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Interpolation Problems in Nest Algebras*

WILLIAM ARVESON

University of California, Berkeley, California 94720

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The main result of this paper is a theorem which allows one to determine when a finitely generated left ideal in certain reflexive operator algebras is trivial (i.e., contains the identity). This is based on a formula which expresses the distance from such an algebra to an arbitrary operator on the underlying Hilbert space. As an application, we are able to deduce an operator-theoretic variant of the Corona theorem. Some applications of the distance formula to quasitriangular operators are given, and we present some new "inner–outer" factorization theorems along the way to the main result.

1. THE DISTANCE FORMULA

Let $\mathcal{A}$ be a (perhaps non-self-adjoint) algebra of operators on a Hilbert space $\mathscr{H}$, and let T be an arbitrary bounded operator on $\mathscr{H}$. If P is a (self-adjoint) projection whose range is invariant under $\mathcal{A}$, then for each $A \in \mathcal{A}$ one has $(1 - P) AP = 0$, hence $\| T - A \| \geqslant \|(1 - P)(T - A)P\| = \|(1 - P) TP\|$. It follows that

$$d(T, \mathcal{A}) \geqslant \sup_P \|(1 - P) TP\|,$$

where $d(T, \mathcal{A})$ is the distance from T to $\mathcal{A}$ and where the supremum is taken over the lattice lat $\mathcal{A}$ of all $\mathcal{A}$-invariant projections. The purpose of this section is to prove that equality holds for a certain class of reflexive operator algebras. This distance formula is essential for the results of Section 4, and appears to be useful in other contexts as well; for instance, some applications to quasitriangular operators and algebras are presented in the following section.

Every set $\mathscr{L}$ of projections in $\mathscr{L}(\mathscr{H})$ determines an algebra alg $\mathscr{L}$, consisting of all operators A satisfying $(1 - P) AP = 0$ for every

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$P \in \mathscr{L}$, and as usual an algebra $\mathfrak{A}$ is called *reflexive* if it arises in this way; equivalently, $\mathfrak{A} = \operatorname{alg} \operatorname{lat} \mathfrak{A}$ [9]. We begin with two general lemmas.

LEMMA 1. *Let $\mathfrak{A} \subset \mathscr{L}(\mathscr{H})$ be a reflexive algebra and let $T \in \mathscr{L}(\mathscr{H})$. Then*

$$d(T, \mathfrak{A}) = \sup_{\mathscr{F}} d(T, \operatorname{alg} \mathscr{F}),$$

where $\mathscr{F}$ ranges over all finite subsets of lat $\mathfrak{A}$.

Proof. If $\mathscr{F}$ is any finite subset of lat $\mathfrak{A}$, then alg $\mathscr{F}$ contains alg lat $\mathfrak{A} = \mathfrak{A}$, so that $d(T, \mathfrak{A}) \geqslant d(T, \operatorname{alg} \mathscr{F})$, proving the inequality $\geqslant$.

For the opposite inequality, let α denote the right-hand side of the asserted formula and choose $\epsilon > 0$. For each finite subset $\mathscr{F} \subseteq \operatorname{lat} \mathfrak{A}$ choose $A_{\mathscr{F}} \in \operatorname{alg} \mathscr{F}$ such that $\| T - A_{\mathscr{F}} \| \leqslant \alpha + \epsilon$. The $A_{\mathscr{F}}$'s are bounded in norm (because $\| A_{\mathscr{F}} \| \leqslant \| T - A_{\mathscr{F}} \| + \| T \| \leqslant \alpha + \epsilon + \| T \|$) so that the sets of operators $\mathscr{S}_{\mathscr{F}} = \{A_{\mathscr{G}} \colon \mathscr{G} \supseteq \mathscr{F}\}^{-\text{weak}}$ are compact in the weak operator topology. Moreover, these sets have the finite intersection property because the finite subsets of lat $\mathfrak{A}$ form an increasing directed set relative to the usual set inclusion. Hence we may find an operator A_0 in the intersection $\bigcap_{\mathscr{F}} \mathscr{S}_{\mathscr{F}}$.

Now since the norm is lower semicontinuous in the weak operator topology it follows that $\| T - X \| \leqslant \alpha + \epsilon$ for every $X \in \mathscr{S}_{\mathscr{F}}$, for each $\mathscr{F}$, and in particular $\| T - A_0 \| \leqslant \alpha + \epsilon$. Finally, we claim that $A_0 \in \mathfrak{A}$ (since ϵ is arbitrary, this will complete the proof). Because $\mathfrak{A}$ is reflexive, this is the same as proving $A_0 \in \operatorname{alg} \operatorname{lat} \mathfrak{A}$. But for every $P \in \operatorname{lat} \mathfrak{A}$, we have $A_0 \in \mathscr{S}_{\{P\}} \subseteq \operatorname{alg}\{P\}$; hence $(1 - P) A_0 P = 0$, as required. ∎

We shall write $l^2 \otimes \mathscr{H}$ for the Hilbert space direct sum $\mathscr{H} \oplus \mathscr{H} \oplus \cdots$ of denumerably many copies of $\mathscr{H}$, and for each operator $T \in \mathscr{L}(\mathscr{H})$, $1 \otimes T$ will denote the operator $T \oplus T \oplus \cdots \in \mathscr{L}(l^2 \otimes \mathscr{H})$. It is well known that the map $X \mapsto 1 \otimes X$ is an ultraweakly continuous *-isomorphism of $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(l^2 \otimes \mathscr{H})$. Moreover, every ultraweakly continuous linear functional ρ on $\mathscr{L}(\mathscr{H})$ can be expressed in the form $\rho(T) = ((1 \otimes T)\xi, \eta)$, where ξ and η are vectors in $l^2 \otimes \mathscr{H}$ of norm $\| \rho \|^{1/2}$ (see [7]).

We shall also make use of the following bit of lore from the elementary theory of Banach spaces, which we merely state for the reader's convenience. *If M is a weak*-closed linear subspace of the dual E' of a Banach space E, then for every $f \in E'$ one has*

$$\inf_{g \in M} \| f - g \| = \sup\{| f(x)| \colon x \in E, x \perp M, \| x \| \leqslant 1\}.$$

LEMMA 2. *Let $\mathcal{A}$ be an arbitrary ultraweakly closed subalgebra of $\mathcal{L}(\mathcal{H})$ containing* 1, *and let $T \in \mathcal{L}(\mathcal{H})$. Then*

$$d(T, \mathcal{A}) = \sup\{\|(1 - P)(1 \otimes T)P\| : P \in \operatorname{lat}(1 \otimes \mathcal{A})\}.$$

Proof. The inequality $\geqslant$ is trivial, and follows from the argument given at the beginning of this section, provided one observes that $d(T, \mathcal{A}) = d(1 \otimes T, 1 \otimes \mathcal{A})$.

For the opposite inequality, we make use of the fact that $\mathcal{L}(\mathcal{H})$ is the dual of the Banach space $\mathcal{L}(\mathcal{H})_*$ of all ultraweakly continuous linear functionals on $\mathcal{L}(\mathcal{H})$, via the duality $[\rho, T] = \rho(T)$. Since $\mathcal{A}$ is ultraweakly closed, the inequality $\leqslant$ will follow from the remarks preceding the lemma provided we can prove that, for every $\rho \in \mathcal{L}(\mathcal{H})_*$ satisfying $\|\rho\| \leqslant 1$ and $\rho \perp \mathcal{A}$, one has

$$|\rho(T)| \leqslant \sup\{\|(1 - P)(1 \otimes T)P\| : P \in \operatorname{lat}(1 \otimes \mathcal{A})\}.$$

For that, choose such a ρ. As we have already observed, there exist vectors $\xi, \eta \in l^2 \otimes \mathcal{H}$, $\|\xi\| \leqslant 1$, $\|\eta\| \leqslant 1$, such that $\rho(X) = ((1 \otimes X)\xi, \eta)$ for every $X \in \mathcal{L}(\mathcal{H})$. The condition $\rho(\mathcal{A}) = 0$ becomes the condition $\eta \perp [(1 \otimes \mathcal{A})\xi]$; so if P denotes the projection onto $[(1 \otimes \mathcal{A})\xi]$, then clearly $P \in \operatorname{lat}(1 \otimes \mathcal{A})$ and $\eta \in \operatorname{range}(1 - P)$. Moreover, since $\mathcal{A}$ must contain the identity it follows that $\xi \in \operatorname{range}(P)$. Hence,

$$\begin{aligned} |\rho(T)| &= |((1 \otimes T)\xi, \eta)| = |((1 \otimes T) P\xi, (1 - P)\eta)| \\ &\leqslant \|(1 - P)(1 \otimes T)P\| \cdot \|\xi\| \cdot \|\eta\| \\ &\leqslant \sup\{\|(1 - Q)(1 \otimes T)Q\| : Q \in \operatorname{lat}(1 \otimes \mathcal{A})\}, \end{aligned}$$

as required. ∎

The preceding result indicates that, in order to compute $d(T, \mathcal{A})$, one should examine the structure of $\operatorname{lat}(1 \otimes \mathcal{A})$. This program was carried out in detail for a broad class of operator algebras in [4], and in fact one may give a proof of Theorem 1.1 below based on the results of [4]. However, for the algebras of interest in this paper (i.e., nest algebras), Lemma 1 provides enough of a reduction that an elementary analysis of the structure of $\operatorname{lat}(1 \otimes \mathcal{A})$ can be made. This is the content of the following two results.

LEMMA 3. *Let $\mathcal{A}$ be a reflexive subalgebra of $\mathcal{L}(\mathcal{H})$ such that* lat $\mathcal{A}$ *contains an element $P \neq 1$ which dominates every projection in* lat $\mathcal{A}$ *other than* 1.

Then every projection $E \in \mathrm{lat}(1 \otimes \mathcal{A})$ *has a decomposition* $E = R + E_0$, *where* R *commutes with* $1 \otimes \mathcal{L}(\mathcal{H})$, $E_0 \leqslant 1 \otimes P$, *and* $R \perp E_0$.

Proof. Assume first that the largest subprojection of E which reduces $1 \otimes \mathcal{L}(\mathcal{H})$ is 0. We will prove that $E \leqslant 1 \otimes P$.

First, note that $X(1 - P) \in \mathcal{A}$ for every $X \in \mathcal{L}(\mathcal{H})$. Indeed, if $Q \in \mathrm{lat}\,\mathcal{A}$, $Q \neq 1$, then $Q \leqslant P$ so that $(1 - Q)\,X(1 - P)Q = 0$. Hence, $X(1 - P) \in \mathrm{alg\,lat}\,\mathcal{A} = \mathcal{A}$, as asserted.

In particular, $(1 - P)$, and therefore P itself, belongs to $\mathcal{A}$. Therefore E is invariant under $1 \otimes P$, and we conclude that E commutes with $1 \otimes P$.

Let $F = (1 \otimes (1 - P))E$. Since $\mathcal{L}(\mathcal{H})(1 - P) \subseteq \mathcal{A}$ and E is invariant under $1 \otimes \mathcal{A}$, it follows that $(1 - E)(1 \otimes \mathcal{L}(\mathcal{H}))F = 0$; i.e., the closed span of the ranges of all operators of the form $(1 \otimes X)F$, $X \in \mathcal{L}(\mathcal{H})$, is contained in E. The latter subspace clearly reduces $1 \otimes \mathcal{L}(\mathcal{H})$, so by the assumption on E we conclude that $(1 \otimes X)F = 0$ for every $X \in \mathcal{L}(\mathcal{H})$. In particular, $F = (1 - 1 \otimes P)E = 0$, giving the desired conclusion $E \leqslant 1 \otimes P$.

In the case of a general $E \in \mathrm{lat}(1 \otimes \mathcal{A})$, let R be the largest subprojection of E which reduces $1 \otimes \mathcal{L}(\mathcal{H})$, and put $E_0 = E - R$. Clearly $E_0 \in \mathrm{lat}(1 \otimes \mathcal{A})$, and E_0 satisfies the assumption at the beginning of the proof. We conclude from the above that $E_0 \leqslant 1 \otimes P$. ∎

In the following lemma, we shall realize $l^2 \otimes \mathcal{H}$ (defined as a direct sum of copies of $\mathcal{H}$) as the Hilbert space tensor product of l^2 with $\mathcal{H}$. Thus for each bounded operator A on l^2 and each $B \in \mathcal{L}(\mathcal{H})$, we may form the operator $A \otimes B$ on $l^2 \otimes \mathcal{H}$ in the usual sense of tensor products; and of course the two definitions of $1 \otimes B$ agree under the natural identification.

LEMMA 4. *Let* $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ *be a reflexive algebra such that* $\mathrm{lat}\,\mathcal{A}$ *is a finite chain* $\{0 = P_0 < P_1 < \cdots < P_n = 1\}$. *Then every element of* $\mathrm{lat}(1 \otimes \mathcal{A})$ *admits an expression*

$$E = A_0 \otimes P_0 + \cdots + A_n \otimes P_n,$$

where $A_0, \ldots, A_n$ *are mutually orthogonal projections in* $\mathcal{L}(l^2)$ *having sum* 1.

Proof. We use induction on $n \geqslant 1$. The case $n = 1$ is trivial. For then, $\mathcal{A} = \mathcal{L}(\mathcal{H})$, and the expression above follows from the familiar fact that the commuting projections for $1 \otimes \mathcal{L}(\mathcal{H})$ are all of the form $A_1 \otimes 1$, where A_1 is a projection acting on l^2.

Assume now that the lemma is true for $k \leqslant n$, and suppose $\operatorname{lat} \mathfrak{A} = \{0 = P_0 < P_1 < \cdots < P_{n+1} = 1\}$. The projection P_n satisfies the hypothesis of the preceding lemma, and thus E decomposes to the form $E = R + E_0$, where R reduces $1 \otimes \mathscr{L}(\mathscr{H})$ and $E_0 \leqslant 1 \otimes P_n$. Since every operator commuting with $1 \otimes \mathscr{L}(\mathscr{H})$ must have the form $X \otimes 1$, it follows that R must have the form $A_{n+1} \otimes 1 = A_{n+1} \otimes P_{n+1}$, where A_{n+1} is a projection acting on l^2.

Consider now the algebra $\mathfrak{A}_0 = \mathfrak{A} \mid \operatorname{range} P_n$. It is easy to see that $\operatorname{lat} \mathfrak{A}_0 = \{0 = P_0 < \cdots < P_n = 1\}$, and $\mathfrak{A}_0$ is reflexive because $\mathfrak{A}$ is. Moreover, if we consider the Hilbert space $\mathscr{K} = (1 - A_{n+1})\, l^2$, then the range of E_0 is contained in $\mathscr{K} \otimes \operatorname{range} P_n$ (because $E_0 \perp A_{n+1} \otimes 1$ and $E_0 \leqslant 1 \otimes P_n$), and is invariant under the algebra $1_{\mathscr{K}} \otimes A_0$. So by the induction hypothesis we conclude that there are mutually orthogonal subprojections $A_1, \ldots, A_n$ of $1 - A_{n+1} = 1_{\mathscr{K}}$, having sum $1 - A_{n+1}$, such that $E_0 = A_0 \otimes P_0 + \cdots + A_n \otimes P_n$ (strictly speaking, we have only stated the induction hypothesis for the case where $\mathscr{K} \cong l^2$ is infinite dimensional; however, the *arguments* given apply to arbitrary Hilbert spaces). The required formula $E = A_0 \otimes P_0 + \cdots + A_{n+1} \otimes P_{n+1}$ is now immediate. ∎

We come now to the main result of this section, which applies to *nest algebras* [11] (i.e., reflexive algebras $\mathfrak{A}$ such that $\operatorname{lat} \mathfrak{A}$ is totally ordered).

THEOREM 1.1. *Let $\mathfrak{A}$ be a nest algebra on a Hilbert space $\mathscr{H}$, and let $T \in \mathscr{L}(\mathscr{H})$. Then*

$$d(T, \mathfrak{A}) = \sup\{\|(1 - P)\, TP\| : P \in \operatorname{lat} \mathfrak{A}\}.$$

Proof. As we have already pointed out, we need only prove the inequality $\leqslant$.

Suppose first that the theorem has been proved for the special case where $\operatorname{lat} \mathfrak{A}$ is finite. Then for every finite chain $\mathscr{F}$ of projections we have $d(T, \operatorname{alg} \mathscr{F}) \leqslant \sup\{\|(1 - E)\, TE\| : E \in \mathscr{F}\}$ (here we have used the fact that $\operatorname{lat} \operatorname{alg} \mathscr{F} = \mathscr{F} \cup \{0, 1\}$, which can easily be proved directly, or found in [4] or [11]). The theorem now follows from an application of Lemma 1.

To deal with the case where $\operatorname{lat} \mathfrak{A} = \{0 = P_0 < P_1 < \cdots < P_n = 1\}$ is finite, we see by Lemma 2 that it suffices to prove that, for every $E \in \operatorname{lat}(1 \otimes \mathfrak{A})$, one has $\|(1 - E)(1 \otimes T)E\| \leqslant \sup_n \|(1 - P_n)\, TP_n\|$. So choose such an E. By Lemma 4, there exist mutually orthogonal projections $A_0, \ldots, A_n \in \mathscr{L}(l^2)$ with $\sum A_k = 1$ and $E = \sum_k A_k \otimes P_k$. Noting that $1 - E = \sum_k A_k \otimes (1 - P_k)$ (the easiest way to see this is to check that the projection $F = \sum_k A_k \otimes (1 - P_k)$ satisfies the conditions $EF = 0$ and $E + F = 1$), we have

$$\begin{aligned}(1 - E)(1 \otimes T)E &= \sum_{k,j} A_k A_j \otimes (1 - P_k)\, TP_j \\ &= \sum_k A_k \otimes (1 - P_k)\, TP_k\,,\end{aligned}$$

using the fact that $A_i \perp A_j$ if $i \neq j$. Thus, $(1 - E)(1 \otimes T)E$ is a direct sum of the operators $A_0 \otimes (1 - P_0)TP_0, \ldots, A_n \otimes (1 - P_n)TP_n$, and we conclude that

$$\|(1 - E)(1 \otimes T)E\| = \sup_k \| A_k \otimes (1 - P_k)\, TP_k \| \leqslant \sup_k \|(1 - P_k)\, TP_k\|,$$

as required. ∎

Remarks. It is instructive to examine the content of this theorem in the finite-dimensional case. Let $e_1, \ldots, e_n$ be an orthonormal base for a Hilbert space $\mathscr{H}$, let P_k be the projection on $[e_1, \ldots, e_k]$, $1 \leqslant k \leqslant n$, and let $\mathfrak{A} = \operatorname{alg}\{P_1, \ldots, P_n\}$. Relative to the basis (e_k), $\mathfrak{A}$ becomes the algebra of all upper triangular $n \times n$ matrices (a_{ij}), $a_{ij} = 0$ for $i > j$. If $T = (t_{ij})$ is an arbitrary $n \times n$ matrix, then $(1 - P_k)\, TP_k$ has a matrix of the form

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline & \bullet\ (k+1,\ k) \\ t_{ij} & 0 \end{array}\right),$$

and the theorem asserts that the distance from T to the upper triangular matrices is the largest of the norms of these block lower triangular submatrices. One might expect that a more likely measure for $d(T, \mathfrak{A})$ would be $\| T_- \|$, where $T_- = (s_{ij})$ is the lower triangular part of T, defined by $s_{ij} = t_{ij}$ if $i > j$ and $s_{ij} = 0$ if $i \leqslant j$. However, this conjecture fails in an extreme way; it is not very hard to show that if π_n is the linear mapping of $n \times n$ matrices given by $\pi_n : T \mapsto T_-$, then $\| \pi_n \|$ tends to $+\infty$ as $n \to \infty$.

2. Applications to Quasitriangular Operators and Algebras

Let $\mathscr{H}$ be a separable Hilbert space. An operator $A \in \mathscr{L}(\mathscr{H})$ is called *quasitriangular* [8] (resp. *quasidiagonal*) if there is an increasing sequence P_n of finite dimensional projections, such that $P_n \uparrow 1$, and

$\|(1 - P_n) AP_n\| \to 0$ (resp. $\| P_nA - AP_n \| \to 0$). In this section we want to consider the algebra of all operators which are quasitriangular or quasidiagonal relative to a *fixed* sequence P_n. More specifically, we are interested in the four sets of operators

$$\begin{aligned} \mathscr{T} &= \text{alg}\{P_n\}, \\ \mathscr{D} &= \{P_n\}', \\ \mathscr{QT} &= \{A \in \mathscr{L}(\mathscr{H}): \|(1 - P_n) AP_n\| \to 0\}, \\ \mathscr{QD} &= \{A \in \mathscr{L}(\mathscr{H}): \| P_nA - AP_n \| \to 0\}, \end{aligned}$$

where, throughout this section, it will be understood that the sequence $\{P_n\}$ is fixed.

Obviously $\mathscr{T}$ is a weakly closed algebra containing 1, and $\mathscr{D} = \mathscr{T} \cap \mathscr{T}^*$ is its diagonal. Similarly, it is easy to see that $\mathscr{QT}$ is a *norm* closed algebra containing 1, and $\mathscr{QD} = (\mathscr{QT}) \cap (\mathscr{QT})^*$ is a C^*-algebra (the proof that $\mathscr{QT}$ is a Banach algebra is contained in the proof of the main result of [12]). Moreover, since $\lim_n \|(1 - P_n) KP_n\| = 0$ for every compact operator K, we see that both $\mathscr{QT}$ and $\mathscr{QD}$ contain the C^*-algebra $\mathscr{C}(\mathscr{H})$ of all compact operators on $\mathscr{H}$.

It follows from these remarks that the algebra $\mathscr{T} + \mathscr{C}(\mathscr{H})$ of all compact perturbations of operators in $\mathscr{T}$ is contained in $\mathscr{QT}$; and the first nontrivial fact that we shall require is that this perturbed algebra is closed in the norm topology.

PROPOSITION 2.1. *$\mathscr{T} + \mathscr{C}(\mathscr{H})$ is norm-closed, and moreover the natural isomorphism of $\mathscr{T}/\mathscr{T} \cap \mathscr{C}(\mathscr{H})$ onto $\mathscr{T} + \mathscr{C}(\mathscr{H})/\mathscr{C}(\mathscr{H})$ is isometric.*

Proof. Let q be the natural projection of $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{C}(\mathscr{H})$. Since $\mathscr{T} + \mathscr{C}(\mathscr{H})$ is a subalgebra of $\mathscr{L}(\mathscr{H})$, we may regard $q(\mathscr{T} + \mathscr{C}(\mathscr{H})) = \mathscr{T} + \mathscr{C}(\mathscr{H})/\mathscr{C}(\mathscr{H})$ as a subalgebra of the Calkin algebra.

There is a natural homomorphism α of $\mathscr{T}/\mathscr{T} \cap \mathscr{C}(\mathscr{H})$ onto $\mathscr{T} + \mathscr{C}(\mathscr{H})/\mathscr{C}(\mathscr{H})$, defined on cosets as $\alpha: A + \mathscr{T} \cap \mathscr{C}(\mathscr{H}) \to A + \mathscr{C}(\mathscr{H})$, $A \in \mathscr{T}$. Clearly the map is norm-decreasing, and we claim now that it is isometric. Indeed, if A is any operator on $\mathscr{H}$, then the norm of $q(A)$ is given by $\lim_n \|(1 - P_n) A(1 - P_n)\|$ (see [3, Lemma 1, p. 292]). So that if $A \in \mathscr{T}$ then each operator $(1 - P_n) A(1 - P_n)$ can be written $A + K_n$, where each $K_n = -P_nA - AP_n + P_nAP_n$ is a finite rank operator in $\mathscr{T}$. This implies that $\| q(A)\| \geqslant \inf\{\| A + K \|: K \in \mathscr{T} \cap \mathscr{C}(\mathscr{H})\}$, and the claim follows.

Now since $\mathscr{T}/\mathscr{T} \cap \mathscr{C}(\mathscr{H})$ is a quotient of a Banach algebra by a closed ideal, it is complete as a Banach space. Since α is isometric, $\mathscr{T} + \mathscr{C}(\mathscr{H})/\mathscr{C}(\mathscr{H})$ is also complete, and therefore closed in the Calkin algebra. It follows that $\mathscr{T} + \mathscr{C}(\mathscr{H}) = q^{-1}(\mathscr{T} + \mathscr{C}(\mathscr{H})/\mathscr{C}(\mathscr{H}))$ is closed. ∎

We remark that the corresponding properties for $\mathscr{D} + \mathscr{C}(\mathscr{H})$ are also valid, and are well known in the theory of C^*-algebras (see [1, 1.8.4]).

It is known that if A is an operator which is quasitriangular relative to the sequence P_n, then there is an infinite subsequence $P_{n_1}, P_{n_2}, \ldots$ which is left *invariant* under some compact perturbation of A (see [9]; there, the subsequence is chosen to be sparse enough that the corresponding series of norms $\sum_k \|(1 - P_{n_k}) AP_{n_k}\|$ is convergent). We will prove here that it is always possible to find a compact perturbation of A which leaves the *entire* sequence $P_1, P_2, \ldots$ invariant. This is a consequence of the following variation on the distance formula of the preceding section.

THEOREM 2.2. *Let $B \in \mathscr{L}(\mathscr{H})$. Then*

$$d(B, \mathscr{T} + \mathscr{C}(\mathscr{H})) = \limsup_n \|(1 - P_n) BP_n\|.$$

Proof. For the inequality $\geqslant$, choose $A \in \mathscr{T}$ and $K \in \mathscr{C}(\mathscr{H})$. Since $(1 - P_n) AP_n = 0$ for every n and $\|(1 - P_n) KP_n\| \to 0$, we have

$$\begin{aligned}\limsup_n \|(1 - P_n) BP_n\| &= \limsup_n \|(1 - P_n)(B + A + K) P_n\| \\ &\leqslant \|B + A + K\|.\end{aligned}$$

The inequality follows by taking the inf over A and K.

For the opposite inequality, let l denote $\limsup_n \|(1 - P_n) BP_n\|$, and choose $\epsilon > 0$. Now find n_0 such that $\|(1 - P_n) BP_n\| \leqslant l + \epsilon$ for every $n \geqslant n_0$. The distance formula implies that there is an operator A in $\operatorname{alg}\{P_{n_0}, P_{n_0+1}, P_{n_0+2}, \ldots\}$ such that $\|B - A\| \leqslant l + 2\epsilon$. Since one can write A in the form $A_0 + F$, where $A_0 \in \mathscr{T}$ and F is finite rank (for example, one may take $A_0 = A(1 - P_{n_0})$), we conclude that $d(B, \mathscr{T} + \mathscr{C}(\mathscr{H})) \leqslant l + 2\epsilon$. The theorem follows because ϵ was arbitrary. ∎

COROLLARY. $\mathscr{Q}\mathscr{T} = \mathscr{T} + \mathscr{C}(\mathscr{H})$.

Proof. If $A \in \mathscr{Q}\mathscr{T}$, then

$$d(A, \mathscr{T} + \mathscr{C}(\mathscr{H})) = \limsup_n \|(1 - P_n) AP_n\| = 0,$$

and the required conclusion $A \in \mathscr{T} + \mathscr{C}(\mathscr{H})$ follows because $\mathscr{T} + \mathscr{C}(\mathscr{H})$ is norm-closed. ▮

Remarks. As we have already pointed out, this theorem can be regarded as a strengthening of the result which asserts that every quasitriangular operator is a compact perturbation of a triangular operator. Since there is an analogous decomposition for quasidiagonal operators, one might expect the corresponding result to be true for $\mathscr{Q}\mathscr{D}$, namely, $\mathscr{Q}\mathscr{D} = \mathscr{D} + \mathscr{C}(\mathscr{H})$. It is interesting that this fails: Joan Plastiras has shown that $\mathscr{Q}\mathscr{D}$ is always larger than $\mathscr{D} + \mathscr{C}(\mathscr{H})$. Thus, while every operator $A \in \mathscr{Q}\mathscr{D}$ can be written simultaneously in the form $A = B + K = C^* + L$, where $B, C \in \mathscr{T}$ and K, L are compact, it may not be possible to choose $B \in \mathscr{T} \cap \mathscr{T}^*$. At this point, the structure of the class of C^*-algebras $\mathscr{Q}\mathscr{D}$ seems quite mysterious.

Finally, we remark that the lim sup appearing in the preceding theorem cannot be replaced with lim; the reader can easily find unilateral weighted shifts B for which $\lim_n \|(1 - P_n) BP_n\|$ fails to exist.

3. Factorization Theorems

It is known that every nonzero bounded analytic function in the open unit disc has a factorization $f = uF$, where u and F are, respectively, inner and outer functions. A closely related theorem asserts that every real-valued bounded measurable function on the unit circle is the boundary value (almost everywhere) of a function of the form $\log |f|$, where f and its reciprocal are bounded analytic functions in the open unit disc. We shall require analogous factorization theorems relative to nest algebras, and the present section is devoted to this discussion.

We consider only nest algebras of the simplest type, namely, algebras of the form $\mathfrak{A} = \mathrm{alg}\{\mathscr{M}_n\}$, where $\mathscr{M}_n$, $n \in \mathbf{Z}$, is a doubly infinite sequence of closed subspaces of a fixed Hilbert space $\mathscr{H}$, satisfying $\mathscr{M}_n \subseteq \mathscr{M}_{n+1}$, $\bigcap_n \mathscr{M}_n = 0$, and $[\bigcup_n \mathscr{M}_n] = \mathscr{H}$ (it will be convenient in this section to deal with subspaces rather than projections). Thus we allow the possibility that $\mathscr{M}_n = 0$ for $n < 0$, or $\mathscr{M}_n = \mathscr{H}$ for $n > 0$, and $\mathscr{M}_n \ominus \mathscr{M}_{n-1}$ may be 0-, finite-, or infinite-dimensional. All of this structure is fixed throughout the section. P_n will denote the projection on $\mathscr{M}_n$, and for an arbitrary operator $A \in \mathscr{L}(\mathscr{H})$, R_A will denote the projection on the closed subspace $[A\mathscr{H}]$.

DEFINITION 3.1. (i) An operator $A \in \mathcal{A}$ is called outer if R_A commutes with every P_n, and $A\mathcal{M}_n$ is dense in $[A\mathcal{H}] \cap \mathcal{M}_n$ for every $n \in \mathbf{Z}$.

(ii) An operator $U \in \mathcal{A}$ is called inner if U is a partial isometry whose initial projection U^*U commutes with every P_n, $n \in \mathbf{Z}$.

We will be mainly concerned with outer operators, and it seems appropriate to illustrate the scope of the definition with a few examples. Note first that if A is an operator in $\mathcal{A}$ which is invertible in $\mathcal{L}(\mathcal{H})$, then A is outer if, and only if, $A\mathcal{M}_n = \mathcal{M}_n$ for every $n \in \mathbf{Z}$, which asserts simply that A^{-1} belongs to $\mathcal{A}$. In particular, every operator in $\mathcal{A} \cap \mathcal{A}^{-1}$ is outer. Similarly, it is easy to see that every operator in the diagonal $\mathcal{A} \cap \mathcal{A}^*$ of $\mathcal{A}$ is outer, regardless of its invertibility properties.

Consider now the Hardy space H^2, let $e_0, e_1, \ldots$ be the usual orthonormal base $e_n = z^n$, $n \geqslant 0$, and let $\mathcal{A}$ be the nest algebra $\text{alg}\{[e_n, e_{n+1}, \ldots]: n \geqslant 0\}$. Thus, $\mathcal{A}$ consists of all operators on $\mathcal{H}$ whose matrix relative to $\{e_n\}$ is lower triangular (to fit $\mathcal{A}$ into the format of the preceding discussion, just notice that $\mathcal{A} = \text{alg}\{\mathcal{M}_n: n \in \mathbf{Z}\}$, where $\mathcal{M}_n = H^2$ for $n \geqslant 0$ and $\mathcal{M}_n = [e_{-n}, e_{-n+1}, \ldots]$ for $n < 0$). Now each function $f \in H^\infty$ gives rise to an analytic Toeplitz operator T_f, and clearly T_f belongs to $\mathcal{A}$. We claim: T_f is an outer operator if and only if f is an outer function in the traditional sense of the word [10]. Indeed, if f is an outer function, then $f \cdot H^2$ is dense in H^2, hence $f \cdot z^n \cdot H^2$ is dense in $z^n \cdot H^2$ for every $n \geqslant 0$, and this implies the conditions 3.1(i) and (ii). Conversely, suppose T_f is an outer operator and let $f = u \cdot F$ be the factorization of f into its inner part u and outer part F [10]. Then the closed range of T_f is simply $u \cdot H^2$, and the only way the projections on $u \cdot H^2$ and $z^n \cdot H^2$ ($n \geqslant 0$) can all commute is for u to be a monomial in z: $u = \lambda z^r$, $r = 0, 1, \ldots$, $\lambda \in \mathbf{C}$, $|\lambda| = 1$. This implies that T_f maps $[e_n, e_{n+1}, \ldots]$ into $[e_{n+r}, e_{n+r+1}, \ldots]$, which cannot be dense in $[e_n, e_{n+1}, \ldots]$ unless $r = 0$. We conclude that u is a constant, and hence f is an outer function.

As a final example, consider the algebra $\mathcal{A} = \text{alg}\{[e_0, e_1, \ldots, e_n]: n \geqslant 0\}$, where $e_0, e_1, \ldots$ is the orthonormal base for H^2 of the preceding paragraph. While this algebra is simply the adjoint of the preceding example, its outer operators behave rather differently. While a complete discussion of the situation would take us too far afield, we feel it is worthwhile to state at least some of the facts without proof, for the interested reader (we remark that none of this makes essential contact with the rest of the paper). Let A be an operator in $\mathcal{A}$ and let (a_{ij}) be the matrix of A relative to the base $\{e_n\}$. Then of course (a_{ij}) is upper triangular, and it can be shown that A is outer if and only if this matrix has the following property: whenever a diagonal term of (a_{ij}) is zero (say $a_{kk} = 0$), then the entire row through that entry is zero (i.e., $a_{kj} = 0$ for $j = k + 1, k + 2, \ldots$). In particular, every upper triangular $A \sim (a_{ij})$ whose diagonal terms a_{kk} are all nonzero is necessarily an outer operator. $\mathfrak{A}$ also contains the coanalytic Toeplitz operators $T_f^* = T_{\bar{f}}$, $f \in H^\infty$; and in contrast with the preceding example, we see that, for a nonzero function $f \in H^\infty$, T_f^* is an outer operator if and only if $f(0) \neq 0$.

Returning to the main discussion, we begin with a lemma which exhibits the most useful technical feature of outer operators.

LEMMA. *Let $A \in \mathfrak{A}$ be outer and let V be an arbitrary operator in $\mathscr{L}(\mathscr{H})$ such that $VA \in \mathfrak{A}$ and $V = 0$ on $[A\mathscr{H}]^\perp$. Then V belongs to $\mathfrak{A}$.*

Proof. We have to show that $V\mathscr{M}_n \subseteq \mathscr{M}_n$ for every $n \in \mathbf{Z}$. Fix n, and decompose $\mathscr{M}_n$ as an orthogonal sum $\mathscr{M}_n = [A\mathscr{M}_n] \oplus (\mathscr{M}_n \ominus [A\mathscr{M}_n])$. Now since the projections on the two spaces $\mathscr{M}_n$ and $[A\mathscr{H}]$ commute and since $[A\mathscr{H}] \cap \mathscr{M}_n = [A\mathscr{M}_n]$, it follows that $\mathscr{M}_n \ominus [A\mathscr{M}_n]$ is orthogonal to $[A\mathscr{H}]$ as well as $[A\mathscr{M}_n]$. This implies that $V = 0$ on $\mathscr{M}_n \ominus [A\mathscr{M}_n]$, so that $V\mathscr{M}_n$ is contained in $V[A\mathscr{M}_n] = [VA\mathscr{M}_n] \subseteq \mathscr{M}_n$, as asserted. ∎

The following result asserts that outer operators, like outer functions, are essentially uniquely determined by their "modulus."

THEOREM 3.2. *Let A, B be outer operators in $\mathfrak{A}$ such that $A^*A = B^*B$. Then there is a partial isometry V in $\mathfrak{A} \cap \mathfrak{A}^*$ such that $V^*V = R_A$, $VV^* = R_B$, $VA = B$.*

Proof. The hypothesis on A and B implies that $\| Ax \| = \| Bx \|$ for every $x \in \mathscr{H}$. Thus we may define a partial isometry V as the closure of the operator V_0: $Ax \mapsto Bx$, $x \in \mathscr{H}$, where of course V_0 is defined as 0 on $[A\mathscr{H}]^\perp$. Clearly V satisfies $VA = B$, and has the right initial and final spaces.

The equation $VA = B$, together with the preceding lemma, implies that $V \in \mathfrak{A}$; and since $V^*B = V^*VA = A$, the same reasoning shows that $V^* \in \mathfrak{A}$. Hence, V belongs to the diagonal $\mathfrak{A} \cap \mathfrak{A}^*$. ∎

Let T be an arbitrary operator on $\mathscr{H}$. We shall be concerned with the possibility of expressing T in the form $T = UA$, where A is an outer operator in $\mathfrak{A}$ and U is a partial isometry satisfying $U^*U = R_A$. Note that a necessary condition for such a factorization is $\bigcap_n [T\mathscr{M}_n] = 0$. To see this, note that $T\mathscr{M}_n = UA\mathscr{M}_n \subseteq U\mathscr{M}_n$, $n \in \mathbf{Z}$, so that $\bigcap_n [T\mathscr{M}_n] \subseteq \bigcap_n [U\mathscr{M}_n]$. But if x belongs to $\bigcap_n [U\mathscr{M}_n]$, then for every n we have $U^*x \in U^*U\mathscr{M}_n \subseteq \mathscr{M}_n$ because $U^*U = R_A$ commutes with the projections $\{P_n\}$, and hence $U^*x \in \bigcap_n \mathscr{M}_n = 0$. It follows that $x = UU^*x = 0$, as asserted.

The main factorization result of this section asserts that this necessary condition is also sufficient. Before stating this formally, we collect an elementary fact, for which we have been unable to find a convenient reference.

LEMMA. *Let M be a closed subspace of a Banach space E, and let T be a bounded operator from E into a Banach space F such that TE is dense in F. Then*

$$\operatorname{codim}[TM] \leqslant \operatorname{codim} M.$$

Proof. To prove that the dimension of the Banach space $F/[TM]$ does not exceed that of E/M, it suffices to exhibit a bounded operator τ from E/M into $F/[TM]$, which has dense range. For that, define

$$\tau(x + M) = Tx + [TM], \qquad x \in E.$$

It is clear that τ is well defined and linear, and a simple estimate (which we leave for the reader) shows that $\| \tau \| \leqslant \| T \|$. That the range of τ is dense follows from the corresponding property of T. ∎

THEOREM 3.3. *Let T be an operator on $\mathscr{H}$ such that $\bigcap_n [T\mathscr{M}_n] = 0$. Then T admits a factorization $T = UA$, where $A \in \mathfrak{A}$ is outer and U is a partial isometry such that $U^*U = R_A$.*

If $UA = VB$ are two such factorizations of T, then there is a partial isometry W in $\mathfrak{A} \cap \mathfrak{A}^*$ such that $W^*W = R_A$, $WW^* = R_B$, $B = WA$, and $V = UW^*$.

Proof. We first dispose of the uniqueness assertion. Suppose that $UA = VB$ satisfy the above conditions. Then $A^*A = B^*B$, so by the preceding theorem there is a partial isometry W in $\mathfrak{A} \cap \mathfrak{A}^*$ such that $W^*W = R_A$, $WW^* = R_B$, and $WA = B$. We have $VB = VWA = UA$, so that $VW = U$ on $[A\mathscr{H}]$; and since both W and U vanish on $[A\mathscr{H}]^\perp$, this implies $VW = U$. Thus, $V = VWW^* = UW^*$.

Turning now to existence of the factorization, let $\mathscr{R}_n$ denote the subspace $[T\mathscr{M}_n]$, $n \in \mathbf{Z}$. Clearly, $\mathscr{R}_n \subseteq \mathscr{R}_{n+1}$, $\bigcup_n \mathscr{R}_n$ is dense in $[T\mathscr{H}]$, and by the hypothesis on T we have $\bigcap_n \mathscr{R}_n = 0$.

Note first that $\dim(\mathscr{R}_n \ominus \mathscr{R}_{n-1}) \leqslant \dim(\mathscr{M}_n \ominus \mathscr{M}_{n-1})$ for every n; this is immediate from the lemma by considering $T \mid \mathscr{M}_n$ as an operator from $\mathscr{M}_n$ into $[T\mathscr{M}_n]$. Thus there exists a partial isometry V

on $\mathscr{H}$ which maps $\mathscr{R}_n \ominus \mathscr{R}_{n-1}$ isometrically into $\mathscr{M}_n \ominus \mathscr{M}_{n-1}$ for every $n \in \mathbf{Z}$, and which vanishes on the orthocomplement of $\sum_n (\mathscr{R}_n \ominus \mathscr{R}_{n-1})$.

Note that the range projection VV^* of V must commute with each P_n, and moreover we claim that $V\mathscr{R}_n = V\mathscr{H} \cap \mathscr{M}_n$, $n \in \mathbf{Z}$. After the preceding observation, this follows immediately from the obvious formula $V(\mathscr{R}_k \ominus \mathscr{R}_{k-1}) = V\mathscr{H} \cap (\mathscr{M}_k \ominus \mathscr{M}_{k-1})$ by summing over all $k \leqslant n$, noting that $\sum_{k \leqslant n} (\mathscr{R}_k \ominus \mathscr{R}_{k-1}) = \mathscr{R}_n$.

Now define the operator $A = VT$. Since the initial space of V is $\sum_n (\mathscr{R}_n \ominus \mathscr{R}_{n-1}) = [T\mathscr{H}]$, it follows that $[A\mathscr{H}] = V\mathscr{H}$, and in particular $R_A = VV^*$ commutes with $\{P_n : n \in \mathbf{Z}\}$. By the preceding paragraph we see that $[A\mathscr{M}_n] = V\mathscr{R}_n = V\mathscr{H} \cap \mathscr{M}_n = [A\mathscr{H}] \cap \mathscr{M}_n$. Hence, A is an outer operator in $\mathfrak{A}$.

Finally, the required formula $T = UA$ follows by taking $U = V^*$ and noting that $T = V^*VT = V^*A$. ∎

COROLLARY 1. *Every operator T in $\mathfrak{A}$ has a factorization $T = UA$, where U is inner and A is outer.*

Proof. Note first that every operator $T \in \mathfrak{A}$ satisfies $\bigcap_n [T\mathscr{M}_n] = 0$ (an immediate consequence of the fact that $T\mathscr{M}_n \subseteq \mathscr{M}_n$ and $\bigcap_n \mathscr{M}_n = 0$). So the theorem implies that T has a factorization $T = UA$ where $A \in \mathfrak{A}$ is outer and U is a partial isometry with $U^*U = R_A$. R_A commutes with each P_n, by definition of outer operators, and the lemma preceding Theorem 3.2 implies that $U \in \mathfrak{A}$. Hence, U is inner. ∎

Remarks. It is well known [10] that H^∞ is a logmodular algebra of functions on the unit circle. Equivalently, every positive measurable function on the unit circle which is bounded above and away from 0 is the boundary value of a function of the form $|f|^2$, where f and $1/f$ are analytic and bounded in the open unit disc. The following corollary is an exact analog of this theorem for certain nest algebras. First, we need a lemma which asserts that every outer operator in $\mathfrak{A}$ which is left-invertible is related in a very simple way to an operator which is *invertible* in $\mathfrak{A}$.

LEMMA. *Let $A \in \mathfrak{A}$ be an outer operator which is bounded below. Then there exists an isometry U in $\mathfrak{A} \cap \mathfrak{A}^*$ and an operator $B \in \mathfrak{A} \cap \mathfrak{A}^{-1}$ such that $A = UB$.*

Proof. Because A is bounded below, it induces in the obvious way an invertible operator from $\mathscr{M}_n \ominus \mathscr{M}_{n-1}$ onto $[A\mathscr{M}_n] \ominus [A\mathscr{M}_{n-1}]$, and in particular the subspaces $\mathscr{M}_n \ominus \mathscr{M}_{n-1}$ and $[A\mathscr{M}_n] \ominus [A\mathscr{M}_{n-1}]$

have the same dimension, $n \in \mathbf{Z}$. Thus we can find an isometry U on $\mathscr{H}$ which takes each space $\mathscr{M}_n \ominus \mathscr{M}_{n-1}$ onto $[A\mathscr{M}_n] \ominus [A\mathscr{M}_{n-1}]$. Because $\bigcap_n [A\mathscr{M}_n] \subseteq \bigcap_n \mathscr{M}_n = 0$, it follows that U maps $\mathscr{M}_n = \sum_{k\leqslant n} (\mathscr{M}_k \ominus \mathscr{M}_{k-1})$ onto $\sum_{k\leqslant n} ([A\mathscr{M}_k] \ominus [A\mathscr{M}_{k-1}]) = [A\mathscr{M}_k]$, and in particular $UU^* = R_A$. Define $B = U^*A$. The above implies that $[B\mathscr{M}_n] = [\mathscr{M}_n]$, so that $B \in \mathcal{O}\!\!l$, has dense range, and of course B is bounded below. Hence B is invertible. The formula $[B\mathscr{M}_n] = \mathscr{M}_n$ now implies $B^{-1}\mathscr{M}_n = \mathscr{M}_n$, so that $B^{-1} \in \mathcal{O}\!\!l$.

Multiplication of $B = U^*A$ on the left by U gives $A = UB$, and now these two formulas, together with the lemma preceding Theorem 3.2, imply that $U \in \mathcal{O}\!\!l \cap \mathcal{O}\!\!l^*$. ∎

COROLLARY 2. *Every invertible positive operator on $\mathscr{H}$ can be factored in the form A^*A, where A belongs to $\mathcal{O}\!\!l \cap \mathcal{O}\!\!l^{-1}$.*

Proof. Let H be an invertible positive operator on $\mathscr{H}$. Then the same is true of its positive square root $H^{1/2}$, so that $\bigcap_n [H^{1/2}\mathscr{M}_n] = 0$. Theorem 3.3 implies that there is an outer operator $A \in \mathcal{O}\!\!l$ and a partial isometry U with $U^*U = R_A$ and $H^{1/2} = UA$. Hence, $H = (H^{1/2})^2 = A^*A$. Now A is bounded below because H is invertible, so the lemma implies that there is a B in $\mathcal{O}\!\!l \cap \mathcal{O}\!\!l^{-1}$ and an isometry V such that $A = VB$. The required factorization $H = B^*B$ follows. ∎

Remarks. The reader may have noticed that the proofs of Theorem 3.3 and its corollaries used the discrete nature of the chain $\{\mathscr{M}_n\}$ in an essential way, and a natural question here is whether factorization results like these are valid for more general next algebras. In particular, *is Corollary 2 valid for the nest algebra $\mathcal{O}\!\!l = \mathrm{alg}\{\mathscr{M}_t: 0 \leqslant t \leqslant 1\}$ acting on $L^2[0, 1]$, where $\mathscr{M}_t$ stands for the closed subspace $L^2[0, t]$ of $L^2[0, 1]$?* This question is closely connected with the known problem which asks if the multiplicity of a nest must be preserved under similarity. Rather than enter a lengthy discussion of this relationship, we will merely indicate the connection with a very specific question. Let $\mathcal{O}\!\!l = \mathrm{alg}\{\mathscr{M}_t\}$ be the nest algebra on $L^2[0, 1]$ described immediately above, and define a nest algebra $\mathscr{B}$ on the direct sum $L^2[0, 1] \oplus L^2[0, 1]$ as follows: $\mathscr{B} = \mathrm{alg}\{\mathscr{M}_t \oplus \mathscr{M}_t: 0 \leqslant t \leqslant 1\}$. It is easy to see that $\mathcal{O}\!\!l \cap \mathcal{O}\!\!l^*$ is a maximal abelian von Neumann algebra, while $\mathscr{B} \cap \mathscr{B}^*$ is a (noncommutative) von Neumann algebra of type I_2. Thus $\mathcal{O}\!\!l$ and $\mathscr{B}$ cannot be unitarily equivalent. It is not known, however, if $\mathcal{O}\!\!l$ and $\mathscr{B}$ can be *similar*. *Equivalently, is there an invertible operator X from $L^2[0, 1]$ to $L^2[0, 1] \oplus L^2[0, 1]$ which maps the family $\{\mathscr{M}_t\}$ of subspaces onto the family $\{\mathscr{M}_t \oplus \mathscr{M}_t\}$?*

We want to point out that if the answer to the latter question is yes, then Corollary 2 must fail for the nest algebra $\mathcal{A}$. To see this, let X be as above, with $X\mathcal{A}X^{-1} = \mathcal{B}$, and suppose to the contrary that Corollary 2 is valid for the positive invertible operator X^*X on $L^2[0, 1]$. Thus we may write $X^*X = A^*A$, where $A \in \mathcal{A} \cap \mathcal{A}^{-1}$. It follows that $U = XA^{-1}$ is a unitary operator from $L^2[0, 1]$ to $L^2[0, 1] \oplus L^2[0, 1]$, and satisfies $U\mathcal{A}U^{-1} = \mathcal{B}$, a state of affairs which was shown to be impossible in the preceding paragraph.

It seems very likely, of course, that the algebras $\mathcal{A}$ and $\mathcal{B}$ are *not* similar; but while this latter problem has been circulating for several years and makes important contact with other parts of operator theory (e.g., the problem of whether every compact operator is hyper-intransitive), no one has made significant progress on it.

4. The Interpolation Theorem

Let $A_1, \ldots, A_N$ be operators on a Hilbert space $\mathcal{H}$. It is an elementary exercise to prove that there exist operators $B_1, \ldots, B_N$ on $\mathcal{H}$ satisfying $B_1A_1 + \cdots + B_NA_N = 1$ if, and only if, the set $\{A_1, \ldots, A_N\}$ is bounded below in the sense that there exists $\epsilon > 0$ such that $\| A_1x \| + \cdots + \| A_Nx \| \geqslant \epsilon \| x \|$, for every $x \in \mathcal{H}$. For instance, if the latter condition is satisfied, then one may choose $B_k = H^{-1}A_k^*$, where H is the positive invertible operator $A_1^*A_1 + \cdots + A_N^*A_N$.

Now if the given operators $A_1, \ldots, A_N$ belong to a given Banach subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$, then one often wants to know if it is possible to solve the equation $B_1A_1 + \cdots + B_NA_N = 1$ with operators B_k in $\mathcal{A}$. If $\mathcal{A}$ is a C^*-algebra the problem becomes trivial: a few moments' thought shows that the solution is completely described by the criterion of the preceding paragraph. Certain commutative algebras of normal and essentially normal operators were considered in [6], where it was shown that the same criterion (together with a similar condition on the adjoints of the A_k's) is again sufficient.

In this paper, however, we are concerned with nest algebras, which are neither self-adjoint nor commutative. In this case the hypothesis $\| A_1x \| + \cdots + \| A_Nx \| \geqslant \epsilon \| x \|$ is not enough, a phenomenon best illustrated by the following simple example. Let $e_1, e_2, \ldots$ be an orthonormal base for $\mathcal{H}$, and let $\mathcal{A}$ be the nest algebra $\operatorname{alg}\{[e_1, \ldots, e_n]: n = 1, 2, \ldots\}$. We consider a pair of operators $\{A_1, A_2\}$ in $\mathcal{A}$, where A_1 is the "backward shift" (defined on $\{e_n\}$ by $A_1e_1 = 0$ and $A_1e_n = e_{n-1}$ for $n \geqslant 1$), and A_2 is the projection onto the one-dimensional space $[e_1]$. A simple computation shows that $\| A_1x \|^2 + \| A_2x \|^2 = \| x \|^2$

for every $x \in \mathscr{H}$, so that $\| A_1 x \| + \| A_2 x \| \geqslant \epsilon \| x \|$ with $\epsilon = 2^{-1/2}$. On the other hand, we claim that there do not exist $B_1, B_2 \in \mathscr{A}$ such that $B_1 A_1 + B_2 A_2 = 1$. For if such B_i did exist then we could apply this identity to the basis vector e_2 (noting that $A_1 e_2 = e_1$ and $A_2 e_2 = 0$) to obtain $B_1 e_1 = e_2$, contradicting the fact that B_1 leaves $[e_1]$ invariant.

A more stringent necessary condition can be expressed in terms of the invariant projections of the algebra $\mathscr{A}$. Indeed if $A_1, \ldots, A_N \in \mathscr{A}$ are operators such that the equation $B_1 A_1 + \cdots + B_N A_N = 1$ is solvable with B_k in $\mathscr{A}$, then since $(1 - P) B_k = (1 - P) B_k (1 - P)$ holds for every projection $P \in \operatorname{lat} \mathscr{A}$, one may multiply the preceding equation on the left by $1 - P$ to obtain $\sum_k (1 - P) B_k (1 - P) A_k = 1 - P$, and hence

$$\sum_k \|(1 - P) A_k x \| \geqslant \epsilon \|(1 - P) x \| \tag{4.1}$$

for every $x \in \mathscr{H}$ and every $P \in \operatorname{lat} \mathscr{A}$, where ϵ may be taken as $\min(\| B_1 \|^{-1}, \ldots, \| B_N \|^{-1})$. It will turn out to be more convenient to deal not with this inequality but with the equivalent one,

$$\sum_k \|(1 - P) A_k x \|^2 \geqslant \epsilon^2 \|(1 - P) x \|^2, \tag{4.2}$$

assumed to hold for every $x \in \mathscr{H}$, $P \in \operatorname{lat} \mathscr{A}$. The equivalence of (4.1) and (4.2) simply reflects the equivalence of the l^1 and l^2 norms on $\mathbf{C}^N$; and of course one needs to adjust ϵ in passing from one to the other. The main result of this section asserts that if $\mathscr{A}$ is a nest algebra of a certain type and $A_1, \ldots, A_N$ belong to $\mathscr{A}$ and satisfy (4.2), then conversely there exist $B_1, \ldots, B_N \in \mathscr{A}$ with $\sum B_k A_k = 1$. As in [6], this type of result will be called an interpolation theorem, by virtue of the analogy with certain interpolation problems in algebras of bounded analytic functions. Indeed, in Section 6 below we will use this theorem to deduce an operator-theoretic variant of the corona theorem.

For an arbitrary operator $A \in \mathscr{L}(\mathscr{H})$, the projections onto $(\ker A)^\perp$ and $[A\mathscr{H}]$ will be denoted, respectively, by D_A and R_A. Suppose now that $\mathscr{A}$ is a reflexive algebra and A is a *partial isometry* in $\mathscr{A}$ with nontrivial kernel. Then A cannot be left-invertible, and the best one can hope for is an operator $B \in \mathscr{A}$ satisfying $BA = D_A$. Of course the operator $B = A^*$ satisfies the equation, but usually it fails to belong to $\mathscr{A}$. However, if such a B does exist then the projection D_A must belong to $\mathscr{A}$ (which, since $\mathscr{A}$ is reflexive, simply means that

$D_A = D_A{}^*$ commutes with every projection in lat $\mathfrak{A}$), and A itself must satisfy an inequality analogous to (4.1):

$$\|(1 - P) Ax\| \geqslant \epsilon \|(1 - P)x\|, \qquad x \in D_A\mathscr{H},$$

for every projection P in in lat $\mathfrak{A}$. Our first result asserts that these two conditions are sufficient as well, at least in case $\mathfrak{A}$ is a nest algebra.

THEOREM 4.3. *Let A be a partial isometry in a nest algebra $\mathfrak{A}$, whose initial projection D_A commutes with* lat $\mathfrak{A}$, *and assume a positive number ϵ exists such that*

$$\|(1 - P) Ax\| \geqslant \epsilon \|(1 - P)x\|,$$

for every $x \in D_A\mathscr{H}$ and $P \in \operatorname{lat} \mathfrak{A}$. Then there is an operator $B \in \mathfrak{A}$ satisfying $BA = D_A$, and $\| B \| \leqslant 4/\epsilon^2$.

Proof. Clearly, $\epsilon \leqslant 1$ except in the trivial case where $A = 0$. We first want to approximate A^* as nearly as possible with an operator from $\mathfrak{A}$. For that, we claim $\|(1 - P) A^*P\| \leqslant (1 - \epsilon^2)^{1/2}$, for every $P \in \operatorname{lat} \mathfrak{A}$; equivalently, $\| PA(1 - P)x\|^2 \leqslant 1 - \epsilon^2$ for every $x \in \mathscr{H}$ satisfying $\| x \| \leqslant 1$. But for each x in $D_A\mathscr{H}$, we have

$$\|(1 - P) A(1 - P)x\|^2 \geqslant \epsilon^2 \|(1 - P)x\|^2$$

by hypothesis, so that

$$\begin{aligned} \| PA(1 - P)x\|^2 &= \| A(1 - P)x\|^2 - \|(1 - P) A(1 - P)x\|^2 \\ &\leqslant \| A(1 - P)x\|^2 - \epsilon^2 \|(1 - P)x\|^2 \\ &\leqslant (1 - \epsilon^2) \|(1 - P)x\|^2 \leqslant (1 - \epsilon^2) \| x\|^2. \end{aligned}$$

Since $PA(1 - P)$ vanishes on $(D_A\mathscr{H})^\perp$, the assertion follows.

From the distance theorem (1.1), we may conclude that $d(A^*, \mathfrak{A}) \leqslant (1 - \epsilon^2)^{1/2}$. This distance is actually achieved (essentially because of weak compactness of the unit ball of $\mathfrak{A}$, an elementary argument which we omit), and thus we may find an operator $C \in \mathfrak{A}$ such that $\| A^* - C \| \leqslant (1 - \epsilon^2)^{1/2}$. By multiplying on the left by D_A if necessary, we may assume $C = D_AC$, and clearly, $\| C \| \leqslant \| A^* \| + \| C - A^* \| \leqslant 1 + (1 - \epsilon^2)^{1/2}$.

Now recall that if T is an element of any Banach algebra $\mathfrak{A}$ with identity satisfying $\| 1 - T \| \leqslant r < 1$, then familiar manipulations with the Neumann series show that T is invertible in $\mathfrak{A}$ and, moreover,

$\| T^{-1} \| \leqslant (1 - r)^{-1}$. Applying this to the operator $T = 1 - D_A + CA$ in $\mathcal{A}$ and noting that

$$\| 1 - T \| = \| D_A - CA \| = \| A^*A - CA \| \\ \leqslant \| A^* - C \| \leqslant (1 - \epsilon^2)^{1/2},$$

we conclude that $T^{-1} \in \mathcal{A}$ and satisfies $\| T^{-1} \| \leqslant (1 - (1 - \epsilon^2)^{1/2})^{-1}$. Finally, put $B = T^{-1}C$. Then B belongs to $\mathcal{A}$ and has norm at most

$$\| C \| \cdot \| T^{-1} \| \leqslant (1 + (1 - \epsilon^2)^{1/2})(1 - (1 - \epsilon^2)^{1/2})^{-1} \leqslant 4/\epsilon^2.$$

Moreover, $BA = ((1 - D_A) + CA)^{-1} CA = D_A$, as required. ∎

THEOREM 4.3 (Interpolation Theorem). *Let $\{\mathcal{M}_n \colon n \in \mathbf{Z}\}$ be an increasing sequence of subspaces of a Hilbert space $\mathcal{H}$ satisfying $\bigcap_n \mathcal{M}_n = 0$ and $[\bigcup_n \mathcal{M}_n] = \mathcal{H}$, and let $\mathcal{A}$ be the nest algebra $\operatorname{alg}\{P_n \colon n \in \mathbf{Z}\}$, where P_n is the projection on $\mathcal{M}_n$. Let $A_1, \ldots, A_N \in \mathcal{A}$ satisfy*

$$\sum_k \|(1 - P_n) A_k x \|^2 \geqslant \epsilon^2 \|(1 - P_n)x \|^2$$

for every $n \in \mathbf{Z}$, $x \in \mathcal{H}$. Then there exist operators $B_1, \ldots, B_N \in \mathcal{A}$ such that $B_1A_1 + \cdots + B_NA_N = 1$. If the A_k's are normalized so that $\| A_k \| \leqslant 1$ for each k, then the B_k's may be chosen so that $\| B_k \| \leqslant 4N\epsilon^{-3}$.

Proof. By adjusting ϵ if necessary, we may assume $\| A_k \| \leqslant 1$ Consider first the positive invertible operator $\sum_k A_k^*A_k$. By Corollary 2 of Theorem 3.3, there is an operator C in $\mathcal{A} \cap \mathcal{A}^{-1}$ such that $\sum_k A_k^*A_k = C^*C$. Evidently, the operators $A_k' = A_kC^{-1}$ belong to $\mathcal{A}$ and have the property that $\sum_k A_k'^*A_k' = 1$.

We claim that the operators $\{A_k'\}$ satisfy (4.2) with constant $\epsilon N^{-1/2}$. Note first that for every $P \in \operatorname{lat} \mathcal{A}$ and $x \in \mathcal{H}$, one has $\|(1 - P) C^{-1}x \| \geqslant N^{-1/2} \|(1 - P)x \|$. Indeed, we have $1 - P = (1 - P) CC^{-1} = (1 - P) C(1 - P) C^{-1}$, and since

$$\| C \|^2 \leqslant \sum_k \| A_k \|^2 \leqslant N,$$

we conclude that

$$\|(1 - P)x \| \leqslant \|(1 - P)C \| \cdot \|(1 - P) C^{-1}x \| \leqslant N^{1/2} \|(1 - P) C^{-1}x \|,$$

as asserted. Now for each $n \in \mathbf{Z}$, we can write

$$\sum_k \|(1 - P_n) A_k'x \|^2 = \sum_k \|(1 - P_n) A_kC^{-1}x \|^2 \\ \geqslant \epsilon^2 \|(1 - P_n) C^{-1}x \|^2 \geqslant \epsilon^2N^{-1} \|(1 - P_n)x \|^2.$$

Thus, (4.2) holds for every projection in lat $\mathcal{A}$ of the form P_n, $n \in \mathbf{Z}$. Since lat $\mathcal{A} = \{P_n: n \in \mathbf{Z}\} \cup \{0, 1\}$, and since P_n tends strongly to 0 (resp. 1) as n tends to $-\infty$ (resp. $+\infty$), the preceding inequality yields (4.2) with constant $\epsilon N^{-1/2}$.

Now consider the algebra $\mathcal{B} = M_N \otimes \mathcal{A}$ of all $N \times N$ matrices over $\mathcal{A}$, regarded as an algebra of operators on the Hilbert space direct sum $\mathcal{K} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ of N copies of $\mathcal{H}$. Writing $Q_n = P_n \oplus \cdots \oplus P_n$, $n \in \mathbf{Z}$ (alternately, Q_n is the $N \times N$ matrix whose entries are P_n along the diagonal and zeros elsewhere), it is a simple computation to see that $\mathcal{B}$ is a nest algebra whose invariant projection lattice is simply $\{Q_n: n \in \mathbf{Z}\} \cup \{0, 1\}$. We will consider the operator U in $\mathcal{B}$ defined as

$$U = \begin{pmatrix} A_1' & 0 & \cdots & 0 \\ A_2' & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_N' & 0 & \cdots & 0 \end{pmatrix}.$$

From the properties of $\{A_k'\}$ it follows that U^*U is the diagonal projection

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

so that U is a partial isometry in $\mathcal{B}$ whose initial projection belongs to $\mathcal{B}$. Note next that U satisfies $\|(1 - P)Ux\| \geqslant (\epsilon N^{-1/2})\|(1 - P)x\|$ for all $x \in \mathcal{K}$ and every $P \in \operatorname{lat} \mathcal{B}$. A moment's thought and the preceding remarks show that we need only consider P's of the form $Q_n = P_n \oplus \cdots \oplus P_n$, and then the above inequality simply becomes the inequality proved in the preceding paragraph.

Thus we may conclude from Theorem 4.3 that there is an operator $V \in \mathcal{B}$ such that $VU = D_U$ and $\|V\| \leqslant 4N\epsilon^{-2}$. If we denote the first row of the $N \times N$ matrix for V as $B_1, \ldots, B_N$, then in patricular we have $B_1A_1' + \cdots + B_NA_N' = 1$ and $\|B_k\| \leqslant 4N\epsilon^{-2}$ for $k = 1, \ldots, N$. If we define $B_k' = C^{-1}B_k$, then multiplication of the preceding formula on the right by C and on the left by C^{-1} yields $B_1'A_1 + \cdots + B_N'A_N = 1$.

Clearly $\|B_k'\| \leqslant \|C^{-1}\| \cdot \|B_k\|$, so the required estimate on $\|B_k'\|$ follows from the observation that $\|C^{-1}\| \leqslant \epsilon^{-1}$, a consequence of the inequality $\|Cx\|^2 = \sum_k \|A_kx\|^2 \geqslant \epsilon^2 \|x\|^2$. ∎

Let $P_1 \leqslant P_2 \leqslant \cdots$ be an increasing sequence of finite dimensional projections in $\mathscr{L}(\mathscr{H})$ with $P_n \uparrow 1$, and let $\mathscr{Q}\mathscr{T}$ be the associated quasitriangular algebra discussed in Section 2:

$$\mathscr{Q}\mathscr{T} = \{A \in \mathscr{L}(\mathscr{H}): \lim_{n\to\infty} \|(1 - P_n) AP_n\| = 0\}.$$

Let q be the canonical projection of $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{C}(\mathscr{H})$. It follows from Proposition 2.1 that the image of the nest algebra $\mathscr{T} = \text{alg}\{P_n\}$ under q is a Banach subalgebra of $\mathscr{L}(\mathscr{H})/\mathscr{C}(\mathscr{H})$, and moreover, the corollary of Theorem 2.2 implies that $q(\mathscr{T}) = q(\mathscr{Q}\mathscr{T})$. Now let $a_1, \ldots, a_N$ belong to $q(\mathscr{T})$. The interpolation problem here asks for conditions under which there will exist $b_1, \ldots, b_N \in q(\mathscr{T})$ such that $b_1 a_1 + \cdots + b_N a_N = 1$. If operators $A_k \in \mathscr{L}(\mathscr{H})$ are chosen such that $q(A_k) = a_k$, then $A_k \in \mathscr{T} + \mathscr{C}(\mathscr{H}) = \mathscr{Q}\mathscr{T}$, and what is required is a set of operators $B_1, \ldots, B_N \in \mathscr{Q}\mathscr{T}$ such that $1 - \sum_k B_k A_k$ is compact. This is characterized in the following.

THEOREM 4.4. *Let $A_1, \ldots, A_N \in \mathscr{Q}\mathscr{T}$. In order that there exist operators $B_1, \ldots, B_N \in \mathscr{Q}\mathscr{T}$ with $1 - \sum_k B_k A_k$ compact, it is necessary and sufficient that there should exist $\epsilon > 0$ and $n_0 \geqslant 1$ such that*

$$\sum_k \|(1 - P_n) A_k x\|^2 \geqslant \epsilon^2 \|(1 - P_n)x\|^2$$

for every $x \in \mathscr{H}$ and $n \geqslant n_0$.

The proof of necessity is a simple exercise, and sufficiency follows from a routine though somewhat tedious variation on what was done in proving Theorem 4.3 (making use of the distance formula Theorem 2.2 rather than Theorem 1.1). Both are left for the reader.

5. TWO PROJECTION MAPPINGS

In this section we work in the Hilbert space $\mathscr{H} = L^2(\mathbf{T}, m)$, where $\mathbf{T}$ denotes the unit circle with normalized linear measure m. $\{e_n: n \in \mathbf{Z}\}$ will denote the usual orthonormal base for $L^2(\mathbf{T}, m)$ (viz $e_n(z) = z^n$, $z \in \mathbf{T}$, $n \in \mathbf{Z}$), and P_n will denote the projection onto the subspace $[e_n, e_{n+1}, \ldots]$, $n \in \mathbf{Z}$. Each function $\phi \in L^\infty = L^\infty(\mathbf{T}, m)$ gives rise to a multiplication operator on $\mathscr{H}$, which we denote L_ϕ. The associated Toeplitz operator T_ϕ is defined as the compression of L_ϕ to the subspace $H^2 = [e_0, e_1, \ldots]$: $T_\phi = P_0 L_\phi |_{H^2}$. It is known that the multiplication algebra $\mathscr{M} = \{L_\phi: \phi \in L^\infty\}$ is a maximal abelian von Neumann subalgebra of $\mathscr{L}(\mathscr{H})$, and that the set $\{T_\phi: \phi \in L^\infty\}$ of Toeplitz operators is a weakly closed self-adjoint linear space of operators on H^2. The purpose of this section is to construct projections of $\mathscr{L}(\mathscr{H})$ and $\mathscr{L}(H^2)$ onto the multiplication algebra and $\{T_\phi: \phi \in L^\infty\}$, respectively, which have certain properties we will make use of in Section 6. We deal first with the multiplication algebra.

Let $\mathscr{B}$ be a C^*-algebra with unit and let $\mathscr{C}$ be a C^*-subalgebra such that $1 \in \mathscr{C}$. Recall that an *expectation* of $\mathscr{B}$ on $\mathscr{C}$ is a positive linear map $\pi: \mathscr{B} \to \mathscr{C}$ satisfying $\pi(1) = 1$, and $\pi(BC) = \pi(B)C$ for every $B \in \mathscr{B}$, $C \in \mathscr{C}$. There is a known method for constructing expectations of $\mathscr{L}(\mathscr{H})$ onto maximal abelian subalgebras such as $\mathscr{M}$: one defines $\pi(X)$ as the "average" of the function $U \mapsto UXU^*$ (U ranging over the unitary group of $\mathscr{M}$) relative to a Banach mean. Here we make a similar construction, but some care must be exercised in order to bring out the desired properties. As usual, H^∞ will denote the (weak*-closed) subalgebra of L^∞ consisting of all functions $\phi \in L^\infty$ with $\int z^n \phi(z)\, dm(z) = 0$, $n = 1, 2, \ldots$.

Proposition 5.1. *There exists an expectation* $\pi: \mathscr{L}(\mathscr{H}) \to \mathscr{M}$ *having the properties*

(i) $\pi(P_n) = 1$ *for every* $n \in \mathbf{Z}$,

(ii) $\pi(\mathrm{alg}\{P_n\}) \subseteq \{L_\phi: \phi \in H^\infty\}$.

Remarks. Condition (i) asserts that in a sense the mapping π is supported at $+\infty$. Condition (ii) asserts that if A is an operator on $\mathscr{H}$ whose matrix relative to $\{e_n: n \in \mathbf{Z}\}$ is lower triangular, then $\pi(A)$ is a multiplication operator having the same property.

Proof. Let $\mathbf{N}$ be the additive semigroup of all positive integers and let Λ be a Banach limit on $\mathbf{N}$. Thus Λ is a state on the commutative C^*-algebra $l^\infty(\mathbf{N})$ (whose value at a bounded sequence $(a_n)_{n\geq 1}$ is denoted $\Lambda_n a_n$) which has the additional property $\Lambda_n a_{n+1} = \Lambda_n a_n$, $(a_n) \in l^\infty(\mathbf{N})$.

Let U denote the "bilateral shift," defined on the basis $\{e_n\}$ by $Ue_n = e_{n+1}$, $n \in \mathbf{Z}$. It is well known that U is a unitary operator which generates the multiplication algebra $\mathscr{M}$ as a von Neumann algebra. Fix $A \in \mathscr{L}(\mathscr{H})$. Then for $x, y \in \mathscr{H}$, we may define the form $[x, y] = \Lambda_n(U^{*n}AU^n x, y)$, and a straightforward application of the Schwarz lemma yields a unique operator $\pi(A) \in \mathscr{L}(\mathscr{H})$ such that $(\pi(A)x, y) = \Lambda_n(U^{*n}AU^n x, y)$.

It is routine to verify that π is a positive linear map, and moreover a standard separation theorem implies that $\pi(A)$ belongs to the weakly closed convex hull of the set $\{U^{*n}AU^n: n = 1, 2, \ldots\}$, $A \in \mathscr{L}(\mathscr{H})$. An application of translation invariance of Λ shows that

$U^*\pi(A)U = \pi(A)$, i.e., $\pi(A)$ commutes with U (and therefore with $U^* = U^{-1}$) and since $\mathcal{M}$ is the von Neumann algebra generated by U, we see that $\pi(A) \in \mathcal{M}' = \mathcal{M}$. The facts that $\pi(1) = 1$, and that $\pi(AB) = \pi(A)B$ when $B \in \mathcal{M}$, are simple consequences of the definition. Thus, π is an expectation of $\mathcal{L}(\mathcal{H})$ on $\mathcal{M}$.

To verify (i), fix $m \in \mathbf{Z}$. Then for each $n \in \mathbf{N}$ we have $U^{*n}P_mU^n = P_{m-n}$, and as $n \to +\infty$ the projections P_{m-n} increase and tend strongly to the identity. So for each $x, y \in \mathcal{H}$ we have $\lim_n (U^{*n}P_mU^nx, y) = (x, y)$. Since Banach limits must take *convergent* sequences in $l^\infty(\mathbf{N})$ to their limits, we conclude that $(\pi(P_m)x, y) = \Lambda_n(U^{*n}P_mU^nx, y) = (x, y)$, and hence $\pi(P_m) = 1$ because x and y were arbitrary.

Finally, to verify (ii), it suffices to show that π maps $\operatorname{alg}\{P_n\}$ into itself. But if $A \in \operatorname{alg}\{P_n\}$, then so does $U^{*n}AU^n$ for every $n \geq 1$. The assertion now follows from the fact that $\operatorname{alg}\{P_n\}$ is a weakly closed algebra and $\pi(A) \in \overline{\operatorname{co}}\{U^{*n}AU^n: n \in \mathbf{N}\}$. ∎

Remarks. We see in particular that there is a projection of norm 1 from the algebra $\operatorname{alg}\{P_n\}$ of all lower triangular operators onto the subalgebra $\{L_\phi: \phi \in H^\infty\}$ of all lower triangular multiplication operators, having the property $\pi(AL_\phi) = \pi(L_\phi A) = \pi(A)L_\phi$, $A \in \operatorname{alg}\{P_n\}, \phi \in H^\infty$. It is curious that the finite-dimensional analog of this conclusion fails. Indeed, if we denote by $\mathcal{A}_n$ the algebra of all lower triangular $n \times n$ matrices (endowed with the operator norm) and by $\mathcal{B}_n$ the commutative subalgebra of all matrices of the form

$$\begin{pmatrix} a_0 & & 0 \\ a_1 & a_0 & \\ \vdots & \ddots & \ddots \\ a_n & \cdots \; a_1 & a_0 \end{pmatrix},$$

$a_0, a_1, \ldots, a_n \in \mathbf{C}$, then it is not very hard to show that for any sequence of projections $\pi_n: \mathcal{A}_n \to \mathcal{B}_n$ with the property $\pi_n(AB) = \pi_n(A)B$, $A \in \mathcal{A}_n$, $B \in \mathcal{B}_n$, one necessarily has $\|\pi_n\| \to +\infty$ as n tends to ∞.

Finally, we remark that property 5.1(i) implies the stronger condition $\pi(P_nA) = \pi(AP_n) = \pi(A)$, for every $A \in \mathcal{L}(\mathcal{H})$, $n \in \mathbf{Z}$. By taking adjoints, if necessary, it suffices to show that $\pi(A(1 - P_n)) = \pi(A) - \pi(AP_n) = 0$. But by the Schwarz inequality for completely positive maps (see [2]), we have

$$\begin{aligned} \phi(A(1 - P_n))^*\phi(A(1 - P_n)) &\leq \phi((1 - P_n)A^*A(1 - P_n)) \\ &\leq \|A\|^2\phi(1 - P_n) = 0, \end{aligned}$$

as asserted.

We turn now to the Hilbert space H^2. For each $n \geqslant 0$ let P_n be the projection onto the subspace $[e_n, e_{n+1}, \ldots]$. We shall require a projection σ of norm 1 from the algebra $\operatorname{alg}\{P_n\}$ of all lower triangular operators to the algebra $\{T_\phi : \phi \in H^\infty\}$ of all analytic Toeplitz operators, which satisfies $\sigma(T_\phi A) = T_\phi \sigma(A)$, $\phi \in H^\infty$, $A \in \operatorname{alg}\{P_n\}$. This is accomplished in the following.

PROPOSITION 5.2. *There is a positive linear projection σ of $\mathscr{L}(H^2)$ onto the space $\{T_\phi : \phi \in L^\infty\}$ of all Toeplitz operators satisfying*

(i) $\sigma(1) = 1, \|\sigma\| = 1,$

(ii) $\sigma(T_\phi A) = \sigma(AT_\phi) = \sigma(A)\, T_\phi$, *for every* $A \in \mathscr{L}(\mathscr{H}), \phi \in H^\infty$,

(iii) $\sigma(\operatorname{alg}\{P_n\}) \subseteq \{T_\phi : \phi \in H^\infty\}$.

Proof. Regard H^2 as a closed subspace of $L^2(\mathbf{T}, m)$. Then P_0 is the projection of L^2 on H^2, and we may compose an operator A on H^2 with P_0 to obtain an operator AP_0 on L^2. We define σ in terms of the expectation π of Proposition 5.1 as follows.

$$\sigma(A) = P_0 \pi(AP_0)\,|_{H^2}, \qquad A \in \mathscr{L}(H^2).$$

It is clear that σ is a positive linear mapping of norm 1, which carries 1 to 1. Since the range of π is $\mathscr{M}$ and $P_0\mathscr{M}\,|_{H^2}$ is contained in $\{T_\phi : \phi \in L^\infty\}$, it follows that the range of σ is contained in $\{T_\phi : \phi \in L^\infty\}$. We claim first that $\sigma(T_\phi) = T_\phi$, for every $\phi \in L^\infty$. Indeed, since $T_\phi P_0 = P_0 L_\phi P_0$, we see from the remarks following Proposition 5.1 that $\pi(T_\phi P_0) = \pi(L_\phi) = L_\phi$, so that $\sigma(T_\phi) = P_0 L_\phi\,|_{H^2} = T_\phi$. This implies in particular that $\sigma(\mathscr{L}(H^2)) = \{T_\phi : \phi \in L^\infty\}$, and $\sigma \circ \sigma = \sigma$.

To verify (ii), choose $\phi \in H^\infty$, $A \in \mathscr{L}(H^2)$. Then $AT_\phi P_0 = AP_0 L_\phi P_0$, so as in the preceding paragraph we have $\pi(AT_\phi P_0) = \pi(AP_0 L_\phi) = \pi(AP_0) L_\phi$. It follows that $\sigma(AT_\phi) = P_0 \pi(AP_0) L_\phi\,|_{H^2} = \sigma(A)\, T_\phi$, where in the last equality we use the fact that $T_\phi = L_\phi\,|_{H^2}$ for $\phi \in H^\infty$. Similarly,

$$\pi(T_\phi AP_0) = \pi(P_0 L_\phi AP_0) = \pi(L_\phi AP_0) = L_\phi \pi(AP_0) = \pi(AP_0) L_\phi,$$

and we conclude that $\sigma(T_\phi A) = \sigma(A)\, T_\phi$.

The property 5.2(iii) is a simple consequence of its counterpart 5.1(ii). ∎

6. AN APPLICATION

The corona theorem asserts that the open unit disc is dense in the maximal ideal space of H^∞. This is equivalent to the assertion that if

$f_1, \dots, f_N$ are bounded analytic functions in the open unit disc which satisfy

$$|f_1(z)| + \cdots + |f_N(z)| \geqslant \epsilon > 0, \qquad |z| < 1, \tag{6.1}$$

then there exist similar functions $g_1, \dots, g_N$ such that $f_1 g_1 + \cdots + f_N g_N = 1$. This was proved in 1962 by Carleson [5], who also obtained estimates on $\| g_i \|_\infty$ in terms of ϵ and $\| f_i \|_\infty$.

The work behind the present paper was begun partly in the hope of giving a relatively natural operator-theoretic proof of Carleson's theorem. While this has not been completely successful, the interpolation theorem of Section 4 does lead to the following operator-theoretic variant. Suppose that, instead of (6.1), the N functions $f_1, \dots, f_N$ satisfy

$$\| T_{f_1}^* x \| + \cdots + \| T_{f_N}^* x \| \geqslant \epsilon \| x \|, \qquad x \in H^2, \tag{6.2}$$

where T_f is the Toeplitz operator on H^2 associated with f. Then Theorem 6.3 below asserts that there exist $g_1, \dots, g_N \in H^\infty$ such that $f_1 g_1 + \cdots + f_N g_N = 1$. Moreover, the estimate we obtain on $\| g_i \|_\infty$ seems considerably better than that of [5].

The relationship between (6.1) and (6.2) deserves a few comments. For each $\lambda \in \mathbf{C}$, $|\lambda| < 1$, we may form the H^2 function $x_\lambda(z) = (1 - \bar{\lambda} z)^{-1}$. Now it is easy to see that $T_f^* x_\lambda = \overline{f(\lambda)}\, x_\lambda$, for each $f \in H^\infty$, so by taking $x = x_\lambda$ in (6.2) we see immediately that (6.2) implies (6.1). Conversely, one can utilize the Corona theorem to deduce (6.2) from (6.1) (the ϵ of (6.2) has to be made smaller than the ϵ of (6.1)); however, we do not know if the latter implication can be proved directly. Needless to say, such a proof would be very desirable.

Finally, we want to acknowledge that Theorem 6.3 appears as one of the results of [6]. But the proof in [6] makes use of the Corona theorem itself, and so is not related to the present discussion.

THEOREM 6.3. *Let $f_1, \dots, f_N \in H^\infty$ and $\epsilon > 0$ be such that*

$$\| T_{f_1}^* x \|^2 + \cdots + \| T_{f_N}^* x \|^2 \geqslant \epsilon^2 \| x \|^2$$

for every $x \in H^2$. Then there exist functions $g_1, \dots, g_N \in H^\infty$ such that $f_1 g_1 + \cdots + f_N g_N = 1$. If $\| f_i \|_\infty \leqslant 1$ for each i, then $g_1, \dots, g_N$ may be chosen so that $\| g_i \|_\infty \leqslant 4N\epsilon^{-3}$.

Proof. Let $e_0, e_1, \dots$ be the usual orthonormal base for H^2 (i.e., $e_k = z^k$), and let P_n be the projection onto $[e_0, e_1, \dots, e_n]$. Then the algebra $\{T_f^*: f \in H^\infty\}$ of all coanalytic Toeplitz operators is a commutative subalgebra of the nest algebra alg$\{P_n: n \geqslant 0\}$. The idea of the proof is the following. The hypothesis of $f_1, \ldots, f_N$ implies (and in fact is equivalent to) the fact that there exist bounded operators $B_1, \ldots, B_N$ on H^2 with $\sum B_k T_{f_k}^* = 1$ (see the opening paragraph of Section 4). We first use the interpolation theorem to conclude that the B_k's can be found within alg$\{P_n\}$. By making use of the projection mapping of Proposition 5.2, we can then find B_k's of the form $T_{g_k}^*$, $g_k \in H^\infty$, and the required relation $\sum f_k g_k = 1$ will follow

In order to apply Theorem 4.3, we claim that for each $n \geqslant 0$,

$$\sum_k \|(1 - P_n)\, T_{f_k}^* x\|^2 \geqslant \epsilon^2 \|(1 - P_n)x\|, \qquad x \in H^2.$$

For that, let $S = T_z$ be the unilateral shift. Then $1 - P_n = S^n S^{*n}$, $n \geqslant 0$, and since S^* commutes with $T_f{}^*$ for each $f \in H^\infty$, we have

$$\begin{aligned}\|(1 - P_n)\, T_f{}^* x\| &= \| S^n S^{*n} T_f{}^* x\| \\ &= \| S^{*n} T_f{}^* x\| = \| T_f{}^* S^{*n} x\|.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_k \|(1 - P_n)\, T_{f_k}^* x\|^2 &= \sum_k \| T_{f_k}^* S^{*n} x\|^2 \\ &\geqslant \epsilon^2 \| S^{*n} x\|^2 = \epsilon^2 \| S^n S^{*n} x\|^2 = \epsilon^2 \|(1 - P_n)x\|^2,\end{aligned}$$

as asserted. So by Theorem 4.3 we may find operators $B_1, \ldots, B_N \in$ alg$\{P_n\}$ such that $\sum B_k T_{f_k}^* = 1$. Moreover, if $\| f_i \|_\infty \leqslant 1$ for each i, then $\| T_{f_i}^* \| \leqslant 1$, and we may even assume $\| B_i \| \leqslant 4N\epsilon^{-3}$.

By taking adjoints we obtain $T_{f_1} B_1{}^* + \cdots + T_{f_N} B_N{}^* = 1$, and $B_k{}^*$ leaves $[e_n, e_{n+1}, \ldots]$ invariant, for every $1 \leqslant k \leqslant N$, $n \geqslant 0$. So if σ is the projection mapping of Proposition 5.2, then $\sigma(B_k{}^*)$ must have the form T_{g_k}, $g_k \in H^\infty$, and we have $\| g_k \|_\infty = \| T_{g_k} \| = \| \sigma(B_k{}^*)\| \leqslant \| B_k \|$. Applying σ to the above formula we obtain

$$\begin{aligned}1 = \sigma(1) &= \sum_k \sigma(T_{f_k} B_k{}^*) \\ &= \sum \sigma(B_k{}^*)\, T_{f_k} = \sum T_{g_k} T_{f_k} = \sum T_{f_k} T_{g_k}.\end{aligned}$$

The desired conclusion $\sum f_k g_k = 1$ is now immediate from the fact that the map $f \in H^\infty \mapsto T_f$ is an algebra monomorphism. ∎

Note added in proof. Since this paper was written, the author has heard from C. F. Schubert, and later from William Helton, that the application to Toeplitz operators (Theorem 6.3) can also be deduced from results associated with the lifting theorem for pairs of commuting contractions. While these methods do lead to a shorter proof of Theorem 6.3 itself, they are closely tied to properties of unilateral shifts, and hence do not apply to the more general interpolation theorems of Section 4.

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