_\psi Px = U_{\psi * \phi} x = 0 ,$$

since $(\psi * \phi)\hat{} = \hat{\psi}\hat{\phi} = 0$ near $sp(x) \subseteq E \cup F$. It follows that $Px \in M(F)$, establishing the assertion about range P.

The proof that kernel $P = M(E)$ is similar, and we omit it ∎

None of these results allows us to conclude that any of the spectral subspaces of U are nontrivial, that is, different from $\{0\}$ or X. Nor do they give any information about the spectra of the subrepresentations $U|_{M(E)}$, defined by

$$t \mapsto U_t|_{M(E)} , \quad t \in G .$$

In general, it is apparent that

$$sp(U|_{M(E)}) \subseteq E ,$$

but the inclusion may be proper, even for one-parameter unitary groups on Hilbert spaces. Indeed, if λ is any point in the spectrum of such a U which is not in the pure point spectrum (that is, there is no nonzero vector $x \in X$ such that $U_t x = e^{i\lambda t} x$ for all t) then, putting $E = \{\lambda\}$, we see that $M(E)$ is zero and hence

$$sp(U|_{M(E)}) = \emptyset \neq E .$$

Nevertheless, we have

PROPOSITION 3.4. If E is a subset of $sp(U)$ which is the closure of its interior relative to $sp(U)$, then

$$sp(U|_{M(E)}) = E .$$

REMARK. The hypothesis means simply that there is an open set $W \subseteq \hat{G}$ such that E is the closure in $\hat{G}$ of $W \cap sp(U)$.

PROOF. We discuss only the nontrivial inclusion $\supseteq$. For that we claim that if $\phi \in L^1$ is such that $U_\phi M(E) = \{0\}$, then $\hat{\phi} = 0$ on E. This will suffice, for it implies that every point of E is essential for $U|_{M(E)}$. Indeed, if $\lambda \in E$ is not essential then, choosing a neighborhood V of λ for which $U_\psi M(E) = \{0\}$ for every ψ such that $\hat{\psi}$ lives in V, we obtain a contradiction by choosing such a ψ for which $\hat{\psi}(\lambda) \neq 0$.

So choose $\phi \in L^1$ such that $U_\phi M(E) = \{0\}$, and write $E = \overline{(W \cap sp(U))}$ where W is an open set in $\hat{G}$. Now for any $\psi \in L^1$ such that $\hat{\psi}$ lives in some compact set K in W we have

$$sp(U_\psi) \subseteq K \cap sp(U) \subseteq W \cap sp(U) \subseteq E ,$$

hence $U_\phi U_\psi = 0$, and by 2.3 (v), $\hat{\phi}\hat{\psi}$ must vanish on $sp(U)$. We can choose such a ψ so that $\hat{\psi}$ is nonzero at any given point of $W \cap sp(U)$ and hence $\hat{\phi} = 0$ on $W \cap sp(U)$. The conclusion $\hat{\phi} = 0$ on $E = \overline{(W \cap sp(U))}$ follows ∎

<u>COROLLARY 1</u>. If $E \subseteq \hat{G}$ is any closed set in $\hat{G}$ whose interior meets $sp(U)$, then $M(E) \neq \{0\}$.

PROOF. Let W be the interior of E and let E_0 be the closure of $W \cap sp(U)$. By 3.4 we have $sp(U|_{M(E_0)}) = E_0 \neq \emptyset$, hence $M(E) \supseteq M(E_0) \neq \{0\}$ ∎

Spectral subspaces (for strongly continuous representations on Banach spaces) were first defined and studied by Godement [13]. His paper points out the connections with harmonic analysis, establishes the assertions of proposition 3.3, and contains the important result that if U is not of the trivial form

$$U_t = \lambda(t)1 ,$$

where λ is some fixed character of G, then there is a spectral subspace $M(E)$ satisfying $\{0\} \neq M(E) \neq X$. Actually, the latter result was restricted to the case where G was compact or a vector group because it was not known in 1947 if one-point sets admit synthesis for general LCA groups. The fact that this was true for $\mathbb{R}^n$ had been established (due to earlier work of Beurling [3]) and it followed for compact groups from the theory of almost periodic functions.

We digress from the main discussion to present a variation of this result. While our method is to deduce Godement's result from proposition 3.4 (which is, so far as we know, a new result), it is possible to proceed more directly to the proof of corollary 2.

COROLLARY 2. Let U be a representaiton of G on X,

$\dim X \geq 2$. Then U has a nontrivial weakly closed invariant subspace.

PROOF. We know that $sp(U) \neq \emptyset$. So assume first that $sp(U)$ contains at least two points. Then there are open sets W_1, W_2 in $\hat{G}$ whose closures are disjoint which satisfy $W_i \cap sp(U) \neq \emptyset$, $i = 1, 2$. Let E_i be the closure of $W_i \cap sp(U)$. By Corollary 1 we have $M(E_i) \neq \{0\}$ and by 3.3(ii), $M(E_1) \cap M(E_2) = M(E_1 \cap E_2) = \{0\}$. Thus $M(E_1)$ is a nontrivial hyperinvariant subspace.

If $sp(U)$ is a singleton $\{\lambda\}$, $\lambda \in \hat{G}$, then since $\{\lambda\}$ is a set of synthesis it follows from the results of section 1 that for every $\rho \in X_*$, $x \in X$, we have

$$\rho(U_t x) = \lambda(t)\rho(x) ,$$

for all t in G . We conclude that $U_t = \lambda(t)1$ is a scalar for every $t \in G$, and every weakly closed subspace of X is invariant under $\{U_t\}$ ∎

Now let U be a unitary representation of G on a Hilbert space H . U implements an action α of G on $L(H)$ as $*$-automorphisms

$$\alpha_t(A) = U_t A U_t^{-1}$$

and, since α defines a representation of G on $X = L(H)$, it has spectral subspaces just as U does. The main results of this section imply that the spectral subspaces of α can be related to those of U in a subtle but effective way (3.5 and 3.7).

We have departed from the treatment in [1] in order to give proofs which are more conceptual. While we have not been entirely successful in this (the proof of 3.7 is still probably not the "right" one), we feel that this discussion improves significantly on the many changes-of-variables in [1].

Let X, Y be Banach spaces as above and let $L_w(X, Y)$ denote the Banach space of all weakly continuous operators from X to Y . Let $F: G \to L_w(X, Y)$ be a bounded function such that

$$f_{\rho,x}(t) = \rho(F(t)x)$$

is measurable in t for each $x \in X$, $\rho \in Y_*$. If E is a closed set in $\hat{G}$, we will say that $sp(F) \subseteq E$ if $sp(f_{\rho,x}) \subseteq E$ for every ρ and x . There is clearly a smallest such set E , which will be denoted $sp(F)$. Of course, $sp(F)$ is the spectrum of the family $\{f_{\rho,x}: \rho \in Y_* , x \in X\} \subseteq L^\infty(G)$. We will also eliminate superfluous definitions by using the notation $\langle F(t) \rangle$ to denote the function $t \in G \mapsto F(t) \in L_w(X, Y,$.

If f and g are two scalar functions in $L^{\infty}(G)$, then the spectrum of the product is contained in the closure of $\{\lambda + \mu: \lambda \in sp(f), \mu \in sp(g)\}$ (according to our convention about writing group operations, $\lambda + \mu$ denotes the character $t \mapsto \lambda(t)\mu(t)$) . The formal reasons for this conclusion are that the Fourier transform of a pointwise product is the convolution of the respective transforms, and the closed support of a convolution is contained in the closed sum of the individual supports. The proof of this assertion about $sp(fg)$ can be based on that formal argument, but it is nontrivial since neither f nor g need be integrable. We require the following generalization of this fact.

Let U (resp. V) be a representation of G on X (resp. Y).

THEOREM 3.5. Let $A \in L_w(X, Y)$ and let E and F be closed sets in $\hat{G}$ satisfying $E + sp\langle V_t A U_t^{-1}\rangle \subseteq F$. Then

$$AM^U(E) \subseteq M^V(F) .$$

PROOF. Applying 2.12 to the restriction of U to $M^U(E)$, we see that $M^U(E)$ is the weakly closed span of U-invariant subspaces M such that, for each α , $sp(U|_{\mu_\alpha})$ is a compact subset of E . Since A is weakly continuous, it therefore suffices to prove the following assertion: if $sp(U)$ is compact, then AX is contained in

$$M^V(sp(U) + sp\langle V_t A U_t^{-1}\rangle) .$$

Let $\phi \in L^1$ be such that $\hat{\phi}$ vanishes on an open set W containing $sp(U) + sp\langle V_t A U_t^{-1}\rangle$ and choose $x \in X$, $\rho \in Y_*$. We have to show that

$$\int_G \phi(t)\rho(V_t^{-1} A U_t x)dt = 0 .$$

Choose a compact set $V \subseteq \hat{G}$, containing $sp(U)$ in its interior, such that $V + sp\langle V_t A U_t^{-1}\rangle \subseteq W$. By 2.13, there is a net $B_\alpha: V \to L_w(X)$ of norm continuous functions such that the net of inverse Fourier transforms

$$\hat{B}_\alpha(t) = \int_V \lambda(t) B_\alpha(\lambda)d\lambda$$

is uniformly bounded and converges in operator norm to the function $t \mapsto U_t$ uniformly on compact subsets of G . Thus the net of scalar valued functions

$$f_\alpha(t) = \rho(V_t A U_t^{-1} \hat{B}_\alpha(t) x) = \int_V \lambda(t)\rho(V_t A U_t^{-1} B_\alpha(\lambda)x)\,d\lambda$$

converges to $t \mapsto \rho(V_t A x)$ in the weak* topology of $L^\infty(G)$. So to prove that

$$\int_G \phi(t)\rho(V_t^{-1} A x)\,dt = 0 \ ,$$

it suffices to prove that

$$\int_G \phi(t) f_\alpha(-t)\,dt = 0$$

for every α. But for α fixed we have by Fubini's theorem

$$\int_G \phi(t) f_\alpha(-t)\,dt = \int_V \int_G \overline{\lambda(t)}\phi(t)\rho(V_t^{-1} A U_t B_\alpha(\lambda)x)\,dt\,d\lambda \ .$$

For λ fixed in V, the Fourier transform of $\bar{\lambda}\cdot\phi$ vanishes on $W - \lambda$, which contains $\mathrm{sp}\langle V_t A U_t^{-1}\rangle$ because $V + \mathrm{sp}\langle V_t A U_t^{-1}\rangle \subseteq W$. Since the spectrum of the function $t \mapsto \rho(V_t A U_t^{-1} B_\alpha(\lambda)x)$ is contained in that of $\langle V_t A U_t^{-1}\rangle$, we see that

$$\int_G \overline{\lambda(t)}\phi(t)\rho(V_t^{-1} A U_t B_\alpha(\lambda)x)\,dt = 0$$

for each λ in V, and the desired conclusion

$$\int_V \int_G \overline{\lambda(t)}\phi(t)\rho(V_t^{-1} A U_t B_\alpha(\lambda)x)\,dt\,d\lambda = 0$$

follows ∎

The following result is the principal one of this section, and provides a converse to 3.5. In order to discuss this, consider the case where U and V are unitary representations of G on Hilbert spaces X and Y. Let E and F be closed sets in $\hat{G}$. The most optimistic converse to 3.5 would assert that if A is any operator from X to Y such that $AM^U(E) \subseteq M^V(F)$, then

3.6 $$E + \mathrm{sp}\langle V_t A U_t^{-1}\rangle \subseteq F \ .$$

This assertion is false of course; if P is the projection of Y onto $M^V(F)$ and $T\colon X \to Y$ is arbitrary, then $A = PT$ satisfies the hypothesis $AM^U(E) \subseteq M^V(F)$. On the other hand, A bears no particular relation to the representation U and so there is nothing to be said about the spectrum of the function $\langle V_t A U_t^{-1} \rangle$. Nevertheless, if the hypothesis is appropriately strengthened then we are able to draw the desired conclusion in general.

THEOREM 3.7. Let E, F be closed sets in $\hat{G}$ such that E is the closure of its interior, and let $A \in L_w(X, Y)$ satisfy

$$AM^U(E + \omega) \subseteq M^V(F + \omega)$$

for every $\omega \in \hat{G}$. Then

$$E + \operatorname{sp}\langle V_t A U_t^{-1} \rangle \subseteq F .$$

The proof requires the following two elementary lemmas.

LEMMA 1. If V is an open set in $\hat{G}$ containing λ, then there is a function $\phi \in L^1 \cap L^\infty$ such that $\hat{\phi}$ lives in V and $\hat{\phi}(\lambda) \neq 0$.

PROOF. Choose $\phi_1 \in L^1$ such that $\hat{\phi}_1$ lives in V and $\hat{\phi}_1(\lambda) \neq 0$ (lemma B), and let ϕ_2 be any compactly supported continuous function on G such that $\hat{\phi}_2(\lambda) \neq 0$. Then the convolution $\phi_1 * \phi_2$ has the stated properties ■

LEMMA 2. Let $u\colon G \times G \to \mathbb{C}$ be a bounded uniformly continuous function and let $\phi \in L^1$, $\psi \in L^\infty$. Then

$$f(x) = \int_G u(t, t - x)\psi(t)\phi(x - t)\,dt$$

is uniformly continuous.

PROOF. The argument is a routine variation of the proof that the convolution of a bounded function with an integrable function is uniformly continuous ([23], p. 4) and is left for the reader ■

PROOF OF 3.7. Choose any two functions ϕ, ψ in L^1 such that $\hat{\phi}$ lives in E and $\hat{\psi} = 0$ near F. Then the range of U is contained in $M^U(E)$ and V_ψ annihilates $M^V(F)$. Thus,

$$V_\psi A U_\phi = 0 .$$

For any ω in $\hat{G}$ and $u \in L^1$, the pointwise product $\omega \cdot u$ belongs to L^1 and we have $(\omega \cdot u)^\wedge(\lambda) = \hat{u}(\lambda - \omega)$. Thus the range of $U_{\omega \cdot \phi}$ is contained in $M^U(E + \omega)$ and $V_{\omega \cdot \psi}$ annihilates $M^V(F + \omega)$. As above, we conclude that

$$V_{\omega \cdot \psi} A U_{\omega \cdot \phi} = 0$$

for every ω in $\hat{G}$.

Fixing $\rho \in Y_*$ and $x \in X$, we see that

$$\iint_{G\times G} \omega(s + t)\rho(V_t^{-1}AU_s^{-1}AU_s^{-1}x)\psi(t)\phi(s)ds\ dt$$
$$= \rho(V_{\omega\cdot\psi}AU_{\omega\cdot\phi}x) = 0 .$$

By the change of variables $r = s + t$, $t = t$, the above becomes

$$\iint_{G\times G} \omega(r)\rho(V_t^{-1}AU_{t-r}x)\psi(t)\phi(r - t)dr\ dt = 0 ,$$

for every ω in $\hat{G}$. So if we put

$$f(r) = \int_G \rho(V_t^{-1}AU_{t-r}x)\psi(t)\phi(r - t)dt ,$$

then by an elementary application of Fubini's theorem we have $f \in L^1$, and the preceding equation asserts that $\hat{f} = 0$. Hence, f vanishes almost everywhere on G .

Now if ϕ is chosen so that, in addition to the requirement support $\hat{\phi} \subseteq E$ we have $\phi \in L^1 \cap L^\infty$, then by lemma 2 f is continuous, and hence f vanishes identically. Thus,

$$\int_G \rho(V_t^{-1}AU_t x)\psi(t)\phi(-t)dt = f(0) = 0 .$$

The same argument apples if we replace ϕ with any translate of itself, so that

$$\int_G \rho(V_t^{-1}AU_t x)\psi(t)\phi(s - t)dt = 0$$

for every s in G . The Fourier transform of this function of s is the product of $\hat{\phi}$ with the Fourier transform of the function $t \mapsto \psi(t)\rho(V_t^{-1}AU_t x)$, and this product of transforms vanishes identically.

Now for a fixed ω in the interior of E , lemma 1 allows us to find $\phi \in L^1 \cap L^\infty$ such that $\hat{\phi}$ lives in E and $\hat{\phi}(\omega) \neq 0$. It follows from the preceding discussion that

3.8 $$\int_G \overline{\omega(t)}\psi(t)\rho(V_t AU_t^{-1}x)dt = 0$$

for every ω in the interior of E , and hence for all ω in E by continuity of Fourier transforms of L^1 functions. Since $\bar{\omega}\cdot\psi$ represents the most general L^1 function whose transform vanishes near $F - \omega$, 3.8 makes the following assertion: if $\omega \in E$ and $\psi \in L^1$ is such that $\hat{\psi}$ vanishes near $F - \omega$, then

$$\int_G \psi(t)\rho(V_t^{-1}AU_tx)dt = 0 ,$$

for every $x \in X$, $\rho \in Y_*$. Thus, the spectrum of the function $\langle V_tAU_t^{-1}\rangle$ is contained in $F - \omega$ for every ω in E . Since

$$\bigcap_{\omega\in E} F - \omega = \{\lambda \in \hat{G}: E + \lambda \subseteq F\} ,$$

we have the desired conclusion ■

Taken together, 3.6 and 3.7 constitute a generalization of theorem 2.3 of [1]. The latter was originally suggested to me by a theorem of Frank Forelli ([11], theorem 1), which asserts that a commutation formula for triples of one-parameter groups on Banach spaces is equivalent to certain mapping properties of their spectral subspaces associated with semi-infinite intervals in $\hat{\mathbb{R}}$. The reader should consult [11] for a precise stateemnt of Forelli's result.

To indicate what it is that 3.7 has to do with commutation formulas, we present the following discussion. Let X be a Banach space and let U (resp. V) be a representation of G (resp. $\hat{G}$) as isometries of X . The pair (U, V) is said to satisfy the Weyl commutation relations if

3.9 $$U_tV_\omega = \omega(t)V_\omega U_t$$

for every $t \in G$ and $\omega \in \hat{G}$. We show that the algebraic relation 3.9 can be completely described by the behavior of the operators V_ω on the spectral subspaces of U .

COROLLARY 1. In order that (U, V) should satisfy 3.9, it is necessary and sufficient that

$$V_\omega M^U(E) \subseteq M^U(E + \omega)$$

for every compact set $E \subseteq \hat{G}$.

PROOF. If (U, V) satisfies 3.9, then $U_tV_\omega U_t^{-1} = \omega(t)V_\omega$, and hence the spectrum of the function $\langle U_tV_\omega U_t^{-1}\rangle$ is contained in the singleton $\{\omega\}$. 3.5 implies that $V_\omega M^U(E) \subseteq M^U(E + \omega)$.

Conversely, assume that the V_ω's satisfy the stated condition and let E be a compact neighborhood of 0 in $\hat{G}$. Then for every $\lambda \in \hat{G}$ we have

$$V_\omega M^U(E + \lambda) \subseteq M^U(E + \lambda + \omega) .$$

So 3.7 implies that the spectrum Σ of $\langle U_tV_\omega U_t^{-1}\rangle$ satisfies

$$\Sigma + E \subseteq \omega + E .$$

Since E can be chosen arbitrarily small this implies $\Sigma \subseteq \{\omega\}$. Hence

$$U_t V_\omega U_t^{-1} = \omega(t) U_0 V_\omega U_0^{-1} = \omega(t) V_\omega ,$$

as required ■

We remind the reader that pairs of unitary representations satisfying 3.9 occur naturally in quantum mechanics [20], and similar pairs of automorphism groups of C^*-algebras are a basic component of the duality theory of crossed products [36].

If U and V are two unitary representations of G on a Hilbert space which have the same spectral subspaces in the sense that $M^U(E) = M^V(E)$ for every closed set E in $\hat{G}$, then by an earlier discussion which identified $M^U(E)$ with the range of the spectral measure for U on the set E, we see that U and V have the same spectral measure, and a fortiori $U_t = V_t$ for every t. The following result from [1] generalizes this to the context we have been discussing.

COROLLARY 2: If U and V are two representations of G on X for which $M^U(E) \subseteq M^V(E)$ for every compact set E in $\hat{G}$, then $U_t = V_t$ for all $t \in G$.

PROOF. Let V be any precompact open set containing 0 in $\hat{G}$ and let E be its closure. Then $M^U(E + \omega) \subseteq M^V(E + \omega)$ for every $\omega \in \hat{G}$ and, taking A to be the identity operator in X we see from 3.7 that

$$E + \mathrm{sp}\langle V_t U_t^{-1} \rangle \subseteq E$$

for every such E. Again, since E can be shrunk to $\{0\}$ we must have $\mathrm{sp}\langle V_t U_t^{-1} \rangle \subseteq \{0\}$ and hence $V_t U_t^{-1}$ is constant in t ■

4. AUTOMORPHISM GROUPS AND DERIVATIONS. We now take up the application of the preceding theory to automorphism groups of operator algebras. Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space H and let $t \mapsto \alpha_t$ be a representation of G as a group of $*$-automorphisms of $\mathcal{R}$ such that

$$t \mapsto \alpha_t(x)\xi, \eta$$

is continuous for every $x \in \mathcal{R}$ and $\xi, \eta \in H$. It is a fact that α defines a representation of G on $\mathcal{R}$ in the sense of section 3, relative to the ultraweak topology on $\mathcal{R}$. This is proved in [1] (proposition 3.0) for one-parameter groups, and follows in general by similar methods.

The fact that each map α_t preserves the $*$-operation of $\mathcal{R}$ implies that the spectrum of α and the spectral subspaces of α posses symmetry properties not shared by more general representations:

4.1
$$\mathrm{sp}(\alpha) = -\mathrm{sp}(\alpha) \;, \quad \underline{\text{and}}$$
$$\mathcal{R}^{\alpha}(E)^{*} = \mathcal{R}^{\alpha}(-E)$$

for every closed set E in $\hat{G}$. We also have the following description of the way spectral subspaces of automorphism groups behave under operator multiplication.

PROPOSITION 4.2. If E, F are closed sets in $\hat{G}$, then

$$\mathcal{R}^{\alpha}(E)\mathcal{R}^{\alpha}(F) \subseteq \mathcal{R}^{\alpha}(\overline{E + F}) \;.$$

PROOF. Let $x \in \mathcal{R}^{\alpha}(E)$. The operator mapping $L_x\colon y \mapsto xy$ is ultraweakly continuous on $\mathcal{R}$ and we have

$$\alpha_t L_x \alpha_t^{-1} = L_{\alpha_t(x)} \;.$$

It follows that for every $y \in \mathcal{R}$ and $\rho \in \mathcal{R}_*$, the function $t \mapsto \rho(\alpha_t L_x \alpha_t^{-1}(y)) = \rho(\alpha_t(x)y)$ has its spectrum in E, and hence $\mathrm{sp}\ \alpha_t L_x \alpha_t^{-1} \subseteq E$. By 3.5, we conclude that the operator mapping L_x carries $\mathcal{R}^{\alpha}(E)$ with $\mathcal{R}^{\alpha}(\overline{E + F})$ ■

We first present a convenient criterion for determining when a given unitary group on H implements the action of G on $\mathcal{R}$. By a <u>cone</u> in $\hat{G}$ we mean a closed set $\Sigma \subseteq \hat{G}$ satisfying

4.3
$$\text{(i)} \quad \Sigma + \Sigma \subseteq \Sigma$$
$$\text{(ii)} \quad \Sigma \cap -\Sigma = \{0\} \;.$$

If, in addition, Σ is the closure of its interior, then Σ will be called <u>proper</u>. Both the positive octant in $\mathbb{R}^n$

$$\Sigma = \{(x_1, \ldots, x_n)\colon x_1 \geqslant 0, \ldots, x_n \geqslant 0\}$$

and the forward light cone in $\mathbb{R}^{n+1}$

$$\Sigma = \{(x_1, \ldots, x_n, t)\colon t \geqslant (x_1^2 + \ldots + x_n^2)^{1/2}\} \;,$$

provide examples of proper cones in vector groups. We shall require the following elementary fact.

LEMMA. Let Σ be a proper cone in $\hat{G}$ and let Σ' be the closure of the complement of Σ in $\hat{G}$. Then

$$\{\lambda \in \hat{G}\colon \lambda + \Sigma' \subseteq \Sigma'\} = -\Sigma \;.$$

PROOF. Let $\lambda \in \hat{G}$ and consider the transformation T_λ of $\hat{G}$ given by $T_\lambda \omega = \omega + \lambda$. Then $T_\lambda(\Sigma') = \Sigma'$ iff T_λ transforms the interior of Σ' into itself, i.e.,

$$T_\lambda(\hat{G} \setminus \Sigma) \subseteq \hat{G} \setminus \Sigma \; .$$

Since $T_\lambda(E) \subseteq F$ iff $T_\lambda^{-1}(\hat{G} \setminus F) \subseteq \hat{G} \setminus E$, for any subsets E, F of $\hat{G}$, the letter is equivalent to the assertion

$$\Sigma - \lambda \subseteq \Sigma$$

which, since Σ is a semigroup containing 0, simply means that $-\lambda \in \Sigma$ ∎

THEOREM 4.4. Let U be a unitary representation of G on H and let Σ be a proper cone in $\hat{G}$. Then the following are equivalent:

(i) $\alpha_t(x) = U_t x U_t^{-1}$ for each $x \in R$, $t \in G$,

(ii) $R^\alpha(\Sigma + \lambda)P(\Sigma + \mu)H \subseteq P(\Sigma + \lambda + \mu)H$,

where P is the spectral measure of U.

PROOF. (i) ⇒ (ii). Fix $\lambda \in \hat{G}$ and choose $x \in R^\alpha(\Sigma + \lambda)$. For each $\xi, \eta \in H$ we have $\langle U_t \times U_t^{-1}\xi, \eta\rangle = \langle \alpha_t(x)\xi, \eta\rangle$, hence the spectrum of the function $t \mapsto \langle U_t \times U_t^{-1}\xi, \eta\rangle$ is contained in $sp_\alpha(x) \subseteq \Sigma + \lambda$. Thus

$$sp\langle U_t \times U_t^{-1}\rangle \subseteq \Sigma + \lambda \; ,$$

and since $(\Sigma + \lambda) + (\Sigma + \mu) \subseteq \Sigma + \lambda + \mu$, it follows from 3.5 that x must map each spectral subspace $M^U(\Sigma + \mu)$ into $M^U(\Sigma + \lambda + \mu)$.

(ii) ⇒ (i). This argument proceeds somewhat differently. U induces an automorphism group $\beta_t = U_t \cdot U_t^{-1}$ of the von Neumann algebra $L(H)$. Let $i\colon R \to L(H)$ be the inclusion map. Then i is ultraweakly continuous, and we want to show that the function $t \mapsto \beta_t i \alpha_t^{-1} \in L_w(R, L(H))$ is constant. Since $\Sigma \cap -\Sigma = \{0\}$, it suffices to establish the two assertions

$$sp\langle \beta_t i \alpha_t^{-1}\rangle \subseteq \Sigma \; , \quad \text{and}$$

$$sp\langle \beta_t i \alpha_t^{-1}\rangle \subseteq -\Sigma \; .$$

For the first, fix $\lambda \in \hat{G}$ and x in $R^\alpha(\Sigma + \lambda)$. Considering x as an operator on H, the hypothesis (ii) asserts that x maps $M^U(\Sigma + \mu)$ into $M^U(\Sigma + \lambda + \mu)$ for every $\mu \in \hat{G}$. Applying 3.6 to $X = Y = H$, $E = \Sigma$, and $F = \Sigma + \lambda$, we see that

$$sp\langle U_t x U_t^{-1}\rangle + \Sigma \subseteq \Sigma + \lambda \; .$$

Since $0 \in \Sigma$ we conclude that $sp\langle U_t$
$\Sigma + \lambda$.

This means that the inclusion i carries $R^{\alpha}(\Sigma + \lambda)$ into the corresponding spectral subspace of β :

$$R^{\beta}(\Sigma + \lambda) = \{y \in L(H): \operatorname{sp}\langle U_t y U_t^{-1}\rangle \subseteq \Sigma + \lambda\}.$$

Since λ is arbitrary in $\hat{G}$, we may apply 3.6 once again to $X = R$, $Y = L(H)$, $A = i$, $E = F = \Sigma$ to conclude that

$$\operatorname{sp}\langle \beta_t i \alpha_t^{-1}\rangle + \Sigma \subseteq \Sigma .$$

Again, since $0 \in \Sigma$, we have the assertion $\operatorname{sp}\langle \beta_t i \alpha_t^{-1}\rangle \subseteq \Sigma$.

To show that $\operatorname{sp}\langle \beta_t i \alpha_t^{-1}\rangle$ is contained in $-\Sigma$, we first claim that if $\tau, \sigma \in \hat{G}$ and K is any compact set in $\hat{G} \setminus \Sigma$, then

4.5. $$R^{\alpha}(-\Sigma + \sigma)P(K + \tau)H \subseteq P((\hat{G} \setminus \Sigma)^{-} + \sigma + \tau)H .$$

Indeed, we know by (ii) that

$$R^{\alpha}(\Sigma + \lambda)P(\Sigma + \mu)H \subseteq P(\Sigma + \lambda + \mu)H .$$

If an operator T on H maps a subspace H_1 into H_2 , then T^* maps $H_2^{\perp}$ into $H_1^{\perp}$. Since $P(E)H^{\perp} = P(\hat{G} \setminus E)H$ for every Borel set $E \subseteq \hat{G}$ and since $R^{\alpha}(\Sigma + \lambda)^* = R^{\alpha}(-\Sigma - \lambda)$, we infer from the preceding inclusion that

$$R^{\alpha}(-\Sigma + \lambda)P(\hat{G} \setminus \Sigma + \lambda + \mu)H \subseteq P(\hat{G} \setminus \Sigma + \mu)H$$

for every λ, μ in $\hat{G}$. Choosing λ and μ so that $\lambda = \sigma$ and $\lambda + \mu = \tau$ and noting that

$$P(K + \tau)H \subseteq P(\hat{G} \setminus \Sigma + \tau)H \quad \text{and}$$

$$P(\hat{G} \setminus \Sigma + \sigma + \tau)H \subseteq P((\hat{G} \setminus \Sigma)^{-} + \sigma + \tau)H ,$$

the assertion 4.5 follows.

Now fix $\sigma \in \hat{G}$ and $x \in R^{\alpha}(-\Sigma + \sigma)$. We claim:

4.6 $$\operatorname{sp}\langle U_t x U_t^{-1}\rangle \subseteq -\Sigma + \sigma .$$

Indeed, taking any open set in $\hat{G} \setminus \Sigma$ having compact closure K in $\hat{G} \setminus \Sigma$, we may apply theorem 3.6 to the formula 4.5, taking $U_t = V_t$, $X = Y = H$, $A = x$, $E = K$, and $F = (\hat{G} \setminus \Sigma)^{-} + \sigma$ to conclude that

$$\operatorname{sp}\langle U_t x U_t^{-1}\rangle + K \subseteq (\hat{G} \setminus \Sigma)^{-} + \sigma .$$

Taking the union of all such K's in $\hat{G} \setminus \Sigma$ we see that we may replace K by $\hat{G} \setminus \Sigma$ in the preceding inclusion and, by taking limits of elements in $\hat{G} \setminus \Sigma$, we even have

$$\operatorname{sp}\langle U_t x U_t^{-1}\rangle + (\hat{G} \setminus \Sigma)^{-} \subseteq (\hat{G} \setminus \Sigma)^{-} + \sigma .$$

So if ω is any element of $\mathrm{sp}\langle U_t x U_t^{-1}\rangle$, then translation by $\omega - \sigma$ maps the closure of $\hat{G} \setminus \Sigma$ into itself. By the lemma we have

$$\omega - \sigma \in -\Sigma ,$$

from which the assertion 4.6 is evident.

Now 4.6 implies that the inclusion $i: R \to L(H)$ maps each spectral subspace $R^{\alpha}(-\Sigma + \sigma)$ into the corresponding spectral subspace for $\beta_t = U_t \cdot U_t^{-1}$:

$$\{y \in L(H): \mathrm{sp}\langle \beta_t(y)\rangle \subseteq -\Sigma + \sigma\} .$$

Taking $X = R$, $Y = L(H)$, $U = \alpha$, $V = \beta$, $A = i$, and $E = -\Sigma$ in 3.6 we conclude that

$$\mathrm{sp}\langle \beta_t i \alpha_t^{-1}\rangle - \Sigma \subseteq -\Sigma ,$$

and hence $\mathrm{sp}\langle \beta_t i \alpha_t^{-1}\rangle \subseteq -\Sigma$ because 0 belongs to $-\Sigma$ ∎

REMARKS. We want to point out that there is a natural C^*-variant of 4.4, in which R is replaced by a C^*-subalgebra of $L(H)$ and the automorphism group α satisfies

$$\lim_{t\to 0} \|\alpha_t(x) - x\| = 0$$

$x \in R$. The proof is the same: one need only check that the inclusion map $i: R \to L(H)$ is continuous relative to the weak (i.e., R'-) topology on R and the ultraweak topology on $L(H)$.

Condition 4.4(ii) appears to differ from the "spectrum condition" studied extensively by Kraus [19], though it seems likely that the two are equivalent.

The theory of spectral subspaces was applied to automorphism groups of operator algebras in [1]. It is perhaps worth pointing out that we were led to this application by a problem concerning non self-adjoint operator algebras, namely a non-commutative version of the classical theorem of F. and M. Riesz. The F. and M. Riesz theorem asserts that any nonzero complex measure on the unit circle which annihilates all trigonometric polynomials

$$p(e^{i\theta}) = a_1 e^{i\theta} + a_2 e^{2i\theta} + \dots + a_n e^{ni\theta}$$

must be mutually absolutely continuous with respect to Lebesgue measure. The solution to this problem ([1], theorem 5.3) was inspired by a generalization of the F. and M. Riesz theorem given by Forelli [11], which has as its setting a flow on a compact Hausdorff space.

Forelli's theorem represents the end of a line of development that was initiated by work of de Leeuw and Glicksberg [9] on the F. & M. Riesz theorem, and especially by the work of Helson and Lowdenslager on invariant subspaces [14]. The objective of the latter was to generalize Beurling's description of the invariant subspaces of the unilateral shift [5]. This paper of Helson and Lowdenslager appears to be the first place in which a one-parameter unitary group is constructed out of a nested one-parameter family $\{H_\lambda : \lambda \in \hat{\mathbb{R}}\}$ of subspaces of a Hilbert space, where the family $\{H_\lambda\}$ is defined in terms of an action of $\mathbb{R}$ on a certain algebra of functions, and where the unitary group is supposed to have certian "implementation" properties relative to the automorphism group. The reader should consult [14] for a more explicit description of the Helson-Lowdenslager program, which involves an intimate relationship between cocycles and invariant subspaces.

It is also true, of course, that the idea of using methods of harmonic analysis to study groups of isometries has been in the air for a long time; it has been a technique known to mathematical physicists as well as workers in partial differential equations and semi-groups. After all, Godement's paper [13] appeared in 1947. But it is a long step from [13] to the Helson-Lowdenslager program in [14], which made essential use of the idea that one can construct a unitary "implementing" group on a Hilbert space out of the spectral subspaces of an action of that group on a different Banach space.

In order to illustrate the use of spectral subspaces in the theory of operator algebras, we include the proof from [1] that every derivation of a von Neumann algebra is inner. This problem was originally taken up by Kaplansky in [18], where it was settled affirmatively in the type I case. The general problem remained open from some time. After considerable preliminary work and encouraging results in a series of special cases (due mainly to Kadison, and Ringrose [15], [16]) the problem was settled by Sakai [24]. The proof we present was discovered somewhat later (in 1972), but it has the advantage that one has a constructive procedure for obtaining the implementing operator for a skew-adjoint derivation in a fairly explicit way, in terms of the spectral subspaces of an associated one-parameter group of automorphisms.

A <u>derivation</u> of a C^*-algebra A is a linear map D of A into itself satisfying

$$D(xy) = xD(y) + D(x)y$$

for all $x, y \in A$. Significantly, every derivation of a C^*-algebra is bounded (this is a theorem of Sakai [25]). D is said to be <u>inner</u> if there is an element b in A for which

$$D(x) = bx - xb \ , \quad x \in A \ .$$

A simple argument allows one to decompose an arbitrary derivation of A into a sum

$$D = D_1 + iD_2$$

where D_1 and D_2 are derivations which are <u>skew-adjoint</u> in the sense that $D_j(x^*) = -D_j(x)^*$. If both D_1 and D_2 are inner then of course so is D, and so the following result implies that every derivation of a von Neumann algebra is inner.

THEOREM 4.7. Let A be a C^*-algebra of operators on a Hilbert space H, having trivial nullspace, and let D be a bounded derivation of A satisfying $D(x^*) = -D(x)^*$, $x \in A$.

Then there is a self-adjoint operator h in the weak closure of A satisfying

$$\text{(i)} \quad \|h\| = \frac{1}{2}\|D\|$$

$$\text{(ii)} \quad D(x) = hx - xh \ , \quad x \in A \ .$$

PROOF. For each $t \in \mathbb{R}$, itD is a bounded self-adjoint derivation of A and therefore

$$\alpha_t = \exp itD$$

defines a uniformly continuous one-parameter group of $*$-automorphisms of A.

For each $\lambda \in \hat{\mathbb{R}}$, define a closed subspace H_λ of H by

$$H_\lambda = [a\xi : a \in A^\alpha[\lambda, \infty) \ , \quad \xi \in H] \ ,$$

the square brackets denoting closed linear space. We have $\lambda \leq \mu \Rightarrow H_\lambda \supseteq H_\mu$, and each subspace H_λ is invariant under the commutant of A.

We claim: $H_\lambda = H$ if $\lambda < 0$. Indeed, choose $\xi \perp H_\lambda$ and let ϕ be a nonnegative continuous function in L^1 such that $\phi(0) > 0$ and $\hat{\phi}$ lives in the interval $[\frac{1}{2}\lambda, -\frac{1}{2}\lambda]$. Then for every $x \in A$ we have

$$sp(\alpha_\phi(x^*x)) \subseteq [\tfrac{1}{2}\lambda, -\tfrac{1}{2}\lambda] \subseteq [\lambda, +\infty) \ ,$$

and hence $\alpha_\phi(x^*x)\xi \in H_\lambda$. Thus, $\langle \alpha_\phi(x^*x)\xi, \xi\rangle = 0$ and we have

$$\int \phi(t) \|\alpha_t(x)\xi\|^2 dt = \langle \alpha_\phi(x^*x)\xi, \xi \rangle = 0 .$$

We conclude that $\phi(t)\|\alpha_t(x)\xi\|^2 = 0$ almost everywhere and so by continuity and the fact that $\phi(0) > 0$ we obtain $\|x\xi\|^2 = \|\alpha_0(x)\xi\|^2 = 0$. Since x is arbitrary in A and A has trivial nullspace, the desired conclusion $\xi = 0$ follows.

Before proceeding further, we require the following

LEMMA. $sp(\alpha) = \sigma(D)$.

PROOF. We know that $sp(\alpha)$ is compact (2.13) and, by corollary 2 of 2.12, $\{\omega_\lambda : \lambda \in sp(\alpha)\}$ is the maximal ideal space of $\alpha(L^1)$. The latter is identified with $\sigma(D)$ by the map $\omega_\lambda \leftrightarrow \lambda \in \sigma(D)$ defined by

$$\omega_\lambda(\alpha_\phi) = \hat{\phi}(\lambda) ,$$

$\phi \in L^1(\mathbb{R})$. Thus, $sp(\alpha)$ and $\sigma(D)$ coincide as subsets of $\hat{\mathbb{R}}$ ■

Returning to the proof of 4.7, note that $H_\lambda = 0$ if $\lambda > \|D\|$. Indeed, by the lemma we have

$$sp(\alpha) = \sigma(D) \subseteq [-\|D\|, \|D\|] ,$$

and so $A^\alpha[\lambda, \infty) = \{0\}$ because $[\lambda, +\infty)$ is disjoint from $sp(\alpha)$. Thus, $H_\lambda = 0$ as well.

Thus the one-parameter family of projections P_λ defined by

$$P_\lambda H = \bigcap_{t<\lambda} H_t$$

is continuous from the left, decreases as λ decreases, and satisfies $P_\lambda = 1$ for $\lambda < 0$ and $P_\lambda = 0$ for $\lambda > \|D\|$. Moreover, by the double commutant theorem, each P_λ belongs to the weak closure of A.

Thus there is a projection valued measure P defined on the Borel sets in $\hat{\mathbb{R}}$ which is uniquely determined by the condition

$$P[\lambda, \infty) = P_\lambda , \quad \lambda \in \hat{\mathbb{R}} .$$

The above conditions imply that P is concentrated on the interval $[0, \|D\|]$, and thus the self-adjoint operator

$$h_0 = \int_{\hat{\mathbb{R}}} \lambda dP(\lambda)$$

is positive, has norm at most $\|D\|$, and belongs to the weak closure of A.

To complete the proof it suffices to show that

4.8 $$\alpha_t(x) = e^{ith_0} x e^{-ith_0} , \quad x \in A ;$$

for this implies after a routine differentiation that

$$D(x) = h_0 x - x h_0 , \quad x \in A ,$$

and thus we may take $h = h_0 - \frac{1}{2} \|D\| \cdot 1$.

To prove 4.8, fix λ and μ in $\mathbb{R}$. Since $A^{\alpha}[\lambda, \infty) A^{\alpha}[\mu, \infty) \subseteq A^{\alpha}[\lambda + \mu , \infty)$ (proposition 4.2), and hence

$$A^{\alpha}[\lambda, \infty) P[\mu, \infty) H \subseteq P[\lambda + \mu, \infty) H .$$

Taking $\Sigma = [0, +\infty)$ in 4.4 and noting that P is the spectral measure of the unitary group $t \mapsto e^{ith_0}$, we see that the action of α on A is implemented by e^{ith_0} ■

The technique of the preceding proof can be adapted to a variety of problems involving automorphism groups. The reader may consult [1], [36] for a "C^*" version of the F. and M. Riesz theorem and an application to Borcher's theorem in one space dimension. The latter has been generalized significantly by Kraus [19], after various partial results in this direction had been obtained by Olesen. Chapter 8 of Pedersen's book [36] contains a more systematic account of the subject.

We conclude this section with a brief discussion of the spectral invariant $\Gamma(\alpha)$ of Connes [8]. Let R be a von Neumann algebra. With every faithful normal state ρ of R there is an associated one-parameter automorphism group $\{\sigma_t^{\rho}: t \in \mathbb{R}\}$, called the <u>modular automorphism group</u> of ρ . While there is no known direct way of defining the group σ^{ρ} in terms of the state ρ (for that, one needs the Tomita-Takesaki theory [36]), it is known that σ^{ρ} is uniquely determined by the KMS boundary conditions. More precisely, σ^{ρ} is the unique one-parameter automorphism group $\{\alpha_t\}$ of R with the property that for every pair of elements x, y in R , there is a continuous function $F = F_{x,y}$ defined on the horizontal strip $\{x + iy: 0 \leqslant y \leqslant 1\}$, which is analytic in the interior of the strip and has boundary values

$$F(t) = \rho(y \alpha_t(x))$$

$$F(t + i) = \rho(\alpha_t(x) y) .$$

Different states ρ_1, ρ_2 as above give rise to different modular automorphism groups σ^1 , σ^2 , but Connes has shown that σ^1 and σ^2 are always <u>exterior equivalent</u> in the sense that there is a strongly continuous mapping $t \mapsto U_t$ of $\mathbb{R}$ into the unitary group of R such that

$$\sigma_t^2(x) = U_t \sigma_t^1(x) U_t^{-1} , \quad t \in \mathbb{R} , \quad x \in R$$

and which satisfies an appropriate cocycle condition $U_{s+t} = U_s\sigma_s^1(U_t)$ [8].

Now the spectrum of a one-parameter automorphism group is not invariant under exterior equivalence. Nevertheless, this instability can be dealt with as follows. Let $t \in G \mapsto \alpha_t \in \text{aut}\, R$ be an automorphism group of R. If e is a nonzero projection in R which is fixed under the action of α, then there is a compressed action α^e of G on the local von Neumann algebra eRe given by

$$\alpha_t^e(exe) = e\alpha_t(x)e ,$$

$t \in G$, $x \in R$. A character $\lambda \in \hat{G}$ is said to be <u>stably essential</u> for α if for every neighborhood V of λ and every nonzero α-fixed projection e in R, there is a $\phi \in L^1$ whose Fourier transform lives in V such that

$$\alpha_\phi^e \neq 0 .$$

$\Gamma(\alpha)$ is defined as the set of all stably essential points. It is clear that λ is stably essential if and only if it is an essential point of every compressed representation α^e; thus

$$\Gamma(\alpha) = \bigcap_e \text{sp}(\alpha^e) ,$$

the intersection taken over all nonzero fixed projections e in R.

It is not hard to see that $\Gamma(\alpha)$ is a closed subgroup of $\hat{G}$. More significantly, if α^1 and α^2 are two actions of G on R which are exterior equivalent, then (8, théorème 2.2.4)

4.9 $$\Gamma(\alpha_1) = \Gamma(\alpha_2) .$$

Moreover, if R is a factor and $e \neq 0$ is any projection fixed under the action of α, then there is an action β of G on R which is exterior equivalent to α and which satisfies

4.10 $$\text{sp}(\beta) \subseteq \text{sp}(\alpha^e) .$$

Taken together, 4.9 and 4.10 imply that

$$\Gamma(\alpha) = \bigcap_{\beta\sim\alpha} \text{sp}(\beta) ,$$

where the intersection is taken over all actions β of G on R <u>which are exterior equivalent to</u> α.

In any case, if one chooses a faithful normal state ρ on a given von Neumann algebra R, then the group $\Gamma(\sigma^\rho)$ does not depend on ρ and therefore it defines an algebraic invariant of R.

For further results on the computation of this important invariant and for its role in the classification of type III factors, see [8].

The invariant Γ has been extended to C^*-dynamical systems by Olesen; for this and further developments, the reader is referred to [36].

5. BOUNDED DERIVATIONS AND THE TANGENT ALGEBRA. Let $\{\alpha_t\}$ be a one-parameter group of $*$-automorphisms of a von Neumann algebra $\mathcal{R}$. There is always a densely defined closed skew-adjoint derivation D of $\mathcal{R}$ which satisfies the formal condition

$$D(x) = \frac{1}{i}\frac{d}{dt}\alpha_t(x)\Big|_{t=0} ,$$

is an appropriate sense. Every self-adjoint operator on a Hilbert space can be approximated in the strong operator topology by a sequence of bounded self-adjoint operators. This is a straightforward consequence of the spectral theorem. Similarly, it is natural to try to approximate D with bounded derivations. Such a program calls for the study of bounded derivations from $\mathcal{R}$ into $\mathcal{L}(H)$, that is, bounded linear mappings $\delta: \mathcal{R} \to \mathcal{L}(H)$ satisfying

$$\delta(xy) = x\delta(y) + \delta(x)y , \quad x, y \in \mathcal{R} .$$

To see why this is so, we consider a simple example. Let $\mathcal{R}$ be the multiplication algebra acting on $H = L^2(\mathbb{R})$, and for each $t \in \mathbb{R}$ let α_t be the automorphism which transforms the multiplication operator L_f, $f \in L^\infty$, to L_{f_t} where

$$f_t(x) = f(x - t) , \quad x \in \mathbb{R} .$$

Since $\mathcal{R}$ is abelian, it admits no nonzero bounded derivations (by theorem 4.7) and, in particular, the generator of $\{\alpha_t\}$ cannot be approximated by bounded derivations of $\mathcal{R}$ into itself.

Nevertheless, we can do almost as well. For if U_t is the one-parameter unitary group on $L^2(\mathbb{R})$ defined by

$$U_t\xi(x) = \xi(x - t) , \quad \xi \in L^2(\mathbb{R})$$

then there is a unique self-adjoint operator X on $L^2(\mathbb{R})$ such that

$$U_t = e^{itX} , \quad t \in \mathbb{R} .$$

The Plancherel theorem implies that X is unitarily equivalent to the operator on $L^2(\hat{\mathbb{R}})$ given by

$$(X\xi)(\lambda) = c\lambda\xi(\lambda) ,$$

$\xi \in L^2(\hat{\mathbb{R}})$, $\lambda \in \hat{\mathbb{R}}$, where c is an appropriate positive constant involving 2π . In any case, we may truncate X at $\pm n$ to obtain a sequence X_n of commuting bounded self-adjoint operators which converges appropriately to X .

In more detail, if we define one-parameter unitary groups U^n by

$$U_t^n = e^{itX_n} ,$$

then the groups U^n converge to U in the sense that

5.1. $$\sup_{t \in K} \|U_t^n \xi - U_t \xi\| \to 0 \quad \text{as} \quad n \to \infty$$

for every $\xi \in L^2(\mathbb{R})$ and every compact set $K \subseteq \mathbb{R}$. It follows that the one-parameter family of mappings $\alpha_t^n : R \to L(H)$ defined by

$$\alpha_t^n(x) = U_t^n \times U_t^{n*} ,$$

converges to the original one-parameter group α_t in the sense that

5.2. $$\sup_{t \in K} \|\alpha_t^n(x)\xi - \alpha_t(x)\xi\| \to 0 \quad \text{as} \quad n \to \infty ,$$

for every $\xi \in H$ and every compact set $K \subseteq \mathbb{R}$. In particular, 5.2 gives a sense in which the sequence $D_n : R \to L(H)$ defined by

$$D_n(z) = X_n z - zX_n$$

converges to the generator D of α .

We conclude that the study of one-parameter automorphism groups of von Neumann algebras suggests the study of <u>bounded</u> derivations of R into $L(H)$.

Another context in which such derivations arise is the perturbation theory of Kadison and Kastler [17]. There is a natural metric on the set of von Neumann algebras acting on H which expresses the "distance" from one to the other in operator norm. This metric has the property that if U is a unitary operator on H which is close to the identity in operator norm, and R is a fixed von Neumann algebra, then R is close to URU^* . A basic unsolved problem in the subject asks if the converse is true: if R_1 and R_2 are close, then is R_1 unitarily equivalent to R_2 via a unitary operator U for which $\|U - 1\|$ is small? This question turns out to be equivalent to the question of whether or not every derivation D of a von Neumann algebra into $L(H)$ is implemented by an operator $x \in L(H)$:

$$D(z) = xz - zx , \quad z \in \mathcal{R} .$$

While this latter problem remains unsolved in general, there are many partial results in the affirmative, the principal ones being the work of Christensen ([6], [7]).

In order to discuss derivations in an appropriate way, it is necessary to adopt a perspective which is more invariant. Let A, B be C^*-algebras. By a derivation from A to B we shall mean a pair (π, D) consisting of a $*$-homomorphism $\pi: A \to B$ and a linear map $D: A \to B$ such that

5.3. $$D(xy) = \pi(x)D(y) + D(x)\pi(y)$$

for all $x, y \in A$. The second component D decomposes uniquely into a sum $D = D_1 + iD_2$, where each D_j is self-adjoint

It is not hard to deduce from 5.3 that D vanishes on the kernel of π, and thus there is a unique linear map $\tilde{D}: \pi(A) \to B$ satisfying $\tilde{D} \circ \pi = D$. If $i: \pi(A) \to B$ denotes the inclusion map, then clearly $(i, \tilde{D})$ is a derivation of the subalgebra $\pi(A)$ of B into B. Thus, in studying a single derivation from A to B, one may always reduce to the case where $A \subseteq B$ and $D: A \to B$ satisfies

$$D(xy) = xD(y) + D(x)y .$$

A theorem of Sakai [24] can be adapted so as to imply that the second component D of any derivation (π, D) is a bounded linear map. It follows that the set $\mathcal{D}(A, B)$ of all derivations of A into B is a closed subset of the cartesian product of metric spaces

$$\hom(A, B) \times \mathcal{B}(A, B) ,$$

$\mathcal{B}(A, B)$ denoting the Banach space of all bounded linear maps of A into B where, for example, the distance between two homomorphisms π_1 and π_2 is defined by

$$d(\pi_1, \pi_2) = \sup_{\substack{x \in A \\ \|x\| \leq 1}} \|\pi_1(x) - \pi_2(x)\| .$$

Thus, $\mathcal{D}(A, B)$ is a complete metric space. The map $p: \mathcal{D}(A, B) \to \hom(A, B)$ defined by

$$p(\pi, D) = \pi$$

defines a surjective continuous function with the property that the fiber $p^{-1}(\pi)$ over each point π $\hom(A, B)$ is a real Banach space. Thus,

$$p: \mathcal{D}(A, B) \to \hom(A, B)$$

defines a family of Banach spaces over the base space $\hom(A, B)$ [2]. It is a fact that if A is a nuclear C^*-algebra, then this family is locally trivial, and therefore $p: \mathcal{D}(A, B) \to \hom(A, B)$

defines a <u>bundle</u> of Banach spaces on hom(A, B) . We omit the proof, since this fact is not relevant to our aim here. For each $r \geq 0$ we can define

$$\mathcal{D}_r(A, B) = \{(\pi, D) \in \mathcal{D}(A, B) : \|D\| \leq r\} .$$

It is clear that

$$\mathcal{D}(A, B) = \bigcup_{r>0} \mathcal{D}_r(A, B) .$$

We will describe an object, to be naturally associated to any C^*-algebra A , which provides an appropriate vehicle for the study of bounded derivations of A into arbitrary C^*-algebras. This construction is analogous to the functor which associates to each compact smooth manifold M its tangent space TM . We begin by discussing briefly a universal property of tangent spaces which is normally not emphasized in differential geometry.

Let M be smooth compact manifold and let TM be its tangent bundle. TM is an appropriately topological space whose underlying point set consists of all ordered pairs (p, v) , where p is a point of M and v is a (continuous) linear functional on $C^\infty(M)$ which satisfies

$$v(fg) = f(p)v(g) + v(f)g(p) ,$$

for all $f, g \in C^\infty(M)$. The set of all such pairs (p, v) where p is a fixed point of M forms a finite dimensional vector space T_pM , called the <u>tangent space</u> at p . We want to consider not the manifolds M and TM , but their associated algebras of smooth (complex-valued) functions

$$A = C^\infty(M) , \quad \underline{\text{and}}$$

$$TA = C^\infty(TM) .$$

For any smooth n-dimensional manifold M there is a natural topology on the algebra $C^\infty(M)$ which makes it into a topological (Frechet) algebra. Briefly, a sequence $f_n \in C^\infty(M)$ converges to zero iff for every point p of M there is a (coordinatized) neighborhood U of p such that f_n and all partial derivatives of f_n converge uniformly to zero on U .

Vector fields on M are identified with (continuous) derivations of the algebra $C^\infty(M)$ into itself. Let N be another smooth manifold By a <u>derivation</u> of $C^\infty(M)$ to $C^\infty(N)$ we mean a pair (π, X) consisting of a continuous algebra homomcrphsim $\pi: C^\infty(M) \to C^\infty(N)$ and a continuous linear map X of $C^\infty(M)$ to

$C^\infty(N)$, satisfying

$$X(fg) = \pi(x)X(g) + X(x)\pi(g) .$$

The homomorphism π corresponds to a smooth map $\phi: N \to M$, and X corresponds to a function ξ from N to the tangent space TM which satisfies

$$\xi(q) \in T_{\phi(q)}M ,$$

for every $q \in N$. Thus, derivations of $C^\infty(M)$ to $C^\infty(N)$ correspond to "vector fields" (ϕ, ξ) from N to M.

Now there is a distinguished derivation from $C^\infty(M)$ to $C^\infty(TM)$ which is defined as follows. For each $f \in C^\infty(M)$, define $i(f)$ and $d(f)$ in $C^\infty(TM)$ by

$$i(f)(p, v) = f(p)$$

$$d(f)(p, v) = v(f) .$$

i is a monomorphism of the algebra structure and d is a linear map satisfying $d(fg) = i(f)d(g) + d(f)i(g)$. $d(f)$ is, of course, the differential of f. The linear space of all functions on TM having the form

$$\sum_{k=1}^{n} i(f_k)d(g_k)$$

is a module over $C^\infty(M)$ which is naturally identified with the space of 1-forms. We write this module as dA, $A = C^\infty(M)$. The linear space of functions

$$i(A) + dA$$

separates points of TM and determines its topology in a natural way. The algebra of functions on TM generated by $i(A) + dA$ is dense in $C^\infty(TM)$.

Now it is a fact that the distinguished derivation $(i, d): C^\infty(M) \to C^\infty(TM)$ has the following universal property: for every derivation (π, X) of $C^\infty(M)$ to $C^\infty(N)$, there is a unique homomorphism $\sigma: C^\infty(TM) \to C^\infty(N)$ satisfying

5.4
$$\sigma \circ i = \pi , \quad \text{and}$$
$$\sigma \circ d = X$$

(to see this, let (ϕ, ξ) be the vector field from N to M corresponding to (π, X), define $\psi: N \to TM$ by

$$\psi(q) = (\phi(q), \xi(q)) ,$$

$q \in N$, and put $\sigma(h) = h \circ \psi$, $h \in C^\infty(TM))$. This universal property, or rather its analogue for C^*-algebras, is basic to what follows.

As a final note, observe that the tangent space of $\mathbb{T}M$ is never compact, but merely σ-compact, even if M is compact. Similarly, we will see that the tangent algebra of a C^*-algebra is not a C^*-algebra but merely a "σ-C^*-algebra".

In order to define σ-C^*-algebras we first consider the commutative case. Let X be a Hausdorff topological space which is $\underline{\sigma\text{-compact}}$ in the sense that there is a sequence $K_1, K_2, \ldots$ of compact subspaces of X such that

$$X = \bigcup_{n=1}^{\infty} K_n .$$

We may obviously assume that $K_n \subseteq K_{n+1}$ for eveery n. By enlarging the given topology if necessary, we may assume that the topology of X is $\underline{\text{inductive}}$ in the sense that a set $U \subseteq X$ is closed (resp. open) iff $U \cap K$ is closed (resp. open in K) for every compact subset $K \subseteq X$. Since every compact subset of X is contained in some K_n, we may confine attention to K's of the form K_n, $n = 1, 2, \ldots$. It can be seen that if the original topology on X is metrizable or locally compact, then it coincides with the inductive topology. In general, however, the inductive topology can be strictly stronger.

A complex function $f: X \to \mathbb{C}$ is continuous in this topology iff $f|K_n$ is continuous on K_n for every $n \geqslant 1$. Let $C(X)$ denote the commutative algebra of all continuous complex functions on X. $C(X)$ has a unit and a natural involution. Notice that $C(X)$ contains unbounded functions if X is not compact, and hence there is no natural norm in sight. But for each $n \geqslant 1$ we have a seminorm $\|\cdot\|_n$ on $C(X)$, defined by

$$\|f\|_n = \sup_{p \in K_n} |f(p)| .$$

Each $\|\cdot\|_n$ is a C^*-seminorm and we have $\|f\|_n \leqslant \|f\|_{n+1}$ for all f. Moreover, we have a natural translation invariant metric defined by

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f-g\|_n}{1+\|f-g\|_n} ,$$

which makes $C(X)$ into a complete metric space. A sequence f_n converges in this metric iff f_n converges uniformly cn every compact subset of X.

More generally, a $\underline{\sigma\text{-}C^*\text{-algebra}}$ is a complex $*$-algebra A, together with a sequence $\|\cdot\|_1$, $\|\cdot\|_2$, ... of C^*-seminorms on A, satisfying the conditions

5.5
(i) $\|x\|_n = 0$ for all $n \geq 1 \Rightarrow x = 0$

(ii) A is complete relative to the metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|x-y\|_n}{1+\|x-y\|_n} .$$

Though it is not necessary to do so, it will be convenient to assume that all σ-C*algebras have units. We remark that if (i) and (ii) are not satisfied for the given sequence of seminorms, then one can always arrange (i) by passing from A to an appropriate quotient algebra, and one may then arrange (ii) by passing to an appropriate completion of this quotient. Second, by replacing $\|\cdot\|_n$ with

$$\|x\|_n' = \max(\|x\|_1, \ldots, \|x\|_n) ,$$

we may assume that $\|x\|_n \leq \|x\|_{n+1}$ for all $n \geq 1$ and all $x \in A$.

Now if $\|\cdot\|$ is any continuous C^*-seminorm on A, then by the general properties of countably normed spaces [12], there is an $n \geq 1$ and a $c > 0$ so that $\|x\| \leq c\|x\|_n$ for every $x \in A$. Because $*$-homomorphisms of C^*-algebras must have norm 1, we may even take $c = 1$, and thus $\|x\| \leq \|x\|_n$, $x \in A$. In any case, this observation implies that the completeness property 5.5(ii) depends not on the particular choice of seminorms $\|\cdot\|_n$, or even on the metric d of 4.5(ii), but only on the topology of A. This is to say that if two sequences of seminorms $\{\|\cdot\|_n\}$ and $\{\|\cdot\|_n'\}$ determine the same topology, and d, d' are the associated metrics, then 5.5(ii) is valid for d iff it is valid for d'.

It is a fact that σ-C^*-algebras enjoy many of the properties of σ-C^*-algebras. Of course, they form a substantially larger category. For instance, using 5.7 below, it can be shown that the most general commutative σ-C^*-algebra is topologically $*$-isomorphic to the algebra $C(X)$ of a σ-compact Hausdorff space X, endowed with its inductive topology as above.

In general, σ-C^*-algebras are projective limits of inverse sequences of C^*-algebras. More precisely, a projective system of C^*-algebras is a sequence $A_1, A_2, \ldots$ of C^*-algebras together with a sequence of surjective $*$-homomorphisms

$$\pi_n: A_{n+1} \to A_n .$$

Every σ-C^*-algebra A gives rise to a projective system

$\pi_n: A_{n+1} \to A_n$ in the following way. Choose a sequence $\|\cdot\|_n \leq \|\cdot\|_{n+1}$ of seminorms on A as in 5.5, let A_n be the completion of A in the norm $\|\cdot\|_n$, and let

$$p_n: A \to A_n$$

be the natural $*$-homomorphism. Each A_n is clearly a unital C^*-algebra. Because $\|x\|_n \leq \|x\|_{n+1}$ and since $p_n(A)$ is dense in A_n (actually, the following discussion implies that $p_n(A) = A_n$), there is a unique surjection $*$-homomorphism $\pi_n: A_{n+1} \to A_n$ satisfying

$$\pi_n \circ p_{n+1} = p_n .$$

Thus, we have a projective system $\pi_n: A_{n+1} \to A_n$, as well as a system of maps $p_n: A \to A_n$ connecting A to $\{A_n\}$. A sequence of elements $a_n \in A_n$ is said to be <u>coherent</u> if $\pi_n(a_{n+1}) = a_n$ for each n. An example of a coherent sequence is obtained by choosing $a_n = p_n(x)$, $n \geq 1$, for a fixed element $x \in A$. Significantly, we have

PROPOSITION 5.6. Every coherent sequence $a_n \in A_n$ has the form $a_n = p_n(x)$ for a unique element $x \in A$.

PROOF. For each $n \geq 1$, we may choose $x_n \in A$ so that $\|p_n(x_n) - a_n\| < 1/n$. For $m, n > k$ we have

$$\begin{aligned}\|x_m - x_n\|_k &= \|p_k(x_m - x_n)\| \\ &= \|\pi_k(p_m(x_m) - a_m) - \pi_k(p_n(x_n) - a_n)\| \\ &\leq \frac{1}{m} + \frac{1}{n},\end{aligned}$$

and thus

$$\lim_{m,n\to\infty} \|x_m - x_n\|_k = 0 , \quad k = 1, 2, \ldots .$$

It follows that $\{x_n\}$ is a Cauchy sequence relative to the metric of 5.5(ii), and hence there is an $x \in A$ such that

$$\lim_{n\to\infty} \|x_n - x\|_k = 0 , \quad k = 1, 2, \ldots .$$

It follows that $a_n = \lim_{m\to\infty} p_n(x_m) = p_n(x)$.

The uniqueness of x follows from 5.5(i); we omit this argument ■

The following result describes a universal property which characterizes A as the projective limit

$$\varprojlim A_n$$

of the inverse sequence $\pi_n: A_{n+1} \to A_n$.

PROPOSITION 5.7. Let B be any σ-C^*-algebra and let $q_n: B \to A_n$ be a sequence of continuous $*$-homomorphisms satisfying $\pi_n \circ q_{n+1} = q_n$, $n \geq 1$. Then there is a unique $*$-homomorphism $\sigma: B \to A$ such that

$$q_n = p_n \circ \sigma , \quad n = 1, 2, \ldots .$$

PROOF. For each $b \in B$, we obtain a coherent sequence $q_n(b) \in A_n$. By the preceding proposition, there is a unique element $\sigma(b) \in A$ satisfying $p_n(\sigma(b)) = q_n(b)$ for all n . This defines a $*$-homomorphism $\sigma: B \to A$.

If $b_k \to 0$ in B , then for every $n \geq 1$, $q_n(b_k) \to 0$ as $k \to \infty$, hence

$$\|\sigma(b_k)\|_n = \|q_n(\sigma(b_k))\| = \|q_n(b_k)\| \to 0$$

as $k \to \infty$. Thus $\sigma(b_k) \to 0$ because the seminorms $\{\|\cdot\|_n\}$ determine the topology of A . That proves σ is continuous.

The uniqueness of σ follows from the uniqueness assertion of 5.6 ■

Thus, every σ-C^*-algebra is a projective limit $\varprojlim A_n$. We now want to point out that <u>every</u> projective system of C^*-algebras $\pi_n: A_{n+1} \to A_n$ arises in this way from a σ-C^*-algebra

$$A_\infty = \varprojlim A_n .$$

To sketch this construction briefly, let A_∞ be the $*$-subalgebra of the infinite Cartesian product of $*$-algebras $A_1 \times A_2 \times \ldots$ consisting of all sequences

$$(a_1, a_2, \ldots) \in A_1 \times A_2 \times \ldots$$

satisfying $\pi_n(a_{n+1}) = a_n$ for each $n \geq 1$. $A_1 \times A_2 \times \ldots$ is a σ-C^*-algebra relative to the product topology, and A_∞ is a closed $*$-subalgebra. Thus A_∞ is a σ-C^*-algebra. Defining $p_n: A_\infty \to A_n$ by

$$p_n(a_1, a_2, \ldots) = a_n ,$$

it is plain that $\pi_n \circ p_{n+1} = p_n$, and one may easily check that the sequence of continuous $*$-homomorphisms $p_n: A_\infty \to A_n$ has the universal property described in proposition 5.7.

Returning now to the main discussion, let A be a fixed C^*-algebra. If (π, D) is a derivation of A into another C^*-algebra B , then we can generate derivations of A into other C^*-algebras as follows. Let B' be another C^*-algebra and let

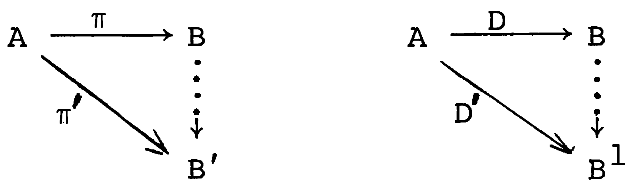
$\sigma: B \to B'$ be a $*$-homomorphism. Then we obtain a derivation $(\sigma\pi, \sigma D)$ of A to B' by composition:

5.8
$$\sigma\pi(x) = \sigma(\pi(x))$$
$$\sigma D(x) = \sigma(D(x)) \ .$$

It is natural to ask if every derivation $(\pi', D') \in \mathcal{D}(A, B')$ arises in this way from an appropriately chosen $\sigma \in \hom(B, B')$. The answer is no; for if (π', D') has the form $(\sigma\pi, \sigma D)$ as above, then since $\|\sigma\| \leqslant 1$ we have $\|D'\| = \|\sigma \circ D\| \leqslant \|D\|$, and hence no derivation (π', D') for which $\|D'\| > \|D\|$ can arise in this way.

In order to obtain a solution of this universal problem we have to consider derivations of A into σ-C^*-algebras; these are pairs (π, D) where π is a (continuous) $*$-homomorphism of A into a σ-C^*-algebra B and D is a self-adjoint continuous linear map of A to B satisfying 5.3.

DEFINITION 5.9. Let A be a C^*-algebra. A triple consisting of a σ-C^*-algebra B and a derivation (π, D) of A into B is said to be universal if, for every C^*-algebra B' and every $(\pi', D') \in \mathcal{D}(A, B')$, there is a unique continuous $*$-homomorphism $\sigma: B \to B'$ which completes both diagrams



as in 5.8.

We will give a construction for universal triples $B, (\pi, D)$ presently. Clearly $B, (\pi, D)$ is unique in the sense that if $B', (\pi', D')$ is another universal triple, then there is a unique isomorphism of σ-C^*-algebras $\sigma: B \to B'$ satisfying $\sigma \circ \pi = \pi'$ and $\sigma \circ D = D'$. We will write TA for B, and (i, d) for (π, D), or (i_A, d_A) if it is necessary to indicate A in the notation. Finally, we will write expressions such as $i(a)d(x) + i(b)d(y)$ more briefly as $adx + bdy$, $a, b, x, y \in A$.

PROPOSITION 5.10. For every unital C^*-algebra A, there is a universal triple TA, (i, d).

PROOF. We first construct a Banach space A-bimodule which is analogous to the module of 1-forms on a smooth n-dimensional manifold.

Let $A \otimes A \otimes A$ be the projective tensor product of three copies of the complex Banach space A. Every element of $A \otimes A \otimes A$ has a representation

$$\text{(i)} \quad \xi = \sum_{n=1}^{\infty} x_n \otimes y_n \otimes z_n , \quad x_n, y_n, z_n \in A ,$$

where

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \|x_n\| \cdot \|y_n\| \cdot \|z_n\| < \infty .$$

The norm of ξ is the infimum of all expressions (ii), where x_n, y_n, z_n satisfy (i). Define a bimodule structure on $A \otimes A \otimes A$ by

$$a(x \otimes y \otimes z)b = ax \otimes y \otimes zb$$

and an involution $\xi \mapsto \xi^*$ by

$$(x \otimes y \otimes z)^* = z^* \otimes y^* \otimes x^* .$$

We obviously have

5.11 $$(a\xi b)^* = b^*\xi^*a^* ,$$

for $a, b \in A$, $\xi \in A \otimes A \otimes A$. Let K be the closed linear subspace of $A \otimes A \otimes A$ generated by elements of the form

$$a \otimes xy \otimes b - ax \otimes y \otimes b - a \otimes x \otimes yb .$$

K is stable under the left and right action of A as well as the $*$-operation, and therefore the quotient

$$dA = A \otimes A \otimes A/K$$

is an involutory Banach A-bimodule whose involuiton satisfies 5.11.

By construction and the universal property of the projective tensor product of Banach spaces, we may conclude that dA has the following property: if B is any C^*-algebra and $\rho: A \times A \times A \to B$ is a trilinear map satisfying

$$\|\rho(x, y, z)\| \leqslant M\|x\| \cdot \|y\| \cdot \|z\|$$
$$\rho(x, y, z)^* = \rho(z^*, y^*, x^*)$$
$$\rho(a, xy, b) = \rho(ax, y, b,) + \rho(a, x, yb) ,$$

then there is a unique bounded linear mapping $R: dA \to B$ satisfying

$$R(a \otimes x \otimes b + K) = \rho(a, x, b) .$$

In addition, one has $R(\xi^*) = R(\xi)^*$ and $\|R\| \leqslant M$.

Now define a linear map d of A into dA by

$$d(a) = 1 \otimes x \otimes 1 + K ,$$

1 denoting the unit of A . Notice that $\|d\| \leqslant 1$, $d(x)^* = d(x^*)$, and by definition of K we have

$$d(xy) = xd(y) + d(x)y .$$

Moreover, the relation between R and ρ above becomes

5.12 $$R(ad(x)b) = \rho(a, x, b) .$$

We now construct the tensor algebra over the module dA . First, if M is any Banach A-bimodule over A , then we may define the n-fold tensor product M^n as follows:

$$M^n = M \underset{A}{\otimes} M \underset{A}{\otimes} \cdots \underset{A}{\otimes} M .$$

More precisely, one first form the n-fold projective tensor product of Banach spaces $M \otimes M \otimes \cdots \otimes M$. This is a bimodule over A in a natural way

$$a(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n)b = a\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n b .$$

Letting K_n denote the closed linear subspace of $M \otimes M \otimes \cdots \otimes M$ generated by elements of the form

$$\xi_1 \otimes \cdots \otimes \xi_k a \otimes \xi_{k+1} \otimes \cdots \otimes \xi_n - \xi_1 \otimes \cdots \otimes \xi_k \otimes a\xi_{k+1} \otimes \cdots \otimes \xi_n ,$$

$a \in A$, $\xi_i \in M_i$, $1 \leqslant k \leqslant n-1$, then M^n is defined as the quotient

$$M_1 \otimes M_2 \otimes \cdots \otimes M_n/K_n .$$

If M has an isometric involution $\xi \mapsto \xi^*$ satisfying 5.11 then we may define an isometric involution in M^n by

$$(\xi_1 \otimes \cdots \otimes \xi_n + K_n)^* = \xi_n^* \otimes \xi_{n-1}^* \otimes \cdots \otimes \xi_1^* + K_n .$$

For each n-tuple $\xi_1, \ldots, \xi_n \in M$ we can define a "product" $\xi_1 \cdot \ldots \xi_n \in M^n$ by

$$\xi_1 \cdot \ldots \cdot \xi_n = \xi_1 \otimes \cdots \otimes \xi_n + K_n .$$

Finally, this product module M^n has the following universal property: if B is any C^*-algebra and $\lambda\colon M \times \cdots \times M \to B$ is an n-variate multilinear mapping satisfying

$$\|\lambda(\xi_1, \ldots, \xi_n)\| \leqslant M\|\xi_1\| \cdot \ldots \cdot \|\xi_n\|$$
$$\lambda(\xi_1, \ldots, \xi_n)^* = \lambda(\xi_n^*, \ldots, \xi_1^*)$$

and

$$\lambda(\xi_1, \ldots, \xi_k a, \xi_{k+1}, \ldots, \xi_n) = \lambda(\xi_1, \ldots, \xi_k, a\xi_{k+1}, \ldots, \xi_n)$$

for every $a \in A$, $\xi_j \in M$, $1 \leqslant k \leqslant n-1$, then there is a unique bounded linear mapping $L: M^n \to B$ satisfying

5.13 $$L(\xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_n) = \lambda(\xi_1, \ldots, \xi_n) .$$

Moreover, one has $\|L\| \leq M$ and $L(\zeta^*) = L(\zeta)^*$ for every $\zeta \in M^n$.

Now for each $n \geq 0$, we can define an involutive Banach A-bimodule dA^n by

$$dA^0 = A , \quad \text{and}$$

$$dA^n = \underbrace{dA \cdot dA \ \ldots \ dA}_{n} , \quad n \geq 1 .$$

Let $\sum dA^n$ denote the (algebraic) direct sum of vector spaces

$$\sum dA^n = A + dA + dA^2 + \ldots .$$

Thus, a generic element of $\sum dA^n$ is a sequence $(a, \xi_1, \xi_2, \ldots)$, where $a \in A$, $\xi_k \in dA^k$, and $\xi_k = 0$ for sufficiently large k. For each nonnegative real number $r \geq 0$ we have a seminorm $\|\cdot\|_r$ on $\sum dA^n$ defined by

$$\|(a, \xi_1, \xi_2, \ldots)\|_r = \|a\| + \sum_1^\infty r^n \|\xi_k\| .$$

Note that $\|\cdot\|_1$ restricts to the given norm on each summand dA^n, and we have $\|\xi\|_{r_1} \leq \|\xi\|_{r_2}$ if $r_1 \leq r_2$.

Now for each $m, n \geq 0$, we have a bounded bilinear mapping

$$(\xi, \eta) \in dA^m \times dA^n \mapsto \xi \cdot \eta \in dA^{m+n}$$

defined on generators $\xi = u_1 \ \ldots \ U_m$, $\eta = v_1 \ \ldots \ v_n$ by

$$(u_1 \cdot \ldots \cdot u_m) \cdot (v_1 \cdot \ldots \cdot v_n) = u_1 \cdot \ldots \cdot u_m \cdot v_1 \cdot \ldots \cdot v_n .$$

This operation extends linearly so as to make $\sum dA^n$ into an associative complex algebra, and we have

$$\|\xi \cdot \eta\|_r \leq \|\xi\|_r \|\eta\|_r$$

for all $r \geq 0$, $\xi, \eta \in \sum dA^n$. The unit of A gives rise to a unit e of $\sum dA^n$ defined by

$$e = (1, 0, 0, \ldots) ,$$

and the map $(a, \xi_1, \xi_2, \ldots) \mapsto (a^*, \xi_1^*, \xi_2^*, \ldots)$ defines an involution of the algebra structure on $\sum dA^n$ such that $\|\xi^*\|_r = \|\xi\|_r$ for all r, ξ.

Finally, let $i, d: A \to \sum dA^n$ denote the linear maps

$$i(a) = (a, 0, 0, \ldots)$$

$$d(a) = (0, d(a), 0, \ldots) .$$

Both i and d are self-adjoint linear maps satisfying

$\|i(x)\|_r \leq \|x\|$, $\|d(x)\|_r \leq r\|x\|$, i is multiplicative, and d satisfies

$$d(xy) = i(x)d(y) + d(x)i(y) .$$

Now if B is any C^*-algebra and $(\pi, D) \in \mathcal{D}(A, B)$, we claim that there is a unique $*$-homomorphism $\sigma: \sum dA^n \to B$ satisfying

$$\sigma(d(x)) = D(x)$$

$$\sigma(i(x)) = \pi(x) ,$$

$x \in A$, and $\|\sigma(\xi)\| \leq \|\xi\|_r$ for all ξ , where $r = \|D\|$. To construct σ , first consider the trilinear map

$$(a, x, b) \to \pi(a)D(x)\pi(b)$$

of $A \times A \times A$ into B . Then there is a linear map $L_1: dA \to B$ of norm at most r , satisfying $L(\xi^*) = L(\xi)^*$ and the formula

5.14 $$L_1(ad(x)b) = \pi(a)D(x)\pi(b) .$$

Now for each $n \geq 2$ we can utilize the universal property 5.13 to obtain a self-adjoint linear map $L_n: dA^n \to B$ such that $\|L_n(\xi)\| \leq r^n\|\xi\|_1$, $\xi \in dA^n$, and

$$L_n(\xi_1 \cdot \ldots \cdot \xi_n) = L_1(\xi_1) \cdot \ldots \cdot L_1(\xi_n) ,$$

$\xi_1, \ldots, \xi_n \in dA$. Finally, we can define σ by

$$\sigma(a, \zeta_1, \zeta_2, \ldots) = \pi(a) + \sum_{n=1}^{\infty} L_n(\zeta_n) .$$

σ is a $*$-homomorphism having all the assorted properties. The proof of uniqueness of σ is left for the reader.

The norms $\|\cdot\|_r$ on $\sum dA^n$ are not C^*-seminorms, but this can be remedied by a familiar device. For each $r \geq 0$, we can define a C^*-seminorm $\|\cdot\|_r'$ on $\sum dA^n$ by

$$\|\zeta\|_r' = \sup\|\sigma(\zeta)\| ,$$

the supremum taken over all $*$-homomoprhisms σ of $\sum dA^n$ into the algebra of all bounded operators on some Hilbert space H_σ , satisfying $\|\sigma(\zeta)\| \leq \|\zeta\|_r$. Clearly $\|\zeta\|_r \leq \|\zeta\|_r'$ for every $\zeta \in \sum dA^n$. Moreover, it is easy to see that the $*$-homomorphism $\sigma: \sum dA^n \to B$ constructed above from (π, D) actually satisfies

$$\|\sigma(\zeta)\| \leq \|\zeta\|_r' , \quad \zeta \in \sum dA^n ,$$

for $r = \|D\|$. If we choose any sequence of reals $r_1 < r_2 < \ldots$ such that $r_k \uparrow \infty$, then we can define the σ-C^*-algebra TA as the completion of $\sum dA^n$ relative to the sequence of C^*-seminorms

$\|\cdot\|'_{r_1} \leq \|\cdot\|'_{r_2} \leq \ldots$. The corresponding derivation

$$(i, d): A \to TA$$

plainly has the right universal properties ■

This construction of TA from A can be modified so that it works for any σ-C^*-algebra A . Moreover, in this setting, the natural derivation $(i, d) \in \mathcal{D}(A, TA)$ has the stronger property that if $(\pi, D) \in \mathcal{D}(A, B)$ is any continuous derivation of σ-C^*-algebras, then there is a unique homomoprhism of σ-C^*-algebras σ: TA B such that $h = \sigma \circ i$, $D = \sigma \circ d$. Here we merely sketch the details.

We claim first that the tangent algebra of any C^*-algebra A has the stronger property: if (π, D) is a continous self-adjoint derivation of A into any σ-C^*-algebra B , then there is a unique homomorphism of σ-C^*-algebras σ hom(TA, B) such that

$$\sigma \circ i = \pi$$

$$\sigma \circ d = D .$$

For this, let $\|\cdot\|_n \leq \|\cdot\|_{n+1}$ be C^*-seminorms on B which determine its topology, let $\pi_n: B_{n+1} \to B_n$ be the associated inverse system of C^*-algebras, and let $p_n: B \to B_n$ be the connecting maps. For each n we have a derivation $(p_n \circ \pi , p_n \circ D) \in \mathcal{D}(A, B_n)$ and, since B_n is a C^*-algebra there is a unique $*$-homomorphism $\sigma_n: TA \to B_n$ satisfying $\sigma_n \circ i = p_n \circ \pi$, $\sigma_n \circ d = p_n \circ D$. For each $\xi \in TA$, the sequence

$$(\sigma_1(\xi), \sigma_2(\xi), \ldots)$$

is coherent in the sense that $\pi_n\sigma_{n+1}(\xi) = \sigma_n(\xi)$, and so by 5.6 there is a unique element $\sigma(\xi) \in B$ for which $p_n\sigma(\xi) = \sigma_n(\xi)$. This defines a $*$-homomorphism $\sigma: TA \to B$ for which $p_n\sigma(i(a)) = \sigma_n i(a) = p_n\pi(a)$, and hence $\sigma(i(a)) = \pi(a)$, $a \in A$. Similarly, $\sigma \circ d = D$. Continuity of σ follows from continuity of each map $p_n \circ \sigma$. The proof of uniqueness of σ is left for the reader.

Second, we indicate how the construction of TA must be modified when A is merely a σ-C^*-algebra. Again, let $\|\cdot\|_n \leq \|\cdot\|_{n+1}$ be C^*-seminorms on A with associated sequences $\pi_n: A_{n+1} \to A_n$, $p_n: A \to A_n$. For each n we form the tangent algebra TA_n and its universal derivation $(i_n, d_n) \in \mathcal{D}(A_n, TA_n)$. Make $\{TA_n\}$ into a projective sequence as follows. For each n , we have a derivation

$$(i_n \circ \pi_n, d_n \circ \pi_n) \in \mathcal{D}(A_{n+1}, TA_n) .$$

By the preceding paragraph, there is a unique $\tilde{\pi}_n \in \hom(TA_{n+1}, TA_n)$ satisfying

$$\tilde{\pi}_n \circ i_{n+1} = i_n \circ \pi_n$$
$$\tilde{\pi}_n \circ d_{n+1} = d_n \circ \pi_n \ .$$

Thus we have an inverse system of σ-C^*-algebras $\tilde{\pi}_n: TA_{n+1} \to TA_n$ and, exactly as we did for C^*-algebras, we may construct an inverse limit

$$TA = \varprojlim TA_n$$

and connecting maps $r_n: TA \to TA_n$, satisfying $\tilde{\pi}_n \circ r_{n+1} = r_n$. TA is a σ-C^*-algebra. Now define (i, d) in $\mathcal{D}(A, TA)$ as follows. For each $a \in A$ we have coherent sequences

$$(i_1p_1(a), i_2p_2(a), \ldots)$$
$$(d_1p_1(a), d_2 0_2(a), \ldots)$$

in $TA_1 \times TA_2 \ \ldots$. Thus there are unique elements $i(a)$, $d(a)$ in TA satisfying

$$r_n i(a) = i_n p_n(a)$$
$$r_n d(a) = d_n p_n(a) \ .$$

It is easily checked that (i, d) is a derivation of A into TA which has the required universal property for derivations of σ-C^*-algebras.

Every smooth mapping $\phi: M \to N$ of manifolds induces a smooth mapping of their tangent spaces $d\phi: TM \to TN$. In this setting, the situation is described as follows. Let A, B be σ-C^*-algebras and let $\pi \in \hom(A, B)$ be a continuous $*$-homomorphism. Then the pair $(i_B \circ \pi, d_B \circ \pi)$ is a self-adjoint derivation of A into TB , so by the universal property of (i_A, d_A) there is a unique $d\pi \in \hom(TA, TB)$ satisfying

5.15
$$d\pi \circ i_A = i_B \circ \pi$$
$$d\pi \circ d_A = d_B \circ \pi \ .$$

Naturally, we call $d\pi$ the <u>differential</u> of π . Using 5.15, the reader can check easily that for composable maps $\pi_2 \in \hom(A, B)$, $\pi_1 \in \hom(B, C)$ we have the <u>chain rule</u>:

$$d(\pi_1 \circ \pi_2) = d\pi_1 \circ d\pi_2 \ .$$

We conclude that

$$T: \begin{matrix} A \to TA \\ \pi \to d\pi \end{matrix}$$

is a <u>covariant functor from</u> σ-C^*-<u>algebras to</u> σ-C^*-<u>algebras</u>. This functor is in fact <u>continuous</u> for appropriate topologies on the spaces hom(A, B) (analogous to the point-norm topology).

If A is a separable C^*-algebra then one can see from its construction that TA is a separable σ-C^*-algebra. TA is never commutative, even when $A = C(X)$, except in the trivial case $A = \mathbb{C}$. The reason for this is clear because, while a commutative C^*-algebra has no nontrivial derivations into itself, it certainly has nontrivial derivations into other C^*-algebras. For instance, if $A \subseteq \mathcal{L}(H)$ is any C^*-algebra of operators and x is any skew-adjoint operator on H which is not in the commutant of A, then $D_x(a) = xa - ax$ defines a nontrivial derivation of A into $\mathcal{L}(H)$.

Nevertheless, we will see that in all cases, TA is <u>homotopically</u> equivalent to A By a <u>homotopy equivalence</u> of σ-C^*-algebras A, B we mean a pair of maps $\pi \in \hom(A, B)$, $\sigma \in \hom(B, A)$ such that both endomorphisms $\sigma \circ \pi$ and $\pi \circ \sigma$ can be joined to the respective identity maps by an arc of $*$-endomorphisms; this means that there is a function

$$t \in [0, 1] \mapsto \theta_t \in \mathrm{end}(A)$$

such that $t \mapsto \theta_t(a)$ is continuous for each $a \in A$, for which $\theta_0 = \sigma \circ \pi$ and $\theta_1 = \mathrm{id}_A$.

Let A be a fixed σ-C^*-algebra. We show that TA and A are homotopy equivalent. We already have a map $i \in \hom(A, TA)$ and we can define a map $\sigma \in \hom(TA, A)$ by the universal property

$$\sigma \circ i = \mathrm{id}_A$$
$$\sigma \circ d = 0 ,$$

because $(\mathrm{id}_A, 0)$ is a derivation of A into itself. We already have $\sigma \circ i = \mathrm{id}_A$, so to show that i, σ is a homotopy equivalence we need only construct an arc $\theta_t \in \mathrm{end}(TA)$ satisfying $\theta_0 = i \circ \sigma$ and $\theta_1 = \mathrm{id}_{TA}$. For each $t \in [0, 1]$, consider the derivation (i, td) of A into TA. By the (extended) universal property we have a unique $\theta_t \in \mathrm{end}(TA)$ satisfying

5.16 $$\theta_t \circ i = i$$
$$\theta_t \circ d = td .$$

Notice that $\lambda = \theta_0$ and $\lambda = i \circ \sigma$ both satisfy the conditions

$$\lambda \circ i = i$$

$$\lambda \circ d = 0$$

and hence $\theta_0 = i \circ \sigma$ by the uniqueness assertion of the universal property for (i, d). Similarly, both $\lambda = \theta_1$ and $\lambda = id_{TA}$ satisfy

$$\lambda \circ i = i$$

$$\lambda \circ d = d ,$$

and hence $\theta_1 = id_{TA}$. The fact that $t \mapsto \theta_t(\xi)$ is continuous for every $\xi \in TA$ follows from the derinition 5.16 and some simple facts from C^*-algebra theory (using the resolution of TA into a projective system of C^*-algebras), and is left for the reader.

In conclusion, we want to point out that the tangent algebra of a C^*-algebra A has been introduced in order to deal with bounded derivations of A whose norms can be arbitrarily large, but <u>finite</u>. In order to deal with unbounded derivations, more precisely, with the generators of one-parameter automorphism groups, one cannot hope to work within the category of σ-C^*-algebras. The reason is clear if one considers the commutative situation. A vector field on a smooth manifold M may be considered as a derivation of the algebra $C^\infty(M)$ of all complex-valued smooth functions on M. $C^\infty(M)$ is a commutative "σ-Banach" $*$-algebra but it is not a σ-C^*-algebra. It does have a sequence $\|\cdot\|_n \leqslant \|\cdot\|_{n+1}$ of continuous C^*-seminorms

$$\|f\|_n = \sup_{p \in K_n} |f(p)|$$

($K_1, K_2, \ldots$ being compact subsets of M such that $M = \cup K_n$), and the completion of $C^\infty(M)$ relative to this sequence $\{\|\cdot\|_n\}$ is a commutative σ-C^*-algebra $C(M)$. But of course derivations of $C^\infty(M)$ cannot be extended to derivations of $C(M)$, and one must work not directly with the σ-C^*-algebra $C(M)$ but with the σ-Banach algebra $C^\infty(M)$.

Similarly, in order to properly formulate non-commutative differential geometry (by this we mean the study of generators of one parameter groups of $*$-automorphisms of a given C^*-algebra) one should work not with a σ-C^*-algebra but with a σ-Banach $*$-algebra which has a σ-C^*-algebra as a natural completion. These developments are still tentative, but it is clear that some variation of Grothendieck's notion of nuclear space is fundamental to the subject.

For some specific and suggestive results, the reader is referred to recent work of A. Connes and S. Sakai.

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