Operator algebras and invariant subspaces

By WILLIAM ARVESON[1]

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Introduction

This paper is concerned with the relation between certain non-self-adjoint algebras of operators on Hilbert space and their invariant subspace lattices. Given an algebra $\mathcal{A}$ of operators on a Hilbert space, the notation lat $\mathcal{A}$ will mean the lattice of all (self-adjoint) projections whose ranges are invariant under every operator in $\mathcal{A}$; dually, with every lattice $\mathcal{L}$ of projections, one associates the algebra, alg $\mathcal{L}$, of all operators which leave each element of $\mathcal{L}$ invariant. An algebra $\mathcal{A}$ is called *reflexive* [16] if it satisfies the condition $\mathcal{A} = \text{alg lat } \mathcal{A}$ (lattices of projections with the corresponding property, $\mathcal{L} = \text{lat alg } \mathcal{L}$, are also called reflexive). It seems appropriate to

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think of this property as generalizing the "double commutant" property of weakly closed self-adjoint algebras.

In practice, the knowledge that a given algebra $\mathcal{A}$ is reflexive allows one to reduce many questions about the algebra to questions about the lattice, lat $\mathcal{A}$. So one is lead to consider two general problems: first, how does one determine if $\mathcal{A}$ is reflexive, and second, how does one approach the analysis of lat $\mathcal{A}$? This paper is the result of a study of these two problems for a special class of operator algebras.

In Chapter I we discuss *commutative subspace lattices*, that is, lattices of mutually commuting self-adjoint projections which are closed in the strong operator topology. It is shown that such lattices are always reflexive. This depends on a "spectral theorem" which gives a description of such lattices in concrete terms, and on a procedure for constructing certain types of operators associated with these lattices. Basic to the results of Chapter I, and to much of the rest of the paper as well, is the notion of a *partially ordered measure space* introduced in Section 1.1.

Chapter II concerns operator algebras which contain a maximal abelian von Neumann algebra. This category of algebras can be regarded as dual to the category of commutative subspace lattices in the sense that a projection lattice $\mathcal{L}$ is commutative if, and only if, the corresponding operator algebra alg $\mathcal{L}$ contains a maximal abelian von Neumann algebra. It is shown that every such algebra $\mathcal{A}$ (which is closed in, say, the weak operator topology) is *pre-reflexive* in the sense that $\mathcal{A}$ and alg lat $\mathcal{A}$ have the same diagonal (Section 2.1); however $\mathcal{A}$ need *not* be reflexive in general (Section 2.5). The problem of determining when $\mathcal{A}$ is reflexive can be reduced to a problem about the commutative subspace lattice lat $\mathcal{A}$, which, in turn, is shown to be closely analogous to the problem of spectral synthesis in commutative harmonic analysis (see Section 2.2). As it turns out, it is possible to characterize the desired property in a usable way, and this leads to a number of new results on reflexive algebras. Some further applications of these techniques are given in Sections 2.3 and 2.4.

In Chapter III we discuss certain aspects of the general problem of classifying (infinite) complete distributive lattices. This is closely connected with the problem of classifying certain reflexive algebras with respect to similarity; indeed, if $\mathcal{A}$ and $\mathcal{B}$ are operator algebras whose invariant subspace lattices can be shown to be non-isomorphic as lattices, then of course one concludes that $\mathcal{A}$ and $\mathcal{B}$ cannot be similar as operator algebras. The main results are a unique factorization theorem (Section 3.3), and the existence of a numerical invariant for these lattices which is somewhat reminiscent of the entropy invariant of ergodic theory (Section 3.5). These results lead to the solution of classification problems for certain reflexive operator algebras.

Finally, it is essential for the techniques of this paper that *all Hilbert spaces be separable or finite-dimensional.* Throughout the paper, the notation $\mathcal{L}(\mathcal{H})$ (resp. $\mathcal{C}(\mathcal{H})$) denotes the algebra of all bounded (resp. compact) operators on $\mathcal{H}$.

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Chapter I. Reflexive Lattices

1.1. Partiallly ordered measure spaces

The theory of (commutative) Boolean σ-algebras of projections on Hilbert spaces is well understood, and is intimately connected with measure theory. The standard technique for analyzing a Boolean algebra $\mathcal{B}$ of projections on a Hilbert space $\mathcal{H}$ is to realize $\mathcal{H}$ as the space $L^2(X, m)$ (where (X, m) is a positive measure space) in such a way that $\mathcal{B}$ becomes an algebra of multiplications by characteristic functions of certain measurable subsets of X. When $\mathcal{H}$ is finite-dimensional, this procedure of course amounts to nothing more than finding an orthonormal basis for $\mathcal{H}$ with respect to which the matrices of the projections in $\mathcal{B}$ are all diagonal. In general, the Boolean algebra may not be multiplicity-free, and therefore the subsets of X that represent elements of $\mathcal{B}$ may form a *proper* sub σ-field $\mathcal{F}_{\mathcal{B}}$ of the σ-field of all measurable subsets of X. In turn, this gives rise to an equivalence relation in X: $x \sim y$ iff $\chi_E(x) = \chi_E(y)$ for all $E \in \mathcal{F}_{\mathcal{B}}$, where χ_E denotes the characteristic function of E.

Now suppose that, instead of a Boolean σ-algebra, one starts with a commutative σ-lattice $\mathcal{L}$ of projections on $\mathcal{H}$. We will see in Section 3 that it is possible to introduce "coordinates" for $\mathcal{H}$ with analogous properties, except that in place of the equivalence relation $\sim$ one ends up with a *quasi-ordering* of the points of X. Now for most purposes, the equivalence relation described in the preceding paragraph is more or less incidental to the analysis of the Boolean algebra $\mathcal{B}$; for instance when $\mathcal{B}$ is multiplicity-free then $\sim$ is the trivial relation $x \sim y$ iff $x = y$. On the other hand, in the case of lattices the ordering is intimately connected with the structure of the lattice, and it will occupy a central position throughout most of this paper.

A *partially ordered Borel space* is a pair $(X, \leqq)$ consisting of a Borel space X (that is, a set X along with a distinguished σ-field of subsets of X, whose elements will be called *Borel sets*) and a relation $\leqq$ in X which is transitive and symmetric, but we allow the possibility that $x \leqq y$ and $y \leqq x$ for distinct elements x and y. This terminology is somewhat at odds with [8] (in [8], $\leqq$ would be called a *quasi*-ordering), but it is convenient for our purposes. Note for instance that an equivalence relation qualifies as a partial ordering. If, on the other hand, $x \leqq y \leqq x$ does imply that $x = y$, then $\leqq$ will be called *strict*.

A subset $E \subseteqq X$ is called *increasing* if, for each $x \in E$ and $y \in X$, $x \leqq y$ implies $y \in E$; *decreasing* sets are defined in the obvious way. We define $L(X, \leqq)$ to be the family of all increasing *Borel* subsets of X. $L(X, \leqq)$ is a *σ-lattice* of subsets of X in the following sense:

(i) *$\varnothing$ and X belong to $L(X, \leqq)$.*

(ii) *If $E_1, E_2, \cdots \in L(X, \leqq)$, then $\bigcup_n E_n \in L(X, \leqq)$ and $\bigcap_n E_n \in L(X, \leqq)$.*
Note that σ-lattices behave much like σ-fields, the difference being that one is unable to take complements in σ-lattices. Note also that when $\leqq$ is the "indiscrete" partial order, in which $x \leqq y$ for all $x, y \in X$, then $L(X, \leqq) = \{\varnothing, X\}$ is trivial. It can happen that $L(X, \leqq)$ is trivial for other partial orderings as well; we now want to introduce a very tractable class of partially ordered Borel spaces which, among other things, does not exhibit this pathology.

DEFINITION 1.1.1. *A partially ordered Borel space $(X, \leqq)$ is called standard if X is standard as a Borel space, and there exists a sequence $f_1, f_2, \cdots$ of real-valued Borel functions on X such that, for all $x, y \in X$, $x \leqq y$ iff $f_n(x) \leqq f_n(y)$ for every $n \geqq 1$,*

Recall, incidentally, that a Borel space X is called *standard* if X is (Borel) isomorphic to a Borel subset of some separable complete metric space in its relative Borel structure (in this terminology we will follow Chapter 3 of [3]).

Now if $(X, \leqq)$ is standard, then the σ-lattice $L(X, \leqq)$ completely determines the order relation $\leqq$ in the sense that $x \leqq y$ if and only if, $\chi_E(x) \leqq \chi_E(y)$ for every $E \in L(X, \leqq)$. Indeed, if $x \leqq y$ then by definition of increasing sets we have $\chi_E(x) \leqq \chi_E(y)$ for every $E \in L(X, \leqq)$; and the converse results from the following:

PROPOSITION 1.1.2. *If $(X, \leqq)$ is standard, then $L(X, \leqq)$ contains a sequence $E_1, E_2, \cdots$ such that $x \leqq y$ iff $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for every n.*

Proof. Let $f_1, f_2, \cdots$ be a sequence of functions satisfying Definition 1.1. Fix $n \geqq 1$. For each rational real number r, the Borel set $A_{n,r} =$

$\{x: f_n(x) \geqq r\}$ is increasing, because if $y \geqq x$ and $f_n(x) \geqq r$ then $f_n(y) \geqq f_n(x) \geqq r$. Thus $A_{n,r} \in L(X, \leqq)$. On the other hand, for each n and for every pair of points $x, y \in X$, we have $f_n(x) \leqq f_n(y)$ iff $\chi_{A_{n,r}}(x) \leqq \chi_{A_{n,r}}(y)$ for every rational r. The required sets $E_1, E_2, \cdots$ are obtained by enumerating the elements of $\{A_{n,r}: n \geqq 1, r \in \mathbf{Q}\}$. □

Example 1.1.3. The simplest example of a standard partially ordered Borel space is $X = \mathbf{R}^n$, $n \geqq 1$, with the partial order

$$(x_1, \cdots, x_n) \leqq (y_1, \cdots, y_n) \text{ iff } x_i \leqq y_i \text{ for all } i .$$

The required functions $f_1, \cdots, f_n$ are of course the n coordinate functions.

Example 1.1.4. Let $\mathbf{2} = \{0, 1\}$ be the two element partially ordered set with its usual ordering ($0 \leqq 0 \leqq 1 \leqq 1$), and let $\mathbf{2}^\infty = \mathbf{2} \times \mathbf{2} \times \cdots$ be the product of countably many copies of $\mathbf{2}$. With the usual product Borel structure, 2^∞ becomes a standard Borel space. For every nonvoid set S of positive integers, define a partial ordering $\leqq_S$ by

$$(x_1, x_2, \cdots) \leqq_S (y_1, y_2, \cdots) \text{ iff } x_i \leqq y_i \text{ for every } i \in S .$$

Again, this order is standard since it is determined by the Borel functions $\{f_i: i \in S\}$, where f_i is the i^{th} coordinate function. In the case where S consists of the even integers 2, 4, 6, $\cdots$, $\leqq_S$ will be called the *even* ordering on $\mathbf{2}^\infty$.

Example 1.1.5. Let E be a (real) separable Banach space. Then the Borel structure generated by its norm topology makes E into a standard Borel space, and thus the Cartesian product $X = E \times \mathbf{R}$ of E with the real line is standard in its product Borel structure. For $(x, s), (y, t) \in X$ (with $x, y \in E$, $s, t \in \mathbf{R}$), define $(x, s) \leqq (y, t)$ to mean $\|y - x\| \leqq t - s$. The required sequence $f_1, f_2, \cdots$ is defined as follows. Since X is separable, the unit ball of the dual of X is compact and metrizable in its relative weak* topology, and thus has a countable dense set $\rho_1, \rho_2, \cdots$. Define $f_1, f_2, \cdots$ on X by $f_n(x, s) = \rho_n(x) + s$. Noting that $\sup_n \rho_n(x) = \|x\|$, a routine verification will now show that $(x, s) \leqq (y, t)$ if, and only if, $f_n(x, s) \leqq f_n(y, t)$ for all $n \geqq 1$. Thus, $(X, \leqq)$ is standard. Of particular interest in special relativity is the example obtained from $E = \mathbf{R}^n$ with its Euclidean norm; here the order $(x, s) \leqq (y, t)$ expresses the fact that the vector (y, t) is *space-like* relative to (x, s) and occurs later in time (this example is discussed further at the end of Section 2.2).

Example 1.1.6. Let $\mathcal{H}$ be a separable Hilbert space and let $(X, \leqq)$ be the partially ordered set of all (self-adjoint) projections on $\mathcal{H}$. The unit ball of $\mathfrak{L}(\mathcal{H})$ is a separable complete metric space in the strong operator topology, the projections form a closed subspace, and therefore X is a standard Borel space in the Borel structure generated by its relative strong operator topology. To define the sequence f_n, let $\xi_1, \xi_2, \cdots$ be a countable dense set in $\mathcal{H}$, and define $f_n(P) = (P\xi_n, \xi_n)$, $n = 1, 2, \cdots$, for every projection P. Each f_n is a Borel function (in fact, a continuous function) on X, and clearly $P \leqq Q$ iff $f_n(P) \leqq f_n(Q)$ for every n. Thus, $(X, \leqq)$ becomes a standard partially ordered Borel space. In this example, X is even a complete lattice (though it fails to be a topological lattice [8] since the lattice operations are not strongly continuous).

Remarks. The reader will be able to supply additional examples of his own. We now want to illustrate the type of pathology that occurs with *non-standard* partial orderings by giving two examples. The first simply imitates the known misbehavior of equivalence relations, while the second shows how even innocent-appearing (non-standard) partial orderings can exhibit related pathology.

For the first example, let X be the real line with its usual Borel structure and define $x \leqq y$ to mean $y - x$ is rational. Then $\leqq$ is an equivalence relation and $L(\mathbf{R}, \leqq)$ consists of all Borel sets which are invariant under all rational translations. Since the rational translations act ergodically on $\mathbf{R}$ with respect to Lebesgue measure, we see that for every $E \in L(\mathbf{R}, \leqq)$, either E or its complement has Lebesgue measure zero. Thus the partial order $\leqq$ is trivial in the sense that the image of $L(\mathbf{R}, \leqq)$ in the measure algebra of Lebesgue measure is the trivial lattice $\{0, 1\}$; in other words, $L(\mathbf{R}, \leqq)$ is too "thin". On the other hand, we note that $\leqq$ is *not* standard. Indeed, the equivalence class $\{y \in \mathbf{R}: x \leqq y\}$ of each $x \in \mathbf{R}$ is countable, and therefore has Lebesgue measure zero. This, together with the preceding observation and known results about Borel structures, implies that there is no sequence $E_1, E_2, \cdots$ of Borel sets such that $x \leqq y$ iff $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for all n (e.g., [3, Theorem 3.3.5]).

As a second and more interesting example, let $X = \mathbf{R}^2$ (usual Borel structure) and define $(x, y) \leqq (x', y')$ in $\mathbf{R}^2$ to be the dictionary order $x < x'$, or $x = x'$ and $y \leqq y'$. Note that this partial order is *strict*, and is in fact a linear ordering of $\mathbf{R}^2$. We show first that $(\mathbf{R}^2, \leqq)$ is not standard.

By 1.1.2, it suffices to show that no sequence of sets in $L(\mathbf{R}, \leqq)$ can separate points. Let E_n be such a sequence. A few moments, thought shows that each E_n must have the form $E_n = (x_n, \infty) \times \mathbf{R} \cup \{x_n\} \times F_n$, where F_n is a Borel subset of $\mathbf{R}$. So if x is any element of $\mathbf{R}$ such that $x \neq x_n$ for every n, then no pair of points (x, y) and (x, z) can be distinguished by the E_n.

The reason this ordering of $\mathbf{R}^2$ is not standard is essentially because its vertical component is too "weak". To illustrate this in concrete terms, let m be two-dimensional Lebesgue measure. Define a new order, $\leqq_m$ of $\mathbf{R}^2$ by $(x, y) \leqq_m (x', y')$ iff $x \leqq x'$. Then $\leqq_m$ is clearly standard, $\leqq_m$ is a weakening of $\leqq$ in the sense that $p \leqq q$ implies $p \leqq_m q$, and the preceding paragraph shows that $L(\mathbf{R}^2, \leqq)$ and $L(\mathbf{R}^2, \leqq_m)$ determine the same lattice in the measure algebra of m. Thus the vertical component of $\leqq$ is irrelevant with respect to Lebesgue measure. Note that this "standardization" $\leqq_m$ of $\leqq$, relative to m, is uniquely determined by the third condition above in the sense of the corollary of 1.1.11 below.

This standardization process can be carried out in general, in the following way. Let X be a standard Borel space and let $\leqq$ be an arbitrary partial ordering of X. Suppose $\leqq$ is not standard. Then for every σ-finite measure m on X there is a standard ordering $\leqq_m$ of X such that $x \leqq y$ implies $x \leqq_m y$, which is equivalent to $\leqq$ in the sense that $L(X, \leqq)$ and $L(X, \leqq_m)$ agree modulo m, and which is *unique* in the sense of the corollary of 1.1.11 below (the existence of $\leqq_m$ can be deduced by the techniques employed later on in this chapter, the exact details of which we omit). So in this sense *one may regard non-standard partial orderings as having been obtained from standard orderings by adding conditions which are measure-theoretically irrelevant* (note, however, that the standardization $\leqq_m$ will in general depend on the particular measure m). In any case, these remarks show that for measure-theoretic purposes *all* partial orderings can be taken to be standard.

Finally, we remark that in the first example where $X = \mathbf{R}$ and $x \leqq y$ means $y - x \in \mathbf{Q}$, the standardization of $\leqq$ relative to Lebesgue measure m is the indiscrete order, in which $x \leqq_m y$ holds for all $x, y \in \mathbf{R}$.

We now return to the general discussion. By an *order isomorphism* of one partially ordered Borel space $(X, \leqq)$ onto another $(Y, \leqq)$, we mean a bijection $\phi: X \rightarrow Y$ such that:

(i) *$x \leqq y$ iff $\phi(x) \leqq \phi(y)$, and*

(ii) *ϕ induces an isomorphism between the Borel structures of Y and X.*

$(X, \leqq)$ and $(Y, \leqq)$ are said to be *order isomorphic* whenever such a map ϕ exists. According to the following result, condition (i) above can usually be replaced with an equivalent condition on the set mapping induced by ϕ.

PROPOSITION 1.1.7. *Let $(X, \leqq)$ and $(Y, \leqq)$ be standard partially ordered Borel spaces, and let $\phi: X \rightarrow Y$ be a Borel isomorphism. Then ϕ is an order isomorphism if, and only if it induces an isomorphism between the σ-lattices $L(Y, \leqq)$ and $L(X, \leqq)$.*

The proof, a routine application of Proposition 1.1.2, is left for the reader.

A partially ordered Borel space $(X, \leqq)$ is called a *subspace* of $(Y, \leqq)$ if X is a subspace of Y in the sense of Borel spaces, and the order on X is inherited from that of Y.

LEMMA 1.1.8. *Every standard partially ordered Borel space is order isomorphic to a Borel subspace of* $(\mathbf{2}^\infty, \leqq)$, *where* $\leqq$ *is the even order on* $\mathbf{2}^\infty$.

Proof. By 1.1.2, we may find Borel sets $E_1, E_2, \cdots$ in the given space $(X, \leqq)$ such that $x \leqq y$ iff $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for all $n = 1, 2, \cdots$. Moreover, since the Borel structure of X is standard we may find a sequence $F_1, F_2, \cdots$ of Borel sets in X which generates the Borel structure of X and separates points. Now define a map $\phi: X \to \mathbf{2}^\infty$ by

$$\phi(x) = (\chi_{F_1}(x), \chi_{E_1}(x), \chi_{F_2}(x), \chi_{E_2}(x), \cdots) .$$

Letting $C_k = \{(y_1, y_2, \cdots) \in \mathbf{2}^\infty: y_k = 1\}$, we have $\phi^{-1}(C_{2k}) = E_k$ and $\phi^{-1}(C_{2k-1}) = F_k$, $k = 1, 2, \cdots$, so that ϕ is a Borel map. Since the sets $\{F_k\}$ separate points in X, ϕ is also 1-1. Thus, $\phi(X)$ is a Borel subset of $\mathbf{2}^\infty$ and ϕ is a Borel isomorphism of X onto $\phi(X)$ in its relative Borel structure ([3, Theorem 3.3.2]).

Now by definition of the sets E_i we have $x \leqq y$ in X iff $\chi_{E_i}(x) \leqq \chi_{E_i}(y)$ for all $i \geqq 1$. This is equivalent to $\phi(x) \leqq \phi(y)$, by definition of ϕ and the even order on $\mathbf{2}^\infty$. □

If E is a subset of a partially ordered set $(X, \leqq)$, we will write E^+ (resp. E^-) for the set of all $y \in X$ for which there is an element $x = x_y \in E$ with $x \leqq y$ (resp. $x \geqq y$). Note that E is increasing iff $E = E^+$, and similarly for decreasing sets. The maps $E \mapsto E^+$ and $E \mapsto E^-$ preserve arbitrary unions, but *not* intersections. When E is singleton $\{x\}$, we will write E^+ as $[x, +\infty)$ and E^- as $(-\infty, x]$; note that in general E^+ is the union of all "intervals" $[x, +\infty)$ as x runs over E.

The essential idea behind the proof of the following lemma is due to Paul Chernoff, who kindly consented to its inclusion here.

LEMMA 1.1.9. *Let* $(\mathbf{2}^\infty, \leqq)$ *be the Cantor space with its usual Borel structure and the even ordering. Let* $L_0(\mathbf{2}^\infty, \leqq)$ *be the* σ*-lattice generated by* $\varnothing$, X, *and the sets* C_{2k}, $k = 1, 2, \cdots$, *where* $C_k = \{(x_1, x_2, \cdots) \in \mathbf{2}^\infty: x_k = 1\}$.

Then for every finite positive Borel measure μ *on* $\mathbf{2}^\infty$ *and every* μ*-measurable increasing set* $E \subseteqq \mathbf{2}^\infty$, *there is an element* $F \in L_0(\mathbf{2}^\infty, \leqq)$ *such that* $F \subseteqq E$, *and* $\bar{\mu}(E \backslash F) = 0$, $\bar{\mu}$ *denoting the completion of* μ.

Remark. We have made use of the usual terminology above, in that a μ-measurable set is one which differs from a Borel set by a subset of a Borel set of μ-measure zero, and $\bar{\mu}$ denotes the usual extension of μ to the σ-field of μ measurable sets.

Proof. $\mathbf{2}^\infty$ becomes a compact metric space in its usual product topology. Note that this topology generates the given Borel structure.

First, we claim that E^+ is closed whenever E is a closed set in $\mathbf{2}^\infty$. Indeed, the set $G = \{(x, y) \in \mathbf{2}^\infty \times \mathbf{2}^\infty : y \leqq x\}$ is clearly closed in the product topology of $\mathbf{2}^\infty \times \mathbf{2}^\infty$, and is therefore compact. Let $P_1(x, y) = x$ be the canonical projection of $\mathbf{2}^\infty \times \mathbf{2}^\infty$ onto the first coordinate space. The conclusion follows since E^+ can be expressed as the image of the compact set $G \cap \mathbf{2}^\infty \times E$ under the continuous function P_1.

Second, we claim that every *closed* increasing set belongs to $L_0(\mathbf{2}^\infty, \leqq)$; equivalently, every open decreasing set belongs to the σ-lattice generated by $\emptyset$, $\mathbf{2}^\infty$, and the complements of $C_2, C_4, C_6, \cdots$. For that, fix U, a nonvoid open decreasing set. Now for each $n \geqq 1$ and every sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{2n})$ of 0's and 1's, let $E(\varepsilon) = \{(x_1, x_2, \cdots) \in \mathbf{2}^\infty : x_i = \varepsilon_i,\ 1 \leqq i \leqq 2n\}$. The family $\{E(\varepsilon)\}$, as ε runs over all such finite sequences with an even number of terms, becomes a base for the topology on $\mathbf{2}^\infty$. Now for $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{2n})$, note that

1.1.10 $$E(\varepsilon)^- = \{x \in \mathbf{2}^\infty : x_2 \leqq \varepsilon_2,\ x_4 \leqq \varepsilon_4, \cdots, x_{2n} \leqq \varepsilon_{2n}\} .$$

Indeed, the inclusion $\subseteqq$ is clear from the fact that the set on the right is decreasing and contains $E(\varepsilon)$. Conversely, if $x \in \mathbf{2}^\infty$ is such that $x_{2k} \leqq \varepsilon_{2k}$ for $1 \leqq k \leqq n$, define $y \in \mathbf{2}^\infty$ by $y_i = \varepsilon_i$ for $1 \leqq i \leqq 2n$, and $y_i = 1$ for $i > 2n$. Then $y \in E(\varepsilon)$, and $x \leqq y$ by definition of the even order; this proves $x \in E(\varepsilon)^-$. Note also that the set on the right is an intersection of n sets, each of which is of the form $\mathbf{2}^\infty$ or $\mathbf{2}^\infty \backslash C_{2k}$, $1 \leqq k \leqq n$. We conclude that $E(\varepsilon)^-$ belongs to the required σ-lattice, for every such ε. Returning now to the open decreasing set U, we can find a sequence $\varepsilon_1, \varepsilon_2, \cdots$ such that $U = \bigcup_j E(\varepsilon_j)$. Thus, $U = U^- = \bigcup_j (E(\varepsilon_j)^-)$, and the claim is immediate from this.

Turning now to the proof of the lemma, let E be a μ-measurable increasing set in $\mathbf{2}^\infty$. By regularity of $\bar{\mu}$, there is a sequence of compact sets K_n such that $K_n \subseteqq K_{n+1} \subseteqq E$ and $\bar{\mu}(E \backslash K_n) \to 0$ as $n \to \infty$. Since E is increasing we have $K_n^+ \subseteqq E$ for every n, and by the first claim, each K_n^+ is closed. Since $K_n \subseteqq K_n^+$ we have $\bar{\mu}(E \backslash K_n^+) \to 0$, and thus $\bar{\mu}(E \backslash \bigcup_n K_n^+) = 0$. By the second claim, each K_n^+ belongs to $L_0(\mathbf{2}^\infty, \leqq)$ and therefore so does their union. This proves that the set $F = \bigcup_n K_n^+$ has all the stated properties. $\square$

We come now to the central concept of this section. By a *partially ordered measure space* we mean a triple $(X, \leqq, m)$ consisting of a partially ordered Borel space $(X, \leqq)$ and a *σ-finite* positive Borel measure m on X; $(X, \leqq, m)$ is called *standard* if $(X, \leqq)$ is standard. Now with each measure m on $(X, \leqq)$ we may define a new σ-lattice $L_m(X, \leqq)$ of subsets of X, consisting of all Borel subsets $E \subseteqq X$ which are *almost increasing* in the sense that there is an m-null Borel set $N \subseteqq X$ (depending on E) such that, for all $x, y \in X \backslash N$, $x \in E$ and $y \leqq x$ imply $y \in E$. $L_m(X, \leqq)$ clearly contains $L(X, \leqq)$, and the inclusion may be proper.

On the other hand, suppose $\mathcal{E}$ is a countable subfamily of $L(X, \leqq)$ such that $x \leqq y$ iff $\chi_E(x) \leqq \chi_E(y)$ for all $E \in \mathcal{E}$. Then we define $L_{\mathcal{E}}(X, \leqq)$ to be the σ-lattice generated by $\mathcal{E}$, $\varnothing$, and X. We therefore have

$$L_{\mathcal{E}}(X, \leqq) \subseteqq L(X, \leqq) \subseteqq L_m(X, \leqq)\,,$$

and it will be useful for us to know when these three σ-lattices determine the same lattice in the measure algebra of m. According to the following theorem, this is always true in the "reasonable" cases.

THEOREM 1.1.11 *Let $(X, \leqq, m)$ be a standard partially ordered measure space, and let $\mathcal{E} = \{E_1, E_2, \cdots\}$ be a countable subset of $L(X, \leqq)$ such that $x \leqq y$ iff $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for all n. Then for every set $B \in L_m(X, \leqq)$, there is an $A \in L_{\mathcal{E}}(X, \leqq)$ such that $m(A \Delta B) = 0$.*

Proof. Since the theorem only involves null sets of m and since every σ-finite measure is equivalent to a finite measure, there is no loss if we assume m is finite.

Now consider the Cantor space $\mathbf{2}^\infty$, with the even partial ordering, and let C_k be the k^{th} cylinder $\{(x_1, x_2, \cdots) \in \mathbf{2}^\infty : x_k = 1\}$. By 1.1.8, there is an order isomorphism ϕ of X onto a Borel subspace of $(\mathbf{2}^\infty, \leqq)$. Moreover, if $\{E_1, E_2, \cdots\}$ is the given sequence in $L(X, \leqq)$, then the proof of 1.1.8 shows that ϕ can be chosen so that $E_k = \phi^{-1}(C_{2k})$ and $\phi(E_k) = C_{2k} \cap \phi(X)$. So after the obvious identification, we may take X as a Borel subset of $\mathbf{2}^\infty$, E_k as $C_{2k} \cap X$, and $\leqq$ as the (relative) even ordering on X.

Now let B be a Borel set in X and let N be an m-null Borel set in X such that $x, y \in X \backslash N$, $x \in B$, $y \geqq x$ implies $y \in B$. Now the mapping $A \subseteqq \mathbf{2}^\infty \mapsto A \cap X$ carries the σ-lattice generated by $\varnothing$, $\mathbf{2}^\infty$, $C_2, C_4, C_6, \cdots$ onto the σ-lattice generated by $\varnothing, X, C_2 \cap X, C_4 \cap X, C_6 \cap X, \cdots$, namely $L_{\mathcal{E}}(X, \leqq)$ (this requires a very simple argument, which we leave for the reader).

Thus the theorem will be proved if we produce a set A_1 in the σ-lattice generated by $\emptyset$, $\mathbf{2}^\infty$, $C_2, C_4, C_6, \cdots$ such that B and $A_1 \cap X$ agree almost everywhere (dm).

A_1 is constructed as follows. Define $B_1 \subseteq \mathbf{2}^\infty$ as the set of all points $y \in \mathbf{2}^\infty$ which are $\geqq$ some point of $B \backslash N$. It is clear that B_1 is an increasing subset of $\mathbf{2}^\infty$. Observe next that $B_1 \cap X \subseteq B \cup N$. For if $y \in B_1 \cap X$ then there is an $x \in B \backslash N$ such that $x \leqq y$; if y does *not* belong to N then $y \in B$ because of the properties of B and N. It follows that

$$B \backslash N \subseteq B_1 \subseteq B \cup N \cup (\mathbf{2}^\infty \backslash X) .$$

Thus, if we define a measure μ on $\mathbf{2}^\infty$ by $\mu(S) = m(S \cap X)$, then the preceding formula implies that B_1 differs from the Borel set B by a set of μ-measure zero (note that $\mu(N) = m(N) = 0$ and $\mu(\mathbf{2}^\infty \backslash X) = 0$), so that B_1 is μ-measurable. By Lemma 1.1.9 there exists A_1 in the σ-lattice generated by $\emptyset$, $\mathbf{2}^\infty$, C_{2k} such that $A_1 \subseteq B_1$ and $B_1 \backslash A_1$ is contained in a Borel set of μ-measure zero. Clearly $A_1 \cap X \subseteq B_1 \cap X \subseteq B \cup N$, and the preceding implies that the difference $(B \cup N) \backslash (A_1 \cap X)$ is a set of m-measure zero. □

Problem. Let $(X, \leqq)$ be a standard partially ordered Borel space, and let $E_1, E_2, \cdots$ be Borel sets such that $x \leqq y$ if and only if $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for all n. Then does $\{\emptyset, X, E_1, E_2, \cdots\}$ generate $L(X, \leqq)$ as a σ-lattice? This is a lattice theoretic counterpart of a known result about σ-fields (e.g., see [21], or Theorem 3.3.5 of [3]). Note also that 1.1.11 above implies that the answer is yes up to sets of measure zero (relative to any σ-finite measure on X).

If $\mathfrak{A}$, $\mathfrak{B}$ are two families of Borel sets in a measure space (X, m), we say that $\mathfrak{A} \sim \mathfrak{B}$ (mod m) if every element of $\mathfrak{A}$ differs by an m-null set from some element of $\mathfrak{B}$, and vice versa.

COROLLARY. *Let X be a standard Borel space, let $\leqq_1$ and $\leqq_2$ be two standard partial orderings of X, and let m be a σ-finite measure on X. In order that $L(X, \leqq_1) \sim L(X, \leqq_2)$ (mod m), it is necessary and sufficient that there exist a Borel set N of measure zero such that, for all $x, y \in X \backslash N$, $x \leqq_1 y$ iff $x \leqq_2 y$.*

Proof. For sufficiency, note that the hypothesis implies that $L_m(X, \leqq_1) = L_m(X, \leqq_2)$. From 1.1.11 we know that $L(X, \leqq_i) \sim L_m(X, \leqq_i)$ (mod m), for $i = 1, 2$, hence the desired conclusion.

Conversely, suppose that $L(X, \leqq_1) \sim L(X, \leqq_2)$ (mod m). We will produce a Borel set N of measure zero such that, for all $x, y \in X \backslash N$, $x \leqq_1 y$ implies $x \leqq_2 y$ (the full conclusion clearly follows from this by symmetry).

Since $\leqq_2$ is standard, there is a sequence $F_1, F_2, \cdots$ in $L(X, \leqq_2)$ such that $x \leqq_2 y$ iff $\chi_{F_n}(x) \leqq \chi_{F_n}(y)$ for all n. By hypothesis, we can find $E_n \in L(X, \leqq_1)$ such that $m(E_n \Delta F_n) = 0$. Let N be the null set $\bigcup_{n=1}^\infty E_n \Delta F_n$, and note that $\chi_{E_n}(x) = \chi_{F_n}(x)$, $n = 1, 2, \cdots$, for every $x \in X \backslash N$. Thus, if $x, y \in X \backslash N$ and $x \leqq_1 y$, then for every $n \geqq 1$ we have $\chi_{F_n}(x) = \chi_{E_n}(x) \leqq \chi_{E_n}(y) = \chi_{F_n}(y)$, hence $x \leqq_2 y$. □

Remarks. This corollary is false if one of the two orderings fails to be standard. For example, let $X = \mathbf{R}^2$, let $(x, y) \leqq_1 (x', y')$ mean $x \leqq x'$, and let $(x, y) \leqq_2 (x', y')$ be the dictionary order: $x < x'$, or $x = x'$ and $y \leqq y'$. Then $\leqq_1$ is standard while $\leqq_2$ is not. Taking Lebesgue measure on $\mathbf{R}^2$ as m, we see from the remarks following 1.1.6 that $L(\mathbf{R}^2, \leqq_1) \sim L(\mathbf{R}^2, \leqq_2)$ (mod m). On the other hand, we claim that if $E \subseteqq \mathbf{R}^2$ is a Borel set for which $\leqq_1$ and $\leqq_2$ agree on E, then $m(E) = 0$. For that, consider the intersection of E with any vertical line L. Now $\leqq_1$ identifies all points of $E \cap L$, while $\leqq_2$ is *strict*; since $\leqq_1$ and $\leqq_2$ agree on E we conclude that $E \cap L$ contains at most one point, and in particular has linear measure zero. The conclusion $m(E) = 0$ now follows from Fubini's theorem.

We conclude this section by describing a broad class of standard partially ordered Borel spaces.

THEOREM 1.1.12. *Let X be a separable locally compact metric space and let $\leqq$ be a partial order on X whose graph $G = \{(x, y): y \leqq x\}$ is closed in $X \times X$. Then $(X, \leqq)$ is standard in the Borel structure generated by its topology.*

Proof. Since the Borel structure of X is clearly standard, it suffices to produce a sequence $F_1, F_2, \cdots$ of increasing Borel sets such that $\chi_{F_n}(x) \leqq \chi_{F_n}(y)$, $n \geqq 1$, implies $x \leqq y$.

Since the complement of G is open we may find, for each point $(x, y) \in G$, compact neighborhoods K_x and L_y of x and y, respectively, such that $K_x \times L_y \subseteqq X \times X \backslash G$. By the Lindelöf property, there is a subsequence of $\{K_x \times L_y\}$ whose interiors cover $X \times X \backslash G$, and thus we obtain a sequence of compact rectangles $K_n \times L_n$ whose union fills out the complement of G.

Now define $F_n = L_n^+$. Since X itself is σ-compact, an argument very similar to the first part of the proof of 1.1.9 shows that F_n is σ-compact, and in particular is a Borel set. Choose $x, y \in X$ such that $x \not\leqq y$; we want to show that there is an n such that $\chi_{F_n}(x) = 1$ and $\chi_{F_n}(y) = 0$. Indeed, since $(y, x) \notin G$ there is an n such that $y \in K_n$ and $x \in L_n$. Noting that $K_n \cap L_n^+ = \varnothing$ (because $K_n \times L_n$ misses G), we see that $y \notin L_n^+$ and $x \in L_n^+$, as required. □

1.2. The subspace lattice $\mathfrak{L}(X, \leqq, m)$

By a *subspace lattice* we mean a family of self-adjoint projections on a Hilbert space which contains 0 and 1, is closed under the lattice operations $\vee$ and $\wedge$, and is closed in the strong operator topology. It is easy to see that subspace lattices are complete as lattices, and that every reflexive lattice is a subspace lattice (i.e., is strongly closed). A subspace lattice is called *commutative* if its elements all commute with each other.

Throughout this section, $(X, \leqq, m)$ will be a fixed standard partially ordered measure space. We shall introduce a commutative subspace lattice associated with $(X, \leqq, m)$ and describe its relation to the partial ordering on X. Let $\mathcal{B}(X)$ denote the σ-field of all Borel sets in X. Each set E in $\mathcal{B}(X)$ gives rise to a projection P_E acting in $L^2(X, m)$, namely the operator which multiplies functions in $L^2(X, m)$ by the characteristic function of E. The map $E \mapsto P_E$ is of course a projection-valued measure, and the family $\{P_E\colon E \in \mathcal{B}(X)\}$ is a *maximal abelian* family of projections, in the sense that it coincides with the family of all projections which commute with each of its members. In particular, $\{P_E;\ E \in \mathcal{B}(X)\}$ is closed in the strong operator topology, and is therefore a commutative subspace lattice. We define

$$\mathfrak{L}(X, \leqq, m) = \{P_E\colon E \in L(X, \leqq)\}\ .$$

Clearly $\mathfrak{L}(X, \leqq, m)$ is a sublattice of $\{P_E\colon E \in \mathcal{B}(X)\}$, which contains 0 and 1.

PROPOSITION 1.2.1. *$\mathfrak{L}(X, \leqq, m)$ is a subspace lattice.*

Proof. We have to show that $\mathfrak{L}(X, \leqq, m)$ is strongly closed. Since $L^2(X, m)$ is a separable Hilbert space, we need only prove that if P_{E_n} is a sequence in $\mathfrak{L}(X, \leqq, m)$ which converges strongly to an operator T, then $T \in \mathfrak{L}(X, \leqq, m)$.

Clearly T must be a projection, and so the preceding remarks show that there is a Borel set E such that $T = P_E$. Let ξ be any strictly positive function in $L^2(X, m)$. Then the finite measure $n(S) = (P_S\xi, \xi)$ is equivalent to m, and moreover $n(E\Delta E_n) = \|P_E\xi - P_{E_n}\xi\|^2 \to 0$ as $n \to \infty$. By passing to a subsequence of $\{E_n\}$ we can assume that $n(E\Delta E_n) \leqq 2^{-n}$, $n = 1, 2, \cdots$. Now put $E_\infty = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k$. A familiar estimate shows that $n(E\Delta E_\infty) = 0$, while clearly $E_\infty \in L(X, \leqq)$. This proves that $P_E = P_{E_\infty} \in \mathfrak{L}(X, \leqq, m)$. $\square$

The following result, closely related to the corollary of 1.1.11, will be useful below.

THEOREM 1.2.2. *Let $\mathcal{E} = \{E_1, E_2, \cdots\}$ be a sequence of Borel sets in X and let $\mathfrak{L}_{\mathcal{E}}$ be the subspace lattice generated by $\{P_E\colon E \in \mathcal{E}\}$, 0, and 1. Then*

$\mathfrak{L}_{\mathfrak{S}} = \mathfrak{L}(X, \leqq, m)$ if, and only if, there is a Borel set N of measure zero such that, for all $x, y \in X \backslash N$, $x \leqq y$ iff $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for all $n = 1, 2, \cdots$.

Proof. Define a new standard order $\leqq_2$ in X by $x \leqq_2 y$ iff $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for all $n \geqq 1$. Assume first that $\mathfrak{L}_{\mathfrak{S}} = \mathfrak{L}(X, \leqq, m)$. This clearly implies that $L(X, \leqq_2) \sim L(X, \leqq)$ (mod m), so that the conclusion follows by the corollary of 1.1.11.

Conversely, assume there is a null set N such that $\leqq$ and $\leqq_2$ agree on $X \backslash N$. The same corollary implies that $L(X, \leqq_2) \sim L(X, \leqq)$ (mod m), and hence $\mathfrak{L}(X, \leqq_2, m) = \mathfrak{L}(X, \leqq, m)$. By 1.1 itself, we conclude that $\mathfrak{L}(X, \leqq_2, m)$ is generated as a *σ-lattice* by 0, 1, and $\{P_E \colon E \in \mathfrak{S}\}$, and in particular this sequence of projections generates $\mathfrak{L}(X, \leqq_2, m) = \mathfrak{L}(X, \leqq, m)$ as a subspace lattice. □

COROLLARY 1. *$\mathfrak{L}(X, \leqq, m)$ is linearly ordered iff there is a Borel set N of measure zero such that $\leqq$ induces a linear ordering of $X \backslash N$.*

Proof. Let $P_1, P_2, \cdots$ be a sequence of projections in $\mathfrak{L}(X, \leqq, m)$ which is strongly dense in $\mathfrak{L}(X, \leqq, m)$ (here we again use separability of $L^2(X, m)$). Choose $E_n \in L(X, \leqq)$ such that $P_n = P_{E_n}$.

Assume that $\mathfrak{L}(X, \leqq, m)$ is a chain. Then we claim that the E_n's can be chosen so as to form a chain of sets. Indeed, for each n, define $F_n = \bigcup \{E_k \colon P_k \leqq P_n\}$. Clearly $F_n \in L(X, \leqq)$, $E_n \subseteqq F_n$ for every n, and $P_k \leqq P_n$ implies $F_k \subseteqq F_n$ (hence $\{F_n\}$ is a chain). Moreover $P_{F_n} = \bigvee \{P_k \colon P_k \leqq P_n\} = P_n = P_{E_n}$. The claim follows by replacing E_n with F_n.

Now $\{P_{F_n} \colon n = 1, 2, \cdots\}$ clearly generates $\mathfrak{L}(X, \leqq, m)$ as a subspace lattice, and the order $x \leqq_2 y$ iff $\chi_{F_n}(x) \leqq \chi_{F_n}(y)$ is clearly a linear ordering on X. The conclusion now follows from 1.2.2.

The converse is routine, and is left for the reader. □

COROLLARY 2. *$\mathfrak{L}(X, \leqq, m)$ is a complemented lattice iff there is a Borel set N of measure zero such that $\leqq$ induces an equivalence relation in $X \backslash N$.*

Proof. Assume that $\mathfrak{L}(X, \leqq, m)$ is complemented. Now $\mathfrak{L}(X, \leqq, m)$ is a sublattice of the Boolean algebra $\{P_E \colon E \in \mathfrak{B}(X)\}$, and recall that Boolean algebras have *unique* complements. This implies that $\mathfrak{L}(X, \leqq, m)$ is closed under the operation $P_E \mapsto P_{E^c}$, where $E^c = X \backslash E$.

As before, choose a countable strongly dense set $P_{E_1}, P_{E_2}, \cdots$ in $\mathfrak{L}(X, \leqq, m)$. Then the sequence $P_{E_1}, P_{E_1^c}, P_{E_2}, P_{E_2^c}, \cdots$ is also dense in $\mathfrak{L}(X, \leqq, m)$. Now the partial order $\leqq_2$ determined by the sets $\{E_1, E_1^c, E_2, E_2^c, \cdots\}$ is clearly an equivalence relation; indeed $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$

and $\chi_{E_n^c}(x) \leqq \chi_{E_n^c}(y)$ imply $\chi_{E_n}(x) = \chi_{E_n}(y)$, so that $x \leqq_2 y$ iff $\chi_{E_n}(x) = \chi_{E_n}(y)$ for every $n \geqq 1$. Now apply 1.2.2 as in Corollary 1.

Again, the converse is left for the reader. □

Recall that a self-adjoint family of operators $\mathcal{S}$ is called *multiplicity-free* if the commutant of $\mathcal{S}$ is abelian (see Chapter 2 of [3], for example). Since $\mathcal{L}(X, \leqq, m)$ is contained in the multiplication algebra M of (X, m) (a maximal abelian von Neumann algebra), $\mathcal{L}(X, \leqq, m)$ will be multiplicity-free iff it generates M as a von Neumann algebra. Now since the projection lattice in the von Neumann algebra generated by $\mathcal{L}(X, \leqq, m)$ is the *complete Boolean algebra* generated by $\mathcal{L}(X, \leqq, m)$ (this follows easily from the lore of abelian von Neumann algebras, and we omit the proof), it follows that *$\mathcal{L}(X, \leqq, m)$ is multiplicity-free if, and only if, $\mathcal{L}(X, \leqq, m) \cup \mathcal{L}(X, \leqq, m)^{\perp}$ generates $\{P_E: E \in \mathcal{B}(X)\}$ as a subspace lattice.* What follows is a more useful criterion.

THEOREM 1.2.3. *$\mathcal{L}(X, \leqq, m)$ is multiplicity-free if, and only if, there is a Borel set N of measure zero such that $\leqq$ induces a strict partial ordering on $X \backslash N$.*

Proof. Assume first that $\leqq$ induces a strict partial ordering on the complement of a null Borel set N. Now if $\mathcal{B}_0$ denotes the σ-field generated by $L(X, \leqq)$, then any operator which commutes with $\mathcal{L}(X, \leqq, m)$ must also commute with $\{P_E: E \in \mathcal{B}_0\}$. We will show that the latter contains all projections P_F, where F is an arbitrary Borel set. This will imply that the commutant of $\mathcal{L}(X, \leqq, m)$ is contained in the (abelian) multiplication algebra of (X, m).

Choose a sequence $E_1, E_2, \cdots$ in $L(X, m)$ such that $x \leqq y$ if and only if $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for all n. Note that $\{E_n\}$ separates points of $X \backslash N$; indeed, if $x, y \in X \backslash N$ and $\chi_{E_n}(x) = \chi_{E_n}(y)$ for all n, then $x \leqq y \leqq x$, hence $x = y$ since $\leqq$ is strict on $X \backslash N$. By [3, Theorem 3.3.5], the sequence of sets $E_n \cap (X \backslash N)$ generates the full Borel field on $X \backslash N$. Thus, for each $E \in \mathcal{B}(X)$ there is an $E_0 \in \mathcal{B}_0$ such that $E \cap (X \backslash N) = E_0 \cap (X \backslash N)$. Since N has measure zero, we conclude that $\{P_E: E \in \mathcal{B}_0\}$ contains $\{P_E: E \in \mathcal{B}(X)\}$, as required.

Conversely, suppose $\mathcal{L}(X, \leqq, m)$ is multiplicity-free. Let $E_1, E_2, \cdots \in L(X, \leqq)$ be such that $x \leqq y$ iff $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for every $n \geqq 1$. We will produce a Borel set N of measure zero such that the sets $\{E_n\}$ separate points of $X \backslash N$ (note that this implies $\leqq$ is *strict* on $X \backslash N$).

Now 1.2.2 implies that $\mathcal{L}(X, \leqq, m)$ is generated as a subspace lattice by $\{P_{E_1}, P_{E_2}, \cdots\}$. By the remarks preceding this theorem we know that

$\mathfrak{L}(X, \leqq, m) \cup \mathfrak{L}(X, \leqq, m)^{\perp}$ generates $\{P_E\colon E \in \mathfrak{B}(X)\}$ as a subspace lattice. Thus, $\{P_E\colon E \in \mathfrak{B}(X)\}$ is generated by the sequence $\{P_{E_1}, P_{E_1^c}, P_{E_2}, P_{E_2^c}, \cdots\}$. We again apply 1.2.2, but this time to the (standard) ordering $x \leqq_1 y$ iff $x = y$, and to the sequence $\mathcal{E} = \{E_1, E_1^c, E_2, E_2^c, \cdots\}$. We conclude that there is a Borel set N of measure zero such that, for $x, y \in X\backslash N$, $\chi_{E_n}(x) = \chi_{E_n}(y)$ for every $n \geqq 1$ implies $x \leqq_1 y$, i.e., $x = y$. This proves that $\{E_n\}$ separates points in $X\backslash N$. □

Two partially ordered measure spaces $(X, \leqq, m)$ and $(Y, \leqq, n)$ are said to be *isomorphic* if there are Borel sets, $M \subseteq X$, $N \subseteq Y$, of measure zero respectively, and an order isomorphism $\phi\colon (X\backslash M, \leqq) \rightarrow (Y\backslash N, \leqq)$ such that $m\phi^{-1}$ and n induce equivalent measures on $Y\backslash N$.

THEOREM 1.2.4. *Let $(X, \leqq, m)$ and $(Y, \leqq, n)$ be standard partially ordered measure spaces. If $(X, \leqq, m)$ and $(Y, \leqq, n)$ are isomorphic, then there is a unitary operator $U\colon L^2(X, m) \rightarrow L^2(Y, n)$ such that $U\mathfrak{L}(X, \leqq, m)U^{-1} = \mathfrak{L}(Y, \leqq, n)$.*

Conversely, if $\mathfrak{L}(X, \leqq, m)$ and $\mathfrak{L}(Y, \leqq, n)$ are unitarily equivalent and, say, $\mathfrak{L}(X, \leqq, m)$ is multiplicity-free, then $(X, \leqq, m)$ and $(Y, \leqq, n)$ are isomorphic.

Proof. Let $M \subseteq X$, $N \subseteq Y$ and $\phi\colon X\backslash M \rightarrow Y\backslash N$ satisfy the conditions for an isomorphism of $(X, \leqq, m)$ on $(Y, \leqq, n)$.

Let m_0 be the restriction of m to $X\backslash M$, and let $\leqq_0$ be the restriction of the partial order $\leqq$ (of X) to $X\backslash M$. We claim first that $\mathfrak{L}(X, \leqq, m)$ and $\mathfrak{L}(X\backslash M, \leqq_0, m_0)$ are unitarily equivalent. Indeed, the restriction map $f \mapsto f_{X\backslash M}$ defines a unitary map V of $L^2(X, m)$ onto $L^2(X\backslash M, m_0)$ (because M has measure zero), and for every multiplication operator L_f (where f is a bounded Borel function on X) we have $VL_fV^{-1} = L_g$, where $g = f|_{X\backslash M}$. Let $E_1, E_2, \cdots \in L(X, \leqq)$ be such that $\{P_{E_1}, P_{E_2}, \cdots\} \cup \{0, 1\}$ generates $\mathfrak{L}(X, \leqq, m)$ as a subspace lattice. Then it suffices to show that $\{VP_{E_1}V^{-1}, VP_{E_2}V^{-1}, \cdots\} \cup \{0, 1\}$ generates $\mathfrak{L}(X\backslash M, \leqq_0, m_0)$ as a subspace lattice. Now $VP_{E_n}V^{-1}$ is multiplication by the characteristic function of $E_n \cap (X\backslash M)$; so the desired conclusion now follows by an obvious application of 1.2.2 to both sequences $\{E_n\}$ (in (X, m)) and $\{E_n \cap (X\backslash M)\}$ (in $(X\backslash M, m_0)$).

By symmetry $\mathfrak{L}(Y, \leqq, n)$ is unitarily equivalent to the "relativized" lattice $\mathfrak{L}(Y\backslash N, \leqq_0, n_0)$, so we are reduced to proving that $\mathfrak{L}(X\backslash M, \leqq_0, m)$ and $\mathfrak{L}(Y\backslash N, \leqq_0, n_0)$ are unitarily equivalent; equivalently, modulo a change in notation, we may assume that M and N are empty, and ϕ is an order isomorphism of $(X, \leqq)$ on $(Y, \leqq)$ for which $m\phi^{-1} \sim n$.

Let w be the Radon-Nikodym derivative $(dm\phi^{-1}/dn)$. Then by a familiar change-of-variables formula, the mapping $Uf(y) = w(y)^{1/2}f(\phi^{-1}y)$ defines a unitary operator from $L^2(X, m)$ onto $L^2(Y, n)$. Moreover, for every bounded Borel function f on X one has $UL_f U^{-1} = L_{f\phi^{-1}}$. Applying 1.1.7 to the inverse map $\phi^{-1}: Y \to X$, we see that the map $E \subseteq X \mapsto \phi(E) \subseteq Y$ carries $L(X, \leqq)$ onto $L(Y, \leqq)$; it follows that $U\mathfrak{L}(X, \leqq, m)U^{-1} = \mathfrak{L}(Y, \leqq, m)$, as required.

Conversely, suppose there is a unitary operator U such that $U\mathfrak{L}(X, \leqq, m)U^{-1} = \mathfrak{L}(Y, \leqq, n)$, and assume one of the two lattices (and therefore both of them) is multiplicity-free. Let $\mathcal{P}(X, m)$ denote the family $\{P_E: E \in \mathcal{B}(X)\}$ of all projections in the multiplication algebra of $L^2(X, m)$, and define $\mathcal{P}(Y, n)$ similarly. By the remarks preceding 1.2.3, $\mathcal{P}(X, m)$ and $\mathcal{P}(Y, n)$ are, respectively, the subspace lattices generated by $\mathfrak{L}(X, \leqq, m) \cup \mathfrak{L}(X, \leqq, m)^{\perp}$ and $\mathfrak{L}(Y, \leqq, n) \cup \mathfrak{L}(Y, \leqq, n)^{\perp}$. It follows that $U\mathcal{P}(X, m)U^{-1} = \mathcal{P}(Y, n)$.

Now, in the terminology of [22], both $\mathcal{P}(X, m)$ and $\mathcal{P}(Y, n)$ are standard σ-Boolean algebras, and the map $P \to UPU^{-1}$ is an isomorphism of $\mathcal{P}(X, m)$ onto $P(Y, n)$. Assume first that both spaces X and Y are uncountable. Then from the argument on p. 130 of [22], it follows that there is a Borel isomorphism $\phi: X \to Y$ such that $UP_E U^{-1} = P_{\phi(E)}$, for every Borel set $E \subseteq X$.

First, note that the measures n and $m\phi^{-1}$ are equivalent. Indeed, for every Borel set $F \subseteq Y$, we have $n(F) = 0$ iff $P_F = 0$ iff $P_{\phi^{-1}(F)} = U^{-1}P_F U = 0$ iff $m\phi^{-1}(F) = 0$.

Next, choose a sequence $E_1, E_2, \cdots$ in $L(X, \leqq)$ which generates $L(X, \leqq)$ as a σ-lattice. By 1.2.2, $(P_{E_1}, P_{E_2}, \cdots\}$ generates $\mathfrak{L}(X, \leqq, m)$ as a subspace lattice, and therefore $\{P_{\phi(E_1)}, P_{\phi(E_2)}, \cdots\} = U\{P_{E_1}, P_{E_2}, \cdots\}U^{-1}$ generates $\mathfrak{L}(Y, \leqq, n) = U\mathfrak{L}(X, \leqq, m)U^{-1}$ as a subspace lattice. Applying 1.2.2 again, there is a Borel set $N \subseteq Y$ of n-measure zero such that, on $Y\backslash N$, $w \leqq z$ iff $\chi_{\phi(E_n)}(w) \leqq \chi_{\phi(E_n)}(z)$ for every $n = 1, 2, \cdots$. That is, $w \leqq z$ iff $\phi^{-1}(w) \leqq \phi^{-1}(z)$, for all $w, z \in Y\backslash N$. Now define $M \subseteq X$ by $M = \phi^{-1}(N)$. Then $m(M) = 0$ (because $n(N) = 0$ and $m\phi^{-1}$ is equivalent to n), and the preceding shows that ϕ induces an order isomorphism of $(X\backslash M, \leqq)$ onto $(Y\backslash N, \leqq)$, completing the proof in case both X and Y are uncountable.

If, say, X is countable, then m is an atomic measure (it follows from this that n is also atomic), and the conclusion follows from a rather trivial adaptation of the preceding argument, which we leave for the reader. □

Remarks. The preceding two theorems show that, at least in the case of *strict* partial orderings, the subspace lattices $\mathfrak{L}(X, \leqq, m)$ are classified to unitary equivalence in terms of the isomorphism type of their associated partially ordered measure spaces $(X, \leqq, m)$. This conclusion (and related ones) will be of considerable use for technical purposes. However, it should not be regarded as a "classification theorem" for commutative subspace lattices.

To see why, consider the case where $(X, \leqq) = (Y, \leqq)$ are both the unit interval with its usual order. Let m be Lebesgue measure, and let n be any nonatomic probability measure, singular relative to m. Then $\mathcal{L}(X, \leqq, m)$ and $\mathcal{L}(Y, \leqq, n)$ are both totally ordered, and one might hope to show that they are unitarily equivalent by exhibiting an isomorphism of $(X, \leqq, m)$ onto $(Y, \leqq, n)$ (that $\mathcal{L}(X, \leqq, m)$ and $\mathcal{L}(Y, \leqq, n)$ *are* unitarily equivalent follows from some work of Kadison and Singer [17]). However, while the identity map defines an order isomorphism between $(X, \leqq)$ and $(Y, \leqq)$, it does *not* give rise to an isomorphism of the partially ordered measure spaces $(X, \leqq, m)$ and $(Y, \leqq, n)$ because of the singularity of n. Indeed, there is no "obvious" isomorphism of these spaces, and in particular, 1.2.4 does not readily lead to an alternate proof that $\mathcal{L}(X, \leqq, m)$ and $\mathcal{L}(Y, \leqq, n)$ are unitarily equivalent.

Here is an example of a more typical classification problem. As in the preceding section, let $\mathbf{2}^\infty$ be the Cantor space of all sequences (x_i) of zeros and ones, and define $(x_i) \leqq (y_i)$ to mean $x_i \leqq y_i$ for every $i = 1, 2, \cdots$. For each real number p, $0 < p < 1$, let m_p be the infinite product measure $m_0 \times m_0 \times \cdots$, where m_0 assigns mass p to $\{1\}$ and mass $1 - p$ to $\{0\}$. Since $\leqq$ is strict, we see from 1.2.3 that the subspace lattice $\mathcal{L}(\mathbf{2}, \leqq, m_p)$ is multiplicity-free, $0 < p < 1$. One might guess that $(\mathbf{2}^\infty, \leqq, m_p)$ and $(\mathbf{2}^\infty, \leqq, m_q)$ are isomorphic for all p, q (and so the corresponding subspace lattices would be unitarily equivalent). But again, while the identity map defines an order isomorphism of $(\mathbf{2}^\infty, \leqq)$ onto itself, it does not give rise to an isomorphism of $(\mathbf{2}^\infty, \leqq, m_p)$ and $(\mathbf{2}^\infty, \leqq, m_q)$ when $p \neq q$; because in that case, the measures m_p and m_q are mutually singular. The solution of this type of classification problem requires a deeper analysis of the structure of subspace lattices, and will be taken up later in Chapter 3. In particular, it turns out that $\mathcal{L}(\mathbf{2}^\infty, \leqq, m_p)$ and $\mathcal{L}(\mathbf{2}^\infty, \leqq, m_q)$ are *not* unitarily equivalent (or even isomorphic as abstract lattices) when $p \neq q$ (see the discussion following 3.5.3).

1.3. A spectral theorem for commutative subspace lattices

In this brief section we show that, given a commutative subspace lattice $\mathcal{L}$ on a separable Hilbert space $\mathcal{H}$, it is always possible to introduce coordinates for $\mathcal{H}$ in such a way that $\mathcal{L}$ becomes the lattice $\mathcal{L}(X, \leqq, m)$ associated with a standard partially ordered measure space $(X, \leqq, m)$. This is an appropriate analogue of the abstract spectral theorem, which asserts that every commutative C^*-algebra of operators on a Hilbert space is unitarily equivalent to an algebra of multiplication operators acting on L^2 of some measure space.

A partial ordering $\leqq$ of a topological space X is said to be *closed* if the graph of $\leqq$, $G = \{(y, x) \in X \times X: x \leqq y\}$, is a closed subset of the Cartesian product $X \times X$. Recall that a closed partial ordering of a locally compact separable metric space is always standard (1.1.12).

THEOREM 1.3.1. *For every separably acting commutative subspace lattice $\mathcal{L}$, there is a compact metric space X, a closed partial order $\leqq$ on X, and a finite Borel measure m on X such that $\mathcal{L}$ is unitarily equivalent to $\mathcal{L}(X, \leqq, m)$.*

Proof. Let $\mathcal{H}$ be the underlying Hilbert space. Because $\mathcal{H}$ is separable, $\mathcal{L}$ contains a strongly dense sequence $\{P_1, P_2, \cdots\}$. Now imbed $\mathcal{L}$ in a maximal abelian von Neumann algebra $\mathcal{R}$, and choose a countable strongly dense subset $\{Q_1, Q_2, \cdots\}$ of the projection lattice of $\mathcal{R}$ (we may also assume that $\{Q_n\}$ contains 1). Let $\mathcal{A}$ be the C^*-algebra generated by $\{P_n\} \cup \{Q_m\}$. Then $\mathcal{A}$ is a commutative C^*-algebra with identity, which is separable in its norm topology, and which is strongly dense in $\mathcal{R}$. Finally, because $\mathcal{R}$ is maximal abelian and $\mathcal{H}$ is separable, there is a unit vector $\xi \in \mathcal{H}$ such that $[\mathcal{R}\xi] = \mathcal{H}$ (see, for example, the corollary of 4.1.4 in [3] for a discussion of this familiar result).

Let X be the spectrum of $\mathcal{A}$. Then X is a compact Hausdorff space, and is second-countable because $\mathcal{A} \cong C(X)$ is norm-separable. By a standard theorem of Urysohn, X is metrizable. Let $\pi: C(X) \to \mathcal{A}$ be the inverse Gelfand map. Then there are sequences $\{E_n\}$, $\{F_n\}$ of closed and open sets in X such that $\pi(\chi_{E_n}) = P_n$ and $\pi(\chi_{F_n}) = Q_n$, $n = 1, 2, \cdots$. Define a partial order $\leqq$ in X to mean $x \leqq y$ if and only if $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for every $n \geqq 1$. Note that the graph $G = \{(y, x): x \leqq y\}$ is closed in $X \times X$ because its complement can be expressed as a union $\bigcup_{n=1}^{\infty} (X \backslash E_n) \times E_n$ of open rectangles. Thus, $\leqq$ is a closed partial ordering of X.

By the Riesz-Markov theorem, there is a Borel probability measure m on X such that

$$\int_X f\,dm = (\pi(f)\xi, \xi)$$

for every $f \in C(X)$. Define a linear map U_0 of $C(X)$ into $\mathcal{H}$ by $U_0: f \to$

$\pi(f)\xi$. By definition of m, it follows that $\| U_0 f \|^2 = \int_X |f|^2 \, dm$, so that U_0 extends uniquely to an isometry U of $L^2(X, m)$ onto $[\mathfrak{A}\xi]$. Because $\mathfrak{A}$ is strongly dense in $\mathcal{R}$ and $[\mathcal{R}\xi] = \mathcal{H}$, it follows that U is unitary. Thus, it remains to show that $U\mathfrak{L}(X, \leqq, m)U^{-1} = \mathfrak{L}$.

Now by definition of U it follows (after a routine calculation) that $UL_f U^{-1} = \pi(f)$ for every $f \in C(X)$, where L_f as usual denotes "multiplication by f". In particular, letting $P(E)$ be multiplication by the characteristic function of $E \subseteqq X$, we see that $UP(E_n)U^{-1} = \pi(\chi_{E_n}) = P_n$, $n = 1, 2, \cdots$. Now by 1.2.2, $\mathfrak{L}(X, \leqq, m)$ is generated as a subspace lattice by $\{P(E_1), P(E_2), \cdots\}$. Thus, $U \cdot U^{-1}$ carries $\mathfrak{L}(X, \leqq, m)$ onto the subspace lattice generated by $\{P_1, P_2, \cdots\}$. Since the latter is clearly $\mathfrak{L}$, we are done. □

Remarks. Since $\mathfrak{L}(X, \leqq, m)$ is isomorphic to the image of $L(X, \leqq)$ (a σ-lattice of sets) in the measure algebra of (X, m), it it "essentially" a lattice of sets. Thus, one may regard this representation theorem as an appropriate analogue, for subspace lattices, of a representation theorem in lattice theory which asserts that every distributive lattice is isomorphic to a lattice of sets ([8, p. 140]; also, see Section 3.4).

There is also a connection with ordinal products which seems worth mentioning. Recall that the *ordinal product* $Y^{\check{X}}$ of two partially ordered sets $(Y, \leqq)$ and $(X, \leqq)$ is defined as the subset of the Cartesian product Y^X consisting of all functions $f: X \to Y$ which are *increasing* in the sense that $x \leqq y$ implies $f(x) \leqq f(y)$. The partial ordering on $Y^{\check{X}}$ is defined coordinatewise: $f \leqq g$ iff $f(x) \leqq g(x)$ for all $x \in X$. Taking $Y = \mathbf{2}$, we see that the map $\chi_E \mapsto E$ identifies $\mathbf{2}^{\check{X}}$ with the lattice of all increasing subsets of X. Now if $(X, \leqq)$ and $(Y, \leqq)$ are partially ordered Borel spaces, we may define an ordinal product $Y^{\check{X}}$ in a similar way except that now we only admit *Borel functions* $f: X \to Y$ as points of $Y^{\check{X}}$. Note that $\mathbf{2}^{\check{X}}$ is now identified as the σ-lattice $L(X, \leqq)$. So given any σ-finite measure m on X, we may define an equivalence relation $\sim$ in $\mathbf{2}^{\check{X}}$ by $f \sim g$ iff $f(x) = g(x)$ almost everywhere (dm). Thus, the subspace lattice $\mathfrak{L}(X, \leqq, m)$ is naturally isomorphic with the quotient of the ordinal product $\mathbf{2}^{\check{X}}$ by the equivalence relation $\sim$.

1.4 The null set theorem

Until now, we have been concerned with the problem of representing certain subspace lattices as the lattice $\mathfrak{L}(X, \leqq, m)$ associated with a partially ordered measure space. In the remainder of this chapter we will take up the problem of constructing "enough" operators in alg $\mathfrak{L}(X, \leqq, m)$ to prove that $\mathfrak{L}(X, \leqq, m)$ is reflexive. Ultimately, the solution of this problem (as well as various problems of Chapter 2 concerning reflexive algebras) depends in an essential way on a theorem which characterizes certain types of null sets in the Cartesian product of two measure spaces. The present section is devoted to a discussion and proof of this measure-theoretic result.

Let (X, m) and (Y, n) be two measure spaces, and let S be a subset of the Cartesian product $X \times Y$. S is called *marginally null* if there are Borel sets $M \subseteqq X$ and $N \subseteqq Y$ such that

(i) $m(M) = n(N) = 0$, and

(ii) $S \subseteqq M \times Y \cup X \times N$.

It is clear that the marginally null subsets of $X \times Y$ form a hereditary σ-ideal of sets. We will say that two functions $f, g: X \times Y \to \mathbf{C}$ agree *marginally almost everywhere* (abbreviated m.a.e.) if $\{(x, y): f(x, y) \neq g(x, y)\}$ is a marginally null set. Thus, $f(x, y) = 0$ (m.a.e.) iff one can modify f in each variable *separately*, on sets of respective measure zero, so as to make f vanish identically.

We first want to point out a simple description of the class of measures on $X \times Y$ which annihilate the class of all marginally null sets. Every finite positive measure σ on $X \times Y$ (the domain of σ is of course the product sigma-field on $X \times Y$) gives rise to measures σ_1, σ_2 on X and Y respectively, defined by

$$\sigma_1(E) = \sigma(E \times Y) ,$$
$$\sigma_2(F) = \sigma(X \times F) ,$$

where E (resp. F) is an arbitrary Borel set in X (resp. Y). These measures σ_i will be called the *marginal* measures of σ, by analogy with the marginal distributions of pairs of random variables in probability theory.

PROPOSITION 1.4.1. *Let (X, m) and (Y, n) be two measure spaces, and let σ be a finite positive measure on $X \times Y$. Then $\sigma(S) = 0$ for every marginally null Borel set $S \subseteqq X \times Y$ if, and only if, $\sigma_1 \ll m$ and $\sigma_2 \ll n$.*

Proof. Suppose first that $\sigma_1 \ll m$ and $\sigma_2 \ll n$. Let S be marginally null. Then there are Borel sets $M \subseteqq X$, $N \subseteqq Y$ such that $S \subseteqq M \times Y \cup X \times N$ and $m(M) = n(N) = 0$. Then $\sigma(S) \leqq \sigma(M \times Y) + \sigma(X \times N) = \sigma_1(M) + \sigma_2(N) = 0 + 0 = 0$. The converse is equally trivial. □

In regard to this class of measures, note first that any finite positive measure σ on $X \times Y$, which is absolutely continuous with respect to the product measure $m \times n$, clearly satisfies $\sigma_1 \ll m$ and $\sigma_2 \ll n$. As a more typical example, let $X = Y = [0, 1]$ and take $m = n$ to be Lebesgue measure. Define σ on the square $X \times Y$ to be $\sigma(S) = m\{x \in X: (x, x) \in S\}$. Then $\sigma_1 = \sigma_2 = m$, while on the other hand σ is concentrated on the diagonal $\{(x, x): x \in X\}$, a set of product measure zero. This example and variants of it show that the structure of this class of measures is very complex.

Now fix $S \subseteq X \times Y$. Then 1.4.1 implies that, if S is marginally null, then $\sigma(S) = 0$ for every finite positive measure σ on $X \times Y$ having absolutely continuous marginals. The main result of this section gives a converse of that assertion, and can be regarded as a type of duality theorem for marginally null sets. We begin by recalling some terminology. Let A be a linear subspace of the real Banach space $C(X)$ of all *real valued* continuous functions on a compact Hausdorff space X. We will always assume that A contains the constants. Let ϕ be a state of A, that is, a linear functional on A, satisfying $\|\phi\| = \phi(1) = 1$; then M_ϕ will denote the set of all norm-preserving linear extensions of ϕ to $C(X)$. M_ϕ is a nonvoid convex subset of the dual of $C(X)$ which is compact in the weak* topology. Moreover, the Riesz-Markov theorem allows us to identify M_ϕ with the set of all *representing measures* of ϕ, namely, all regular Borel probability measures μ on X which satisfy

$$\phi(f) = \int_X f\,d\mu\,, \qquad f \in A\,.$$

Finally, for such a measure μ, we will make the traditional abuse of notation by using the same symbol for the linear functional determined by μ:

$$\mu(f) = \int f\,d\mu\,, \qquad f \in C(X)\,.$$

The following lemma is a minor variation of the result on page 110 of [14]. We sketch the proof for completeness.

LEMMA. *Let X be a compact Hausdorff space, let $A \subseteq C(X)$ be a linear space containing* 1, *and let ϕ be a state of A. Then for every $u \in C(X)$*:

$$\sup\{\sigma(u): \sigma \in M_\phi\} = \inf\{\phi(f): f \in A,\ f \geqq u\}\,.$$

Proof. The inequality $\leqq$ is trivial, for if $\sigma \in M_\phi$ and $f \in A$ dominates u, then $\sigma(u) \leqq \sigma(f) = \phi(f)$. For $\geqq$, let α denote the right side of the asserted equation. We want to construct $\sigma \in M_\phi$ for which $\sigma(u) = \alpha$. Define σ_0 on $\{f + tu: f \in A, t \in \mathbf{R}\}$ by $\sigma_0(f + tu) = \phi(f) + t\alpha$. The inequality already proved implies that $\sigma_0 \geqq 0$, so an extension theorem of M. G. Krein

([25, p. 227]) provides a positive linear extension σ of σ_0 to $C(X)$. Clearly $\sigma \in M_\phi$, and $\sigma(u) = \alpha$ by construction. □

THEOREM 1.4.2. *Let X, Y be compact Hausdorff spaces, let m, n be regular Borel probability measures on X and Y respectively, and let K be a closed subset of $X \times Y$. Assume $\sigma(K) = 0$ for every regular Borel probability measure σ on $X \times Y$ satisfying $\sigma_1 = m$ and $\sigma_2 = n$. Then K is marginally null.*

Proof. Let $\mathcal{F} = \{u \in C(X \times Y): \chi_K \leqq u\}$. Then $\mathcal{F}$ is a decreasing directed subset of $C(X)$ in the usual order, and every regular Borel probability measure σ on $X \times Y$ gives rise to a net $u \in \mathcal{F} \mapsto \int u\, d\sigma$. Note first that, for fixed σ, $\lim_u \int u\, d\sigma = \sigma(K)$. Indeed, the limit on the left exists because the net is monotonic, and clearly $\int u\, d\sigma \geqq \int \chi_K\, d\sigma = \sigma(K)$ for every $u \in \mathcal{F}$. On the other hand, for each $\varepsilon > 0$ we can find (by regularity of σ) an open set G containing K for which $\sigma(G) \leqq \sigma(K) + \varepsilon$. Urysohn's lemma provides a function $u \in C(X \times Y)$ with $u = 1$ on K, $u = 0$ off G, and $0 \leqq u \leqq 1$ in between. Thus $u \in \mathcal{F}$ and

$$\int u\, d\sigma \leqq \int \chi_G\, d\sigma \leqq \sigma(K) + \varepsilon ,$$

proving the claim.

Now let $A \subseteqq C(X \times Y)$ be the space of all functions of the form $h(x, y) = f(x) + g(x)$, $f \in C(X)$, $g \in C(Y)$, and let ϕ be the state of A defined by $\phi(h) = \int h\, dm \times n$. Noting that M_ϕ consists of all regular Borel probability measures σ on $X \times Y$ satisfying $\sigma_1 = m$ and $\sigma_2 = n$, we see from the hypothesis that $\sigma(K) = 0$ for every $\sigma \in M_\phi$. From the preceding paragraph we conclude that $\lim_u \int u\, d\sigma = 0$ for every $\sigma \in M_\phi$. In fact, this convergence to 0 is *uniform* over M_ϕ; for since M_ϕ is weak* compact and since the net $\hat{u}(\sigma) = \int u\, d\sigma$ of continuous functions on M_ϕ converges monotonically to zero at every point of M_ϕ, Dini's theorem implies that $\sup_{\sigma \in M_\phi} \int u\, d\sigma$ tends to 0 as $u \in \mathcal{F}$ decreases to χ_K.

Hence we may find a sequence $u_1, u_2, \cdots$ in $C(X \times Y)$ satisfying the following conditions:

(i) $\chi_K \leqq u_{k+1} \leqq u_k$,

(ii) $\sup_{\sigma \in M_\phi} \int u_k\, d\sigma \leqq 2^{-k-2}$.

Now apply the lemma to obtain a sequence $h_k(x, y) = f_k(x) + g_k(y)$ in A with $u_k \leqq h_k$ and

$$\int f_k\,dm + \int g_k\,dn = \rho(h_k) \leqq 2^{-k-1}\ .$$

Put $a_k = \inf_x f_k(x)$, $b_k = \inf_y g_k(y)$, $f'_k = f_k - a_k$, $g'_k = g_k - b_k$. Then f'_k and g'_k are nonnegative, and we claim $0 \leqq a_k + b_k \leqq 2^{-k-1}$. Indeed, choosing $x_k \in X$ and $y_k \in Y$ such that $a_k = f(x_k)$ and $b_k = g(y_k)$ we see that

$$0 \leqq \chi_k(x_k, y_k) \leqq u_k(x_k, y_k) \leqq a_k + b_k \leqq f(x) + g(y)\ ,$$

for all $x \in X$, $y \in Y$, so the claim follows by integrating these inequalities against the probability measure $m \times n$. We conclude that $\chi_K(x, y) \leqq f'_k(x) + g'_k(y) + 2^{-k-1}$, and the nonnegative functions f'_k, g'_k satisfy

$$\int f'_k\,dm, \int g'_k\,dn \leqq \int f'_k\,dm + \int g'_k\,dn$$
$$= \int f_k\,dm + \int g_k\,dn - (a_k + b_k) \leqq 2^{-k-1}\ .$$

Now define nonnegative functions $F_k \in L^1(X, m)$ and $G_k \in L^1(Y, n)$ by $F_k = \sum_{l=k}^{\infty} f'_l$, $G_k = \sum_{l=k}^{\infty} g'_l$. Then we have

(iii) $\int_X F_k\,dm, \int_Y G_k\,dn \leqq 2^{-k}$,

(iv) $F_k \geqq F_{k+1}$, $G_k \geqq G_{k+1}$,

(v) $\chi_K(x, y) \leqq F_k(x) + G_k(y) + 2^{-k}$.

By (iv), the nonnegative limit functions $F(x) = \lim_k F_k(x)$ and $G(y) = \lim_k G_k(y)$ exist, and by (iii) and the monotone convergence theorem we have $\int F\,dm = \int G\,dn = 0$. In particular, if $M = \{x \in X: F(x) \neq 0\}$ and $N = \{y \in Y: G(y) \neq 0\}$, then $m(M) = 0$ and $n(N) = 0$. On the other hand, (v) implies that $\chi_K(x, y) \leqq F(x) + G(y)$, from which we conclude that $K \subseteqq M \times Y \cup X \times N$, as required. □

Problem. A natural question here is whether this theorem is valid for more general subsets K of $X \times Y$. In particular, if K is an *arbitrary* Borel set in $X \times Y$ for which $\sigma(K) = 0$ for every regular Borel probability measure σ with $\sigma_1 = m$ and $\sigma_2 = n$, does it follow that K is marginally null? The theorem itself implies in this case that every compact subset of K is marginally null, so that the answer is yes if K is sigma-compact. The proof of the following theorem asserts that the answer is yes (in the metrizable case) provided the complement of K can be expressed as a countable union of Borel rectangles; that includes the case where K is closed, and is quite adequate for our purposes here. However, the answer for arbitrary Borel sets K is unknown to us.

NULL SET THEOREM 1.4.3. *Let X, Y be standard Borel spaces and let μ, ν be sigma-finite measures on X and Y respectively. Let $\mathcal{A}$ be the class* *of all Borel probability measures σ on $X \times Y$ for which there is a positive constant $c = c_\sigma$ such that $\sigma_1 \leqq c\mu$ and $\sigma_2 \leqq c\nu$.*

Let $S \subseteqq X \times Y$ have the form $S = \bigcap_{n=1}^\infty S_n$, where each S_n belongs to the Boolean algebra generated by all Borel rectangles $E \times F$, $E \subseteqq X$, $F \subseteqq Y$. If $\sigma(S) = 0$ for every $\sigma \in \mathcal{A}$, then S is marginally null.

Proof. We may clearly assume that neither μ nor ν is zero. Because μ is sigma-finite, an elementary argument shows that there is a Borel function $w: X \to \mathbf{R}$ such that $o < w(x) \leqq c < \infty$ for all x, and $\int w\, d\mu = 1$. Thus $m(E) = \int_E w(x)\, d\mu(x)$ defines a probability measure on X such that $\mu \ll m \leqq c\mu$. Similarly, we may find a probability measure n on Y and a positive constant c' such that $\nu \ll n \leqq c'\nu$. Let $\mathcal{A}_0$ be the class of all probability measures σ on $X \times Y$ for which $\sigma_1 = m$ and $\sigma_2 = n$. Then $\mathcal{A}_0 \subseteqq \mathcal{A}$, so it suffices to prove the following assertion, for $S \subseteqq X \times Y$ as above: if $\sigma(S) = 0$ for every $\sigma \in \mathcal{A}_0$, then S is marginally null (relative to the measures m, n).

First, write $S = \bigcap_n S_n$, where each S_n belongs to the Boolean algebra generated by Borel rectangles $E \times F$, $E \subseteqq X$, $F \subseteqq Y$. Now for each n, $X \times Y \backslash S_n$ is a finite union of Borel rectangles, and thus we may find sequences $E_1, E_2, \cdots \subseteqq X$, $F_1, F_2, \cdots \subseteqq Y$ of Borel sets such that $X \times Y \backslash S = \bigcup_{k=1}^\infty E_k \times F_k$. We now map $X \times Y$ into the Cantor square $\mathbf{2}^\infty \times \mathbf{2}^\infty$ as follows. Choose sequences $A_n \subseteqq X$, $B_n \subseteqq Y$ of Borel sets such that $\{A_n\}$ (resp. $\{B_n\}$) generates the Borel structure of X (resp. Y). Define $f: X \to \mathbf{2}^\infty$ and $g: Y \to \mathbf{2}^\infty$ by

$$f(x) = (\chi_{A_1}(x), \chi_{E_1}(x), \chi_{A_2}(x), \chi_{E_2}(x), \cdots),$$
$$g(y) = (\chi_{B_1}(y), \chi_{F_1}(y), \chi_{B_2}(y), \chi_{F_2}(y), \cdots).$$

Then (see [3, Theorem 3.3.4 and its Corollary 2]) $f(X)$ (resp. $g(Y)$) is a Borel set of $\mathbf{2}^\infty$ and f (resp. g) is a Borel isomorphism of X (resp. Y) onto its range. Note also that the product mapping $f \times g: X \times Y \to f(X) \times g(Y) \subseteqq \mathbf{2}^\infty \times \mathbf{2}^\infty$ carries S onto $K \cap (f(X) \times g(Y))$, where $K \subseteqq \mathbf{2}^\infty \times \mathbf{2}^\infty$ is the set consisting of all ordered pairs of sequences (ξ, η) satisfying $\xi_{2n} = 0$ or $\eta_{2n} = 0$ for every $n = 1, 2, \cdots$. Finally note that K is closed (and therefore compact) in the usual product topology on $\mathbf{2}^\infty \times \mathbf{2}^\infty$.

Now define probability measures p, q on $\mathbf{2}^\infty$ by $p = mf^{-1}$ and $q = ng^{-1}$. Let σ be a Borel probability measure on $\mathbf{2}^\infty \times \mathbf{2}^\infty$ such that $\sigma_1 = p$ and $\sigma_2 = q$. We claim: $\sigma(K) = 0$. Indeed, since $p(f(X)) = m(X) = 1$ we see that $\sigma(f(X) \times \mathbf{2}^\infty) = \sigma_1(f(X)) = 1$ and similarly $\sigma(\mathbf{2}^\infty \times g(Y)) = 1$. This implies that σ is concentrated on

$$f(X) \times g(Y) = (\mathbf{2}^\infty \times g(Y)) \cap (f(X) \times \mathbf{2}^\infty) .$$

Now define a positive measure σ' on $X \times Y$ by $\sigma'(A) = \sigma(f \times g(A))$. Then for $E \subseteq X$,

$$\sigma_1'(E) = \sigma'(E \times Y) = \sigma(f(E) \times g(Y)) = \sigma(f(E) \times \mathbf{2}^\infty) = p(f(E)) = m(E) .$$

Similarly, $\sigma_2'(F) = n(F)$ for $F \subseteq Y$. Therefore $\sigma' \in \mathcal{C}_0$, so by the hypothesis on S we have $\sigma'(S) = 0$. In terms of σ, we conclude that

$$\sigma(K) = \sigma(K \cap (f(X) \times g(Y))) = \sigma(f(S)) = \sigma'(S) = 0 ,$$

as asserted.

The compact set $K \subseteq \mathbf{2}^\infty \times \mathbf{2}^\infty$ therefore satisfies the hypotheses of 1.4.2 (relative to the measures p and q), and we conclude that there are Borel sets M, $N \subseteq \mathbf{2}^\infty$ such that $p(M) = q(N) = 0$, and $K \subseteq M \times \mathbf{2}^\infty \cup \mathbf{2}^\infty \times N$. Pulling this back under $(f \times g)^{-1}$, we obtain $S \subseteq f^{-1}(M) \times Y \cup X \times g^{-1}(N)$, where $m(f^{-1}(M)) = p(M) = 0$ and $n(g^{-1}(N)) = q(N) = 0$. Thus S is marginally null, and we are done. □

1.5. Pseudo integral operators

Let (X, m) be a positive measure space. Then certain Borel functions $k: X \times X \to \mathrm{C}$ give rise to (bounded) integral operators T_k on the L^p spaces of (X, m), defined by

$$T_k f(x) = \int_X k(x, y) f(y) m(dy) , \qquad x \in X$$

(it will be covenient in this section to use the notation $m(dy)$ rather than $dm(y)$, in such integrals). For instance, if $k \in L^2(X \times X, m \times m)$, then T_k defines a Hilbert-Schmidt operator on $L^2(X, m)$. In this section we want to introduce a considerably broader class of operators associated with measures, rather than functions, defined on $X \times X$.

Let (X, m), (Y, n) be two positive measures spaces, and give $X \times Y$ the product Borel structure. For every complex-valued Borel measure μ on $X \times Y$ of finite total variation, let $|\mu|(E)$ denote the variation of μ on the Borel set $E \subseteq X \times Y$. Then $|\mu|$ is a finite positive Borel measure and, in turn, we obtain marginal measures $|\mu|_1$, $|\mu|_2$ on X and Y, defined by $|\mu|_1(E) = |\mu|(E \times Y)$ and $|\mu|_2(F) = |\mu|(X \times F)$. $A(X \times Y, m, n)$ will denote the set of all such measures μ on $X \times Y$, for which there is a positive constant c (depending on μ) satisfying

$$|\mu|_1(E) \leqq cm(E) , \quad E \subseteq X , \quad \text{and}$$
$$|\mu|_2(F) \leqq cn(F) , \quad F \subseteq Y .$$

Thus, the marginal measures of $|\mu|$ are absolutely continuous with respect to m and n, and have bounded Radon-Nikodym derivatives. For $\mu \in A(X \times Y, m, n)$, we define $\|\mu\|$ as the smallest constant c satisfying the above inequalities. A trivial verification shows that $\|\cdot\|$ is a *norm* on the vector space $A(X \times Y, m, n)$. Note also that there is no relation between this norm and the variation norm $|\mu|(X \times Y)$, for elements μ in this vector space of measures.

THEOREM 1.5.1. *Let (X, m), (Y, n) be sigma-finite measure spaces, fix p, $1 < p < \infty$, and let q be the conjugate index of p, $p^{-1} + q^{-1} = 1$. Let $\mu \in A(X \times Y, m, n)$. Then for each $f \in L^p(Y, n)$, $g \in L^q(X, m)$, the function $h(x, y) = f(y)g(x)$ belongs to $L^1(X \times Y, |\mu|)$. Moreover, there is an operator T_μ from $L^p(Y, n)$ into $L^p(X, m)$ satisfying*

(i) $\|T_\mu\| \leqq \|\mu\|$.

(ii) $\langle T_\mu f, g\rangle = \iint_{X \times Y} f(y)g(x)\mu(dx, dy), \qquad f \in L^p(Y, n),\ g \in L^q(X, m),$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing of $L^p(X, m)$ and $L^q(X, m)$.

Proof. Now for every bilinear form $[\cdot, \cdot]$ on $L^p(Y, n) \times L^q(X, m)$ satisfying $|[f, g]| \leqq a \|f\| \|g\|$, there is an operator T from $L^p(Y, n)$ into $L^p(X, m)$, of norm at most a, satisfying $[f, g] = \langle Tf, g\rangle$ (this is a simple consequence of the canonical identification of $L^p(X, m)$ as the dual of $L^q(X, m)$). Thus, all of the assertions will follow if we prove that, for $f \in L^p(Y, n)$, $g \in L^q(X, m)$,

$$\iint_{X \times Y} |f(y)| \cdot |g(x)| \cdot |\mu|(dx, dy) \leqq \|\mu\| \cdot \|f\|_p \|g\|_q .$$

Applying Hölder's inequality to the left side, we obtain

$$\begin{aligned}
&\iint_{X \times Y} |f(y)| \cdot |g(x)| \cdot |\mu|(dx, dy) \\
&\leqq \left(\iint_{X \times Y} |f(y)|^p |\mu|(dx, dy)\right)^{1/p} \left(\iint_{X \times Y} |g(x)|^q |\mu|(dx, dy)\right)^{1/q} \\
&= \left(\int_Y |f(y)|^p |\mu|_2(dy)\right)^{1/p} \left(\int_X |g(x)|^q |\mu|_1(dx)\right)^{1/q} \\
&\leqq \|\mu\|^{1/p} \|f\|_p \|\mu\|^{1/q} \|g\|_q = \|\mu\| \cdot \|f\|_p \cdot \|g\|_q
\end{aligned}$$

because $|\mu|_2(dy) \leqq \|\mu\| n(dy)$ and $|\mu|_1(dx) \leqq \|\mu\| m(dx)$. □

Remarks. We will give a more concrete expression for these operators T_μ below. While one is primarily concerned with the case $p = 2$, it is of interest to consider other values of p. To see why, consider the case where both measure spaces (X, m) and (Y, n) are standard, and μ is a *positive* measure in $A(X \times Y, m, n)$. If we let $\|T_\mu\|_p$ denote the norm of T_μ as an operator from $L^p(Y, n)$ into $L^p(X, m)$, then the preceding theorem implies that $\sup_{1<p<\infty} \| T_\mu \|_p \leqq \| \mu \|$. It is not hard to see that this inequality is actually equality, that T_μ defines an operator in the two extreme cases $p = 1$ and $p = +\infty$, and moreover $\| \mu \|$ is the larger of $\| T_\mu \|_1$ and $\| T_\mu \|_\infty$ (the latter can be deduced from the Riesz convexity theorem). Moreover, T_μ is *positive* in the sense that it maps nonnegative functions in $\bigcap_{1 \leqq p \leqq \infty} L^p(Y, n)$ to nonnegative functions in $\bigcap_{1 \leqq p \leqq \infty} L^p(X, m)$. Conversely, it can be shown that every positive linear transformation T of $\bigcap_p L^p(Y, n)$ into $\bigcap_p L^p(X, m)$, for which $c = \sup_p \| T \|_p < \infty$, determines a positive measure μ on $X \times Y$ via $\iint f(y)g(x)\mu(dx, dy) = \langle Tf, g \rangle$. Moreover, it can also be shown that μ satisfies $\mu_1 \leqq cm$ and $\mu_2 \leqq cn$. Now if m, say, is a finite measure, then the condition $\mu_1 \leqq cm$ implies $\mu(X \times Y) \leqq cm(X) < \infty$, so that $\mu \in A(X \times Y, m, n)$. So in this case, we infer that the operators from $L^2(Y, n)$ to $L^2(X, m)$ which come from positive measures in $A(X \times Y, m, n)$ are precisely those operators which map positive functions to positive functions, and which also determine bounded operators from $L^p(Y, n)$ to $L^p(X, m)$ for *every* value of p, $1 \leqq p \leqq +\infty$.

In the remainder of this section, we will consider the case where (X, m) is a *standard* (sigma-finite) measure space, $Y = X$, and $n = m$. In place of $A(X \times X, m, m)$, we will employ the shorter notation $A(X \times X, m)$. We first want to define a multiplication in $A(X \times X, m)$ which will make it into a normed algebra, and which is appropriately related to operator multiplication. In the special case where X is a set with n elements, say $X = \{1, 2, \cdots, n\}$, and $m(\{i\}) = m_i > 0$, then the elements μ of $A(X \times X, m)$ are identified with $n \times n$ matrices (a_{ij}) via

$$\mu(S) = \sum \{a_{ij} \colon (i, j) \in S\} .$$

Here, the marginals $|\mu|_1$ and $|\mu|_2$ are given by

$$|\mu|_1(\{i\}) = \sum_{j=1}^{n} |a_{ij}| ,$$
$$|\mu|_2(\{j\}) = \sum_{i=1}^{n} |a_{ij}| .$$

and the norm $\| \mu \|$ is the larger of the two numbers $\max_i (1/m_i) \sum_j |a_{ij}|$ and $\max_j (1/m_j) \sum_i |a_{ij}|$. In this case, the product of $\mu \sim (a_{ij})$ and $\nu \sim (b_{ij})$ will turn out to be the matrix (c_{ij}),

$$c_{ij} = \sum_{k=1}^{n} \frac{1}{m_k} a_{ik} b_{kj} .$$

Thus in the case where $m_i = 1$ for all i, we have the usual algebra of $n \times n$ matrices.

In order to define this "matrix" multiplication of measures, we shall have to make use of the following bit of lore on the disintegration of positive measures.

LEMMA. *Let X be a standard Borel space, let μ be a finite positive measure on $X \times X$, and let $\mu_1(E) = \mu(E \times X)$ be the first marginal of μ. Then there is a map $x \mapsto \mu^x$ of X into the space of all Borel probability measures on X satisfying*

(i) *$x \mapsto \mu^x(E)$ is a Borel function, for every Borel set $E \subseteq X$,*

(ii) *$\mu(S) = \int_X \left(\int_X \chi_S(x, y)\mu^x(dy)\right)\mu_1(dx)$ for every Borel set $S \subseteq X \times X$.*

Since we do not have a reference for this result, we will sketch how it can be deduced from a result in [9]. Since every standard Borel space is Borel isomorphic to a compact metric space (see [21]) and since (i) and (ii) involve only measures and Borel structures, we may assume that X is a compact metric space and the Borel structure on X is generated by its topology. Since the projection $p_1(x, y) = x$ of $X \times X$ onto its first coordinate space is continuous and since $\mu_1 = \mu p_1^{-1}$, we may infer from page 59 of [9] that there is a map $x \mapsto \mu^x$ of X into the set of probability measures on X for which

(i) $x \mapsto \int_X f(y)\mu^x(dy)$ is a Borel function for every $f \in C(X)$,

(ii) $\int_{X \times X} h\, d\mu = \iint h(x, y)\mu^x(dy)\mu_1(dx)$ for every $h \in C(X \times X)$.

Since the class of all Borel functions f (resp. h) for which (i) (resp. (ii)) holds is closed under bounded pointwise sequential convergence, it follows that (i) (resp. (ii)) holds for arbitrary bounded Borel functions f (resp. h). The required assertions are now immediate. □

The disintegration formula (ii) will be expressed by the notation $\mu(dx, dy) = \mu^x(dy)\mu_1(dx)$. There is a similar disintegration of μ relative to the second marginal $\mu_2(F) = \mu(X \times F)$, written $\mu(dx, dy) = \mu_y(dx)\mu_2(dy)$.

Now let (X, m) be a standard sigma-finite measure space and let $\mu \in A(X \times X, m)$. We want to employ related disintegrations of μ, except that in place of the marginals of μ (or $|\mu|$), we will use the measure m.

PROPOSITION 1.5.3. *For every $\mu \in A(X \times X, m)$, there exist maps $x \mapsto \mu_x$, $y \mapsto \mu^y$, of X into the set of all complex Borel measures on X of finite total variation, such that*

(i) $|\mu^x|(X) \leqq \|\mu\|$, $|\mu_y|(X) \leqq \|\mu\|$, *for all $x, y \in X$,*

(ii) *$\mu^x(E)$ and $\mu_y(E)$ define Borel functions for every fixed Borel set $E \subseteq X$,*

(iii) $\mu(dx, dy) = \mu^x(dy)m(dx) = \mu_y(dx)m(dy)$,

(iv) $|\mu|(dx, dy) = |\mu^x|(dy)m(dx) = |\mu_y|(dx)m(dy)$.

Moreover, if ν^x *and* ν_y *are two other families of measures satisfying* (i), (ii), *and* (iii), *then* $\nu^x = \mu^x$ a.e. (m) *and* $\nu_y = \mu_y$ a.e. (m).

Proof. For existence, let $\sigma = |\mu|$ be the variation measure of μ. Applying the lemma to σ, we obtain disintegrations $\sigma(dx, dy) = \sigma^x(dy)\sigma_1(dx) = \sigma_y(dx)\sigma_2(dy)$, where each σ^x and σ_y is a probability measure on X. Since $\mu \in A(X \times X, m)$, we have $\sigma_1 \leqq \|\mu\|\cdot m$ and $\sigma_2 \leqq \|\mu\|\cdot m$. So by the Radon-Nikodym theorem there are Borel functions w_1, w_2 on X such that $0 \leqq w_i(x) \leqq \|\mu\|$, $x \in X$, and $\sigma_i(dx) = w_i(x)m(dx)$. Now since σ is the variation of μ, we may find a Borel function u on $X \times Y$ such that $|u(x, y)| = 1$ for all x, y and $\mu(dx, dy) = u(x, y)\sigma(dx, dy)$. Finally, define μ^x and μ_y by

$$\mu^x(dy) = u(x, y)w_1(x)\sigma^x(dy)$$
$$\mu_y(dx) = u(x, y)w_2(y)\sigma_y(dx)\ .$$

A routine check shows that these measures satisfy (i) and (ii), and since $\sigma_i(dz) = w_i(z)m(dz)$, we see that $\sigma(dx, dy) = w_1(x)\sigma^x(dy)m(dx) = w_2(y)\sigma_y(dx)$ $m(dx)$, from which (iii) is evident. Finally, since u has unit modulus, the measure $|\mu^x|$ is given by $|\mu^x|(dy) = w_1(x)\sigma^x(dy)$, so that

$$|\mu^x|(dy)m(dx) = w_1(x)\sigma^x(dy)m(dx) = \sigma^x(dy)\sigma_1(dx) = \sigma(dx, dy)\ .$$

Similarly, $|\mu_y|(dx)m(dy) = \sigma(dx, dy)$, proving (iv).

For uniqueness, let ν^x be a family of measures having the same properties (i), (ii), (iii) as μ^x. We will produce a Borel set $N \subseteqq X$ such that $m(N) = 0$ and $\nu^x = \mu^x$ for all $x \notin N$. Choose a countable Boolean algebra $\{F_1, F_2, \cdots\}$ of Borel sets in X which generates the Borel structure on X. Now fix $j = 1, 2, \cdots$, and consider the bounded Borel functions $f_j(x) = \mu^x(F_j)$ and $g_j(x) = \nu^x(F_j)$. Because

$$\mu^x(dy)m(dx) = \nu^x(dy)m(dx)\ \big(= \mu(dx, dy)\big)\ ,$$

we see that, for every Borel set $E \subseteqq X$ with $m(E) < \infty$,

$$\int_E f_j(x)m(dx) = \int_E g_j(x)m(dx) = \mu(E \times F_j)\ .$$

Thus $f_j = g_j$ a.e. (m). Let N_j be an m-null set so that $f_j(x) = g_j(x)$ for all $x \notin N_j$, and put $N = \bigcup_{j=1}^{\infty} N_j$. Then for $x \notin N$, we have $\mu^x(F_j) = \nu^x(F_j)$ for every $j = 1, 2, \cdots$. Because both μ^x and ν^x are countably additive (for fixed x) and because the Boolean algebra $\{F_1, F_2, \cdots\}$ generates the Borel field on X, we conclude that $\mu^x = \nu^x$ for every $x \in N$. The uniqueness of the μ_y constituents follows by symmetry. $\square$

We can now define the product $\mu*\nu$ of two measures $\mu, \nu \in A(X\times X, m)$ when (X, m) is a standard sigma-finite measure space. Choose disintegrations

$$\mu(dx, dy) = \mu_y(dx)m(dy) \text{ and } \nu(dx, dy) = \nu^x(dy)m(dx)$$

according to 1.5.2. For each $z \in X$, $\mu_z \times \nu^z$ defines a complex measure on $X \times X$ of finite total variation. We claim first that there is a nonnegative function w in $L^1(X, m)$ such that $|\mu_z \times \nu^z(S)| \leqq w(z)$ for every Borel set $S \subseteq X \times X$. For that, put $w(z) = \|\mu\|\cdot|\nu^z|(X)$. Then clearly $w \geqq 0$, w is measurable, and w is integrable because

$$\int w(z)m(dz) = \|\mu\|\cdot\int|\nu^z|(X)m(dz) = \|\mu\|\cdot|\nu|(X\times X) < \infty$$

(here we use $|\nu^z|(dx)m(dz) = |\nu|(dx, dz)$ and the fact that ν has finite total variation). Moreover, for every Borel set $S \subseteq X \times X$,

$$|\mu_z \times \nu^z(S)| \leqq |\mu_z| \times |\nu^z|(S) \leqq |\mu_z|(X)\cdot|\nu^z|(X) \leqq \|\mu\|\cdot|\nu^z|(X) = w(z)\,,$$

as asserted. In particular, the function $z \mapsto \mu_z \times \nu^z(S)$ is in $L^1(X, m)$ for each Borel set $S \subseteq X \times X$, and we may define a complex valued set function $\mu*\nu$ by

$$\mu*\nu(S) = \int_X \mu_z \times \nu^z(S)m(dz)$$

for every Borel set $S \subseteq X \times X$. The fact that $\mu*\nu$ is a measure of finite total variation also follows from the preceding claim, along with the dominated convergence theorem. Note finally that, by the uniqueness statement of 1.5.2, the possible arbitrariness in the choice of μ_z and ν_z does not affect $\mu*\nu$, so that $\mu*\nu$ is well-defined by μ and ν.

PROPOSITION 1.5.3. *The multiplication $\mu*\nu$ makes $A(X\times X, m)$ into a normed algebra.*

Proof. We shall only verify that $\mu*\nu$ belongs to $A(X \times X, m)$ and satisfies the inequality $\|\mu*\nu\| \leqq \|\mu\|\cdot\|\nu\|$. For this, note that by definition of $\mu*\nu$ we have $|\mu*\nu(S)| \leqq \int|\mu_z| \times |\nu^z|(S)m(dz)$, so that $|\mu*\nu|$ is dominated by the measure $p(S) = \int|\mu_z| \times |\nu^z|(S)m(dz)$. Hence it suffices to show that $p_1(dx) \leqq \|\mu\|\cdot\|\nu\|\cdot m(dx)$ and $p_2(dy) \leqq \|\mu\|\cdot\|\nu\|\cdot m(dy)$. Now for $E \subseteq X$, we have

$$p_1(E) = p(E \times X) = \int_X |\mu_z|(E)\,|\nu^z|(X)m(dz) \leqq \|\nu\| \int_X |\mu_z|(E)m(dz)$$
$$= \|\nu\|\cdot|\mu|(E \times X) = \|\nu\|\cdot|\mu|_1(E) \leqq \|\nu\|\cdot\|\mu\|\cdot m(E)\,,$$

because $|\nu^z|(X) \leqq \|\nu\|$ for all z and $|\mu|_1(E) \leqq \|\mu\|\cdot m(E)$. Similarly $p_2 \leqq \|\mu\|\cdot\|\nu\|\cdot m$. □

Now let G be a Borel subset of $X \times X$. $A(G, m)$ will denote the subspace of all measures μ which are concentrated on G. We will be most concerned with those G's for which $A(G, m)$ is also closed under multiplication. To this end, given $G \subseteqq X \times X$, we define $G \circ G$ as the set of all pairs $(x, y) \in X \times X$ for which there is a $z \in X$ such that $(x, z) \in G$ and $(z, y) \in G$. To say that $G \circ G \subseteqq G$ means that G is the *graph* of a transitive relation R on X: $G = \{(x, y): yRx\}$. Hence, in order that a subset G of $X \times X$ be the graph of a partial ordering of X, it is necessary and sufficient that G should contain the diagonal $\Delta = \{(x, x): x \in X\}$ and satisfy $G \circ G \subseteqq G$.

PROPOSITION 1.5.4. *Let G be a Borel set in $X \times X$ such that $\Delta \subseteqq G$ and $G \circ G \subseteqq G$. Then $A(G, m)$ is a subalgebra of $A(X \times X, m)$. If $0 < m(X) < \infty$ then $A(G, m)$ contains a multiplicative identity of unit norm.*

Proof. Choose $\mu \in A(G, m)$, and let $\mu(dx, dy) = \mu_x(dy)m(dx)$ be the disintegration of μ via 1.5.2. For each $x \in X$ and $E \subseteqq X$, let $E_x = \{z \in X: (x, z) \in E\}$. We claim that $|\mu_x|$ is concentrated on G_x for almost every $x(m)$. Indeed, by 1.5.2 we know that $|\mu|(dx, dy) = |\mu_x|(dy)m(dx)$, so that if $A = X \times X\backslash G$, then

$$0 = |\mu|(A) = \int \chi_A(x, y)\,|\mu|\,(dx, dy) = \int_X \Big(\int_X \chi_A(x, y)\,|\mu_x|\,(dy)\Big)m(dx)$$
$$= \int_X |\mu_x|\,(A_x)m(dx)\,.$$

This implies that $|\mu_x|\,(A_x) = 0$ for almost every x, hence the assertion.

Now choose $\mu, \nu \in A(G, m)$. We want to show that $\mu*\nu$ belongs to $A(G, m)$. Write $\mu(dx, dy) = \mu_y(dy)m(dx)$, $\nu(dx, dy) = \nu^y(dx)m(dy)$. We want to show that $\mu*\nu = \int \mu_z \times \nu^z m(dz)$ is concentrated on G; it is clearly sufficient to prove that $|\mu_z| \times |\nu^z|$ is concentrated on G for almost every z. Now by the preceding paragraph we can find an m-null set $M \subseteqq X$ such that, for all $x \notin M$, $|\mu_x|$ lives in G_x. Similarly, there is an m-null set $N \subseteqq X$ such that $|\nu^y|$ lives in $G^y = \{z: (z, y) \in G\}$ for every $y \notin N$. We conclude that if $z \notin M \cup N$, then $|\mu_z| \times |\nu^z|$ lives in $G_z \times G^z \subseteqq G \circ G \subseteqq G$, as required.

Finally, assume $0 < m(X) < \infty$. Define a positive finite measure δ on

$X \times X$ by $\delta(S) = m\{x: (x, x) \in S\}$. Clearly $\delta_1 = \delta_2 = m$, so that $\delta \in A(X \times X, m)$, and $\|\delta\| = 1$. Moreover, since δ is concentrated on $\Delta \subseteqq G$, we have $\delta \in A(G, m)$. The canonical disintegrations of δ (see 1.5.2) are given by $\delta(dx, dy) = \delta^x(dy)m(dx) = \delta_y(dx)m(dy)$, where $\delta_z = \delta^z$ can be taken as the unit point mass at $\{z\}$. So for any measure $\nu \in A(X \times X, m)$, say $\nu(dx, dy) = \nu^x(dy)m(dx)$, and any pair E, F of Borel sets in X, we have

$$\delta * \nu(E \times F) = \int_X \delta_x(E)\nu^x(F)m(dx) = \int_E \nu^x(F)m(dx) = \nu(E \times F) .$$

Since $\delta * \nu$ and ν agree on all Borel rectangles in $X \times X$ they must be identical measures. The proof that $\nu = \nu * \delta$ is similar. □

For p fixed, $1 < p < \infty$, and for $\mu \in A(X \times X, m)$, we will write T_μ for the bounded operator on $L^p(X, m)$ determined by μ as in 1.5.1. Let q be the conjugate index of p, $p^{-1} + q^{-1} = 1$, and let T'_μ be the adjoint of T_μ, defined as an operator on $L^q(X, m)$ by $\langle Tf, g\rangle = \langle f, T'g\rangle$, $f \in L^p$, $g \in L^q$. We want to remark that $T'_\mu = T_{\tilde{\mu}}$, where $\tilde{\mu}$ is the "transposed" measure $\tilde{\mu} = \mu\phi$, ϕ denoting the reflection $\phi: (x, y) \mapsto (y, x)$ in $X \times Y$. Indeed, $\mu \mapsto \tilde{\mu}$ defines an isometric anti-automorphism of the normed algebra $A(X \times X, m)$; we omit these routine verifications.

We also want to point out a rather more transparent formula for $T_\mu f$, $f \in L^p(X, m)$, $\mu \in A(X \times X, m)$. For this, write $\mu(dx, dy) = \mu^x(dy)m(dx)$ as in 1.5.2. Then we claim:

$$T_\mu f(x) = \int_X f(y)\mu^x(dy) ,$$

for almost every $x \in X$. For by definition of T_μ we have, for every $g \in L^q(X, m)$,

$$\langle Tf, g\rangle = \iint f(y)g(x)\mu^x(dy)m(dx) ,$$

and the above formula follows by a routine application of Fubini's theorem. As for T'_μ, we see either by a direct verification or by the preceding paragraph that, for every $g \in L^q(X, m)$,

$$T'_\mu g(y) = \int_X g(x)\mu_y(dx) .$$

PROPOSITION 1.5.5. *Fix* p, $1 < p < \infty$. *Then* $T_{\mu*\nu} = T_\mu T_\nu$, *for all* $\mu, \nu \in A(X \times X, m)$.

Proof. Fix $f \in L^p(X, m)$, $g \in L^q(X, m)$. Then we have $\langle T_{\mu*\nu}f, g\rangle = \iint f(y)g(x)\mu*\nu(dx, dy)$. Using Fubini's theorem and the definition of $\mu*\nu$ we can write

$$\langle T_{\mu*\nu}f, g\rangle = \iiint f(y)g(x)\mu_z(dx)\nu^z(dy)m(dz) = \langle F, G\rangle ,$$

where $F(z) = \int f(y)\nu^z(dy)$ and $G(z) = \int g(x)\mu_z(dx)$. Note that, by the preceding remarks, $F = T_\nu f \in L^p$ and $G = T'_\mu g \in L^q$. It follows that $\langle T_{\mu*\nu}f, g\rangle = \langle T_\nu f, T'_\mu g\rangle = \langle T_\mu T_\nu f, g\rangle$. Since $f \in L^p$ and $g \in L^q$ were arbitrary, we are done. □

For each function $f \in L^\infty(X, m)$ and each p, $1 < p < \infty$, we write L_f for the bounded operator on $L^p(X, m)$ given by $(L_f g)(x) = f(x)g(x)$, $x \in X$.

THEOREM 1.5.6. *Let (X, m) be a sigma-finite standard measure space and let G be a Borel set in $X \times X$ which contains the diagonal and satisfies $G \circ G \subseteqq G$.*

Then for every p, $1 < p < \infty$, $\mu \mapsto T_\mu$ defines a contractive representation of $A(G, m)$ onto an algebra of bounded operators on $L^p(X, m)$, whose range contains all multiplication operators L_f, $f \in L^\infty(X, m) \cap L^1(X, m)$.

Proof. The only assertion that remains to be proved is that every multiplication operator L_f, $f \in L^\infty \cap L^1$, belongs to the range of $\{T_\mu : \mu \in A(G, m)\}$. Fix such an f, and define a measure μ on $X \times X$ by

$$\mu(S) = \int_X \chi_S(x, x)f(x)m(dx) .$$

Clearly $|\mu|(X \times X) \leqq \int_X |f(x)|\, m(dx) < \infty$, and

$$|\mu|_1(dz) = |\mu|_2(dz) = |f(z)|\, m(dz) .$$

Thus $\mu \in A(X \times X, m)$ and, moreover, $\|\mu\| \leqq \|f\|_\infty$. The canonical decomposition of μ can be expressed as $\mu(dx, dy) = f(x)\delta^x(dy)m(dx)$, where δ^x is the unit point mass at $\{x\}$. By the remarks preceding 1.5.5, we see that, for g in $L^p(X, m)$,

$$T_\mu g(x) = \int_X g(y)f(x)\delta^x(dy) = f(x)g(x) ,$$

for almost every $x \in X$, so that $T_\mu = L_f$, as required. □

1.6. Commutative lattices are reflexive

We now assemble some of the results of the preceding sections into a proof that every separably acting commutative subspace lattice is reflexive. Let $(X, , \leqq, m)$ be a standard sigma-finite partially ordered measure space. Let $\mathfrak{L}(X, \leqq, m)$ be the subspace lattice (acting on $L^2(X, m)$) defined in Section 1.2. We also want to define an operator algebra on $L^2(X, m)$, as follows.

Let $G = \{(x, y): y \leqq x\} \subseteq X \times X$ be the graph of the given partial order on X. We define $\mathcal{A}_{\min}(X, \leqq, m)$ to be the ultraweak closure of the set of operators $\{T_\mu: \mu \in A(G, m)\}$. By 1.5.6, *$\mathcal{A}_{\min}(X, \leqq, m)$ is an algebra of operators on $L^2(X, m)$ which contains the multiplication algebra of $L^\infty(X, m)$.* The reason for the subscript min in this notation is explained in remark 1.6.2. We first want to point out an alternate description of the pseudo-integral operators in $\mathcal{A}_{\min}(X, \leqq, m)$ (also, see 2.2.6).

Proposition 1.6.0. *Let $\mu \in A(X \times X, m)$, and let T_μ be the corresponding pseudo-integral operator. Then $T_\mu \in \operatorname{alg} \mathcal{L}(X, \leqq, m)$ if, and only if, μ is concentrated on G.*

Proof. Assume first that μ lives on G, and let $E \subseteq X$ be an increasing Borel set. We have to show that T leaves the range of the projection P_E invariant; equivalently,

$$(T_\mu f, g) = \int_{X \times X} f(y)\overline{g(x)}d\mu(x, y) = 0$$

for all f, $g \in L^2(X, m)$ such that f lives in E and g lives in $X \backslash E$. But $(X \backslash E) \times E$ is disjoint from G (because E is an increasing set), and supports the function $f(y)\overline{g(x)}$. The assertion follows since μ lives on G.

Conversely, assume $T_\mu \in \operatorname{alg} \mathcal{L}(X, \leqq, m)$. By 1.1.2, we may find a sequence E_n of increasing Borel sets such that the complement of G is the union $\bigcup_n (X \backslash E_n) \times E_n$. So to prove that $\mu \in A(G, \mu)$, it suffices to show that $|\mu| (X \backslash E_n \times E_n) = 0$, $n \geqq 1$, or equivalently,

$$(T_\mu f, g) = \int_{X \times X} f(y)\bar{g}(x)d\mu(x, y) = 0$$

for every pair f, $g \in L^2(X, m) \cap L^\infty(X, m)$ such that f lives in E_n and g lives in $X \backslash E_n$. This clearly follows from the fact that each projection P_{E_n} is invariant under T_μ. □

THEOREM 1.6.1.

$$\mathcal{L}(X, \leqq, m) = \operatorname{lat} \mathcal{A}_{\min} (X, \leqq, m) .$$

Proof. The inclusion $\subseteq$ follows from the preceding proposition. Conversely, let P be an invariant projection for the algebra $\mathcal{A}_{\min}(X, \leqq, m)$. We have to produce an increasing Borel set E such that $P = P_E$. Because $\mathcal{A}_{\min}(X, \leqq, m)$ contains the multiplication algebra M, a maximal abelian von Neumann algebra, it follows that $P \in M$, and hence there is a Borel set $A \subseteq X$ such that $P = P_A$. We claim that there is a Borel set $N \subseteq X$ such that $m(N) = 0$ and, for all x, $y \notin N$, $x \in A$ and $y \geqq x$ imply $y \in A$ (from this and 1.1.11, it will follow that there is an $E \in L(X, \leqq)$ such that $P_A = P_E$, as required).

To prove the claim, consider the set $S = G \cap (X \backslash A) \times A \subseteqq X \times X$. We will show that S is marginally null. Note first that the complement of G is a countable union of Borel rectangles; for if we choose $E_1, E_2, \cdots$ in $L(X, \leqq)$ such that $x \leqq y$ iff $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for all $n \geqq 1$ (cf. 1.1.2), then $X \times X \backslash G$ has the form $\bigcup_{n=1}^{\infty} (X \backslash E_n) \times E_n$. Thus S satisfies the structural hypothesis of the Null Set Theorem 1.4.3. So by that theorem, to prove S is marginally null, it suffices to show that the space of measures $A(S, m)$ contains only the zero measure. For that, choose $\mu \in A(S, m)$. Because $S \subseteqq G$, the associated operator T_μ belongs to $\mathfrak{A}_{\min}(X, \leqq, m)$, and therefore T_μ leaves $P = P_A$ invariant. Note that this imples μ annihilates every Borel rectangle $C \times D$ contained in $(X \backslash A) \times A$ satisfying $m(C)$, $m(D) < \infty$. Indeed, the functions $f = \chi_D$ and $g = \chi_C$ belong to $L^2(X, m)$, f is in the range of P_A, and g is orthogonal to the range of P_A, so that $\mu(C \times D) = (T_\mu f, g) = 0$. It follows that μ vanishes on the sigma ring generated by such rectangles; i.e., $\mu(H) = 0$ for every Borel set $H \subseteqq (X \backslash A) \times A$, or equivalently, $|\mu|((X \backslash A) \times A) = 0$. Because μ is concentrated on the subset S of $(X \backslash A) \times A$, the claim $\mu = 0$ follows.

Therefore S is marginally null, and we can find Borel sets N_1, N_2 such that $S \subseteqq N_1 \times X \cup X \times N_2$ and $m(N_1) = m(N_2) = 0$. This means that, if $x, y \notin N_1 \cup N_2$, $y \in A$, and $y \leqq x$, then necessarily $x \in A$. The null set $N = N_1 \cup N_2$ therefore has the desired property. □

Remark 1.6.2. We will see later in Chapter 2 that the algebra $\mathfrak{A}_{\min}(X, \leqq, m)$ has a crucial minimality property, namely that for every ultraweakly closed algebra $\mathfrak{B}$ of operators on $L^2(X, m)$ such that

(i) $\mathfrak{B}$ contains the multiplication algebra of (X, m), and

(ii) lat $\mathfrak{B} = \mathfrak{L}(X, \leqq, m)$,

then $\mathfrak{B}$ necessarily contains $\mathfrak{A}_{\min}(X, \leqq, m)$. Thus, $\mathfrak{A}_{\min}(X, \leqq, m)$ is characterized abstractly by the fact that it is the smallest ultraweakly closed algebra having properties (i) and (ii). On the other hand, the algebra alg $\mathfrak{L}(X, \leqq, m)$ is *weakly* closed and is clearly the largest algebra having these properties; we conclude that all such algebras $\mathfrak{B}$ are trapped between $\mathfrak{A}_{\min}(X, \leqq, m)$ and alg $\mathfrak{L}(X, \leqq, m)$. We will also see in Chapter 2 that $\mathfrak{A}_{\min}(X, \leqq, m)$ is *not* necessarily weakly dense in alg $\mathfrak{L}(X, \leqq, m)$.

THEOREM 1.6.3. *Every separably acting commutative subspace lattice is reflexive.*

Proof. By the spectral Theorem 1.3.1, such a lattice is unitarily equivalent to the lattice $\mathcal{L}(X, \leqq, m)$ associated with a standard sigma-finite partially ordered measure space $(X, \leqq, m)$. Now apply 1.6.1. □

Problem. Let us call a subspace lattice $\mathcal{L}$ (acting on a Hilbert space $\mathcal{H}$) *strongly reflexive* if there is a set $\mathcal{S}$ of compact operators on $\mathcal{H}$ such that $\mathcal{L} = \operatorname{lat} \mathcal{S}$. It would be of considerable interest to have a characterization of the separably-acting commutative subspace lattices which are strongly reflexive. For example, under what conditions on the standard partially ordered measure space $(X, \leqq, m)$ will $\mathcal{L}(X, \leqq, m)$ be strongly reflexive? When do there exist *no* nonzero compact operators in alg $\mathcal{L}(X, \leqq, m)$? As a more concrete question, let p be a positive real number less than 1 and let $(\mathbf{2}^\infty, \leqq, m_p)$ be the partially ordered measure space described in the last paragraph of Section 1.2. Then it is easy to see that the graph $G \subseteqq \mathbf{2}^\infty \times \mathbf{2}^\infty$ of the partial order $\leqq$ has measure zero relative to the product measure $m_p \times m_p$; from this it follows easily that there are no Hilbert-Schmidt operators in alg $\mathcal{L}(2^\infty, \leqq, m_p)$. But we do not know if alg $\mathcal{L}(2^\infty, \leqq, m_p)$ contains more "singular" compact operators; in particular, is there a single nonzero compact operator in alg $\mathcal{L}(2^\infty, \leqq, m_p)$?

Problem. The above proof of 1.6.3 used the properties of partially ordered measure spaces in an essential way. Naturally, one would prefer a "coordinate-free" proof. Presumably, this would involve constructing the operator algebra $\mathcal{A}_{\min}(X, \leqq, m)$ in an abstract manner, and there seems to be no obvious way to go about doing this. As a test problem for such techniques (of minimal intrinsic interest, perhaps), we pose the following: Is 1.6.3 valid for inseparable Hilbert spaces?

Chapter 2. Reflexive Algebras

2.1. The minimal algebra of a partially ordered measure space

Let $(X, \leqq, m)$ be a partially ordered standard sigma-finite measure space, which will be fixed throughout this section. We have already encountered the operator algebra $\mathcal{A}_{\min}(X, \leqq, m)$ in Section 1.6, defined as the ultraweak closure of the algebra of all pseudo-integral operators T_μ, where μ runs over the appropriate class of all measures on $X \times X$ which are concentrated on the graph of the given partial order on X. Unless there is cause for confusion, we will employ the shorter notation $\mathcal{A}_{\min}$ for this algebra; $\mathcal{A}_{\min}$ will be called the *minimal algebra* of $(X, \leqq, m)$.

We have also shown in Section 1.6 that $\mathcal{A}_{\min}$ contains the multiplication algebra of (X, m), and has $\mathcal{L}(X, \leqq, m)$ as its invariant projection lattice. The principal result of this section implies that $\mathcal{A}_{\min}$ is the *smallest* ultraweakly closed operator algebra on $L^2(X, m)$ which has these two properties, a fact which leads to a number of general conclusions about operator algebras on Hilbert spaces.

At this point, we feel that a word of explanation may be in order concerning our use of the "ultra" topologies rather than the more traditional weak and strong operator topologies. Perhaps the main benefit of this formulation is that theorems about utraweakly closed operator algebras usually have interesting corollaries about *norm*-closed algebras of compact operators (see Corollary 2 of 2.1.8 for an illustration of this). This bonus is not available if one works with weakly closed algebras. As another important, though less tangible reason, one knows that the full algebra $\mathcal{L}(\mathcal{H})$ is the dual of a certain Banach space (namely the space of all ultraweakly continuous linear functionals), and the canonical pairing of $\mathcal{L}(\mathcal{H})$ with its predual identifies the ultraweak topology with the weak* topology. So in this sense the ultraweak topology is a very natural one. In any case, the principal results of this chapter all have counterparts in the smaller category of weakly closed algebras.

We begin by restating some familiar facts about the ultraweak topology in a form convenient for our purposes. Let $\mathcal{K}$ be a separable infinite-dimensional Hilbert space, which will be fixed throughout this section. We shall realize the Hilbert space tensor product $\mathcal{K} \otimes L^2(X, m)$ as the space of all weakly measurable functions $F: X \to \mathcal{K}$ for which the norm

$$\|F\| = \left(\int_X \|F(x)\|^2 dm(x)\right)^{1/2}$$

is finite (one of course identifies functions which agree almost everywhere). Note first that a linear functional ρ on $\mathcal{L}(L^2(X, m))$ is ultraweakly continuous if and only if it has a representation $\rho(T) = ((I_{\mathcal{K}} \otimes T)F, G)$, where $I_{\mathcal{K}}$ denotes the identity operator on $\mathcal{K}$ and $F, G \in \mathcal{K} \otimes L^2(X, m)$. Indeed, the fact that the linear map $T \mapsto I_{\mathcal{K}} \otimes T$ is ultraweakly continuous insures that every such functional $T \mapsto (I_{\mathcal{K}} \otimes TF, G)$ is ultraweakly continuous. Conversely, every ultraweakly continuous linear functional ρ on $\mathcal{L}(L^2(X, m))$ admits a representation

$$\rho(T) = \sum_1^\infty (Tf_n, g_n)$$

where $f_n, g_n \in L^2(X, m)$ satisfy $\sum_1^\infty \|f_n\|^2 < \infty$ and $\sum_1^\infty \|g_n\|^2 < \infty$ ([10, p. 38]). So if we choose an orthonormal basis $e_1, e_2, \cdots$ for $\mathcal{K}$ then we may define functions F, G in $\mathcal{K} \otimes L^2(X, m)$ by

$$F(x) = \sum_1^\infty f_n(x)e_n ,$$
$$G(x) = \sum_1^\infty g_n(x)e_n$$

(note, for example, that $F(x)$ is well-defined almost everywhere because the condition $\int_X \sum_1^\infty |f_n(x)|^2 dm(x) = \sum_1^\infty ||f_n||^2 < \infty$ implies that $\sum_1^\infty |f_n(x)|^2 < \infty$ almost everywhere). Equivalently, one has $F = \sum_1^\infty e_n \otimes f_n$, $G = \sum_1^\infty e_n \otimes g_n$. In either case, a simple computation shows that $\rho(T) = ((I_{\mathcal{K}} \otimes T)\, F, G)$, as asserted.

Now let $\mathcal{B}$ be an arbitrary algebra of operators on $L^2(X, m)$ which contains the identity, and let $T \in \mathfrak{L}(L^2(X, m))$. Then notice that T belongs to the ultraweak closure of $\mathcal{B}$ if, and only if, lat $(I_{\mathcal{K}} \otimes \mathcal{B})$ is contained in lat $(I_{\mathcal{K}} \otimes T)$. Indeed, suppose $I_{\mathcal{K}} \otimes T$ leaves every element of lat $(I_{\mathcal{K}} \otimes \mathcal{B})$ invariant. To see that $T \in \mathcal{B}^-$, choose any ultraweakly continuous linear functional ρ which annihilates $\mathcal{B}$, say $\rho(A) = ((I_{\mathcal{K}} \otimes A)F, G)$, F, $G \in \mathcal{K} \otimes L^2(X, m)$. Then G is orthogonal to $[I_{\mathcal{K}} \otimes \mathcal{B}F]$, an $I_{\mathcal{K}} \otimes \mathcal{B}$-invariant subspace. Since $\mathcal{B}$ contains the identity, F must belong to $[I_{\mathcal{K}} \otimes \mathcal{B}F]$, so by hypothesis $I_{\mathcal{K}} \otimes TF \in [I_{\mathcal{K}} \otimes \mathcal{B}F]$. The required conclusion $\rho(T) = ((I_{\mathcal{K}} \otimes T)F, G) = 0$ is now evident. The converse implication is trivial. The point is that, if one is able to determine the invariant subspaces of $I_{\mathcal{K}} \otimes \mathcal{B}$ from the information given about $\mathcal{B}$, then one knows the ultraweak closure of $\mathcal{B}$.

The following simple result gives a convenient representation for operators $I_{\mathcal{K}} \otimes T$ when T is a pseudo-integral operator.

PROPOSITION 2.1.1. *Let $\mu \in A(X, m)$ and let $F, G \in \mathcal{K} \otimes L^2(X, m)$. Then the function $h(x, y) = (F(y), G(x))$ is $|\mu|$-integrable over $X \times X$, and*

$$((I_{\mathcal{K}} \otimes T_\mu)F, G) = \int_{X \times X} (F(y), G(x))d\mu(x, y)\ .$$

Proof. Since $|(F(y), G(x))| \leqq ||F(y)|| \cdot ||G(x)||$, the estimate used in the proof of Theorem 1.5.1 shows that

$$\int_{X \times X} |(F(y), G(x))|\, d\, |\mu| \leqq ||\mu|| \cdot ||F|| \cdot ||G||\ .$$

The integrability of h is immediate from this, and in fact

$$[F, G] = \int_{X \times X} (F(y), G(x))d\mu(x, y)$$

defines a sesquilinear form on $\mathcal{K} \otimes L^2(X, m)$ of norm at most $||\mu||$. Let $\langle \cdot, \cdot \rangle$ be the bounded sesquilinear form $\langle F, G \rangle = (I_{\mathcal{K}} \otimes T_\mu F, G)$. In order to show that these two forms agree, it therefore suffices to show that they agree on the fundamental set of all functions of the form $(e \otimes f)(x) = f(x)e$, $e \in \mathcal{K}$, $f \in L^2(X, m)$. But if e, $e' \in \mathcal{K}$ and f, $f' \in L^2(X, m)$, then $\langle e \otimes f, e' \otimes f' \rangle = (e \otimes T_\mu f, e' \otimes f') = (e, e')(T_\mu f, f')$, while

$$[e \otimes f, e' \otimes f'] = \int (f(y)e, f'(x)e')d\mu(x, y) = (e, e') \int f(y)\overline{f'(x)}d\mu(x, y)$$
$$= (e, e')(T_\mu f, f').$$

The conclusion follows. □

Remark. In the remainder of this section, M will denote the usual multiplication algebra $\{L_f: f \in L^\infty(X, m)\}$ of operators on $L^2(X, m)$ associated with bounded complex-valued Borel functions on X. Then $I_{\mathcal{K}} \otimes M$ is a commutative von Neumann algebra acting on $\mathcal{K} \otimes L^2(X, m)$, and we want to summarize briefly a convenient (and well-known) description of $\operatorname{lat}(I_{\mathcal{K}} \otimes M)$. Let $F: X \to \mathfrak{L}(\mathcal{K})$ be a bounded operator-valued function which is such that $x \mapsto (F(x)e, e')$ is a Borel function, for each e, e' in $\mathcal{K}$ (we shall refer to such an F simply as an operator-valued Borel function). F gives rise to a bounded operator L_F on $\mathcal{K} \otimes L^2(X, m)$ defined by $(L_F G)(x) = F(x)G(x)$, $G \in \mathcal{K} \otimes L^2(X, m)$. Clearly L_F commutes with the algebra $I_{\mathcal{K}} \otimes M$ of all scalar-valued multiplications. Conversely, it is a familiar fact that every operator in the commutant of $I_{\mathcal{K}} \otimes M$ has this form L_F for some bounded Borel function $F: X \to \mathfrak{L}(\mathcal{K})$ (e.g., [3, Section 4.2]). Moreover, if L_F is a projection then one may choose F so that $F(x)$ is a projection in $\mathfrak{L}(\mathcal{K})$ for every $x \in X$. Now since $I_{\mathcal{K}} \otimes M$ is a self-adjoint algebra, each of its invariant subspaces is a reducing subspace, and therefore the corresponding projection must commute with $I_{\mathcal{K}} \otimes M$. We conclude that $\operatorname{lat}(I_{\mathcal{K}} \otimes M)$ consists precisely of all multiplications L_P, where $P: X \to \mathfrak{L}(\mathcal{K})$ is a *projection-valued* Borel function.

The following result describes a broad class of invariant subspaces for the algebra $I_{\mathcal{K}} \otimes \mathfrak{A}_{\min}$. A function F from X into some other partially ordered set (for example, the self-adjoint operators on $\mathcal{K}$ with the usual ordering) is said to be *essentially increasing* if there is a Borel set $N \subseteq X$ of measure zero such that, for all $x, y \in X \backslash N$, $x \leqq y$ implies $F(x) \leqq F(y)$.

PROPOSITION 2.1.2. *Let $P: X \to \mathfrak{L}(\mathcal{K})$ be a projection-valued Borel function which is essentially increasing. Then $L_P \in \operatorname{lat}(I_{\mathcal{K}} \otimes \mathfrak{A}_{\min})$.*

Proof. Let $P^\perp$ be the projection valued Borel function $P^\perp(x) = I_{\mathcal{K}} - P(x)$, $x \in X$. Then clearly $L_{P^\perp} = (L_P)^\perp$, and it suffices to show that, if $F, G \in \mathcal{K} \otimes L^2(X, m)$ are such that $F(x) = P(x)F(x)$ and $G(x) = P^\perp(x)G(x)$ for all x, then $(I_{\mathcal{K}} \otimes TF, G) = 0$ for all $T \in \mathfrak{A}_{\min}$. Since the pseudo-integral operators T_μ, $\mu \in A(G, m)$ (G denoting the graph $\{(x, y): y \leqq x\}$ of $\leqq$) are utraweakly dense in $\mathfrak{A}_{\min}$, it suffices to show that $(I_{\mathcal{K}} \otimes T_\mu F, G) = 0$ for every $\mu \in A(G, m)$. But by 2.1.1 we know that

$$(I_{\mathcal{K}} \otimes T_\mu F, G) = \int_G (F(y), G(x))d\mu(x, y)$$

(the integration is extended over G rather than $X \times X$ because μ is concentrated on G). Now since P is essentially increasing, we may find a Borel set $N \subseteqq X$ such that $m(N) = 0$ and P restricted to $X \backslash N$ is increasing. So if $(x, y) \in G \backslash (X \times N \cup N \times X)$, then $y \leqq x$ and $P(y) \leqq P(x)$; in particular, $P^\perp(x)P(y) = 0$. It follows that $(F(y), G(x)) = (P(y)F(y), P^\perp(x)G(x)) = 0$ everywhere on G except perhaps on $X \times N \cup N \times X$. Since the latter set has $|\mu|$-measure zero (because $|\mu|$ has absolutely continuous marginals), we conclude that

$$\int (F(y), G(x))d\mu(x, y) = 0 \, ,$$

as required. □

We remark that 2.1.2 actually gives a complete description of lat $(I_{\mathcal{K}} \otimes \mathfrak{A}_{\min})$; see Corollary 1 below. We come now to the principal considerations of this section.

PROPOSITION 2.1.3. *The weakly closed convex hull of* $\mathfrak{L}(X, \leqq, m)$ *consists of all multiplications* L_f *such that* f *is an essentially increasing Borel function with* $0 \leqq f \leqq 1$.

Proof. Let $\mathcal{C}$ be the weakly closed convex hull of $\mathfrak{L}(X, \leqq, m)$, and choose an operator A in $\mathcal{C}$. Clearly A belongs to the multiplication algebra M, and $0 \leqq A \leqq I$. Thus there is a function $f \in L^\infty(X, m)$, $0 \leqq f \leqq 1$, such that $A = L_f$. We want to show that f is essentially increasing.

Now since the weak and strong operator topologies have the same continuous linear functionals, they also have the same closed convex sets. Moreover, since $L^2(X, m)$ is separable, the strong topology on the unit ball of $\mathfrak{L}(L^2(X, m))$ is metrizable. We conclude that there is a sequence A_n of operators, each of which is a convex linear combination of elements of $\mathfrak{L}(X, \leqq, m)$, which converges strongly to L_f. By definition of $\mathfrak{L}(X, \leqq, m)$, each A_n has the form L_{f_n}, where each f_n is a finite convex combination of characteristic functions of *increasing* Borel sets; in particular, each f_n is an increasing function.

Now since $L_{f_n} \to L_f$ strongly, we have

$$\int |f_n(x) - f(x)|^2 \, |g(x)|^2 dm(x) \longrightarrow 0$$

for every nonvanishing function $g \in L^2(X, m)$; fixing such a function g, a familiar argument shows that we may extract a subsequence $f_{n'}$ of f_n which converges to f on the complement of some m-null set N. Now if $x, y \in X \backslash N$ and $x \leqq y$, then

$$f(x) = \lim_{n'} f_{n'}(x) \leqq \lim_{n'} f_{n'}(y) = f(y) ,$$

which shows that f is essentially increasing.

Conversely, let f be an essentially increasing Borel function for which $0 \leqq f \leqq 1$. To show that $L_f \in \mathcal{C}$, we make use of the spectral theorem as follows. For every Borel set E in the unit interval, let $P(E)$ be the projection of $L^2(X, m)$ onto the subspace of all functions which live in $\{x: f(x) \in E\}$. Then the spectral theorem asserts that $L_f = \int_0^1 tP(dt)$, in the sense that $(L_f\xi, \eta) = \int_0^1 t(P(dt)\xi, \eta)$ for all ξ, $\eta \in L^2(X, m)$. Now fix ξ, η, and consider the complex measure σ on $[0, 1]$ defined by $\sigma(dt) = (P(dt)\xi, \eta)$. Since the function $t \mapsto \sigma([0, t))$ is bounded and left-continuous, the usual integration-by-parts formula implies that

$$\int_0^1 t\sigma(dt) + \int_0^1 \sigma([0, t))dt = \sigma([0, 1]) ;$$

or equivalently, $\int_0^1 t\sigma(dt) = \int_0^1 \sigma([t, 1])dt$. We conclude that

$$(L_f\xi, \eta) = \int_0^1 (P([t, 1])\xi, \eta)dt ,$$

for every ξ, $\eta \in L^2(X, m)$. A standard separation theorem now shows that L_f belongs to the weakly closed convex hull of the set of projections $\{P([t, 1]): 0 \leqq t \leqq 1\}$.

Thus, it suffices to show that each projection $P([t, 1])$ belongs to $\mathfrak{L}(X, \leqq, m)$, But since f is essentially increasing, there is a null set $N \subseteqq X$ such that, for all x, $y \in X \backslash N$, $x \leqq y$ implies $f(x) \leqq f(y)$. Fixing t, we see that if x, $y \in X\backslash N$, $x \leqq y$, and $x \in \{f \geqq t\}$, then $y \in \{f \geqq t\}$. By 1.1.11, the projection defined by the set $\{f \geqq t\}$, namely $P([t, 1])$, belongs to $\mathfrak{L}(X, \leqq, m)$. □

Remarks. The above convex set is weakly compact, and therefore it is of interest to know its extreme points. Clearly every projection in $\mathfrak{L}(X, \leqq, m)$ is extreme (as it is an extreme point of the entire unit ball of M), and conversely it is not hard to see that these are the only extreme points. It follows that the extreme points of the convex set of all essentially increasing functions $f \in L^\infty(X, m)$, for which $0 \leqq f \leqq 1$, are precisely the characteristic functions of increasing Borel sets. We shall return to these and related considerations in Chapter 3 (Section 3.5).

LEMMA 2.1.4. *Let $\mathfrak{A}$ be an algebra of operators on $L^2(X, m)$ such that $M\mathfrak{A}M \subseteqq \mathfrak{A}$ and which satisfies* lat $\mathfrak{A} = \mathfrak{L}(X, \leqq, m)$. *Then for every function $f \in L^\infty(X, m)$, $0 \leqq f \leqq 1$, the following are equivalent:*

(i) *f is essentially increasing,*

(ii) *$TL_fT^* \leqq L_f$ for every T in the unit ball of $\mathfrak{A}$.*

Proof. (i) *implies* (ii). Let f be essentially increasing, $0 \leqq f \leqq 1$. Now the set of all operators A, $0 \leqq A \leqq I$, for which $TAT^* \leqq A$ for every T in the unit ball of $\mathfrak{A}$, is clearly weakly closed and convex. By 2.1.3, to show that L_f belongs to this set it suffices to show that $TPT^* \leqq P$ for every $P \in \mathfrak{L}(X, \leqq, m)$, $T \in$ ball $\mathfrak{A}$. But such a P is $\mathfrak{A}$-invariant, so that $TPT^* = PTPT^*P \leqq \|T\|^2P^2 \leqq P$, as required.

(ii) *implies* (i). Let $f \in L^\infty(X, m)$ satisfy the condition (ii). First, we claim that $TL_{f^n}T^* \leqq L_{f^n}$ for every integer $n \geqq 1$. For that, choose $\varepsilon > 0$, and put $g(x) = (f(x) + \varepsilon)^{1/2}$. Then L_g is a positive invertible element of the multiplication algebra, and for each $T \in \mathfrak{A}$, $\|T\| \leqq 1$, we assert that $\|L_g^{-1}TL_g\| \leqq 1$. Indeed, $TL_g^2T^* = T(L_f + \varepsilon I)T^* \leqq L_f + \varepsilon TT^* \leqq L_f + \varepsilon I = L_g^2$, so that $L_g^{-1}TL_g^2T^*L_g^{-1} \leqq I$; i.e., $L_g^{-1}TL_g$ is a contraction. Of course, $L_g^{-1}TL_g$ also belongs to $\mathfrak{A}$. Now replace T with $L_g^{-1}TL_g$ and argue the same way to deduce that $L_g^{-2}TL_g^2$ is a contraction in $\mathfrak{A}$. Continuing in this way inductively, we see in particular that $\|L_g^{-n}TL_g^n\| \leqq 1$ for every $n \geqq 1$; equivalently, $L_g^{-n}TL_g^{2n}T^*L_g^{-n} \leqq I$. Multiplication on left and right by L_g^n gives $TL_{g^{2n}}T^* \leqq L_{g^{2n}}$, or $TL_{(f+\varepsilon)^n}T^* \leqq L_{(f+\varepsilon)^n}$, for every $n \geqq 1$. The claim follows by allowing ε to decrease to 0.

We claim next that, for each $t > 0$, the set $\{x: f(x) > t\}$ defines an $\mathfrak{A}$-invariant subspace of $L^2(X, m)$; or, what is the same, that the space of all $L^2(X, m)$ functions which live in $\{f \leqq t\}$ is invariant under $\mathfrak{A}^*$. For that, choose $g \in L^2(X, m)$ such that $g = 0$ off $\{f \leqq t\}$. Then $|f(x)^n g(x)| \leqq t^n|g(x)|$ for every $n \geqq 1$, $x \in X$, so that by the preceding paragraph we have $\|L_{f^n}T^*g\|^2 \leqq \|L_{f^n}g\|^2 \leqq t^{2n}\|g\|^2$, $n = 1, 2, \cdots$. Thus, dividing by t^{2n} and taking n^{th} roots of both sides we obtain

$$\left(\int_X \left(\frac{f(x)}{t}\right)^{2n} |T^*g(x)|^2 dm(x)\right)^{1/n} \leqq \|g\|^{2/n},$$

$n = 1, 2, \cdots$. As $n \to \infty$, the right side tends to 1 (or 0 if $g = 0$), and the left side tends to the essential supremum of the function $(f(x)/t)^2$ relative to the finite positive measure $|T^*g(x)|^2dm(x)$. Thus $(f(x)/t) \leqq 1$ for almost every x in the set $\{x \in X: |T^*g(x)| \neq 0\}$; equivalently, T^*g lives essentially in the set $\{f \leqq t)$, which is what we wanted to show.

We can now show that f is essentially increasing. Let R denote the countable set of all positive rational real numbers. For each $t \in R$, the projection in M determined by the set $\{f > t\}$ is $\mathfrak{A}$-invariant, so by hypothesis it belongs to $\mathfrak{L}(X, \leqq, m)$. By definition of $\mathfrak{L}(X, \leqq, m)$ there is an *increasing* Borel set $E_t \subseteqq X$ which gives rise to the same projection, and therefore the symmetric difference $N_t = E_t \Delta \{f > t\}$ has measure zero. Thus, $N = \bigcup_{t \in R} N_t$ is a Borel set of measure zero. We claim: if $x \leqq y$ and neither x nor y belongs to N, then $f(x) \leqq f(y)$ (this will complete the proof). Indeed, suppose $f(x) > f(y)$. Then we may choose $t \in R$ such that $f(x) > t > f(y)$, and hence $x \in \{f > t\}$ while $y \notin \{f > t\}$. But $\{f > t\}\backslash N = E_t \backslash N$. Since x belongs to the left side and $y \geqq x$ we have $y \in E_t$, and thus $y \in E_t \backslash N = \{f > t\} \backslash N \subseteqq \{f > t\}$, contradicting the preceding sentence. □

Remark. It may be of interest to isolate a fact contained in the proof of the above two results. Let $\mathfrak{A}$ be an algebra of operators on a Hilbert space $\mathcal{H}$ and let $A \in \mathfrak{L}(\mathcal{H})$ be such that $0 \leqq A \leqq I$. Let $A = \int_0^1 tP(dt)$ be the spectral resolution of A. Then the second half of the preceding proof that (ii) $\Rightarrow$ (i) shows essentially that if $TA^n T^* \leqq \| T \|^2 A^n$ for every $T \in \mathfrak{A}$ and every $n \geqq 1$, then every projection $P([t, 1])$, $0 \leqq t \leqq 1$, belongs to lat $\mathfrak{A}$ (the converse is also true, and follows from a minor variation of the "integration by parts" device in the proof of 2.1.3). But more significantly, the proof also implies that if the operator A happens to belong to $\mathfrak{A}$, then the same conclusion follows from the much weaker hypothesis that $TAT^* \leqq \| T \|^2 A$, for every $T \in \mathfrak{A}$. This stronger form is essential for the proof of the following key theorem.

THEOREM 2.1.5. *Let $\mathfrak{A}$ be an algebra of operators on $L^2(X, m)$ which contains the multiplication algebra and satisfies* $\operatorname{lat} \mathfrak{A} = \mathfrak{L}(X, \leqq, m)$. *Then every element of* $\operatorname{lat}(I_{\mathcal{K}} \otimes \mathfrak{A})$ *has the form L_P, where $P: X \rightarrow \mathfrak{L}(\mathcal{K})$ is an essentially increasing projection-valued Borel function.*

Proof. Now $\operatorname{lat}(I_{\mathcal{K}} \otimes \mathfrak{A})$ is contained in $\operatorname{lat}(I_{\mathcal{K}} \otimes M)$ (because $\mathfrak{A}$ contains M), so by the remarks preceding 2.1.2, every element of $\operatorname{lat}(I_{\mathcal{K}} \otimes \mathfrak{A})$ has the form L_P, where $P: X \rightarrow \mathfrak{L}(\mathcal{K})$ is a projection-valued Borel function. We have to produce a Borel set $N \subseteqq X$ of measure zero such that, for all $x, y \in X \backslash N$, $x \leqq y$ implies $P(x) \leqq P(y)$.

Fix $\xi \in \mathcal{K}$, $\| \xi \| = 1$. We claim first that the scalar valued function $f(x) = (P(x)\xi, \xi)$ is essentially increasing. For this, it suffices to show that $TL_f T^* \leqq L_f$ for every contraction $T \in \mathfrak{A}$, by the preceding lemma. So choose a function $g \in L^2(X, m)$ and a contraction $T \in \mathfrak{A}$. Then

$$\begin{aligned}(TL_f T^* g, g) &= (L_f T^* g, T^* g) \\ &= \int_X (P(x)\xi, \xi) \, | (T^* g)(x) |^2 dm(x) .\end{aligned}$$

We may write the integrand as $\| P(x)\xi \|^2 \cdot | T^* g(x) |^2$, which is the square of the norm of the value of the ($\mathcal{K}$-valued) function $L_P(I_{\mathcal{K}} \otimes T^*)\xi \otimes g$ at the point x (indeed, the latter is simply

$$P(x)[(I_{\mathcal{K}} \otimes T^*)\xi \otimes g](x) = P(x)(\xi \otimes T^*g)(x) = T^*g(x)\cdot(P(x)\xi)) .$$

Thus, we see that

$$(TL_fT^*g, g) = \| L_P(I_{\mathcal{K}} \otimes T^*)\xi \otimes g \|^2 .$$

Now since L_P is invariant under $I_{\mathcal{K}} \otimes T$, we have $L_P(I_{\mathcal{K}} \otimes T^*) = L_P(I_{\mathcal{K}} \otimes T^*)L_P$, so that the right side of the preceding formula is at most $\| L_P(I_{\mathcal{K}} \otimes T^*) \|^2 \cdot \| L_P(\xi \otimes g) \|^2 \leqq \| L_P(\xi \otimes g) \|^2$. Noting finally that

$$\begin{aligned} \| L_P(\xi \otimes g) \|^2 &= \int_X \| P(x)\xi \|^2 \, | g(x) |^2 \, dm(x) \\ &= \int_X (P(x)\xi, \xi) \, | g(x) |^2 \, dm(x) \\ &= (L_f g, g) , \end{aligned}$$

we obtain the asserted inequality $(TL_fT^*g, g) \leqq (L_fg, g)$.

We now prove that P is essentially increasing. Let $\xi_1, \xi_2, \cdots$ be a dense sequence in the unit sphere of $\mathcal{K}$. By the preceding paragraph, we may find for each $n \geqq 1$ a Borel set N_n, such that $m(N_n) = 0$, and such that the restriction of the function $f_n(x) = (P(x)\xi_n, \xi_n)$ to $X\backslash N_n$ is increasing. Let N be the null set $N = \bigcup_{n=1}^{\infty} N_n$. Choose $x, y \in X\backslash N$ such that $x \leqq y$; then note that $P(x) \leqq P(y)$. Indeed, since *every* f_n is increasing on $X\backslash N$ we have $(P(x)\xi_n, \xi_n) \leqq (P(y)\xi_n, \xi_n)$ for every $n \geqq 1$, therefore $(P(x)\xi, \xi) \leqq (P(y)\xi, \xi)$ for every $\xi \in \mathcal{K}$ of unit norm, because of the density of $\xi_1, \xi_2, \cdots$. This clearly implies $P(x) \leqq P(y)$, completing the proof. □

COROLLARY 1. lat $(I_{\mathcal{K}} \otimes \mathcal{A}_{\min})$ *consists precisely of all multiplications* L_P, *where* $P: X \to \mathcal{L}(\mathcal{K})$ *is an essentially increasing projection-valued Borel function.*

Proof. We already know that $\mathcal{A}_{\min}$ contains M and has $\mathcal{L}(X, \leqq, m)$ as its invariant projection lattice. So the conclusion is immediate from 2.1.5 and 2.1.2. □

COROLLARY 2. *Let* $\mathcal{A}$ *be any ultraweakly closed algebra of operators on* $L^2(X, m)$ *which contains the multiplication algebra and has* $\mathcal{L}(X, \leqq, m)$ *as its invariant subspace lattice. Then* $\mathcal{A}$ *contains* $\mathcal{A}_{\min}$.

Proof. By 2.1.5 and 2.1.2, we know that lat $(I_{\mathcal{K}} \otimes \mathcal{A})$ is contained in lat $(I_{\mathcal{K}} \otimes \mathcal{A}_{\min})$. By the remarks preceding 2.1.1, this implies that $\mathcal{A}_{\min}$ is contained in the ultraweak closure of $\mathcal{A}$, namely $\mathcal{A}$. □

Corollary 2 will enable us to draw a number of general conclusions about operator algebras. First, we need to identify the diagonal of the algebra $\mathcal{A}_{\min}(X, \leqq, m)$ (recall that the *diagonal* of an algebra $\mathcal{A}$ is the self-adjoint subalgebra $\mathcal{A} \cap \mathcal{A}^*$). Let us define an equivalence relation $x \sim y$ in X to mean $x \leqq y$ *and* $y \leqq x$. Since $\leqq$ is a standard partial ordering, so is $\sim$ (indeed, if the functions $f_1, f_2, \cdots$ determine $\leqq$ in the sense of 1.1.1, then the sequence $f_1, -f_1, f_2, -f_2, \cdots$ determines the partial ordering $\sim$). Moreover, observe that the graph of $\sim$ is the "diagonal" $G \cap \tilde{G}$ of the graph G of $\leqq$, where $\tilde{G}$ denotes the image of G under the canonical reflection $(x, y) \mapsto (y, x)$ of $X \times X$; in particular, $\mathcal{A}_{\min}(X, \sim, m)$ is a subalgebra of $\mathcal{A}_{\min}(X, \leqq, m)$. Now the mapping $\mu \mapsto \mu^*$ of measures on $X \times X$, defined by $\mu^*(dx, dy) = \bar{\mu}(dy, dx)$, induces an isometric involution of the algebra $A(G \cap \tilde{G}, m)$ which satisfies $T_{\mu^*} = (T_\mu)^*$ (see the discussion following 1.5.4). It follows that the set of operators $\{T_\mu : \mu \in A(G \cap \tilde{G}, m)\}$ is a self-adjoint algebra, and thus its ultraweak closure $\mathcal{A}_{\min}(X, \sim, m)$ *is a von Neumann algebra*, which of course contains the multiplication algebra.

PROPOSITION 2.1.6. *The three von Neumann algebras* $\mathcal{A}_{\min}(X, \sim, m)$, *the diagonal of* $\mathcal{A}_{\min}(X, \leqq, m)$ *and the commutant of* $\mathcal{L}(X, \leqq, m)$, *are identical.*

Proof. From the preceding remarks we see that $\mathcal{A}_{\min}(X, \sim, m)$ is contained in the diagonal of $\mathcal{A}_{\min}(X, \leqq, m)$ which, in turn, is *reduced* by every projection in $\mathcal{L}(X, \leqq, m)$. Thus, it suffices to show that $\mathcal{L}(X, \leqq, m)'$ is contained in $\mathcal{A}_{\min}(X, \sim, m)$; and by the double commutant theorem for von Neumann algebras, this reduces to showing that lat $\mathcal{A}_{\min}(X, \sim, m)$ is contained in the double commutant of $\mathcal{L}(X, \leqq, m)$.

Now by 1.6.1, we know that lat $\mathcal{A}_{\min}(X, \sim, m) = \mathcal{L}(X, \sim, m)$; and we claim that $\mathcal{L}(X, \sim, m)$ is contained in the von Neumann algebra generated by $\mathcal{L}(X, \leqq, m)$. For that, choose a sequence $E_1, E_2, \cdots$ of Borel sets in X such that $x \leqq y$ if and only if $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for every $n \geqq 1$ (cf. 1.1.2). Letting E_n' denote the complement of E_n, then note that $x \sim y$ if and only if $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ and $\chi_{E_n'}(x) \leqq \chi_{E_n'}(y)$ for every $n \geqq 1$. By 1.2.2, the projections $\{P_{E_1}, P_{E_1'}, P_{E_2}, P_{E_2'}, \cdots\}$ generate $\mathcal{L}(X, \sim, m)$ as a subspace lattice. Since each P_{E_n} and $P_{E_n'} = I - P_{E_n}$ belong to the von Neumann algebra generated by $\mathcal{L}(X, \leqq, m)$ (indeed, $P_{E_n} \in \mathcal{L}(X, \leqq, m)$ for every n), we conclude that $\mathcal{L}(X, \sim, m)$ is also contained in the latter. □

The following property of operator algebras will prove to be useful in the sequel.

DEFINITION 2.1.7. *An operator algebra* $\mathcal{A}$ *is called pre-reflexive if* $\mathcal{A} \cap \mathcal{A}^* = (\text{lat}\, \mathcal{A})'$.

Note that the commutant of lat $\mathcal{A}$ is simply the diagonal of the algebra alg lat $\mathcal{A}$, so that $\mathcal{A}$ is pre-reflexive if and only if it has the same *diagonal* as alg lat $\mathcal{A}$. By 1.6.1 and 2.1.6, we see that the algebras $\mathcal{A}_{\min}(X, \leqq, m)$ are always pre-reflexive (though they are not always reflexive, see Section

2.5). We are now in position to draw two general conclusions about operator algebras.

THEOREM 2.1.8. (i) *Every ultraweakly closed algebra which contains a maximal abelian von Neumann algebra is pre-reflexive.*

(ii) *Let* $\mathfrak{L}$ *be a commutative subspace lattice, and let* $\mathfrak{C}$ *be the class of all ultraweakly closed pre-reflexive algebras* $\mathfrak{A}$ *which satisfy* $\operatorname{lat}\mathfrak{A} = \mathfrak{L}$. *Then* $\mathfrak{C}$ *has a smallest element* $\mathfrak{A}_{\min}$ *(as well as a largest element* $\operatorname{alg}\mathfrak{L}$*), relative to inclusion.*

Proof. For (i), let $\mathfrak{A}$ be an ultraweakly closed algebra which contains a maximal abelian von Neumann algebra M. By a simple adaptation of the proof of 1.3.1, there is a standard partially ordered sigma-finite measure space $(X, \leqq, m)$ and a unitary operator from the underlying space for $\mathfrak{A}$ onto $L^2(X, m)$ which carries M to the multiplication algebra of (X, m) and carries $\operatorname{lat}\mathfrak{A}$ (a sublattice of the projection lattice in M) to $\mathfrak{L}(X, \leqq, m)$ (i.e., just take M for the maximal abelian algebra occurring in the proof of 1.3.1). Letting $\mathfrak{A}_1$ be the image of $\mathfrak{A}$, we know from Corollary 2 that $\mathfrak{A}_{\min}(X, \leqq, m) \subseteqq \mathfrak{A}_1 \subseteqq \operatorname{alg}\mathfrak{L}(X, \leqq, m)$, so that the three diagonals are related in the same way: $\mathfrak{A}_{\min}(X, \leqq, m) \cap \mathfrak{A}_{\min}(X, \leqq, m)^* \subseteqq \mathfrak{A}_1 \cap \mathfrak{A}_1^* \subseteqq \mathfrak{L}(X, \leqq, m)'$. But by 2.1.6, the two extreme members are the same, and hence $\mathfrak{A} \cap \mathfrak{A}^* = (\operatorname{lat}\mathfrak{A})'$.

For (ii), let M be any *particular* maximal abelian von Neumann algebra which contains the lattice $\mathfrak{L}$. As in the preceding paragraph, we may assume that the Hilbert space is $L^2(X, m)$, that $\mathfrak{L} = \mathfrak{L}(X, \leqq, m)$, and that M is the multiplication algebra of (X, m). Note first that since $\operatorname{lat}\mathfrak{A}_{\min}(X, \leqq, m) = \mathfrak{L}(X, \leqq, m)$ and $\mathfrak{A}_{\min}(X, \leqq, m) \cap \mathfrak{A}_{\min}(X, \leqq, m)^* = \mathfrak{L}(X, \leqq, m)'$ (by 1.6.1 and 2.1.6), $\mathfrak{A}_{\min}(X, \leqq, m)$ belongs to $\mathfrak{C}$. On the other hand, if $\mathfrak{A}$ is any other element of $\mathfrak{C}$, then $M \subseteqq \mathfrak{L}(X, \leqq, m)' = \mathfrak{A} \cap \mathfrak{A}^* \subseteqq \mathfrak{A}$. By Corollary 2 we conclude that $\mathfrak{A}_{\min}(X, \leqq, m) \subseteqq \mathfrak{A}$, hence $\mathfrak{A}_{\min}(X, \leqq, m)$ is the smallest element of $\mathfrak{C}$. □

Remarks. Part (ii) of this theorem describes the precise sense in which the algebra $\mathfrak{A}_{\min}(X, \leqq, m)$ is independent of the representation implicit in the symbols $(X, \leqq, m)$. The theorem also suggests what will prove to be a useful division of the problem of determining when a given algebra $\mathfrak{A}$ is reflexive. One first determines whether or not $\mathfrak{A}$ is pre-reflexive. Granting that, one then seeks to prove that there is a *unique* pre-reflexive (ultraweakly closed) algebra which has $\operatorname{lat}\mathfrak{A}$ as its invariant subspace lattice. If the answer is yes, then $\mathfrak{A}$ must be reflexive. While the second step asks for more than we actually need, it has the advantage that, since it involves the lattice lat $\mathfrak{A}$ rather than the algebra $\mathfrak{A}$, it is frequently more convenient to deal with. These matters will be taken up in the following section.

The next result strengthens a theorem of [24] (in the case where the underlying space is separable).

COROLLARY 1. *Let $\mathfrak{A}$ be an ultraweakly closed algebra which contains a maximal abelian von Neumann algebra, such that every $\mathfrak{A}$-invariant subspace reduces $\mathfrak{A}$. Then $\mathfrak{A}$ is self-adjoint.*

Proof. Because lat $\mathfrak{A}$ is closed under orthocomplementation, alg lat $\mathfrak{A} =$ (lat $\mathfrak{A})'$ is self-adjoint. The conclusion now follows from part (i) of 2.1.8. □

Remark. This corollary can be used to strengthen a density theorem of [1] in the following way: *every transitive subalgebra of $\mathfrak{L}(\mathfrak{H})$ which contains a maximal abelian von Neumann algebra is ultraweakly dense in $\mathfrak{L}(\mathfrak{H})$.* Indeed, letting $\mathfrak{A}$ denote the ultraweak closure of the given algebra, we see from the corollary that $\mathfrak{A} = \mathfrak{A}^*$ is an irreducible von Neumann algebra, so that $\mathfrak{A} = \mathfrak{L}(\mathfrak{H})$ (by the double commutant theorem).

The following application to algebras of compact operators would not have been available had we worked with weak, rather than ultraweak, operator topologies. First, a simple observation.

LEMMA. *Let $\mathfrak{B}$ be an algebra of operators on a Hilbert space such that every cyclic invariant subspace of $\mathfrak{B}$ reduces $\mathfrak{B}$, which has trivial null space. Then $\xi \in [\mathfrak{B}\xi]$ for every $\xi \in \mathfrak{H}$.*

Proof. Choose $\xi \in \mathfrak{H}$. The hypothesis means that the subspace $[\mathfrak{B}\xi]$ reduces $\mathfrak{B}$. Let P be the projection onto $[\mathfrak{B}\xi]$. Then P commutes with $\mathfrak{B}$, and for every $B \in \mathfrak{B}$ we have $B(\xi - P\xi) = BP^{\perp}\xi = P^{\perp}B\xi = 0$, since $P^{\perp}$ annihilates $[\mathfrak{B}\xi]$. We conclude that $\xi - P\xi = 0$ because $\mathfrak{B}$ has trivial null space. □

COROLLARY 2. *Let $\mathfrak{A}$ be a norm-closed algebra of compact operators on a Hilbert space $\mathfrak{H}$ such that $\mathfrak{A} \neq \mathfrak{C}(\mathfrak{H})$. Suppose there is a commutative $*$-algebra $\mathfrak{D}$ such that $\mathfrak{D}'$ is commutative and $\mathfrak{D}\mathfrak{A}\mathfrak{D} \subseteqq \mathfrak{A}$. Then $\mathfrak{A}$ has a proper closed invariant subspace.*

Proof. Contrapositively, suppose $\mathfrak{A}$ is transitive. We may clearly assume that $\mathfrak{D}$ contains the identity, and therefore the ultraweak closure $\mathfrak{D}_1$ of $\mathfrak{D}$ is a maximal abelian von Neumann algebra. Let $\mathfrak{A}_1$ be the ultraweak closure of $\mathfrak{A}$. Note that $\mathfrak{A}_1$ has trivial null space (because of the transitivity hypothesis), and we have $\mathfrak{D}_1\mathfrak{A}_1\mathfrak{D}_1 \subseteqq \mathfrak{A}_1$

We claim first that $\mathfrak{A}_1$ contains the identity. For that, let $\mathcal{K}$ be a separable infinite-dimensional Hilbert space. Then the claim amounts to showing that $\xi \in [I_{\mathcal{K}} \otimes \mathfrak{A}_1\xi]$ for every $\xi \in \mathcal{K} \otimes \mathcal{H}$. Since $I_{\mathcal{K}} \otimes \mathfrak{A}_1$ has trivial null space, the lemma leaves us to prove that every subspace $[I_{\mathcal{K}} \otimes \mathfrak{A}_1\xi]$ reduces $I_{\mathcal{K}} \otimes \mathfrak{A}_1$. Now as before, we may assume that $\mathcal{H} = L^2(X, m)$, where (X, m) is a standard measure space and $\mathfrak{D}_1$ is the multiplication algebra of (X, m). Because $\mathfrak{A}_1$ is transitive we have lat $\mathfrak{A}_1 = \mathfrak{L}(X, \leqq, m)$ where $\leqq$ is the indiscrete order, in which $x \leqq y$ for all $x, y \in X$. Now since $\mathfrak{D}_1\mathfrak{A}_1 \subseteqq \mathfrak{A}_1$, it follows that, for each $\xi \in \mathcal{K} \otimes L^2(X, m)$, the subspace $[I_{\mathcal{K}} \otimes \mathfrak{A}_1\xi]$ is invariant under $I_{\mathcal{K}} \otimes \mathfrak{D}_1$. We may now argue exactly as in the proof of 2.1.5 to conclude that the projection onto $[I_{\mathcal{K}} \otimes \mathfrak{A}_1\xi]$ has the form L_P, where $P: X \to \mathfrak{L}(\mathcal{K})$ is an essentially increasing projection-valued Borel function. Since the ordering on X is indiscrete, this means that $P(x)$ is essentially a constant P_0; i.e., $L_P = P_0 \otimes I_{\mathcal{H}}$. The latter projection is invariant under $I_{\mathcal{K}} \otimes \mathfrak{L}(\mathcal{H})$, so in particular it reduces $I_{\mathcal{K}} \otimes \mathfrak{A}_1$.

Now because $I \in \mathfrak{A}_1$ and $\mathfrak{D}_1\mathfrak{A}_1 \subseteqq \mathfrak{A}_1$, it follows that $\mathfrak{A}_1$ contains $\mathfrak{D}_1$. By the remark following Corollary 1 of 2.1.8, $\mathfrak{A}_1 = \mathfrak{L}(\mathcal{H})$.

Finally, we claim that $\mathfrak{A} = \mathcal{C}(\mathcal{H})$ (this contradiction will complete the proof). Let ρ be a bounded linear functional on $\mathcal{C}(\mathcal{H})$ which annihilates $\mathfrak{A}$. We will show that $\rho = 0$. Indeed, it is well known that ρ extends (uniquely) to an ultraweakly continuous linear functional $\tilde{\rho}$ on $\mathfrak{L}(\mathcal{H})$ ([11, §4.1]). Since $\tilde{\rho}(\mathfrak{A}) = \rho(\mathfrak{A}) = 0$, it follows by ultraweak continuity that $\tilde{\rho}(\mathfrak{A}_1) = 0$. But $\mathfrak{A}_1 = \mathfrak{L}(\mathcal{H})$, by the preceding paragraph, and therefore $\tilde{\rho} = 0$. Hence $\rho = 0$, as required. □

We conclude this section with some applications to the triangular algebras of Kadison and Singer [17]. Recall that an operator algebra $\mathfrak{A}$ is called *triangular* if $\mathfrak{A} \cap \mathfrak{A}^*$ is a maximal abelian von Neumann algebra. Triangular algebras are related to the algebras studied here. For example, let $(X, \leqq, m)$ be a partially ordered measure space whose ordering is *strict*. By Theorem 1.2.3 and Proposition 2.1.6, we see that $\mathfrak{A}_{\min}(X, \leqq, m)$ (as well as alg $\mathfrak{L}(X, \leqq, m)$) is triangular.

It was pointed out in [17] that if $\mathfrak{A}$ is a triangular algebra on a finite dimensional space $\mathcal{H}$, then there is an orthonormal basis for $\mathcal{H}$ with respect to which the matrices of the operators in $\mathfrak{A}$ are all upper triangular ($\mathfrak{A}$ need not contain all upper triangular matrices, but it does contain all diagonal matrices). By analogy with this property, we will say an operator algebra $\mathfrak{A}$ is *hyperintransitive* if lat $\mathfrak{A}$ contains a linearly ordered subset whose generated von Neumann algebra is maximal abelian in $\mathfrak{L}(\mathcal{H})$. Kadison and Singer also gave an example of a triangular algebra on a separable space which is not hyperintransitive (in fact, which has no nontrivial invariant subspaces whatsoever). Their example, however, is not closed in any of the natural operator topologies. In [2, p. 101] (also see [4]), some related examples of triangular algebras are described, which have no invariant subspaces and are closed in the *norm* topology. Thus, the following result would seem to be the best possible generalization of the finite-dimensional result cited above.

COROLLARY 3. *Every ultraweakly closed triangular algebra on a separable space is hyperintransitive.*

Proof. Let $\mathfrak{A}$ be the given algebra. By part (i) of 2.1.8, we know that $(\mathrm{lat}\,\mathfrak{A})' = \mathfrak{A} \cap \mathfrak{A}^*$ is a maximal abelian von Neumann algebra. Therefore, by the double commutant theorem, the von Neuman algebra generated by $\mathrm{lat}\,\mathfrak{A}$ is the maximal abelian von Neumann algebra $\mathfrak{A} \cap \mathfrak{A}^*$.

Thus it suffices to establish the following assertion. Let $\mathfrak{L}$ be a subspace lattice on a separable space which generates a maximal abelian von Neumann algebra M; then $\mathfrak{L}$ contains a linearly ordered subset $\mathfrak{L}_0$ which generates M as a von Neumann algebra. For the proof, let $\{P_1, P_2, \cdots\}$ be a countable subset of $\mathfrak{L}$ which contains 0 and I and is strongly dense in $\mathfrak{L}$. By adjoining all finite unions and intersections, if necessary, we may assume that $\{P_n\}$ is closed under the lattice operations $\vee$ and $\wedge$. We assert: there is an increasing sequence $\mathfrak{L}_1 \subseteqq \mathfrak{L}_2 \subseteqq \cdots$ of finite linearly ordered subsets of $\{P_n\}$ such that the Boolean algebra generated by $\mathfrak{L}_n$ contains $0, P_1, P_2, \cdots, P_n, I$, for every $n \geqq 1$. Granting that, the linearly ordered set $\mathfrak{L}_0 = \bigcup_{n=1}^{\infty} \mathfrak{L}_n$ clearly generates the same Boolean algebra as $\{P_n\}$, so that the von Neumann algebra generated by $\mathfrak{L}_0$ is simply M, as required (see the discussion preceding 1.2.3).

$\mathfrak{L}_n$ is constructed inductively as follows. Let $\mathfrak{L}_1 = \{0, P_1, I\}$. Given $\mathfrak{L}_1 \subseteqq \mathfrak{L}_2 \subseteqq \cdots \subseteqq \mathfrak{L}_n$ all satisfying the stated property, let $0 = E_0 \leqq E_1 \leqq \cdots \leqq E_r = I$ be the elements of $\mathfrak{L}_n$. For each k, $0 \leqq k \leqq r-1$, put $F_k = E_k \vee (P_{n+1} \wedge E_{k+1})$. Then for each $k < r$ we have $E_k \leqq F_k \leqq E_{k+1}$, so that $\mathfrak{L}_{n+1} = \{E_0, \cdots, E_r\} \cup \{F_0, \cdots, F_{r-1}\}$ is a linearly ordered subset of $\{P_j\}$ which contains $\mathfrak{L}_n$. Let $\mathfrak{B}_{n+1}$ be the Boolean algebra generated by $\mathfrak{L}_{n+1}$. For each k, $0 \leqq k < r-1$, we have $F_k - E_k = P_{n+1} \wedge (E_{k+1} - E_k)$, and therefore $\bigvee_{k=0}^{r-1} (F_k - E_k) = P_{n+1}$ because $\{E_{k+1} - E_k : 0 \leqq k \leqq r-1\}$ forms a partition of I. This proves that $\mathfrak{B}_{n+1}$ contains P_{n+1}; that it contains $0, P_1, \cdots P_n, I$ is a consequence of the induction hypothesis on $\mathfrak{L}_n$. □

Remark. Corollary 2 of course implies that every (separably acting) weakly closed triangular algebra is hyperintransitive, answering affirmatively a conjecture of Peter Rosenthal [27], [24]. This is of interest because such algebras need not be reflexive in general (see Section 2.5), so that the technique of [27] cannot be applied.

Though Corollary 3 is stated in an abstract form, it can be combined with some results of]17] to give a very concrete infinite-dimensional analogue of the finite-dimensional result which asserts that every triangular algebra is unitarily equivalent to an algebra of upper triangular matrices. For that, let $\mathcal{T}[0,1]$ denote the algebra of all operators on $L^2[0,1]$ which leave all of the subspaces $L^2[0, t]$, $0 \leqq t \leqq 1$, invariant. $\mathcal{T}[0, 1]$ is a hyperintransitive maximal triangular algebra, and occupies a position analogous to that of the algebra of all upper triangular $n \times n$ matrices.

THEOREM. *Every ultraweakly closed triangular algebra having a nonatomic diagonal is unitarily equivalent to a subalgebra of* $\mathcal{T}$ [0, 1].

Proof. Let $\mathfrak{A}$ be the given algebra. By Corollary 2, we may find a linearly ordered subset $\mathfrak{L}$ of lat $\mathfrak{A}$ which generates $\mathfrak{A} \cap \mathfrak{A}^*$ as a von Neumann algebra. Let $\mathfrak{A}_1 = \text{alg}\, \mathfrak{L}$. By [17, Theorem 3.1.1] $\mathfrak{A}_1$ is a hyperintransitive maximal triangular algebra (which of course contains $\mathfrak{A}$), and its diagonal is $\mathfrak{L}' = (\text{lat}\, \mathfrak{A})' = \mathfrak{A} \cap \mathfrak{A}^*$, a nonatomic von Neumann algebra. By [17, Theorem 3.3.1], $\mathfrak{A}_1$ is unitarily equivalent to $\mathcal{T}[0, 1]$. The assertion follows. □

2.2. Synthetic lattices

We now introduce a property of subspace lattices which will turn out to have a useful connection with the problem of determining which algebras are reflexive.

DEFINITION 2.2.1. *A reflexive lattice* $\mathfrak{L}$ *is called synthetic if the only ultraweakly closed pre-reflexive algebra* $\mathfrak{A}$ *satisfying* $\text{lat}\, \mathfrak{A} = \mathfrak{L}$ *is the algebra* $\mathfrak{A} = \text{alg}\, \mathfrak{L}$.

The connection with reflexive algebras arises through the observation that if one is able to prove that a given ultraweakly closed algebra $\mathfrak{A}$ is (a) pre-reflexive and (b) its lattice lat $\mathfrak{A}$ is synthetic, then one concludes that $\mathfrak{A}$ is reflexive.

We shall be concerned mainly with the question: which commutative subspace lattices are synthetic? Now if $\mathfrak{L}$ is a commutative subspace lattice then one knows from 2.1.8 (ii) that there is a *smallest* ultraweakly closed pre-reflexive algebra $\mathfrak{A}_{\min}$ having $\mathfrak{L}$ as its invariant subspace lattice. Thus $\mathfrak{L}$ is synthetic if and only if $\mathfrak{A}_{\min} = \text{alg}\, \mathfrak{L}$ is reflexive. Furthermore, by the preceding results we may assume that $\mathcal{L} = \mathcal{L}(X, \leqq, m)$ is the lattice associated with a standard partially ordered sigma-finite measure space, and that $\mathcal{A}_{\min} = \mathcal{A}_{\min}(X, \leqq, m)$ is the (closure of the) algebra of all pseudo-integral operators associated with $(X, \leqq, m)$. Thus, the problem is to characterize the partially ordered measure spaces $(X, \leqq, m)$ for which $\mathcal{A}_{\min}(X, \leqq, m)$ is reflexive. We will see that such a characterization exists, and involves a property of the *graph* of the relation $\leqq$ which is very closely akin to the sets of spectral synthesis of commutative harmonic analysis (this also explains the choice of terminology in definition 2.2.1).

First, we want to present a result that describes a wide variety of commutative synthetic lattices. Given a family $\{\mathcal{L}_i\}$ of subspace lattices on a Hilbert space $\mathcal{H}$, the notation $\bigvee_i \mathcal{L}_i$ denotes the subspace lattice generated by $\bigcup_i \mathcal{L}_i$, and $\mathcal{L}^\perp$ will denote the lattice $\{I - P: P \in \mathcal{L}\}$. By a *chain* we mean a subspace lattice with the property that any two of its elements are comparable.

DEFINITION 2.2.2. *A subspace lattice $\mathcal{L}$ is said to have finite width if there is a finite set of chains $\mathcal{C}_1, \cdots, \mathcal{C}_n$ such that $\mathcal{L} = \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n$. The width of $\mathcal{L}$ is defined as the smallest such integer n, when $\mathcal{L}$ has finite width, and is defined as ∞ otherwise.*

Remarks. Every finite lattice of course has finite width. We also want to point out that *every (separably acting) commutative Boolean algebra has width* 2 (provided the underlying space has dimension at least 2). To sketch the proof briefly, let $\mathcal{B}$ be a countable Boolean algebra which is strongly dense in the given Boolean algebra. Then it is known that $\mathcal{B}$ contains a linearly ordered subset $\mathcal{L}$ which generates $\mathcal{B}$ as a Boolean algebra (this also follows from the more general construction in the proof of Corollary 3 of 2.1.8). If $\mathcal{C}$ denotes the strong closure of $\mathcal{L}$, it follows that the two chains $\mathcal{C}$ and $\mathcal{C}^\perp$ generate the given Boolean algebra as a subspace lattice, from which the assertion is evident.

THEOREM 2.2.3. *Every commutative subspace lattice of finite width is synthetic.*

It is convenient to divide the proof into two lemmas. We shall write $\mathcal{A}_{\min}(\mathcal{L})$ for the smallest ultraweakly closed pre-reflexive algebra having a given commutative subspace lattice $\mathcal{L}$ as its invariant subspace lattice.

LEMMA 1. *Let $\mathcal{L}$ be a synthetic commutative subspace lattice and let $\mathcal{C}$ be a chain which commutes with $\mathcal{L}$. Then for every $P \in \mathcal{C}$, $P(\operatorname{alg} \mathcal{L})P^\perp \subseteq \mathcal{A}_{\min}(\mathcal{L} \vee \mathcal{C})$.*

Proof. Introduce coordinates so that $\mathcal{L} \vee \mathcal{C}$ becomes the lattice $\mathcal{L}(X, \leqq, m)$ associated with a standard sigma-finite partially ordered measure space. Choose sequences E_n (resp. F_n) of increasing Borel sets such that the projections $\{P_{E_n}\}$ (resp. $\{P_{F_n}\}$) are strongly dense in the sublattice $\mathcal{L}$ (resp. $\mathcal{C}$) of $\mathcal{L} \vee \mathcal{C} = \mathcal{L}(X, \leqq, m)$. We may now define weaker partial orderings $\leqq_1$ and $\leqq_2$ on X by $x \leqq_1 y$ if and only if $\chi_{E_n}(x) \leqq \chi_{E_n}(y)$ for all n, and $x \leqq_2 y$ if and only if $\chi_{F_n}(x) \leqq \chi_{F_n}(y)$ for all n. By construction, we have $\mathcal{L} = \mathcal{L}(X, \leqq_1, m)$ and $\mathcal{C} = \mathcal{L}(X, \leqq_2, m)$. So by 1.2.2 and its first corollary (and by discarding a null set if necessary), we may assume $\leqq_1$ and $\leqq_2$ satisfy $x \leqq y \Leftrightarrow x \leqq_1 y$ and $x \leqq_2 y$, and moreover that $\leqq_2$ is a *linear* ordering of X.

Now choose $P \in \mathcal{C} = \mathcal{L}(X, \leqq_2, m)$. Then there is an increasing set E (relative to $\leqq_2$) such that $P = P_E$. Because $\mathcal{L}(X, \leqq_1, m)$ is assumed synthetic, we know that alg $\mathcal{L} = \mathcal{A}_{\min}(\mathcal{L}) = \mathcal{A}_{\min}(X, \leqq_1, m)$ is the ultraweak closure of all pseudo-integral operators T_μ, where μ lives on the graph $\{(x, y) \in X \times X : y \leqq_1 x\}$ of $\leqq_1$. So to prove $P_E(\text{alg } \mathcal{L})P_E^\perp$ is contained in $\mathcal{A}_{\min}(\mathcal{L} \vee \mathcal{C}) = \mathcal{A}_{\min}(X, \leqq, m)$, it suffices to show that, for every such μ, $P_E T_\mu P_E^\perp$ has the form T_ν where $\nu \in A(X \times X, m)$ is concentrated on the graph of $\leqq$. For that, define a measure ν on $X \times X$ by $\nu(S) = \mu(S \cap E \times E^c)$. Then clearly $\nu \in A(X \times X, m)$, and since $\nu(dx, dy) = \chi_E(x)\chi_{E^c}(y)\mu(dx, dy)$ it is evident that $P_E T_\mu P_E^\perp = T_\nu$. Moreover, if G_i denotes the graph of $\leqq_i$, $i = 1, 2$, then ν is concentrated on $G_1 \cap (E \times E^c)$. Now since E is increasing relative to $\leqq_2$, no point (x, y) of $E \times E^c$ can satisfy $x \leqq_2 y$; and since $\leqq_2$ is *linear* we must therefore have $y \leqq_2 x$. Thus $E \times E^c \subseteqq G_2$, and we deduce that ν is concentrated on $G_1 \cap G_2$, the graph of $\leqq$, as required. □

Remark. Let $\mathcal{L}_1$, $\mathcal{L}_2$ be commutative subspace lattices on $\mathcal{H}$ such that $\mathcal{L}_1 \subseteqq \mathcal{L}_2$ Then we claim: $\mathcal{A}_{\min}(\mathcal{L}_2) \subseteqq \mathcal{A}_{\min}(\mathcal{L}_1)$. One way to see this is to introduce coordinates for $\mathcal{H}$ in such a way that $\mathcal{L}_2$ becomes the lattice $\mathcal{L}(X, \leqq_2, m)$ associated with the usual sort of partially ordered measure space. Arguing as in the proof of the preceding lemma, we may assume that $\mathcal{L}_1$ has the form $\mathcal{L}_1 = \mathcal{L}(X, \leqq_1, m)$ where $\leqq_1$ is a second standard partial ordering which, because $\mathcal{L}_1 \subseteqq \mathcal{L}_2$, can be chosen weaker than $\leqq_2$ in the sense that $x \leqq_2 y \Rightarrow x \leqq_1 y$. This means that the graph of $\leqq_2$ is contained in the graph of $\leqq_1$, so by definition of the algebras $\mathcal{A}_{\min}(X, \leqq_i, m)$, we see that $\mathcal{A}_{\min}(X, \leqq_2, m) \subseteqq \mathcal{A}_{\min}(X, \leqq_1, m)$, as asserted.

We do not know if it is possible to give a "coordinate-free" proof of this assertion.

LEMMA 2. *Let $\mathcal{L}$ be a commutative subspace lattice and let $\mathcal{C}$ be a* *chain which commutes with* $\mathcal{L}$. *Assume that both* $\mathcal{L}$ *and* $\mathcal{L} \vee \mathcal{C} \vee \mathcal{C}^{\perp}$ *are synthetic. Then* $\mathcal{L} \vee \mathcal{C}$ *is synthetic.*

Proof. Choose $T \in \operatorname{alg}(\mathcal{L} \vee \mathcal{C})$; we will prove that $T \in \mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C})$. For that, let $\{E_1, E_2, \cdots\}$ be a countable strongly dense subset of $\mathcal{C}$, with $E_1 = 0, E_2 = I$. Fix $n \geqq 1$. For each n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of ± 1's, define $E^{\alpha} = E_1^{\alpha_1} E_2^{\alpha_2} \cdots E_n^{\alpha_n}$, where $E_i^{\alpha_i}$ is defined as E_i if $\alpha_i = +1$ or $E_i^{\perp}$ if $\alpha_i = -1$. As α runs over all such n-tuples, E^{α} runs over the atoms of the Boolean algebra generated by $\{E_1, \cdots, E_n\}$ (some of the E^{α}'s may be 0, but each nonzero atom appears exactly once.) Thus, we may write $T = R_n + S_n$, where

$$R_n = \sum_{\alpha} E^{\alpha} T E^{\alpha} ,$$
$$S_n = \sum_{\alpha \neq \beta} E^{\alpha} T E^{\beta} .$$

We claim first that $S_n \in \mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C})$. It suffices to show that, for all $\alpha \neq \beta$, $E^{\alpha} T E^{\beta} \in \mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C})$. Now since $\alpha \neq \beta$ we must have $\alpha_i \neq \beta_i$ for some i, $1 \leqq i \leqq n$. If $\alpha_i = -1$ and $\beta_i = +1$, then $E^{\alpha} T E^{\beta}$ has the form $F E_i^{\perp} T E_i G = 0$, since T leaves E_i invariant. On the other hand, if $\alpha_i = +1$ and $\beta_i = -1$, then $E^{\alpha} T E^{\beta}$ has the form $F E_i T E_i^{\perp} G$, where F, G are projections in $\mathcal{C} \vee \mathcal{C}^{\perp}$. By Lemma 1 we know that $E_i T E_i^{\perp} \in \mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C})$, and since $\mathcal{C} \vee \mathcal{C}^{\perp} \subseteqq (\mathcal{L} \vee \mathcal{C})' \subseteqq \mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C})$, the claim follows.

Next, observe that R_n commutes with each atom E^{α}, and thus R_n commutes with $\{E_1, \cdots, E_n\}$. We also have $R_n \in \operatorname{alg} \mathcal{L}$ for every n. Since $\|R\| \leqq \|T\|$, we may (by a standard compactness argument) extract a weakly convergent subsequence $R_{n'}$; and since these operators are uniformly bounded, we in fact have $R_{n'} \to R$ in the ultraweak topology. It follows that $S_{n'} = T - R_{n'}$ converges ultraweakly, to S, say, and $T = R + S$. Because $\mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C})$ is ultraweakly closed, we have $S \in \mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C})$, and clearly $R \in \mathcal{C}' \cap \operatorname{alg} \mathcal{L} = \operatorname{alg}(\mathcal{L} \vee \mathcal{C} \vee \mathcal{C}^{\perp})$. But by hypothesis, $\mathcal{L} \vee \mathcal{C} \vee \mathcal{C}^{\perp}$ is synthetic, so that R belongs to $\mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C} \vee \mathcal{C}^{\perp})$ which, by the preceding remarks, is contained in $\mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C})$. We conclude that $T = R + S$ belongs to $\mathfrak{A}_{\min}(\mathcal{L} \vee \mathcal{C})$. □

Turning now to the proof of 2.2.3, it suffices to prove the following assertion: If $\mathcal{L}$ is any commutative subspace lattice of finite width, then $\mathcal{L} \vee \mathcal{B}$ is synthetic for every commutative Boolean algebra $\mathcal{B}$ such that $\mathcal{B} \cup \mathcal{L}$ is commutative. The proof uses induction on the width of $\mathcal{L}$. Assume first that $\mathcal{L}$ has width 1, i.e., is a chain. Then since both $\mathcal{B}$ and $\mathcal{B} \vee \mathcal{L} \vee \mathcal{L}^{\perp}$ are commutative Boolean algebras, they are synthetic (this is a trivial consequence of the definition of synthetic lattices). So by Lemma 2, $\mathcal{B} \vee \mathcal{L}$ is synthetic.

Assume the assertion is true for lattices of width n, let $\mathcal{L}$ be a commutative lattice of width $n+1$, and let $\mathcal{B}$ be a commutative Boolean algebra commuting with $\mathcal{L}$. Then we can write $\mathcal{L} = \mathcal{L}_0 \vee \mathcal{C}$ where $\mathcal{L}_0$ has width n and $\mathcal{C}$ is a chain. By the induction hypothesis, both $\mathcal{L}_0 \vee \mathcal{B}$ and $\mathcal{L}_0 \vee \mathcal{B} \vee \mathcal{C} \vee \mathcal{C}^{\perp}$ are synthetic. So again we conclude from Lemma 2 that $\mathcal{L}_0 \vee \mathcal{B} \vee \mathcal{C} = \mathcal{L} \vee \mathcal{B}$ is synthetic. □

Problem. It seems reasonable to conjecture that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are (mutually commuting) commutative subspace lattices such that $\mathcal{L}_1$ is synthetic and $\mathcal{L}_2$ has finite width, then $\mathcal{L}_1 \vee \mathcal{L}_2$ is synthetic. Note that Lemma 2 reduces this problem to the case where $\mathcal{L}_2$ is a Boolean algebra.

It is known that if $\mathcal{A}$ is a weakly closed algebra containing a maximal abelian von Neumann algebra, and if lat $\mathcal{A}$ is either a chain [23], or a necessarily commutative Boolean algebra [24], then $\mathcal{A}$ is reflexive. Since both chains and Boolean algebras have finite width, the following improves on both of these results.

COROLLARY. *Every ultraweakly closed algebra $\mathcal{A}$ which contains a maximal abelian von Neumann algebra, and for which* lat $\mathcal{A}$ *has finite width, is reflexive.*

Proof. By 2.1.8 (i), $\mathcal{A}$ is pre-reflexive, and by 2.2.3, lat $\mathcal{A}$ is synthetic. The conclusion $\mathcal{A} =$ alg lat $\mathcal{A}$ is therefore immediate. □

In order to study (commutative) synthetic lattices of infinite width, one must look further into the structure of the lattice $\mathcal{L}(X, \leqq, m)$. Throughout the remainder of this section, (X, m) will denote the standard space consisting of a *separable locally compact metric space* X together with a sigma-finite Borel measure m on X; we shall consider only partial orderings of X whose graph is closed in $X \times X$ (note that by 1.1.12, all such orderings are standard). While this additional topological structure is not actually necessary for carrying out the analysis, the topology provides an extremely convenient context for these somewhat awkward measure-theoretic considerations. Moreover, a simple device allows one to reduce the study of general standard partially ordered measure spaces to this setting.

DEFINITION 2.2.4. *The support of an operator T on $L^2(X, m)$ (written* sup(T)*) is defined as the set of all points $p \in X \times X$ with the following property: for every rectangular open neighborhood $U \times V$ of p, there exist* *functions* f, $g \in L^2(X, m)$, *supported in* U, V *respectively, such that* $(Tg, f) \neq 0$.

Remark. Note that $\sup(T)$ is a closed subset of $X \times X$; and in the special case of a pseudo-integral operator T_μ, $\mu \in A(X \times X, m)$, $\sup(T_\mu)$ is simply the closed support of the measure μ.

Now let F be an arbitrary closed subset of $X \times X$. We associate two linear spaces of operators with F as follows:

$$\mathfrak{A}_{\min}(F) = \{T_\mu : \mu \in A(F, m)\}^- ,$$
$$\mathfrak{A}_{\max}(F) = \{T : \sup(T) \subseteqq F\} ,$$

where the bar denotes ultraweak closure. A simple argument shows that $\mathfrak{A}_{\max}(F)$ is an ultraweakly (in fact, weakly) closed linear space of operators. Note also that, in general, neither $\mathfrak{A}_{\min}(F)$ nor $\mathfrak{A}_{\max}(F)$ is an algebra. However, both are *bimodules* over the multiplication algebra $M = \{L_f : f \in L^\infty(X, m)\}$ in the sense that $M\mathfrak{A}_{\min}(F)M \subseteqq \mathfrak{A}_{\min}(F)$ and $M\mathfrak{A}_{\max}(F)M \subseteqq \mathfrak{A}_{\max}(F)$.

Now suppose we are also given a partial order $\leqq$ on X whose graph $G = \{(x, y) : y \leqq x\}$ is closed. Then $\mathfrak{A}_{\min}(G)$ coincides with the minimal algebra $\mathfrak{A}_{\min}(X, \leqq, m)$ defined already (the slight difference in the two notations is no cause for alarm). In this case, we first want to identify $\mathfrak{A}_{\max}(G)$ with the algebra $\operatorname{alg} \mathfrak{L}(X, \leqq, m)$. As above, for every Borel set $E \subseteqq X$, the notation $L^2(E)$ denotes the space of all functions $f \in L^2(X, m)$ which live on E.

PROPOSITION 2.2.5. *Let T be an operator on $L^2(X, m)$ and let $E \times F$ be a Borel rectangle in $X \times X$ disjoint from* $\sup(T)$. *Then* $P_E T P_F = 0$.

Proof. Now m is mutually absolutely continuous with a finite measure on X; and since finite Borel measures on locally compact metric spaces are always inner regular, it follows that we may find compact sets $K_n \subseteqq K_{n+1} \subseteqq E$, $L_n \subseteqq L_{n+1} \subseteqq F$ such that $m(E \backslash \bigcup_n E_n) = m(F \backslash \bigcup_n L_n) = 0$. The sequences P_{K_n} and P_{L_n} of projections therefore converge strongly to P_E and P_F respectively, so that $P_{K_n} T P_{L_n} \rightarrow P_E T P_F$ strongly. Thus, it suffices to show that $P_{K_n} T P_{L_n} = 0$ for every n; equivalently, we may assume that both E and F are *compact*.

First, fix $x \in E$. We claim that there are open sets U_x, V_x, in X such that $x \in U_x$, $F \subseteqq V_x$ and $TL^2(V_x) \perp L^2(U_x)$. Indeed, for each $y \in F$ the point (x, y) does not belong to $\sup(T)$ so by definition of $\sup(T)$ there is a rectangular open neighborhood $U_y \times V_y$ of (x, y) for which $TL^2(V_y) \perp L^2(U_y)$.

By compactness, we may find $y_1, \cdots, y_n \in F$ such that $F \subseteqq V_{y_1} \cup \cdots \cup V_{y_n}$. Define

$$U_x = U_{y_1} \cap \cdots \cap U_{y_n} \text{ and } V_x = V_{y_1} \cup \cdots \cup V_{y_n} .$$

Then $L^2(U_x) \subseteqq L^2(U_{y_1}) \cap \cdots \cap L^2(U_{y_n})$ and $L^2(V_x)$ is the space spanned by $L^2(V_{y_1}), \cdots, L^2(V_{y_n})$, so that $TL^2(V_x) \perp L^2(U_x)$, as required.

To complete the proof choose, for each $x \in E$, a pair U_x, V_x of open sets having the property asserted above. Again, we may cover E with a finite union $U = U_{x_1} \cup \cdots \cup U_{x_m}$, and, if we define $V = V_{x_1} \cap \cdots \cap V_{x_m}$, then $E \subseteqq U$, $F \subseteqq V$, and $TL^2(V) \perp L^2(U)$ follows by the argument of the preceding paragraph. Since $L^2(E) \subseteqq L^2(U)$ and $L^2(F) \subseteqq L^2(V)$, the desired conclusion $P_E T P_F = 0$ is immediate.

THEOREM 2.2.6. *Let F be a closed subset of $X \times X$. Then the following are equivalent, for every bounded operator T on $L^2(X, m)$*:

(i) $Tf \in [\mathcal{A}_{\min}(F)f]$, *for every* $f \in L^2(X, m)$,

(ii) $\sup(T) \subseteqq F$.

Proof. (i) *implies* (ii). Let $p \in X \times X$ belong to the complement of F. We claim that $p \notin \sup(T)$. Because F is closed, we can find a rectangular open neighborhood $U \times V$ of p which is disjoint from F. Let $f \in L^2(V)$, $g \in L^2(U)$. Then the function $f(y)g(x)$ vanishes on F, so for every measure $\mu \in A(F, m)$ we have

$$(T_\mu f, g) = \int f(y)\bar{g}(x)d\mu(x, y) = 0 .$$

This implies that $g \perp [\mathcal{A}_{\min}(F)f]$, and since $Tf \in [\mathcal{A}_{\min}(F)f]$, we conclude that $(Tf, g) = 0$. Thus, $p \notin \sup(T)$.

(ii) *implies* (i). Assume that T is supported in F, and let $f, g \in L^2(X, m)$ be such that $g \perp [\mathcal{A}_{\min}(F)f]$. Then we claim that $(Tf, g) = 0$. Indeed, by hypothesis the function $\phi(x, y) = f(y)\bar{g}(x)\, \chi_F(x, y)$ has integral zero relative to every measure in $A(X \times X, m)$. By 1.4.3, the set $\{\phi \neq 0\}$ must be marginally null, so that there are Borel sets $M, N \subseteqq X$ of measure zero such that

$$\{g \neq 0\} \times \{f \neq 0\} \cap F \subseteqq M \times X \cup X \times N .$$

Define $A = \{g \neq 0\} \backslash M$, $B = \{f \neq 0\} \backslash N$. Then $A \times B$ is disjoint from F, so by 2.2.5, $P_A T P_B = 0$. Since $f = P_B f$ and $g = P_A g$, we conclude that $(Tf, g) = (P_A T P_B f, g) = 0$. □

COROLLARY. *Let $\leqq$ be a partial ordering of X whose graph G is closed in $X \times X$. Then*

$$\mathcal{A}_{\max}(G) = \operatorname{alg} \mathcal{L}(X, \leqq, m) .$$

Proof. By 2.2.6 we know that $T \in \mathfrak{A}_{\max}(G)$ if and only if $Tf \in [\mathfrak{A}_{\min}(G)f]$ for every $f \in L^2(X, m)$. But this simply means that $T \in \operatorname{alg\,lat} \mathfrak{A}_{\min}(G)$; and since $\operatorname{lat} \mathfrak{A}_{\min}(G) = \mathfrak{L}(X, \leqq, m)$, the conclusion follows. $\square$

We conclude that the problem of determining when the lattice $\mathfrak{L}(X, \leqq, m)$ is synthetic is a special case of the problem of determining when a closed subset F of $X \times X$ has the property that $\mathfrak{A}_{\min}(F) = \mathfrak{A}_{\max}(F)$; equivalently, when can every operator T which is supported in F be ultraweakly approximated by pseudo-integral operators supported in F?

We now want to reformulate some known results from harmonic analysis so as to emphasize an analogy with this problem; in one form or another, this material all can be found in [28]. Let G be a locally compact abelian group and let $\langle f, g\rangle = \int_G f(x)g(x)dx$ be the canonical pairing of $L^\infty(G)$ and $L^1(G)$. Let S be a nonvoid set of functions in $L^\infty(G)$. We define the *spectrum* of S to be the set of all points $\omega \in \hat{G}$ such that, for every neighborhood U of ω, there is a function $g \in L^1(G)$ whose Fourier transform lives in U and which satisfies $\langle S, g\rangle \neq \{0\}$; the spectrum of S is written $\operatorname{sp}(S)$, or $\operatorname{sp}(f)$ if S is a singleton $\{f\}$. If the functions in S happen to be integrable, then $\operatorname{sp}(S)$ is simply the closed support of the set of Fourier transforms of the functions in S. Traditionally, $\operatorname{sp}(S)$ is defined as the set of all characters contained in the weak*-closed span of the set of all translates of the functions in S; and it is a nontrivial fact that the two definitions are equivalent. The above definition, however, is far better suited for our purposes here.

Now let F be a closed subset of $\hat{G}$. Then we may associate two subspaces $S_{\min}(F)$, $S_{\max}(F)$ of $L^\infty(G)$ with F as follows. $S_{\min}(F)$ is defined as the weak*-closed linear span of all characters of F, and $S_{\max}(F) = \{f \in L^\infty(G) \colon \operatorname{sp}(f) \subseteqq F\}$. Using the above definition of $\operatorname{sp}(f)$ it is easy to see that $S_{\max}(F)$ is weak*-closed, contains $S_{\min}(F)$, and of course both subspaces are translation invariant. Let $I_{\min}(F) = S_{\max}(F)^\perp$ and $I_{\max}(F) = S_{\min}(F)^\perp$ be the respective annihilators of these subspaces in $L^1(G)$. The latter are norm-closed translation-invariant subspaces of $L^1(G)$, and therefore they form *ideals* in $L^1(G)$ (regarding $L^1(G)$ as a Banach algebra relative to convolution). It is clear that $I_{\max}(F)$ consists precisely of all functions $f \in L^1(G)$ whose Fourier transform $\hat{f}$ vanishes on F; moreover, using the definition we have given for the spectrum, it is not hard to see that $I_{\min}(F)$ is the norm-closure of all functions $f \in L^1(G)$ for which $\hat{f} = 0$ on an *open neighborhood* U_f of F. The first nontrivial facts about these subspaces are the following:

(i) $\mathrm{sp}\,(S_{\min}(F)) = \mathrm{sp}\,(S_{\max}(F)) = F$, *and if* S *is any other weak*-closed translation invariant subspace of* $L^\infty(G)$ *satisfying* $\mathrm{sp}\,(S) = F$, *then* $S_{\min}(F) \subseteqq S \subseteqq S_{\max}(F)$.

Dually, we have

(i)′ *Both ideals* $I_{\min}(F)$ *and* $I_{\max}(F)$ *have kernel* F, *and if* J *is any other closed ideal in* $L^1(G)$ *satisfying* $\ker(J) = F$, *then* $I_{\min}(F) \subseteqq J \subseteqq I_{\max}(F)$.

Now a basic problem of harmonic analysis is to determine which bounded measurable functions on G can be "synthesized" from their pure harmonic constituents. More precisely, fix $f \in L^\infty(G)$, let S be the weak*-closed linear span of the translates of f, and let K be the set of all characters in S. The problem is to determine conditions under which f belongs to the weak*-closed span of K or equivalently, when $S = S_{\min}(K)$. Now it is easy to see that $\mathrm{sp}\,(S) = K$, so by (i) above we know that, in general, $S_{\min}(K) \subseteqq S \subseteqq S_{\max}(K)$. Therefore the synthesis problem has an affirmative solution provided the set K has the property that $S_{\min}(K) = S_{\max}(K)$.

More generally, a closed set $F \subseteqq \hat{G}$ is said to admit *spectral synthesis* if $S_{\min}(F) = S_{\max}(F)$; equivalently, $I_{\min}(F) = I_{\max}(F)$ in $L^1(G)$. From (i) and (i)′ we see that F admits synthesis if and only if there is a *unique* closed ideal I in $L^1(G)$ (resp. weak*-closed translation invariant subspace S in $L^\infty(G)$) satisfying $\ker(I) = F$ (resp., $\mathrm{sp}\,(S) = F$). This ideal-theoretic property is the traditional definition of sets of spectral synthesis. The two equivalent forms which are technically easiest to work with are the following:

(ii) *Every function* $f \in L^\infty(G)$ *satisfying* $\mathrm{sp}\,(f) \subseteqq F$ *can be weak*-appoximated by finite linear combinations of characters in* F.

(ii)′ *Every function* $f \in L^1(G)$ *such that* $\hat{f} = 0$ *on* F *can be norm-approximated by a sequence* $f_n \in L^1(G)$ *such that* $\hat{f}_n = 0$ *on a neighborhood* U_n *of* F.

Let us now return to the case of a locally compact separable measure space (X, m), and let $\mathcal{H}$ be the Hilbert space $L^2(X, m)$. In place of the function algebra $L^\infty(G)$ we have the operator algebra $\mathfrak{L}(\mathcal{H})$, and in place of $L^1(G)$ we have the Banach space of all ultraweakly continuous linear functionals on $\mathfrak{L}(\mathcal{H})$. The latter is naturally identified with the space $\mathfrak{L}^1(\mathcal{H})$ of all *nuclear* (i.e., trace-class) operators on $\mathcal{H}$, so that it is in a sense a non-commutative L^1 space. Note also that, as in the case of $L^\infty(G)$ and $L^1(G)$, the canonical pairing of $\mathfrak{L}(\mathcal{H})$ and $\mathfrak{L}^1(\mathcal{H})$ identifies $\mathfrak{L}(\mathcal{H})$ with the dual of $\mathfrak{L}^1(\mathcal{H})$, and the ultraweak topology with the weak*-topology. Finally, in place of weak*-closed translation invariant subspaces of $L^\infty(G)$, we consider ultraweakly closed subspaces of $\mathfrak{L}(\mathcal{H})$ which are *bimodules* over the multiplication algebra of (X, m).

It will also be convenient to regard functions on $X \times X$ as being analogous to functions defined on the *dual* of G; in particular, we first want to identify $\mathfrak{L}^1(\mathcal{H})$ with a space of functions of $X \times X$, which will occupy a position very similar to that of the space of all Fourier transforms of functions in $L^1(G)$. Recall first that the *projective tensor product* $E \hat{\otimes} F$ of two (complex) Banach spaces E, F is the completion of the algebraic tensor product $E \otimes F$ in the greatest cross norm $\|\cdot\|_\gamma$, defined on an element $z \in E \otimes F$ by

$$\| z \|_\gamma = \inf \sum_{k=1}^{n} \| x_k \| \cdot \| y_k \| ,$$

the infimum extended over all representations of z as a finite sum $z = \sum_{k=1}^{n} x_k \otimes y_k$, $x_k \in E$, $y_k \in F$ (see [29]). Every element z in the completion can be represented as an absolutely convergent series $z = \sum_{k=1}^{\infty} x_k \otimes y_k$, where $\sum \| x_k \| \cdot \| y_k \| < \infty$, and the norm of z is given by

$$\| z \|_\gamma = \inf \sum \| x_k \| \cdot \| y_k \|$$

where again the infimum is taken over all such expressions for z.

Now it is known that the Banach space of all nuclear operators on $L^2(X, m)$ is identifiable with the projective tensor product $L^2(X, m) \hat{\otimes} L^2(X, m)$ (see [29]; here we will employ the shorter notation $L^2 \hat{\otimes} L^2$), and what we need to do is identify $L^2 \hat{\otimes} L^2$ with a certain space of functions on $X \times X$.

PROPOSITION 2.2.7. *Let $f_n, g_n \in L^2(X, m)$ be such that $\sum_{n=1}^{\infty} \| f_n \| \cdot \| g_n \| < \infty$. Then the infinite series $\phi(x, y) = \sum_{n=1}^{\infty} f_n(y)\overline{g_n(x)}$ is absolutely convergent* (m.a.e.) *on $X \times X$. Moreover, the function ϕ is $|\mu|$-integrable for every measure $\mu \in A(X \times X, m)$, and $\int \phi d\mu = \sum_{n=1}^{\infty} (T_\mu f_n, g_n)$. The ultraweakly continuous linear functional $\rho: A \in \mathfrak{L}(L^2) \mapsto \sum_n (A f_n, g_n)$ vanishes if and only if the function ϕ vanishes marginally almost everywhere.*

Proof. Let S be the set of all points $(x, y) \in X \times X$ for which $\sum_n | f_n(y) | \cdot | g_n(x) | = +\infty$. We claim first that S is marginally null. For that, define

$$f'_n(y) = (\| g_n \| / \| f_n \|)^{1/2} f_n(y) ,$$
$$g'_n(x) = (\| f_n \| / \| g_n \|)^{1/2} g_n(x) .$$

Then

$$\sum_n \int_X |f_n'(y)|^2 dm(y) = \sum_n \|g_n\| \cdot \|f_n\| < \infty ,$$

and similarly

$$\sum_n \int_X |g_n'(x)|^2 dm(x) = \sum \|f_n\| \cdot \|g_n\| < \infty .$$

It follows that there are Borel sets $M, N \subseteq X$ of measure zero such that $\sum_n |f_n'(y)|^2 < \infty$ for all $y \notin N$, and $\sum_n |g_n'(x)|^2 < \infty$ for all $x \notin M$. So by the Schwarz inequality, we have

$$\begin{aligned}\sum_n |f_n(y)\overline{g_n}(x)| &= \sum_n |f_n'(y)| \cdot |g_n'(x)| \\ &\leqq \left(\sum_n |f_n'(y)|^2 \sum_n |g_n'(x)|^2\right)^{1/2} ,\end{aligned}$$

which is finite whenever (x, y) does not belong to the marginally null set $M \times X \cup X \times N$.

For the second assertion of the proposition, choose μ in $A(X \times X, m)$. Then $|\mu| \in A(X \times X, m)$, so that the pseudo-integral operator $T_{|\mu|}$ is bounded, and we have

$$\begin{aligned}\int_{X \times X} |\phi| \, d|\mu| &\leqq \sum_{n=1}^{\infty} \int |f_n(y)| \cdot |g_n(x)| \, d|\mu| \\ &= \sum_{n=1}^{\infty} (T_{|\mu|}|f_n|, |g_n|) \\ &\leqq \| T_{|\mu|} \| \sum_n \|f_n\| \cdot \|g_n\| < \infty ,\end{aligned}$$

which proves that ϕ is $|\mu|$-integrable. The fact that $\int \phi d\mu = \sum_n (T_\mu f_n, g_n)$ follows from a simple application of the dominated convergence theorem.

Now let $\rho(A) = \sum_n (Af_n, g_n)$ be the associated ultraweakly continuous linear functional on $\mathfrak{L}(L^2)$. Assume first that $\phi(x, y) = 0$ m.a.e. Then $\int \phi d\mu = 0$ for every measure $\mu \in A(X \times X, m)$, so by the formula just proved we conclude that

$$\rho(T_\mu) = \sum_n (T_\mu f_n, g_n) = \int \phi d\mu = 0 .$$

Since the pseudo-integral operators are obviously ultraweakly dense in $\mathfrak{L}(L^2)$ (e.g., every Hilbert-Schmidt operator whose kernel function $k(x, y)$ satisfies $\int_X |k(x, y)| \, dm(x) \leqq M < \infty$ for all y and $\int_X |k(x, y)| \, dm(y) \leqq M$ for all x is a pseudo-integral operator), it follows that $\rho = 0$. Conversely, assume $\rho = 0$. We want to show that the set $R = \{(x, y) \in X \times X: \phi(x, y) \neq 0\}$ is marginally null. First, we claim that R is a countable union of Borel rectangles. Let f_n', g_n' be the functions defined in the first part of the proof, satisfying $\sum_n |f_n'|^2 < \infty$ and $\sum_n |g_n'|^2 < \infty$ a.e. (m), and $\phi(x, y) = \sum_n f_n'(y)\overline{g_n'}(x)$. Define functions $F, G: X \to l^2$ by

$$F(x) = (f_1'(x), f_2'(x), \cdots) \text{ and } G(x) = (g_1'(x), g_2'(x), \cdots) .$$

If l^2 is given its usual Borel structure, both F and G become Borel functions, and we have $\phi(x, y) = \langle F(y), G(x) \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the inner product in l^2. Now the set of all pairs $\{(\xi, \eta) \in l^2 \times l^2: \langle \xi, \eta \rangle \neq 0\}$ is an open set in $l^2 \times l^2$, and therefore it is a countable union $\bigcup_n U_n \times V_n$ of open rectangles in $l^2 \times l^2$. So if we define the Borel sets

$$F_n = \{y \in X: F(y) \in U_n\} \text{ and } E_n = \{x \in X: G(x) \in V_n\} ,$$

it follows that $R = \bigcup_n E_n \times F_n$ has the asserted form.

Next, we claim that $m \times m(R) = 0$; equivalently, $\int_{A \times B} |\phi| \, dm \times m = 0$ for every pair of Borel sets $A, B \subseteqq X$ for which $m(A) < \infty$ and $m(B) < \infty$. For that, fix A and B, and define a measure μ on $X \times X$ by

$$\mu(dx, dy) = \frac{|\phi(x, y)|}{\phi(x, y)} \chi_A(x) \chi_B(y) m(dx) m(dy) ,$$

where the quotient is defined as 0 whenever $\phi(x, y) = 0$. Then μ belongs to $A(X \times X, m)$, and we have

$$\int_{A \times B} |\phi| \, dm \times m = \int_{X \times X} \phi d\mu = \sum_n (T_\mu f_n, g_n) = \rho(T_\mu) = 0 ,$$

as asserted.

In particular, we have $m(E_n) m(F_n) = m \times m(E_n \times F_n) = 0$, for every $n \geqq 1$, so that either $m(E_n) = 0$ or $m(F_n) = 0$. If $m(E_n) = 0$, then $E_n \times F_n \subseteqq E_n \times X$ is marginally null, and if $m(F_n) = 0$ then $E_n \times F_n \subseteqq X \times F_n$ is also clearly marginally null. Thus, $R = \bigcup_{n=1}^\infty E_n \times F_n$ must be marginally null, and the proof is complete. □

Let us agree to identify two functions on $X \times X$ which coincide marginally almost everywhere. Then the preceding proposition implies that $L^2 \hat{\otimes} L^2$ may be taken as the space of all functions $\phi: X \times X \to \mathbf{C}$ which admit a representation

$$\phi(x, y) = \sum_{n=1}^\infty f_n(y) \overline{g_n}(x)$$

where $f_n, g_n \in L^2(X, m)$ satisfy $\sum_n \|f_n\| \cdot \|g_n\| < \infty$. Moreover, the $L^2 \hat{\otimes} L^2$-norm of such a function ϕ is

$$\|\phi\|_\gamma = \inf \sum_{n=1}^\infty \|f_n\| \cdot \|g_n\| ,$$

the infimum taken over all such sequences f_n, g_n for which $\sum_{n=1}^\infty f_n(y) \overline{g_n}(x) = \phi(x, y)$ (marginally almost everywhere). The duality between $L^2 \hat{\otimes} L^2$ and $\mathfrak{L}(L^2)$ is defined by $\langle \phi, T \rangle = \sum_n (T f_n, g_n)$, where f_n, g_n satisfy the above conditions, and if T happens to be a pseudo-integral operator T_μ, $\mu \in A(X \times X, m)$, the pairing takes the more convenient form

$$\langle \phi, T_\mu \rangle = \int_{X \times X} \phi d\mu .$$

The following notation will be useful. Let F be a closed subset of $X \times X$. $Z(F)$ will denote the space of all functions $\phi \in L^2 \hat{\otimes} L^2$ which vanish on F (m.a.e.), i.e., such that $\phi\chi_F = 0$ (m.a.e.). It is clear that $Z(F)$ is a linear subspace of $L^2 \hat{\otimes} L^2$, and 2.2.8 below implies that $Z(F)$ is closed in the norm $\|\cdot\|_\gamma$. If $\mathfrak{S}$ is a set of operators on $L^2(X, m)$, $\mathfrak{S}^\perp$ will denote its annihilator $\{\phi \in L^2 \hat{\otimes} L^2 : \langle \phi, \mathfrak{S} \rangle = \{0\}\}$.

THEOREM 2.2.8. *Let $F \subseteqq X \times X$ be closed. Then $\mathfrak{A}_{\min}(F)^\perp = Z(F)$, and $\mathfrak{A}_{\max}(F)^\perp$ is the norm closure of the union $\bigcup Z(E)$, where E runs over all closed sets in $X \times X$ which contain F in their interiors.*

Proof. We first identify $\mathfrak{A}_{\min}(F)^\perp$. If $\phi \in Z(F)$, then for every measure $\mu \in A(F, m)$ we have $\int \phi d\mu = \int \phi \chi_F d\mu = 0$, since $\phi\chi_F = 0$ (m.a.e.). This implies $\langle \phi, T \rangle = 0$ for every pseudo-integral operator $T \in \mathfrak{A}_{\min}(F)$, and since the latter are ultraweakly dense in $\mathfrak{A}_{\min}(F)$, we see that $\phi \perp \mathfrak{A}_{\min}(F)$. Conversely, choose $\phi \in \mathfrak{A}_{\min}(F)^\perp$. Then $\int \phi d\mu = 0$ for every $\mu \in A(F, m)$, and this clearly implies that $\int |\phi| d\mu = 0$ for every positive measure $\mu \in A(F, m)$. Hence $\{\phi \neq 0\} \cap F$ is a null set for every positive measure in $A(X \times X, m)$. Now the proof of the preceding proposition shows that $\{\phi \neq 0\}$ has the form $\bigcup_{n=1}^\infty E_n \times F_n$, where E_n, F_n are Borel sets in X, and so for each fixed $n \geqq 1$, $\mu(E_n \times F_n \cap F) = 0$, $\mu \in A(X \times X, m)$. By the null set theorem (1.4.3), $E_n \times F_n \cap F$ is marginally null, so that $\{\phi \neq 0\} \cap F = \bigcup_n (E_n \times F_n \cap F)$ is marginally null. This proves that $\phi \in Z(F)$.

Next, let $E \subseteqq X \times X$ be a closed set containing F in its interior. We claim: $Z(E) \subseteqq \mathfrak{A}_{\max}(F)^\perp$. Choose $\phi \in Z(E)$, and let T be an operator supported in F. By σ-compactness of X, we may find an increasing sequence $K_n \subseteqq X$ of compact sets such that $\bigcup_n K_n = X$. It follows that the sequence P_{K_n} of projections (on $L^2(X, m)$) converges strongly to the identity, so that the sequence of operators $P_{K_n} T P_{K_n}$ converges ultraweakly to T. Thus it suffices to show that $\langle \phi, P_{K_n} T P_{K_n} \rangle = 0$ for every n; and since $P_{K_n} T P_{K_n}$ is supported in the compact set $F \cap K_n \times K_n$, we are reduced to proving that if $\phi \in Z(E)$ and T has compact support contained in the interior of E, then $\langle \phi, T \rangle = 0$.

For that, we may cover the support of T with a finite union $\bigcup_{j=1}^n U_j \times V_j$ of open rectangles, all contained in E. By the usual "disjointing" process, we may write $\bigcup_{j=1}^{n} U_j \times V_j$ as a finite disjoint union of *Borel* rectangles $\bigcup_{k=1}^{m} A_k \times B_k$, and its complement $X \times X \backslash \bigcup_j U_j \times V_j$ as a similar union $\bigcup_{l=1}^{r} C_l \times D_l$. Now we have

$$\sum_k \chi_{A_k \times B_k} + \sum_l \chi_{C_l \times D_l} = 1 , \qquad \text{on } X \times X,$$

so that

$$\phi = \sum_k \phi \cdot \chi_{A_k \times B_k} + \sum_l \phi \cdot \chi_{C_l \times D_l}$$

(there is of course no problem of convergence, since all sums are finite). Note that $\phi \cdot \chi_{A_k \times B_k} = 0$ (m.a.e.), because $\phi = 0$ on E. Hence $\langle \phi, T \rangle = \sum_l \langle \phi, P_{C_l} T P_{D_l} \rangle$. But for each l, we see from 2.2.5 that $P_{C_l} T P_{D_l} = 0$. It follows that $\langle \phi, T \rangle = 0$, as required.

Finally, we claim that the union $\bigcup_E Z(E)$, as E runs over all closed sets containing F in their interior, is dense in $\mathfrak{A}_{\max}(F)^{\perp}$ in the norm $\| \cdot \|_r$. Let f be a bounded linear functional on $L^2 \hat{\otimes} L^2$ which annihilates $\bigcup_E Z(E)$. We want to prove that $f(\mathfrak{A}_{\max}(F)^{\perp}) = \{0\}$. Now since $\mathfrak{L}(L^2)$ is the dual of $L^2 \hat{\otimes} L^2$, f must have the form $f(\phi) = \langle \phi, T \rangle$ for some operator T on $L^2(X, m)$, and it suffices to prove that $T \in \mathfrak{A}_{\max}(F)$, i.e., $\sup(T) \subseteq F$. So choose any point p not in F. We claim; $p \notin \sup(T)$. Indeed, since F is closed, we may find an open rectangular neighborhood $U \times V$ of p whose *closure* is compact and is disjoint from F. Put $E = X \times X \backslash U \times V$. Then E is closed, and F is contained in the complement of $(U \times V)^{-}$, an open subset of the interior of E. Note now that $P_U T P_V = 0$. Indeed, for every $\phi \in L^2 \hat{\otimes} L^2$, the function $\phi \cdot \chi_{U \times V}$ belongs to $Z(E)$, so that by the hypothesis on f we have $\langle \phi, P_U T P_V \rangle = \langle \phi \cdot \chi_{U \times V}, T \rangle = f(\phi \cdot \chi_{U \times V}) = 0$. Because ϕ was arbitrary it follows that $P_U T P_V = 0$. Since $U \times V$ is a neighborhood of p, we conclude that $p \notin \sup(T)$, by definition of the support of an operator. □

2.2.8 implies that the following two conditions are equivalent, for every closed subset F of $X \times X$:

2.2.9. (i) *Every operator T supported in F can be ultraweakly approximated by pseudo-integral operators supported in F.*

(ii) *For every function $\phi \in L^2 \hat{\otimes} L^2$ which vanishes on F* (m.a.e.), *there is a sequence $\phi_n \in L^2 \hat{\otimes} L^2$, each term of which vanishes* (m.a.e.) *on a closed neighborhood E_n of F, such that $\| \phi - \phi_n \|_r \to 0$ as $n \to \infty$.*

In condition (ii), a closed *neighborhood* of F of course means a closed set which contains F in its interior.

DEFINITION 2.2.10. *A closed set $F \subseteq X \times X$ is called synthetic* (*relative to the sigma-finite measure m*) *if it satisfies the equivalent conditions of* 2.2.9.

THEOREM 2.2.11. *Let $\leqq$ be a partial ordering of X whose graph G is closed in $X \times X$. Then the lattice $\mathfrak{L}(X, \leqq, m)$ is synthetic if and only if G is a synthetic set in $X \times X$.*

Proof. According to 2.2.6, $\mathfrak{A}_{\max}(G) = \text{alg}\, \mathfrak{L}(X, \leqq, m)$, while of course $\mathfrak{A}_{\min}(G) = \mathfrak{A}_{\min}(X, \leqq, m)$. We already know that $\mathfrak{L}(X, \leqq, m)$ is a synthetic lattice if and only if $\mathfrak{A}_{\min}(X, \leqq, m) = \text{alg}\, \mathfrak{L}(X, \leqq, m)$, i.e., $\mathfrak{A}_{\min}(G) = \mathfrak{A}_{\max}(G)$, so the assertion is evident. □

We now want to illustrate how 2.2.11 can be used to establish that certain lattices of infinite width are synthetic. Let Γ be a separable locally compact abelian group (written additively), and let Σ be a *closed* subset of Γ which contains 0 and satisfies $\Sigma + \Sigma \subseteqq \Sigma$. Σ determines a partial ordering of Γ in the obvious way: $x \leqq y \Leftrightarrow y - x \in \Sigma$. Note that this order is translation invariant (i.e., $x \leqq y \Rightarrow x + z \leqq y + z$ for all $z \in \Gamma$), and its graph $G = \{(x, y): x - y \in \Sigma\}$ is clearly closed in $\Gamma \times \Gamma$. Let m denote Haar measure on Γ.

THEOREM 2.2.12. *If 0 belongs to the closure of the interior of Σ, then the lattice $\mathfrak{L}(\Gamma, \leqq, m)$ is synthetic.*

Proof. By 2.2.11, we need to prove that $G = \{(x, y): x - y \in \Sigma\}$ is a synthetic set in $\Gamma \times \Gamma$; for that, we will verify the condition 2.2.9 (ii).

Let $\phi \in L^2 \hat{\otimes} L^2$ be such that $\phi = 0$ m.a.e. on G. Note first that, if t belongs to the interior of Σ, then $\Sigma - t$ is a closed neighborhood of Σ. Indeed, if U is any open neighborhood of 0 in Γ such that $t + U \subseteqq \Sigma$, then $U \subseteqq \Sigma - t$, hence $\Sigma + U \subseteqq \Sigma - t$, and of course $\Sigma + U$ is open. It follows from this remark that $G - (t, 0) = \{(x, y): x - y \in \Sigma - t\}$ is a closed neighborhood of G.

Now let U_t be the regular representation of Γ on $L^2(\Gamma)$, defined by $U_t f(x) = f(x + t)$. $U_t \otimes I$ determines an action of Γ on $L^2 \hat{\otimes} L^2$, defined by $U_t \otimes I\psi(x, y) = \psi(x + t, y)$, $\psi \in L^2 \hat{\otimes} L^2$. Because U is strongly continuous on L^2, the representation $t \mapsto U_t \otimes I$ is strongly continuous on the Banach space $L^2 \hat{\otimes} L^2$ (this routine verification is left for the reader). Now choose a sequence t_n in the interior of Σ such that $t_n \to 0$, and put $\phi_n(x, y) = \phi(x + t_n, y)$. Then $\phi_n = (U_{t_n} \otimes I)\phi$, so that $\| \phi - \phi_n \|_\gamma \to 0$ as $n \to \infty$ because $t_n \to 0$. On the other hand, $\phi_n = 0$ (m.a.e.) on $G - (t_n, 0)$, because $\phi = 0$ (m.a.e.) on G, and by the preceding paragraph $G - (t_n, 0)$ is a closed neighborhood of G. Condition 2.2.9 (ii) follows. □

As a special case of particular interest, let Γ be the vector group $\mathbf{R}^n \times \mathbf{R} = \{(x, t): x \in \mathbf{R}^n, t \in \mathbf{R}\}$, $n \geqq 1$, let m be Lebesgue measure on

$\mathbf{R}^n \times \mathbf{R}$, and let $|x|$ denote the euclidean norm of a vector $x \in \mathbf{R}^n$. We have already discussed the partial ordering $(x, s) \leqq (y, t)$ of $\mathbf{R}^n \times \mathbf{R}$ defined by $|y - x| \leqq t - s$ (see example 1.1.5). We shall refer to its associated lattice $\mathfrak{L}(\mathbf{R}^n \times \mathbf{R}, \leqq, m)$ as the *Causal Lattice*. Note that the operators in alg $\mathfrak{L}(\mathbf{R}^n \times \mathbf{R}, \leqq, m)$ are very closely related to the *causal operators* of Segal and Foures [13] (we are indebted to Arlan Ramsay for calling our attention to this reference), and the following result may be of interest for related questions in special relativity.

COROLLARY. *For each $n \geqq 1$, the Causal Lattice $\mathfrak{L}(\mathbf{R}^n \times \mathbf{R}, \leqq, m)$ is synthetic.*

Proof. Let Σ be the set of all pairs $(x, t) \in \mathbf{R}^n \times \mathbf{R}$ satisfying $|x| \leqq t$. Σ satisfies the hypotheses of 2.2.12, and we have $(x, s) \leqq (y, t)$ if and only if $(y, t) - (x, s) \in \Sigma$. Thus the conclusion is immediate from 2.2.12. □

There are a number of results in Harmonic analysis which suggest conjectures about subspace lattices. For example, Herz ([28, p. 166]) has proved that the Cantor ternary set in the real line admits spectral synthesis. This suggests the following:

Problem. Let $\leqq$ be the "product order" on the Cantor sequence space 2^∞, defined by $(x_i) \leqq (y_i) \Leftrightarrow x_i \leqq y_i$ for every $i = 1, 2, \cdots$. Fix p, $0 < p < 1$, and let m_p be the product measure $\prod_i \mu_i$ on 2^∞, where each μ_i assigns mass p to $\{1\}$ and $1 - p$ to $\{0\}$. Is the associated subspace lattice $\mathfrak{L}(2^\infty, \leqq, m_p)$ synthetic? The case $p = 1/2$ may be special, since $m_{1/2}$ is invariant under the usual group operation in 2^∞.

Another natural class of problems concerns the relation between synthetic lattices and compact operators. As a sample, we state the following:

Problem. Let (X, m) be a sigma-finite locally compact separable measure space, and let $\leqq$ be a strict partial ordering of X having closed graph. Assume that $\mathfrak{L}(X, \leqq, m)$ is *strongly* reflexive in the sense of the discussion following 1.6.3, i.e., $\mathfrak{L}(X, \leqq, m) = \operatorname{lat} \mathcal{C}$, where $\mathcal{C}$ is the algebra of all compact operators in alg $\mathfrak{L}(X, \leqq, m)$. Then is $\mathfrak{L}(X, \leqq, m)$ necessarily synthetic?

We conjecture, incidentally, that alg $\mathfrak{L}(2^\infty, \leqq, m_p)$ contains *no* nonzero compact operators, $\mathfrak{L}(2^\infty, \leqq, m_p)$ being the lattice of the preceding problem.

2.3. Generators

A subset $\mathcal{S}$ of an ultraweakly closed operator algebra $\mathcal{A}$ is said to *generate* $\mathcal{A}$ if $\mathcal{A}$ is the ultraweak closure of the algebra generated by $\mathcal{S}$. W. E. Longstaff [20] has proved that every reflexive algebra $\mathcal{A}$ for which lat $\mathcal{A}$ is a chain is generated (as a weakly closed algebra) by two operators. On the other hand, Peter Rosenthal [20] has given an example of a reflexive algebra which is not finitely generated (even relative to the weak operator topology). Thus it is of interest to determine which reflexive algebras are finitely generated.

In this section we prove that every reflexive algebra $\mathcal{A}$, for which lat $\mathcal{A}$ is commutative and synthetic, is doubly generated. Note that this improves Longstaff's result because, by 2.2.3, lat $\mathcal{A}$ is synthetic whenever it is commutative and has finite width.

Throughout this section, (X, m) will be a standard sigma-finite measure space, and M will denote the multiplication algebra acting on $L^2(X, m)$. The following result gives a generalization of Corollary 2 of 2.1.5 to linear spaces of operators on $L^2(X, m)$ which are bimodules over M.

LEMMA. *Let $\mathcal{S}$ be an ultraweakly closed linear space of operators on $L^2(X, m)$ satisfying $M\mathcal{S}M \subseteq \mathcal{S}$, and let T be a pseudo-integral operator on $L^2(X, m)$ such that $Tf \in [\mathcal{S}f]$ for every $f \in L^2(X, m)$. Then $T \in \mathcal{S}$.*

Proof. This may be deduced from Corollary 2 of 2.1.5 by means of the following device.

Let (X_i, m_i), $i = 1, 2$, be two (disjoint) copies of the measure space (X, m). Then the Hilbert space direct sum $L^2(X, m) \oplus L^2(X, m)$ can be realized as the function space $L^2(X_1 \oplus X_2, m_1 \oplus m_2)$ (here, the symbol $\oplus$ has the obvious meaning), so that the family $\mathcal{A}$ of all operators on $L^2(X, m) \oplus L^2(X, m)$ having a 2×2 matrix representation

$$\begin{pmatrix} A & T \\ 0 & B \end{pmatrix}, \qquad A, B \in M,\ T \in \mathcal{S},$$

becomes associated with an ultraweakly closed linear space of operators on $L^2(X_1 \oplus X_2, m_1 \oplus m_2)$, which contains the multiplication algebra of $(X_1 \oplus X_2, m_1 \oplus m_2)$. Note also that $\mathcal{A}$ is in fact an *algebra*, because $M\mathcal{S}M \subseteq \mathcal{S}$.

Now let T_μ, $\mu \in A(X \times X, m)$, be a pseudo-integral operator on $L^2(X, m)$ for which $T_\mu f \in [\mathcal{S}f]$, $f \in L^2(X, m)$, and define

$$\tilde{T}_\mu = \begin{pmatrix} 0 & T_\mu \\ 0 & 0 \end{pmatrix}.$$

The reader may easily verify that $\tilde{T}_\mu$ satisfies $\tilde{T}_\mu g \in [\mathcal{A}g]$ for every $g \in L^2(X_1 \oplus X_2, m_1 \oplus m_2)$, so that $\tilde{T}_\mu$ leaves every element of lat $\mathcal{A}$ invariant.

Now $\tilde{T}_\mu$ is clearly a pseudo-integral operator on $L^2(X_1 \oplus X_2, m_1 \oplus m_2)$ (it arises from a measure which is a copy of μ on the subset $X_1 \times X_2$ of $(X_1 \oplus X_2) \times (X_1 \oplus X_2)$). So we see from Proposition 1.6.0 that $\tilde{T}_\mu$ belongs to the minimal algebra associated with lat $\mathfrak{A}$. By Corollary 2 of 2.1.5, it follows that $\tilde{T}_\mu \in \mathfrak{A}$, hence $T_\mu \in \mathfrak{S}$. □

Now assume, further, that X is a separable locally compact metric space, and let F be a closed subset of $X \times X$. The following result asserts that $\mathfrak{A}_{\min}(F)$ is singly generated *as a bimodule over M.*

THEOREM 2.3.1. *Let $F \subseteq X \times X$ be closed. Then there is a measure $\sigma \in A(F, m)$ such that $\mathfrak{A}_{\min}(F)$ is the ultraweakly closed linear span of $MT_\sigma M$.*

Proof. Let ball $A(F, m)$ denote the unit ball in the normed linear space $A(F, m)$ (cf. Section 1.5), so that $\{T_\mu: \mu \in \text{ball } A(F, m)\}$ is a set of operators on $L^2(X, m)$ of norm at most 1. Because $L^2(X, m)$ is separable, the unit ball of $\mathfrak{L}(L^2(X, m))$ is separable in the weak operator topology, and we may find a sequence $\mu_1, \mu_2, \cdots$ in ball $A(F, m)$ such that $\{T_\mu: \mu \in \text{ball } A(F, m)\}$ is contained in the weak closure of $\{T_{\mu_n}: n \geqq 1\}$.

For each n, let $a_n = \max(1, |\mu_n|(X \times X), \|\mu_n\|)$, and define a finite positive measure σ on $X \times X$ by $\sigma = \sum_{n=1}^\infty (2^n a_n)^{-1} |\mu_n|$. Clearly σ lives on F, and in terms of the norm on $A(X \times X, m)$ we have $\|\sigma\| \leqq \sum_n (2^n a_n)^{-1} \|\mu_n\| \leqq \sum_n 2^{-n} = 1$. Hence σ belongs to $A(F, m)$.

To complete the proof, we claim that $\mathfrak{A}_{\min}(F)$ is contained in the ultraweakly closed span $\mathfrak{S}$ of $MT_\sigma M$; by the lemma, this will follow if we can prove that, for each $\mu \in A(F, m)$, $T_\mu f \in [\mathfrak{S} f]$ for every $f \in L^2(X, m)$. So choose $f, g \in L^2(X, m)$ such that $g \perp \mathfrak{S} f$. Consider the finite (complex) measure σ' on $X \times X$ defined by $d\sigma'(x, y) = f(y)\bar{g}(x) d\sigma(x, y)$. Then by definition of $\mathfrak{S}$ we see that, for every pair $u, v \in L^\infty(X, m)$,

$$\int_{X \times X} u(y)v(x) d\sigma'(x, y) = (L_v T_\sigma L_u f, g) = 0 ,$$

because $L_v T_\sigma L_u \in MT_\sigma M \subseteq \mathfrak{S}$. From this we infer that σ' is the zero measure, and hence $f(y)\overline{g(x)} = 0$ almost everywhere $(d\sigma)$. Since each measure $|\mu_n|$ is absolutely continuous with respect to σ, we conclude that

$$(T_{\mu_n} f, g) = \int_{X \times X} f(y)\bar{g}(x) d\mu_n(x, y) = 0$$

for every $n = 1, 2, \cdots$. It follows from the choice of $|\mu_n|$ that $g \perp T_\mu f$ for every $\mu \in \text{ball } A(F, m)$, and the assertion is now evident. □

COROLLARY 1. *Let $\mathfrak{L}$ be a commutative subspace lattice and let $\mathfrak{A}_{\min}(\mathfrak{L})$*

be the minimal algebra of $\mathfrak{L}$, *in the sense of* 2.1.8 (ii). *Then* $\mathfrak{A}_{\min}(\mathfrak{L})$ *is doubly generated.*

Proof. By 1.3.1, we may assume that $\mathfrak{L} = \mathfrak{L}(X, \leqq, m)$ is the lattice associated with a standard partially ordered measure space $(X, \leqq, m)$. Note that we may even assume that X is the Cantor space 2^∞ and $\leqq$ is the even order on 2^∞, defined by $(x_i) \leqq (y_i) \Leftrightarrow x_i \leqq y_i$ for every *even* $i \geqq 2$. Indeed, 1.1.7 implies that $(X, \leqq)$ is isomorphic (as a partially ordered Borel space) to a Borel subspace X' of $(2^\infty, \leqq_e)$ ($\leqq_e$ denoting the even ordering of 2^∞), so that we may transfer m to a measure m' on 2^∞, concentrated on X', in such a way that $(X, \leqq, m)$ and $(2^\infty, \leqq_e, m')$ are isomorphic as partially ordered measure spaces. The assertion now follows from 1.2.4.

In particular, we may represent $\mathfrak{L}$ as the lattice $\mathfrak{L}(X, \leqq, m)$, where X is a compact metric space and $\leqq$ is a partial ordering of X whose graph G is *closed* in the product topology of $X \times X$. 2.3.1 now implies that there is a measure σ in $A(G, m)$ such that $\mathfrak{A}_{\min}(\mathfrak{L}) = \mathfrak{A}_{\min}(X, \leqq, m)$ is the ultraweakly closed span of $MT_\sigma M$. Finally, a familiar theorem of von Neumann asserts that there is a self-adjoint operator $A \in M$ (M denoting the multiplication algebra of (X, m)) such that M is the ultraweak closure of all polynomials in A. It follows that the ultraweakly closed algebra generated by the pair $\{A, T_\sigma\}$ contains M, therefore $MT_\sigma M$, and therefore it contains $\mathfrak{A}_{\min}(X, \leqq, m)$. The opposite inclusion is obvious. □

COROLLARY 2. *Every reflexive algebra* $\mathfrak{A}$, *such that* lat $\mathfrak{A}$ *is commutative and synthetic, is doubly generated.*

Proof. Because $\mathfrak{A}$ is reflexive, we have $\mathfrak{A} = \text{alg lat}\, \mathfrak{A}$, and since lat $\mathfrak{A}$ is commutative and synthetic, $\text{alg lat}\, \mathfrak{A} = \mathfrak{A}_{\min}(\text{lat}\, \mathfrak{A})$. Therefore, $\mathfrak{A} = \mathfrak{A}_{\min}(\text{lat}\, \mathfrak{A})$ is doubly generated by Corollary 1. □

Remarks. Since every commutative subspace lattice of finite width is synthetic (2.2.3), we see in particular that every reflexive algebra $\mathfrak{A}$, for which lat $\mathfrak{A}$ is commutative and has finite width, is doubly generated.

2.4 A Tauberian-type theorem, and smooth integral operators

Let G be a locally compact abelian group. Weiner's Tauberian theorem asserts that the translates of an integrable function f on G will span $L^1(G)$ provided that the Fourier transform of f never vanishes [28]. This follows from a more general result which, stated contrapositively, asserts that every closed ideal I in the convolution algebra $L^1(G)$ which is not all of $L^1(G)$ must have nonempty kernel. By considering annihilators relative to the canonical pairing of $L^\infty(G)$ and $L^1(G)$, one arrives at the following equivalent result: *every nonzero weak*-closed translation invariant subspace of $L^\infty(G)$ contains a character of G* (the theorem was formulated in this way by Beurling [5], [6]).

The purpose of this section is to present an analogous result about ultraweakly closed linear spaces of operators, and to point out how it leads to the existence of integral operators (in certain linear spaces of operators on $L^2(\mathbf{R}^n)$, say) whose kernel functions are infinitely differentiable.

In the following two results, (X, m) will denote a standard sigma-finite measure space.

LEMMA. *Let $\mathcal{A}$ be an ultraweakly closed algebra of operators on $L^2(X, m)$ which contains the multiplication algebra M. Assume that the subalgebra of $\mathcal{A}$ consisting of all pseudo-integral operators is self-adjoint. Then $\mathcal{A}$ is self-adjoint.*

Proof. 1.6.0 and Corollary 2 of 2.1.5, we know that every pseudo-integral operator in $\operatorname{alg}\operatorname{lat}\mathcal{A}$ must already belong to $\mathcal{A}$. Thus, the hypothesis amounts to saying that the algebra $\mathcal{A}_0$ of pseudo-integral operators in $\operatorname{alg}\operatorname{lat}\mathcal{A}$ is self-adjoint. Now $\mathcal{A}_{\min}(\operatorname{lat}\mathcal{A})$ is the ultraweak closure of $\mathcal{A}_0$, and has the same invariant subspace lattice as $\mathcal{A}$. Hence $\operatorname{lat}\mathcal{A} = \operatorname{lat}\mathcal{A}_0$, being the lattice of a self-adjoint algebra, must be orthocomplemented. Now apply Corollary 1 of 2.1.8. □

THEOREM 2.4.1. *Let $\mathcal{S}$ be a nonzero ultraweakly closed linear space of operators on $L^2(X, m)$ such that $M\mathcal{S}M \subseteq \mathcal{S}$, M denoting the multiplication algebra. Then $\mathcal{S}$ contains a nonzero pseudo-integral operator.*

Proof. As in the proof of the lemma preceding 2.3.1, we may realize $L^2(X, m) \oplus L^2(X, m)$ as the space $L^2(X_1 \oplus X_2, m_1 \oplus m_2)$ associated with two disjoint copies (X_i, m_i), $i = 1, 2$, of (X, m), and the ultraweakly closed algebra $\mathcal{A}$, defined as the set of all operators on $L^2(X, m) \oplus L^2(X, m)$ which admit a 2×2 matrix representation

$$\begin{pmatrix} A & S \\ 0 & B \end{pmatrix} \qquad A, B \in M,\ S \in \mathcal{S},$$

becomes an ultraweakly closed algebra on $L^2(X_1 \oplus X_2, m_1 \oplus m_2)$ which contains the multiplication algebra of $(X_1 \oplus X_2, m_1 \oplus m_2)$. Since $\mathcal{S} \neq \{0\}$, $\mathcal{A}$ is not self-adjoint, so by the lemma $\mathcal{A}$ must contain a pseudo-integral operator R such that $R^* \notin \mathcal{A}$. Now R must have the form

$$R = \begin{pmatrix} T_\sigma & T_\mu \\ 0 & T_\tau \end{pmatrix}$$

where μ, σ, τ are measures in $A(X \times X, m)$; and since R^* does not belong to $\mathcal{A}$ we must have $T_\mu \neq 0$. Therefore, T_μ is the required pseudo-integral operator in $\mathcal{S}$. □

In the remainder of this section, G will denote a separable locally compact abelian group, with Haar measure m.

DEFINITION 2.4.2. *A bounded operator T on $L^2(G)$ is called a smooth integral operator if there is a bounded continuous function k on $G \times G$ such that*

$$(Tf, g) = \int_{G\times G} f(y)\overline{g(x)}k(x, y)dm(x)dm(y)$$

for every pair f, g of continuous functions with compact support. In the special case $G = \mathbf{R}^n$, $n \geqq 1$, k is required to be infinitely differentiable.

Note that the function k is uniquely determined by T, when it exists. In the following, U_t, $t \in G$, will denote the regular representation of G on $L^2(G)$: $U_t f(x) = f(x + t)$, $f \in L^2(G)$.

COROLLARY. *Let $\mathcal{S}$ be a nonzero ultraweakly closed linear space of operators on $L^2(G)$ such that $U_x \mathcal{S} U_y \subseteqq \mathcal{S}$ for all $x, y \in G$. Then $\mathcal{S}$ contains a nonzero smooth integral operator.*

Proof. Let W be the unitary map of $L^2(G)$ onto $L^2(\hat{G})$ ($\hat{G}$ denoting the dual of G) determined by the Fourier transform, and let $\hat{\mathcal{S}} = W\mathcal{S}W^{-1}$. Then $\hat{\mathcal{S}}$ is a nonzero ultraweakly closed linear space of operators on $L^2(\hat{G})$. The condition $U_x\mathcal{S}U_y \subseteqq \mathcal{S}$, $x, y \in G$, becomes $L_f\hat{\mathcal{S}}L_g \subseteqq \hat{\mathcal{S}}$, whenever f and g are characters of $\hat{G}$. Since the weak*-closed linear span of the characters of $\hat{G}$ exhausts $L^\infty(\hat{G})$, it follows that the multiplication algebra M of $L^2(\hat{G})$ is the ultraweakly closed span of the set of multiplications by all characters of $\hat{G}$. We may therefore conclude that $M\hat{\mathcal{S}}M \subseteqq \hat{\mathcal{S}}$. 2.4.1 now implies that $\hat{\mathcal{S}}$ contains a pseudo-integral operator $T_\mu \neq 0$, where $\mu \in A(\hat{G} \times \hat{G}, \hat{m})$ ($\hat{m}$ denoting Haar measure on $\hat{G}$). By replacing μ with the measure $\mu_K(S) = \mu(K \times K \cap S)$ for K an appropriate compact subset of $\hat{G}$, we may even assume that μ has compact support (note that $T_{\mu_K} = L_{\chi_K} T_\mu L_{\chi_K} \in \hat{\mathcal{S}}$ for every compact set $K \subseteqq G$).

Now put $A = W^{-1}T_\mu W$. For any functions f, $g \in L^2(\hat{G})$, we have

$$(T_\mu f, g) = \int_{\hat{G}\times\hat{G}} f(\eta)\bar{g}(\xi)d\mu(\xi, \eta) .$$

A straightforward computation (which we leave for the reader) shows that A therefore satisfies

$$(Af, g) = \int_{G\times G} f(y)\bar{g}(x)k(x, y)dm(x)dm(y)$$

for continuous functions f, g on G of compact support, where

$$k(x, y) = \int_{\hat{G}\times\hat{G}} \bar{\xi}(x)\eta(y)d\mu(\xi, \eta)$$

is the inverse Fourier transform of the compactly supported measure μ (whose domain is the group $\hat{G} \times \hat{G} = (G \times G)^{\wedge}$). Clearly k is continuous and bounded by the total variation of μ. Finally, in the case $G = \mathbf{R}^n$, $k \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ follows from the well-known fact that the Fourier transform of a compactly supported measure on $\mathbf{R}^n \times \mathbf{R}^n$ is infinitely differentiable (in fact, real-analytic). □

Remarks. There are a number of other results that one can prove along similar lines. For example, *any ultraweakly closed algebra* $\mathcal{A}$ *of operators on* $L^2(G)$, *which contains all translations* $\{U_x: x \in G\}$ *and for which* lat $\mathcal{A}$ *is synthetic, contains an algebra* $\mathcal{A}_0$ *of smooth integral operators which is ultraweakly dense in* $\mathcal{A}$. This may be regarded as a non-commutative generalization of the fact that the convolution operators C_k on $L^2(G)$, associated with continuous functions $k \in L^1(G)$ by

$$C_k f(x) = \int_G f(x - t)k(t)dm(t) , \qquad f \in L^2(G) ,$$

are ultraweakly dense in the von Neumann algebra generated by $\{U_x: x \in G\}$. The proof, which uses the fact that $\mathcal{A}$ is reflexive along with the Fourier transform device of the preceding corollary, is left for the interested reader.

2.5. A non-reflexive operator algebra

In this section we give an example of a weakly closed operator algebra $\mathcal{A}$ which contains a maximal abelian von Neumann algebra, but which is not reflexive; since $\mathcal{A}$ is necessarily pre-reflexive (2.1.8(i)), lat $\mathcal{A}$ also provides an example of a commutative (reflexive) lattice which is not synthetic. This construction is related to an example of *L.* Schwartz [30] which asserts that the 2-sphere in $\mathbf{R}^3$ is not a set of synthesis for the group algebra $L^1(\mathbf{R}^3)$. We begin with a simple result which provides a useful reduction. X will denote a separable locally compact metric space and m is a sigma-finite measure on X.

PROPOSITION 2.5.1. *Let* $F \subseteq X \times X$ *be closed, and suppose there is a bounded operator* T *on* $L^2(X, m)$ *which is supported in* F *but which does not belong to the weak closure of* $\mathcal{A}_{\min}(F)$. *Let* $\mathcal{B}$ *denote the weakly closed* *algebra of all operators on* $L^2(X, m) \oplus L^2(X, m)$ *which admit a* 2×2 *matrix representation*

$$\begin{pmatrix} A & S \\ 0 & B \end{pmatrix},$$

where A, B *belong to the multiplication algebra and* S *belongs to the weak closure of* $\mathcal{A}_{\min}(F)$. *Then* $\mathcal{B}$ *is not reflexive.*

Proof. Choose an operator T on $L^2(X, m)$ such that $\sup(T) \subseteqq F$ and $T \notin \mathcal{A}_{\min}(F)^-$ (where the bar denotes weak closure). Then of course

$$\tilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$$

does not belong to $\mathcal{B}$. On the other hand, we claim that $\operatorname{lat} \mathcal{B} \subseteqq \operatorname{lat} \tilde{T}$. Indeed, by 2.2.6 we know that $Tf \in [\mathcal{A}_{\min}(F)f]$ for every $f \in L^2(X, m)$, and this implies that $\tilde{T}\xi \in [\mathcal{B}\xi]$ for every $\xi \in L^2(X, m) \oplus L^2(X, m)$. The latter condition implies $\operatorname{lat} \mathcal{B} \subseteqq \operatorname{lat} \tilde{T}$. □

Therefore, to produce an example of a weakly closed non-reflexive algebra which contains a maximal abelian von Neumann algebra, it suffices to establish the following. Let dx denote Lebesgue measure in the space $X = \mathbf{R}^3$, and define the closed subset Σ of $\mathbf{R}^3 \times \mathbf{R}^3$ by

$$\Sigma = \{(x, y): |x - y| = 1\},$$

where $|z|$ denotes the Euclidean norm of the vector $z \in \mathbf{R}^3$. We will describe an operator T on $L^2(\mathbf{R}^3)$ with the following properties:

2.5.2. (i) $\sup(T) \subseteqq \Sigma$,

(ii) *T does not belong to the weak closure of* $\mathcal{A}_{\min}(\Sigma)$.

The device of 2.5.1 will then produce the required example. We define T initially as a bilinear form on the space C_0 of all compactly supported infinitely differentiable functions on $\mathbf{R}^3$ as follows:

$$(Tf, g) = \int_{S^2} \frac{\partial}{\partial n}(f * \tilde{g})(x) da(x),$$

where $S^2 = \{x \in \mathbf{R}^3: |x| = 1\}$ denotes the 2-sphere, $da(x)$ represents the usual surface area on S^2, $\partial/\partial n$ denotes differentiation along the outward normal on S^2, $\tilde{g}(x) = \overline{g(-x)}$, and finally $f*g$ is the usual convolution

$$f*g(x) = \int_{\mathbf{R}^3} f(t)g(x - t)dt.$$

That T is bounded is immediate from the following result. We will employ the usual notation

$$\hat{f}(\xi) = (2\pi)^{-3/2}\int_{\mathbf{R}^3} e^{-i\langle \xi, x\rangle} f(x)dx$$

for the Fourier transform of an integrable function f on $\mathbf{R}^3$.

PROPOSITION 2.5.3. *Define* $k \in L^\infty(\mathbf{R}^3)$ *by*

$$k(\xi) = \cos|\xi| - \int_0^1 \cos(|\xi|t)dt .$$

Then there is a positive constant c such that $(Tf, g) = c(L_k\hat{f}, \hat{g})$, *for all* $f, g \in C_0^\infty$.

Remark. If we let W be the closure of the operator defined on the dense subspace $L^1(\mathbf{R}^3) \cap L^2(\mathbf{R}^3)$ by $f \mapsto \hat{f}$, then of course W is unitary, and 2.5.3 implies that T and cW^*L_kW determine the same bilinear form on $C_0^\infty \times C_0^\infty$. Since the multiplication operator L_k is clearly bounded, the same is true of T.

Proof of 2.5.3. Fix f, g in C_0^∞. Then by Green's theorem we have

$$(Tf, g) = \int_{S^2} \frac{\partial}{\partial n}(f*\tilde{g})da = \int_{|x|\leqq 1} \Delta(f*\tilde{g})(x)dx ,$$

where $\Delta = \sum_{i=1}^3 (\partial^2/\partial x_i^2)$ denotes the Laplacian. Now if $h \in C_0^\infty$, then

$$(\Delta h)^\wedge(\xi) = -|\xi|^2\hat{h}(\xi) .$$

Applying this to $h = f*\tilde{g}$ we see that the Fourier transform of $\Delta(f*\tilde{g})$ is the function $-|\xi|^2F(\xi)\overline{G(\xi)}$, where $F = \hat{f}$, $G = \hat{g}$. So if χ denotes the characteristic function of the set $\{x \in \mathbf{R}^3 : |x| \leqq 1\}$, then the right side of the displayed equation can be written

$$(\Delta(f*g), \chi) = -\int_{\mathbf{R}^3} |\xi|^2 F(\xi)\bar{G}(\xi)\bar{\hat{\chi}}(\xi)d\xi = \int_{\mathbf{R}^3} u(\xi)F(\xi)\bar{G}(\xi)d\xi$$

where $u(\xi) = -|\xi|^2\bar{\hat{\chi}}(\xi)$. It remains to show that u has the required form (i.e., is a scalar multiple of k).

The transform of χ has the form

$$\hat{\chi}(\xi) = |\xi|^{-1/2}\int_0^1 J_{1/2}(|\xi|r)r^{3/2}dr ,$$

where $J_{1/2}$ denotes the Bessel function of order 1/2 ([12, p. 203]). Since $J_{1/2}(s)$ has the form $cs^{-1/2}\sin s$, where c is a positive constant, an integration by parts yields

$$\int_0^1 J_{1/2}(|\xi|r)r^{3/2}dr = c|\xi|^{-1/2}\int_0^1 \sin(|\xi|r)r\,dr$$
$$= c|\xi|^{-3/2}\left(\int_0^1 \cos(|\xi|t)dt - \cos|\xi|\right) = -c|\xi|^{-3/2}k(\xi) .$$

The assertion follows. □

PROPOSITION 2.5.4. $\sup(T) \subseteqq \Sigma$.

Proof. Let U be an open set in $\mathbf{R}^3$. Then the space of all functions $f \in C_0^\infty$ which live in U is dense in $L^2(U)$ (in the L^2-norm). So it suffices to verify the following: For every $p \notin \Sigma$, there is a rectangular open neighborhood $U \times V$ of p such that $(Tf, g) = 0$ for every pair of functions $f, g \in C_0^\infty$ which live in V, U respectively. Indeed, for every such p, choose open precompact sets U, V in $\mathbf{R}^3$ such that $p \in U \times V$ and the closure of $U \times V$ is disjoint from the closed set Σ. Now if $f, g \in C_0^\infty$ live in V, U respectively, then $f*\tilde{g}$ lives in $V - U \subseteqq \bar{V} - \bar{U}$. Since the latter set is compact and disjoint from the 2-sphere S^2, $f*\tilde{g}$ vanishes on a neighborhood of S^2. Hence

$$(Tf, g) = \int_{S^2} \frac{\partial}{\partial n}(f*\tilde{g})(x)da(x) = 0 ,$$

as required. □

It remains only to show that T satisfies property 2.5.2(ii).

PROPOSITION 2.5.5. *T does not belong to the weak operator closure of* $\mathfrak{A}_{\min}(\Sigma)$.

Proof. We will exhibit a finite set of functions $f_1, g_1, \cdots, f_n, g_n \in L^2(\mathbf{R}^3)$ (in fact, $n = 10$) such that $\Sigma_i(Tf_i, g_i) \neq 0$ but $\Sigma_i(T_\mu f_i, g_i) = 0$ for every measure $\mu \in A(\Sigma, dx)$.

Fix a function u in $C_0^\infty(\mathbf{R}^3)$, and define a complex-valued function ϕ on $\mathbf{R}^3 \times \mathbf{R}^3$ by

$$\phi(x, y) = u(y)\bar{u}(x)(|x - y|^2 - 1) .$$

ϕ may be rewritten in the form

$$\phi(x, y) = \sum_{1=1}^3 u(y)x_i^2\bar{u}(x) + \sum_{i=1}^3 y_i^2 u(y)\bar{u}(x) - 2\sum_{i=1}^3 y_i u(y)x_i\bar{u}(x) - u(y)\bar{u}(x) ,$$

so that it has the form $\phi(x, y) = \sum_{i=1}^{10} f_i(y)\bar{g}_i(x)$ for appropriate $f_i, g_i \in L^2(\mathbf{R}^3)$ (which, of course, depend on u). In particular, $\phi \in L^2(\mathbf{R}^3) \hat{\otimes} L^2(\mathbf{R}^3)$. We will show that $\int \phi d\mu = 0$ for every $\mu \in A(\Sigma, dx)$, and that if the function u is appropriately chosen, then $\Sigma_i(Tf_i, g_i) \neq 0$.

The first assertion is clear; for ϕ vanishes indentically on Σ, and therefore $\int \phi d\mu = 0$ for every finite measure μ on $\mathbf{R}^3 \times \mathbf{R}^3$ which is concentrated on Σ.

For the second assertion, we claim that

$$\Sigma_i(Tf_i, g_i) = 2\int_{S^2}(u*\tilde{u})(x)da(x) .$$

The desired conclusion follows from this, since one may obviously find $u \in C_0^\infty$ for which $u*\tilde{u}$ is strictly positive on S^2, and therefore $\int_{S^2} u*\tilde{u}da > 0$. Now the functions f_1, g_i appearing in the formula $\phi(x, y) = \sum_i f_i(y)\overline{g_i}(x)$ belong to C_0^∞, so that by definition of T we have

$$\begin{aligned}\sum_i(Tf_i, g_i) &= \sum_i \int_{S^2}\frac{\partial}{\partial n}(f_i*\tilde{g}_i)(x)da(x)\\ &= \int_{S^2}\frac{\partial}{\partial n}\left(\sum_i f_i*\tilde{g}_i(x)\right)da(x) .\end{aligned}$$

Now write

$$\begin{aligned}\sum_i f_i*\tilde{g}_i(x) &= \sum_i \int_{\mathbf{R}^3} f_i(t)\overline{g_i}(t-x)dt\\ &= \int_{\mathbf{R}^3}\sum f_i(t)\overline{g_i}(t-x)dt\\ &= \int_{\mathbf{R}^3}\phi(t-x, t)dt ,\end{aligned}$$

so that by interchanging the order of integration and differentiation we have

$$\begin{aligned}\sum(Tf_i, g_i) &= \int_{S^2}\frac{\partial}{\partial n}\int_{\mathbf{R}^3}\phi(t-x, t)dtda(x)\\ &= \int_{S^2}\int_{\mathbf{R}^3}\sum_{k=1}^3 \frac{x_k}{|x|}\frac{\partial}{\partial x_k}\phi(t-x, t)dtda(x) .\end{aligned}$$

Now $\phi(t-x, t) = u(t)\bar{u}(t-x)(|x|^2-1)$. So if t is fixed and $|x| = 1$, then

$$\frac{\partial}{\partial x_k}\phi(t-x, t) = u(t)\overline{u(t-x)}\cdot 2x_k ,$$

and therefore

$$\Sigma_k \frac{x_k}{|x|}\frac{\partial}{\partial x_k}\phi(t-x, t) = 2u(t)\overline{u(t-x)}|x| = 2u(t)\overline{u(t-x)} .$$

It follows that

$$\Sigma(Tf_i, g_i) = 2\int_{S^2}\int_{\mathbf{R}^3}u(t)\bar{u}(t-x)dt\, da(x) = 2\int_{S^2}u*\tilde{u}(x)da(x) ,$$

as required. □

Chapter 3. Lattice-theoretic invariants for reflexive algebras

Let $(X, \leqq, m)$ and $(Y, \leqq, n)$ be two standard partially ordered measure spaces; for simplicity, assume both partial orderings are strict. Then the reflexive algebras $\operatorname{alg}\mathcal{L}(X, \leqq, m)$ and $\operatorname{alg}\mathcal{L}(Y, \leqq, n)$ are unitarily equivalent if and only if there is a unitary operator U such that $U\mathcal{L}(X, \leqq, m)U^{-1} =$

$\mathcal{L}(Y, \leqq, n)$. In turn, the results of Section 1.2 show that this reduces to the problem of determining whether $(X, \leqq, m)$ and $(Y, \leqq, n)$ are isomorphic as partially ordered measure spaces. As we have pointed out in Section 1.2, this characterization is not a useful one for dealing with even the most common special cases. The purpose of this chapter is to make a coordinate-free analysis of the *structure* of the lattices $\mathcal{L}(X, \leqq, m)$ which will enable one to deal with classification problems of this type.

Two operator algebras $\mathcal{A}$, $\mathcal{B}$ (acting on different spaces perhaps) are said to be *similar* if there is an invertible operator V such that $V\mathcal{A}V^{-1} = \mathcal{B}$. It is easy to see that the mapping of subspaces, $\mathfrak{M} \mapsto V\mathfrak{M}$, induces a bijection of lat $\mathcal{A}$ onto lat $\mathcal{B}$ which preserves the lattice operations $\vee$ and $\wedge$; so to prove that $\mathcal{A}$ and $\mathcal{B}$ are *not* similar, it suffices to show that lat $\mathcal{A}$ and lat $\mathcal{B}$ are not isomorphic as lattices.

We shall consider the following two test problems. Let N be an integer, $N \geqq 2$, and let $\mathbf{N}$ be the finite chain $\{0, 1, 2, \cdots, N-1\}$. $\mathbf{N}^\infty$ will denote the infinite Cartesian product $\mathbf{N} \times \mathbf{N} \times \cdots$ with its obvious product Borel structure, and $\leqq$ will denote the (strict) product order on $\mathbf{N}^\infty$, defined by $(x_i) \leqq (y_i) \Leftrightarrow x_i \leqq y_i$ for every $i = 1, 2, \cdots$.

Example 3.0.1. For each $N \geqq 2$, let μ_N be the product measure $\nu \times \nu \times \cdots$ on $\mathbf{N}^\infty$, where each factor ν assigns uniform mass $1/N$ to each point in the coordinate space $\mathbf{N}$. Let $\mathcal{A}_N$ be the reflexive algebra alg $\mathcal{L}(\mathbf{N}^\infty, \leqq, \mu_N)$. A natural question here is, can $\mathcal{A}_M$ and $\mathcal{A}_N$ be similar if $M \neq N$?

Example 3.0.2. For each p, $0 < p < 1$, define m_p as the product measure $n \times n \times \cdots$ on 2^∞, where each factor n assigns mass p to $\{1\}$ and $1 - p$ to $\{0\}$. Here one asks can alg $\mathcal{L}(2^\infty, \leqq, m_p)$ and alg $\mathcal{L}(2^\infty, \leqq, m_q)$ be similar if $p \neq q$?

As we will see, these two problems illustrate different phenomena, and the answers to both are no. The solution of 3.0.1 depends on a unique factorization theorem for certain distributive lattices (Section 3.3), while 3.0.2 involves a numerical invariant introduced in Section 3.5.

3.1. Primes and Generators

In this chapter, we are concerned with certain properties of commutative subspace lattices which are most conveniently discussed in a more general context. The appropriate setting is a class of distributive lattices, described as follows. Recall [8] that a *valuation* of a lattice L is a real-valued function v defined on L with the property

$$v(x \vee y) + v(x \wedge y) = v(x) + v(y) ,$$

for all $x, y \in L$. v is *positive* if $v(x) < v(y)$ whenever $x \leqq y$, $x \neq y$. A *metric lattice* is a pair (L, v), consisting of a lattice L with a distinguished positive valuation v; we will usually not refer to v explicitly unless there is cause for confusion. In any metric lattice L, we may define a distance function ∂ on L in terms of the distinguished valuation v by

$$\partial(x, y) = v(x \vee y) - v(x \wedge y) .$$

This makes L into a metric space, in which the lattice operations $x \rightarrow a \vee x$ and $x \rightarrow a \wedge x$, for fixed $a \in L$, are contractions ([8, p. 77]). A metric lattice L having a 0 and 1 is complete as a metric space if and only if L is a complete lattice and its valuation v is *normal* in the sense that, for every increasing sequence x_n with least upper bound x (this is written $x_n \uparrow x$), one has

$$v(x_n) \uparrow v(x) ,$$

and dually, $x_n \downarrow x$ implies $v(x_n) \downarrow v(x)$ ([8, pp. 80–81]). Such an L is called a *complete* metric lattice.

Throughout this section, L *will denote a distributive complete metric lattice* (we require that L contain a 0 and 1); moreover, the main results will require that L be *separable* in the sense that it is generated, as a complete lattice, by a countable subset (equivalently, L is separable as a metric space). Such an L is the lattice-theoretic counterpart of a finite measure algebra, the only difference being that one may not be able to take complements in L. For our purposes, the main examples are the subspace lattices $L = \mathfrak{L}(X, \leqq, m)$, where the valuation is defined by

$$v(P_E) = n(E) ,$$

n being a distinguished finite positive measure on X which is mutually absolutely continuous with m.

A nonzero element p of a lattice L is called a *prime* if, for any two elements $x, y \in L$, $x \vee y = p$ implies $x = p$ or $y = p$. Note that if the positive integers are regarded as a lattice, relative to the partial ordering $m \leqq n \Leftrightarrow m$ divides n, then the primes are the elements of the form q^n, where q is an ordinary prime and n is a positive integer. On the other hand, observe that a nonatomic Boolean algebra has no primes whatsoever. A simple argument also shows that in a *finite* lattice, distributive or not, every element is a finite union of primes ([8, p. 20, Ex. 4]. Note also that [8] uses the term *join-irreducible*; we feel, however, that the above terminology is better suited for our purposes).

For any subset $S \subseteq L$, the notation $[S]$ will denote the *complete* sublattice generated by S, 0, and 1. Equivalently, $[S]$ can be described as the metric closure in L of the sublattice generated by S, 0, and 1 (see [8]).

DEFINITION 3.1.1. *By a generator for L we mean a countable subset G of L which contains 0 and 1 and satisfies $[G] = L$.*

If we enumerate the elements of G, say $G = \{a_1, a_2, \cdots\}$ (it is not necessary that G or L be infinite, but we *shall* always require that the labeling satisfy $a_i \neq a_j$ if $i \neq j$), then we may define a "tail" sublattice G of L as follows: $G_\infty = \bigcap_n [a_n, a_{n+1}, \cdots]$ if G is infinite, and $G_\infty = \{0, 1\}$ if G is finite. We leave it for the reader to check that G_∞ does not depend on the particular enumeration of the elements of G.

We come now to the main result of this section, which is basic to much of this chapter

THEOREM 3.1.2. *Assume L is infinite, and let G be a generator for L. Then every prime in L has a representation $p = p_1 \wedge p_\infty$, where p_1 is an intersection of certain elements of G, and p_∞ belongs to G_∞.*

Proof. Let p be a prime in L, and define $p_1 = \bigwedge \{a \in G: a \geqq p\}$. Write $G = \{a_1, a_2, \cdots\}$. Then for each $n \geqq 1$ we define an element $q_n \in [a_{n+1}, a_{n+2}, \cdots]$ by

$$q_n = \bigwedge \{x \in [a_{n+1}, a_{n+2}, \cdots]: x \geqq p\} .$$

We will prove that $p = p_1 \wedge q_n$. Granting that for a moment, note that the conclusion follows. Indeed, we have $q_n \leqq q_{n+1}$, so that if p_∞ denotes $\bigvee_n q_n$, then q_n converges to p_∞ in the metric of L. Since the map $x \mapsto p_1 \wedge x$ is continuous, we see that $p_1 \wedge p_\infty = \lim_n p_1 \wedge q_n = p$; the theorem follows because p_∞ belongs to $\bigcap_n [a_{n+1}, a_{n+2}, \cdots] = G_\infty$.

Fixing n, by definition of p_1 and q_n we have $p \leqq p_1$ and $p \leqq q_n$, so that $p \leqq p_1 \wedge q_n$. Hence, we need to prove that $p \geqq p_1 \wedge q_n$. For that we require a few preliminaries.

LEMMA 1. *Let p be a prime in L. Then $p \leqq a_1 \vee a_2 \vee \cdots \vee a_n$ implies $p \leqq a_i$ for some $i = 1, 2, \cdots, n$.*

Proof. We repeat this simple argument ([8, p. 139]) for completeness. Since L is distributive, the condition $p \leqq \bigvee_{i=1}^m a_i$ implies $p = p \wedge \bigvee_{i=1}^m a_i = \bigvee_{i=1}^m (p \wedge a_i)$. Since p is prime, we must have $p = p \wedge a_i$ for some i; that is, $p \leqq a_i$. □

For any nonvoid subset $S \subseteq L$, write S_σ (resp. S_δ) for the set of all elements of L of the form $\bigvee_{i=1}^\infty x_i$ (resp. $\bigwedge_{i=1}^\infty x_i$), where $x_1, x_2, \cdots$ is an arbitrary sequence in L, perhaps with repetitions.

LEMMA 2. *Let G be a generator for L, and let H be the set of all finite intersections of elements of G. Then $L = H_{\sigma\delta}$.*

Proof. H contains G, and is clearly closed under (finite) intersections. Since L is distributive, this implies that the set H' of all finite unions of elements of H is also closed under finite intersections (this routine detail is left for the reader). Therefore H' is the sublattice generated by G.

Next, note that H' is dense in L. For the closure of H' is *metrically* complete, therefore lattice-theoretically complete ([8, p. 82]; recall that H' contains the 0 and 1 of L), and therefore it contains $[G] = L$.

Finally, to prove that $L = H_{\sigma\delta}$, we show equivalently that $L = (H')_{\sigma\delta}$. Choose $x \in L$. Then there is a sequence $x_n \in H'$ such that $\partial(x_n, x) \leqq 2^{-n}$. It follows that

$$\partial(x_{n+1} \vee \cdots \vee x_{n+k}, x) \leqq \sum_1^k \partial(x_{n+j}, x) \leqq \sum_1^k 2^{-(n+j)} \leqq 2^{-n} ,$$

for every $k \geqq 1$. So put $y_n = x_{n+1} \vee x_{n+2} \vee \cdots = \lim_k (x_{n+1} \vee \cdots \vee x_{n+k})$. Then $y_n \in (H')\sigma$, and the preceding shows that $\partial(y_n, x) \leqq 2^{-n}$. This implies that $y_n \to x$ metrically. On the other hand, since y_n is a decreasing sequence, it must converge to its greatest lower bound. Thus, $x = \wedge_n y_n \in (H')_{\sigma\delta}$. □

Returning to the proof of 3.1.2, we have to show that $p \geqq p_1 \wedge q_n$. For that, define a subset S of L as follows:

$$S = \{x \in L: x \geqq p \text{ implies } x \geqq p_1 \wedge q_n\} ,$$

i.e., $x \in S$ if and only if $x \ngeqq p$, or $x \geqq p$ and $x \geqq p_1 \wedge q_n$. It suffices to show that $S = L$ (for then p will belong to S and, since $p \geqq p$, we must therefore have $p \geqq p_1 \wedge q_n$). Note that S is closed under *arbitrary* intersections. So if we let H be the set of all finite intersections of elements of G, then it suffices to show that S contains H_σ; for we know by Lemma 2 that $L = H_{\sigma\delta}$, and so this will imply that $L = H_{\sigma\delta} \subseteqq S_\delta \subseteqq S$.

To this end, choose an element $x \in H_\sigma$. Let B_n (resp. C_n) be the set of all finite intersections of elements of $\{1, a_1, a_2, \cdots, a_n\}$ (resp., $\{1, a_{n+1}, a_{n+2}, \cdots\}$). Since B_n is a finite set, we may enumerate its elements as $\beta_1, \cdots, \beta_l$. We claim first that x must have the form $x = \bigvee_{j=1}^{l} (\beta_j \wedge c_j)$, where $c_1, \cdots, c_l$ belong to $(C_n)_\sigma$. Indeed, since every element of H has the form $\beta_j \wedge c$, for some $j = 1, \cdots, l$ and some $c \in C_n$, and since x is a countable union of such elements, the assertion follows from the infinite distributivity property $\bigvee_{i=1}^{\infty} (\beta \wedge c_i) = \beta \wedge \bigvee_{i=1}^{\infty} c_i$ ([8, pp. 82, 146]).

To see that $x = \bigvee_{j=1} (\beta_j \wedge c_j)$ belongs to S, assume $x \geqq p$; we have to show that $x \geqq p_1 \wedge q_n$. By Lemma 1, there is a j, $1 \leqq j \leqq l$, such that $\beta_j \wedge c_j \geqq p$, i.e., $\beta_j \geqq p$ and $c_j \geqq p$. Since β_j is an intersection of certain elements of $\{1, a, \cdots, a_n\}$, all of which dominate p, we must have $\beta_j \geqq p_1$. On the other hand, since $c_j \in [a_{n+1}, a_{n+2}, \cdots]$ and $c_j \geqq p$, we have $c_j \geqq q_n$. Therefore $x \geqq \beta_j \wedge c_j \geqq p_1 \wedge q_n$, as required. □

Remark. There is a simpler variant of 3.1.2 which applies to the case where L is finite; that is, every prime is an intersection of certain elements of any given generator G for L. We leave it for the reader to extract the proof from the above.

DEFINITION 3.1.3. *A basis for L is a generator G, all of whose elements (other than 0 or 1) are primes, which satisfies $G_\infty = \{0, 1\}$.*

Most of the results to follow are applicable to lattices with a basis, for instance, the lattices described in the introduction to Chapter 3. The following corollary describes a sense in which bases are unique.

COROLLARY. *Let G_1 and G_2 be two bases for an infinite lattice L. Then every element of G_1 is an intersection of elements of G_2, and vice versa.*

Proof. Choose $a \in G_1$, $a \neq 0, 1$. Since a is prime, 3.1.2 implies $a = b \wedge c$, where b is an intersection of elements of G_2 and $c \in (G_2)_\infty = \{0, 1\}$. The only possibility for c is $c = 1$, hence $a = b$ has the asserted form. The cases $a = 0$ or $a = 1$ are trivial. □

3.2. Primes and factorizations

In this section we introduce the notion of an infinite factorization for a distributive metric lattice, and we identify the primes in such lattices.

Throughout the section, L will denote a separable distributive complete metric lattice. By a *sublattice* of L we mean a subset of L which is closed under the lattice operations, and which contains both the zero and unit of L. 2 will denote the trivial sublattice $\{0, 1\}$.

DEFINITION 3.2.1. *L is called primary if its unit is a prime.*

Remarks. Thus, L is primary if and only if $a \vee b = 1$ implies $a = 1$ or $b = 1$, for all $a, b \in L$. In particular, no nontrivial element of a primary lattice can have a complement in L, so that primary lattices represent an opposite extreme from Boolean algebras.

Note also that a *finite* distributive lattice is primary if and only if it contains a *largest* non-unit e (i.e., $e \neq 1$ and e dominates every other element $a \neq 1$). Equivalently, suppose L is the lattice $L(X, \leqq)$ of all increasing subsets of a finite partially ordered set $(X, \leqq)$ (for simplicity, assume the ordering $\leqq$ is *strict*). Note then that $L(X, \leqq)$ is primary if and only if the set X has a smallest element. More generally, it is not hard to see in this case that the primes of $L(X, \leqq)$ are precisely the "intervals" $[a, 1] = \{y \in X: y \geqq a\}$, a being an arbitrary point of X. Finally, note the connection between primes and primary lattices: *an element* $p \in L$ *is prime if and only if the ideal* $p \wedge L = \{p \wedge x: x \in L\}$ *is a primary lattice in its own right.*

Given a family $\{L_\alpha\}$ of sublattices of L, we write $\bigvee_\alpha L_\alpha$ for the *complete* sublattice generated by $\bigcup_\alpha L_\alpha$. If the family is a sequence $L_1, L_2, \cdots$, we write L_∞ for the "tail" sublattice $\bigcap_{n=1}^\infty (L_n \vee L_{n+1} \vee \cdots)$. Here, it will be convenient to allow repetitions in the sequence $\{L_n\}$; nevertheless, the tail sublattice is independent of the enumeration in the sense that if $M_n = L_{\pi(n)}$, where π is any permutation of the positive integers, then $M_\infty = L_\infty$.

THEOREM 3.2.2. *Let* L_n, $n \geqq 1$, *be a sequence of sublattices of* L *such that* $L = \bigvee_n L_n$ *and which satisfy*

(i) L_n *contains a largest non-unit* e_n, *and*

(ii) *for all* $x \in L_n$, $y \in \bigvee_{k=n+1}^\infty L_k$, $x \vee y = 1$ *implies* $x = 1$ *or* $y = 1$.

Then L *is primary if and only if* L_∞ *is primary.*

Proof. The "only if" part is trivial, since every sublattice of a primary lattice must be primary.

So assume L_∞ is primary, and choose $a, b \in L$ such that $a \vee b = 1$. Let Λ be the sublattice generated by $\bigcup_{n=1}^\infty L_n$. We deal first with the special case where both a, b belong to Λ_σ (see Section 3.1). Letting H denote the set of all finite intersections $x_1 \wedge \cdots \wedge x_m$, $x_i \in \bigcup_{n=1}^\infty L_n$, then every element of Λ is a finite union of elements of H, so that a and b must have the form $a = \bigvee_{n=1}^\infty a_n$, $b = \bigvee_{n=1}^\infty b_n$, where $a_n, b_n \in H$.

Now let e_n be the largest non-unit in L_n. We associate two sequences α_n, α_n' with a as follows:

$$\alpha_n = \bigvee \{a_k: a_k \leqq e_1 \vee \cdots \vee e_n\},$$
$$\alpha_n' = \bigvee \{a_k: a_k \nleqq e_1 \vee \cdots \vee e_n\}.$$

It is clear that $a = \alpha_n \vee \alpha_n'$, that $\alpha_n \leqq \alpha_{n+1}$, and that $\alpha_n' \geqq \alpha_{n+1}'$. Note also that $\alpha_n' \in \bigvee_{j=n+1}^\infty L_j$. For if $a_k \in H$ is such that $a_k \nleqq e_1 \vee \cdots \vee e_n$, say $a_k = x_1 \wedge \cdots \wedge x_r$, $x_i \in \bigcup_n L_n$, then $a_k \leqq x_i$ for all i and hence $x_i \nleqq e_1 \vee \cdots \vee e_n$, $1 \leqq i \leqq r$. Now if x_i is a non-unit then it cannot belong to $L_1 \cup \cdots \cup L_n$, by definition of $e_1, \cdots, e_n$, and therefore $x_i \in \bigcup_{j=n+1}^\infty L_j \subseteqq \bigvee_{j=n+1}^\infty L_j$. If $x_i = 1$ the same conclusion is evident. We conclude that $a_k \in \bigvee_{j=n+1}^\infty L_j$, and since α_n' is a (countable) union of such elements a_k, the claim follows.

Similarly, we may write $b = \beta_n \vee \beta_n'$, where $\beta_n \leqq \beta_{n+1}$, $\beta_n' \geqq \beta_{n+1}'$, and $\beta_n' \in \bigvee_{i=n+1}^\infty L_j$. So we can write

$$1 = a \vee b = (\alpha_n \vee \beta_n) \vee (\alpha'_n \vee \beta'_n) \leqq e_1 \vee \cdots \vee e_n \vee \alpha'_n \vee \beta'_n ,$$

and hence $e_1 \vee \cdots \vee e_n \vee \alpha'_n \vee \beta'_n = 1$ for every $n \geqq 1$. By the hypothesis (ii) we see that $e_2 \vee \cdots \vee e_n \vee \alpha'_n \vee \beta'_n = 1$ and, continuing inductively, we arrive at the conclusion $\alpha'_n \vee \beta'_n = 1$. Put $\alpha'_\infty = \bigwedge_n \alpha'_n$, $\beta'_\infty = \bigwedge_n \beta'_n$. Since both sequences α'_n, β'_n are decreasing, they converge to α'_∞, β'_∞ respectively in the metric of L; and since the map $x, y \mapsto x \vee y$ is metrically continuous ([8]), we conclude that $\alpha'_\infty \vee \beta'_\infty = 1$. Finally, note that since α'_n, β'_n belong to $\bigvee_{j=n+1}^{\infty} L_j$ and decrease to their limits, it follows that both α'_∞ and β'_∞ belong to L_∞ (see the proof of 3.1.2).

Since L_∞ is assumed to be primary, we must have $\alpha'_\infty = 1$ or $\beta'_\infty = 1$, say $\alpha'_\infty = 1$. Since $\alpha'_\infty \leqq \alpha'_n \leqq a$, we conclude that $a = 1$, as required.

Consider now the case of general elements $a, b \in L$ with $a \vee b = 1$. Since $L = \Lambda_{\sigma\delta}$ (see Lemma 2 in the previous section), there exist $a_n, b_n \in \Lambda_\sigma$ such that $a = \bigwedge_n a_n$, $b = \bigwedge_n b_n$. Because Λ_σ is closed under *finite* intersections, we may assume that $a_n \geqq a_{n+1}$, $b_n \geqq b_{n+1}$. Hence $a \vee b \leqq a_n \vee b_n$ for every n, so that $a_n \vee b_n = 1$. By the argument already given, we must have $a_n = 1$, or $b_n = 1$, for every $n \geqq 1$. So one of the two sequences $\{a_n\}$, $\{b_n\}$ must take on the value 1 infinitely often, say $\{a_n\}$; this, however, implies that $a = \lim_n a_n = 1$, completing the proof. □

DEFINITION 3.2.3. *By a factorization of L we mean a sequence L_n, $n \geqq 1$, of finite sublattices of L having the properties*:

(i) $L = \bigvee_n L_n$.

(ii) (*independence*) *For all $a, a' \in L_n$, $b, b' \in \bigvee_{j \neq n} L_j$, $a \wedge b \leqq a' \vee b'$ implies $a \leqq a'$ or $b \leqq b'$, for every $n \geqq 1$.*

(iii) (*zero-one law*) $L_\infty = 2$.

This will be expressed by the notation $L = \bigotimes_n L_n$. Of course, there exist lattices of the type discussed here which have no factorizations (example: the unit interval). However, there are many that do; a class of examples is described after 3.2.4 below. There is a similar definition of factorizations of *finite* distributive lattices $L = L_1 \otimes \cdots \otimes L_n$, obtained by simply deleting condition (iii) from 3.2.3.

Suppose now that L has a factorization $L = \bigotimes_n L_n$, and that $\{E_1, E_2, \cdots\}$ is a partition of the set of positive integers into finite subsets E_i. For each $n \geqq 1$, define a finite sublattice $M_n \subseteq L$ by

$$M_n = \bigoplus_{j \in E_n} L_j .$$

Then a routine (though somewhat tedious) argument establishes that $\{M_n\}$ is another factorization of L. This proof will be left for the reader, since the result is not required below.

We come now to the main result of this section, which identifies the set of primes in any factorable lattice $L = \bigotimes_n L_n$. The notation $\mathcal{P}(L)$ will denote the set of all primes in L. It is well to keep in mind that if M is a sublattice of L, elements of $\mathcal{P}(M)$ need not be primes in the larger lattice L, in general.

THEOREM 3.2.4. *Assume that L admits a factorization $L = \bigotimes_n L_n$. Then the primes of L are precisely the nonzero elements of the form $q = \bigwedge_n p_n$,* where $p_1, p_2, \cdots$ *is any sequence such that $p_n \in \mathcal{P}(L_n)$ for every n.*

Proof. First, let p be a prime in L. To produce a sequence $p_n \in \mathcal{P}(L_n)$ such that $p = \bigwedge_n p_n$, we utilize 3.1.2 as follows. Let $G = \bigcup_{n=1}^{\infty} L_n$. Then G is a (countable) generator for L. Note also that $G_\infty = 2$. For if we enumerate $G = \{g_1, g_2, \cdots\}$, then for each n there is a k such that $\{g_k, g_{k+1}, \cdots\}$ contains no element of the (finite) set $L_1 \cup \cdots \cup L_n$. Therefore $\{g_k, g_{k+1}, \cdots\} \subseteqq \bigcup_{j=n+1}^{\infty} L_j$, and the claim $G_\infty \subseteqq L_\infty = 2$ follows. By 3.1.2, we conclude that there is a sequence $a_1, a_2, \cdots \in G$ such that $p = \bigwedge_k a_k$.

Now define $p_n = \bigwedge \{a_k : a_k \in L_n\}$ (if $\{a_k\}$ contains no element of L_n, put $p_n = 1$). Note that $p = \bigwedge_{n=1}^{\infty} p_n$, and it remains to show that $p_n \in \mathcal{P}(L_n)$. So fix n, and choose $a, b \in L_n$ such that $p_n = a \vee b$. Writing $q_n = \bigwedge_{i \neq n} p_i$, we may write $p = p_n \wedge q_n = (a \vee b) \wedge q_n = (a \wedge q_n) \vee (b \wedge q_n)$. Since p is prime we must have either $p = a \wedge q_n$ or $p = b \wedge q_n$, suppose $p = a \wedge q_n$. Since $a \wedge q_n \leqq a \vee 0$, we may write $p = p_n \wedge q_n = a \wedge q_n \leqq a \vee 0$. Since $p_n, a \in L_n$ and $q_n, 0 \in \bigvee_{i \neq n} L_i$, we see from property 3.2.3(ii) that $p_n \leqq a$ (the case $q_n \leqq 0$ is impossible, since $0 \neq p \leqq q_n$), and since $a \leqq a \vee b = p_n$ we have $p_n = a$. The other case, $p = b \wedge q_n$, leads to $p_n = b$ in the same way.

Conversely, let $p_1, p_2, \cdots$ be a sequence such that $p_n \in \mathcal{P}(L_n)$ and $p = \bigwedge_n p_n$ is not 0. Define a new lattice $M = p \wedge L = \{p \wedge x : x \in L\}$. We have to show that M is primary (note that p is the unit for M). This is deduced from 3.2.2 as follows.

For each $n \geqq 1$, let $M_n = p \wedge L_n$. Then M_n is a finite sublattice of M. Moreover, since the map $x \in L \mapsto p \wedge x \in M$ is a lattice homomorphism preserving arbitrary unions and intersections ([8, pp. 82, 146]), it follows that M is complete and, since $L = \bigvee_n L_n$, we have $M = \bigvee_n M_n$. To check the other hypotheses of 3.2.2, suppose $x \in M_n$ and $y \in M_{n+1} \bigvee M_{n+2} \bigvee \cdots$ satisfy $x \vee y = 1_M = p$. Then $x = p \wedge a$ and $y = p \wedge b$ for appropriate elements $a \in L_n$, $b \in L_{n+1} \bigvee L_{n+2} \bigvee \cdots$. Thus $p = (p \wedge a) \vee (p \wedge b) = p \wedge (a \vee b)$, or $p \leqq a \vee b$. Again, writing $q_n = \bigwedge_{i \neq n} p_i$, the preceding becomes $p_n \wedge q_n = p \leqq a \vee b$. Since p_n, $a \in L_n$ and q_n, $b \in \bigvee_{i \neq n} L_i$, we see from property (ii) of Definition 3.2.3 that either $p_n \leqq a$ or $q_n \leqq b$. If $p_n \leqq a$, then $p \leqq p_n \leqq a$ so that $x = p \wedge a = p = 1_M$, as required. If $q_n \leqq b$, the same reasoning shows that $y = p = 1_M$.

Next, note that M_n is primary. For if $x, y \in M_n$ satisfy $x \vee y = 1_M = p$, say $x = p \wedge a$, $y = p \wedge b$, $a, b \in L_n$, then as in the preceding paragraph we deduce that $p_n \wedge q_n = p \leqq a \vee b = (a \vee b) \vee 0$. Again, since p_n, $a \vee b \in L_n$ and q_n, $0 \in \bigvee_{i \neq n} L_i$, we deduce from 3.2.3(ii) that $p_n \leqq a \vee b$ (since the other alternative, $q_n \leqq 0$, is impossible because $0 \neq p \leqq q_n$). Because p_n is prime in L_n, Lemma 1 of the preceding section implies $p_n \leqq a$ or $p_n \leqq b$. Thus, $x = p \wedge a = p = 1_M$ or $y = p \wedge b = p = 1_M$, as required.

Since each M_n is finite, as well as primary, it must contain a largest non-unit. Therefore, all the hypotheses of 3.2.2 are satisfied. Therefore to show that M is primary, it suffices to show that M_∞ is primary.

In fact, we claim that $M_\infty = \{0, p\}$. Indeed, if x belongs to $M_\infty = \bigcap_n \{p \wedge a : a \in L_n \vee L_{n+1} \vee \cdots\}$, then for each n we can write $x = p \wedge a_n$, $a_n \in L_n \vee L_{n+1} \vee \cdots$. By replacing a_n with $a_n \vee a_{n+1} \vee \cdots$, we can assume $a_n \geqq a_{n+1}$; and therefore a_n converges to its limit $a_\infty = \bigwedge_n a_n$ in the metric of L. By continuity of the map $z \mapsto p \wedge z$, we conclude that $x = p \wedge a_\infty$. Since a_∞ belongs to $L_\infty = 2$, x must be either 0 or p, as required. □

We conclude this section with a discussion of a class of examples. For each $n \geqq 1$, let $(X_n, \leqq)$ be a finite partially ordered set (for simplicity we assume the ordering is strict), and let m_n be a probability distribution on X_n which assigns positive mass to every point. Define X to be the product $\times_n X_n$, with the obvious Borel structure; define a probability measure m on X by $m = \times_n m_n$; and define a (strict) partial order on X by $(x_n) \leqq (y_n) \Leftrightarrow x_n \leqq y_n$ for every coordinate $n \geqq 1$. Finally, let L be the subspace lattice $\mathcal{L}(X, \leqq, m)$. We obtain a positive normal valuation v on $\mathcal{L}(X, \leqq, m)$ by

$$v(P_E) = m(E) ,$$

for every increasing Borel set $E \subseteq X$, and this makes $\mathcal{L}(X, \leqq, m)$ into a complete separable distributive metric lattice.

We associate with the coordinate spaces $\{X_n, \leqq)$ of $(X, \leqq)$ a natural sequence of sublattices L_n, where L_n is defined as the set of all projections P_E, where E is an increasing Borel set which depends only on the n^{th} coordinate (more precisely, E has the form $\{(x_n) \in X : x_n \in A\}$ where A is an arbitrary increasing subset of X_n). It is easy to see that $\bigcup_n L_n$ generates L (this follows from 1.2.2, for example), and the fact that $L_\infty = 2$ is a consequence of the Kolmogorov zero-one law ([19], [15]) for measure algebras based on product measures.

We claim now that $L = \bigotimes_n L_n$. We only need to verify property 3.2.3(ii). For that, let $\mathcal{B}_n$ denote the (finite) sigma-field of all Borel sets E in X of the form $E = \{(x_i) \in X: x_n \in F\}$ where $F \subseteqq X_n$. We claim that if $A, A' \in \mathcal{B}_n$ and if B, B' belong to the sigma-field generated by $\bigcup_{i \neq n} \mathcal{B}_i$, then $A \cap B \subseteqq A' \cup B'$ implies $m(A' \backslash A) = 0$ or $m(B' \backslash B) = 0$ (the assertion follows from this). Now $(A \backslash A') \cap (B \backslash B') = A \cap B \backslash (A' \cup B')$ is empty, so that $m((A \backslash A') \cap (B \backslash B')) = 0$. Because of the way m was defined as a product measure, the sets $A \backslash A'$ and $B \backslash B'$ are (probabilistically) independent, so that

$$m(A \backslash A') m(B \backslash B') = m((A \backslash A') \cap (B \backslash B')) = 0 ;$$

hence either $m(A \backslash A') = 0$ or $m(B \backslash B') = 0$, as required.

We can now describe quite explicitly the primes of L. For every sequence $x = (x_i)$ in X, we write $[x, 1]$ for the interval $\{(y_i) \in X: y_n \geqq x_n$ for every $n\}$. Clearly, $[x, 1]$ is an increasing Borel set, and so it determines a projection $P_{[x,1]} \in \mathfrak{L}(X, \leqq, m) = L$. Note, however, that $P_{[x,1]} = 0$ whenever $[x, 1]$ has measure zero. We assert: *The primes of L are the projections $P_{[x,1]}$, where x runs over all points in X for which $m([x, 1]) > 0$.* We merely sketch how this is deduced from 3.2.4. For $x = (x_i)$ fixed, we may write $[x, 1] = \bigcap_{n=1}^{\infty} A_n$, where $A_n = \{(y_i) \in X: y_n \geqq x_n\}$, and therefore $P_{[x,1]} = \bigwedge_n P_{A_n}$. By 3.2.4, it suffices to verify that the primes of L_n are just the projections P_A, where A has the form $A = \{(y_i): y_n \geqq a\}$, a being an arbitrary point in the n^{th} coordinate space X_n. This, however, follows from the discussion after Definition 3.2.1, and the natural identification of L_n with the lattice $L(X_n, \leqq)$.

We have already pointed out that a nonatomic Boolean algebra has no primes whatsoever. Thus, it is of interest to determine when one of these lattices contains no primes. This can also be dealt with in terms of 3.2.4, and we now outline the argument very briefly. For each $n \geqq 1$, define an invariant γ_n associated with the "coordinate" measure space $(X_n, \leqq, m_n)$ as follows:

$$\gamma_n = \max \{m_n([a, 1]): a \in X_n\} .$$

Alternately, γ_n can be defined in terms of the distinguished valuation v on L as

$$\gamma_n = \max \{v(P): P \in \mathcal{P}(L_n)\} .$$

Now for any sequence $P_n \in \mathcal{P}(L_n)$ we have $v(\bigwedge_n P_n) = \prod_{n=1}^{\infty} v(P_n)$ (reflecting the fact that $m = \bigtimes_n m_n$ is a product measure). Recalling that, for any sequence t_n of real numbers, $0 < t_n \leqq 1$, the infinite product $\prod_n t_n$ is positive if and only if the series $\sum_n (1 - t_n)$ converges, we arrive at the following characterization: *In order that* L *contain at least one prime, it is necessary and sufficient that the series* $\sum_{n=1}^{\infty} (1 - \gamma_n)$ *should converge.*

Finally, let us consider the special case where each $(X_n, \leqq)$ is a fixed finite partially ordered set $(X_1, \leqq)$, and $m_n = m_1$ for each n. Assuming that X_1 has a smallest element, it follows that $L(X_1, \leqq)$ is primary, hence each L_n is primary and so L is primary by 3.2.2. For any sequence $x = (x_i) \in X$, one sees easily that $m([x, 1]) > 0$ if and only if $x_n = 0$ for all but a finite number of coordinates n. Let X_0 be the (countable) subset of all such points $x \in X$. We conclude that the map $x \in X_0 \mapsto P_{[x,1]}$ is a bijection of X_0 onto $\mathcal{P}(L)$. Moreover, one may also verify that this map is an order anti-isomorphism relative to the ordering that X_0 inherits from X. In particular, every prime in L is a *finite* intersection $P_1 \wedge P_2 \wedge \cdots \wedge P_n$, where $P_n \in \mathcal{P}(L_n)$. Of course, these remarks apply to the lattices discussed in the introduction to Chapter 3.

3.3. A unique factorization theorem

We now investigate the problem of uniqueness for factorizations of a given lattice L; as in the preceding section, all lattices are assumed to be separable distributive complete metric lattices having a 0 and 1.

We first want to relate this problem to some known results in the case of finite lattices. Suppose L is realized as the lattice $L(X, \leqq)$ of all increasing subsets of a finite partially ordered set $(X, \leqq)$. Then a known theorem asserts that L has a decomposition into a cardinal product $L_1 L_2 \cdots L_n$, where each L_i has trivial center, and moreover this decomposition is unique up to the order of the factors ([8, pp. 26–27]). In terms of the underlying set $(X, \leqq)$, there is a natural equivalence relation $\sim$ in X defined by $x \sim y$ if and only if there is a sequence $x = t_1, t_2, \cdots, t_r = y$ in X such that for all i, either $t_i \leqq t_{i+1}$ or $t_{i+1} \leqq t_i$. Letting $X_1, X_2, \cdots, X_n$ be the associated equivalence classes and $\leqq_k$ the ordering that X_k inherits from X, then the factors L_k appearing in the cardinal product $L = L_1 L_2 \cdots L_n$ may be taken as $L_k = L(X_k, \leqq_k)$. In particular, a factorization of $L(X, \leqq)$ into a cardinal product corresponds to *partitioning* the space X into disjoint subspaces of a certain type.

Here, on the other hand, we are interested in factorizations of the type $L = L_1 \otimes \cdots \otimes L_n$. In the case $L = L(X, \leqq)$ (for finite X), this amounts to realizing $(X, \leqq)$ as a *Cartesian product* of spaces $(X_k, \leqq_k)$, in the sense that $X = \times_{k=1}^{n} X_k$ and $(x_1, \cdots, x_n) \leqq (y_1, \cdots, y_n)$ if and only if $x_k \leqq_k y_k$ for every $k = 1, 2, \cdots, n$. So far as we can determine, the following material gives new results even in the case of finite lattices (though in that case, needless to say, one may give much simpler arguments).

A finite lattice L is said to be *indecomposable* if the only factorizations of L, of the type $L = L_1 \otimes L_2$, have the property that $L_1 = 2$ or $L_2 = 2$. A factorization of a (perhaps infinite) lattice $L = \bigotimes_n L_n$ is called *indecomposable* if each factor L_n is an indecomposable lattice, and is not the trivial sublattice 2. It is natural to ask the extent to which indecomposable factorizations of a given infinite lattice are unique. Unfortunately, nothing can be said in general. To illustrate this, we consider the case where L is the measure algebra of the unit interval. Let (X, m) be the standard measure space consisting of the Cantor space $X = 2^\infty$ and the measure $m = \times_{k=1}^\infty m_k$, where m_k assigns mass 1/2 to each element 0, 1 of 2. Then the measure algebra of (X, m) admits a corresponding factorization $\bigotimes_n L_n$, where each factor L_n is a four-element Boolean algebra (note therefore that each L_n is indecomposable in the above sense). Because the measure algebra of (X, m) is separable and nonatomic it must be isomorphic, as a complete Boolean algebra, to L (see [15, p. 173]). Therefore, L *admits an indecomposable factorization* $L = \bigotimes_n L_n$ *into four-element Boolean algebras* L_n. Now in a similar way, by considering the measure space (Y, n), where $Y = 3^\infty$ is an infinite Cartesian product of three-element sets and where $n = \times_{k=1}^\infty n_k$, n_k assigning uniform mass 1/3 to each point of 3, we arrive at the conclusion that L *also admits an indecomposable factorization* $L = \bigotimes_n M_n$ *into eight-element Boolean algebras* M_n.

Thus, it may be unexpected that unique factorization does hold for *primary* lattices L, even in the case of lattices $\mathfrak{L}(X, \leqq, m)$ which are based on *nonatomic* measures m. This is a consequence of the following result.

REFINEMENT THEOREM 3.3.1. *Assume that* L *is primary, and let* $L = \bigotimes_n L_n = \bigotimes_n L'_n$ *be two factorizations of* L. *Then there is a double sequence* L_{mn}, $m, n \geqq 1$, *of finite sublattices of* L *having the properties*:

(i) *For each* m (*resp.* n), $L_{mn} = 2$ *for all but finitely many values of* n (*resp.* m), *and*

(ii) $L_m = \bigotimes_n L_{mn}$, $L'_n = \bigotimes_m L_{mn}$.

Proof. Simply put $L_{mn} = L_m \cap L'_n$. Because of the properties of $\{L_m\}$, it is clear that $\{L_{mn}: m \geqq 1\}$ forms a factorization of $\bigvee_{m=1}^\infty L_{mn}$ for each fixed n, i.e., $\bigvee_m L_{mn} = \bigotimes_m L_{mn}$, and similarly for the other variable (by symmetry).

Note next that $L_i \cap L_j = 2$ if $i \neq j$. Indeed, if $a \in L_i \cap L_j$, then since $a \wedge 1 \leqq 0 \vee a$, it follows from the "independence" property of factorizations that $a \leqq 0$ or $1 \leqq a$, as asserted. Now for fixed $n \geqq 1$, this implies that the sets $L_m \cap L'_n$, $m \geqq 1$, induce a partition of $L'_n \backslash 2$. Since L'_n is finite, we conclude that $L_{mn} = L_m \cap L'_n = 2$ for all but finitely many values of m. The property (i) follows by symmetry.

For (ii), we only need to show that $L'_n = \bigvee_{m=1}^{\infty} L_{mn}$, the remainder following by symmetry. So fix $n \geqq 1$. Since the inclusion $\supseteqq$ is obvious and since L'_n is generated by its primes, it suffices to show that $\mathcal{P}(L'_n) \subseteqq \bigcup_{m=1}^{\infty} L_{mn}$. Choose $p \in \mathcal{P}(L'_n)$, and we may assume that $p \neq 1$. First, we claim that p is a prime in L. Since L is primary, so is each L'_i, and therefore $1 \in \mathcal{P}(L'_i)$ for every $i \neq n$. Thus p has the form $p = \bigwedge_r p_r$ where $p_r \in \mathcal{P}(L'_r)$ for every r. By 3.2.4, p must be prime in L.

Since L_k is generated by $\mathcal{P}(L_k)$, $\bigcup_{k=1}^{\infty} \mathcal{P}(L_k)$ is a generator for L. By 3.1.2, there exist elements $q_1, q_2, \cdots \in \bigcup_k \mathcal{P}(L_k)$ such that $p = \bigwedge_k q_k$ (we may also assume that $q_k \neq 1$ for each k). Now fix k. The argument of the preceding paragraph shows that q_k is prime in L, so that we may repeat the same argument with the generator $\bigcup_i \mathcal{P}(L'_i)$ to obtain a sequence $q_{k1}, q_{k2}, \cdots \in \bigcup_i \mathcal{P}(L'_i)$ such that $q_k = \bigwedge_l q_{kl}$, and $q_{kl} \neq 1$ for each l. Now we claim that each q_{kl} belongs to $\mathcal{P}(L'_n)$. Indeed, if $q_{kl} \in \mathcal{P}(L'_i)$ for some $i \neq n$, then because $p \leqq q_k \leqq q_{kl}$ we see that $p \wedge 1 \leqq 0 \vee q_k$. By the "independence" property of factorizations, we conclude that $p \leqq 0$ or $1 \leqq q_{kl}$, both of which are false.

It follows that $q_k = \bigwedge_l q_{kl}$ belongs to $L'_n = [\mathcal{P}(L'_n)]$, for every k. But we also know that $q_k \in \bigcup_j \mathcal{P}(L_j)$, so in particular there is a value of j (depending on k) such that $q_k \in \mathcal{P}(L_j) \cap L'_n \subseteqq L_j \cap L'_n = L_{jn}$. Thus $q_k \in \bigvee_j L_{jn}$ for every k, and we may now conclude that $p = \bigwedge_k q_k$ belongs to $\bigvee_j L_{jn}$, as required. □

UNIQUE FACTORIZATION THEOREM 3.3.2. *Suppose L is primary. Then for any two indecomposable factorizations $L = \bigotimes_n L_n = \bigotimes_n L'_n$ of L, there is a permutation π of the positive integers such that $L'_n = L_{\pi(n)}$ for every n.*

Proof. By the preceding result, we may write $L_m = \bigotimes_n L_{mn}$ and $L'_n = \bigotimes_m L_{mn}$. By the indecomposability property, it follows that the doubly infinite matrix (L_{mn}) has exactly one nontrivial entry in every row and in every column. Thus there is a permutation π of the positive integers such that $L_{n\pi^{-1}(n)}$ is the nontrivial entry which occurs in the n^{th} row, $n \geqq 1$. Thus, $L_n = \bigvee_j L_{nj} = L_{n\pi^{-1}(n)}$, and so $L_{\pi(n)} = L_{\pi(n)n} = \bigvee_j L_{jn} = L'_n$, as asserted. □

Discussion of Example 3.0.1. Let N be an integer, $N \geqq 2$, and let $\mathbf{N}$ be the chain $\{0, 1, \cdots, N-1\}$. Consider the subspace lattice $L_N = \mathfrak{L}(\mathbf{N}^\infty, \leqq, m)$ defined in Section 3.0. Then L_N admits an obvious factorization $L_N = \bigotimes_j L_j$, where each L_j is a chain consisting of $N+1$ elements. By the unique factorization theorem every indecomposable factorization of L consists of factors which are chains containing $N+1$ elements, and in particular, the integer N is an isomorphism invariant of the lattice L_n. Hence, L_N and L_M cannot be isomorphic if $N \neq M$. Since, as subspace lattices, L_M and L_N are reflexive (1.6.3), we conclude that *the operator algebras* alg $\mathfrak{L}(\mathbf{N}^\infty, \leqq, m)$ *and* alg $\mathfrak{L}(\mathbf{M}^\infty, \leqq, m)$ *are not similar if* $N \neq M$.

We also point out, however, that this argument does *not* distinguish between the lattices $\mathfrak{L}(2^\infty, \leqq, m_p)$, $0 < p < 1$, of problem 3.0.2, since they all admit indecomposable factorizations of the form $\bigotimes_j L_j$ where each L_j is a chain of length 3. The latter problem is taken up in Section 3.5 below.

3.4 A representation theorem

In order to establish a key result in the following section, it is necessary to realize a given lattice in an appropriate way as a subspace lattice. The purpose of this section is to present a suitable representation theorem.

It is well-known that every separable complete Boolean algebra B, which admits a positive finite sigma-additive "measure," is isomorphic to the measure algebra of a (finite) standard measure space. The main result below gives a strengthened generalization of this theorem to separable complete distributive metric lattices. Given such a lattice L, we shall make use of the following notation. Recall ([8, p. 74]) that the *variation* of a valuation μ of L is defined as follows:

$$\|\mu\| = \sup \textstyle\sum_{k=1}^n |\mu(a_k) - \mu(a_{k-1})| ,$$

the supremum taken over all finite chains $0 = a_0 \leqq a_1 \leqq \cdots \leqq a_n = 1$ in L. We shall write val(L) for the real vector space of all valuations μ on L satisfying $\mu(0) = 0$ and $\|\mu\| < \infty$. It is very easy to see that the norm $\|\cdot\|$ makes val(L) into a Banach space. Moreover, if we define lattice operations $\vee$ and $\wedge$ in val(L) by

$$\mu \vee \nu(x) = \sup \textstyle\sum_{k=1}^n \max\left(\mu(a_k) - \mu(a_{k-1}), \nu(a_k) - \nu(a_{k-1})\right) ,$$

where the supremum is taken over all finite chains $0 = a_0 \leqq a_1 \leqq \cdots \leqq a_n = x$ connecting 0 and x ($\mu \wedge \nu$ is of course defined dually), then val(L) becomes a *Banach lattice* (i.e., a vector lattice in which $|\mu| \leqq |\nu|$ implies $\|\mu\| \leqq \|\nu\|$, $|\mu|$ being defined as $\mu \vee (-\mu)$ (see [8, pp. 84, 246]; indeed, this all remains true for *arbitrary* lattices L). We shall be concerned primarily with the following two subspaces of val (L).

Note first that an element $\mu \in \text{val}(L)$ satisfies $\mu \geqq 0$ (in the natural order of val(L)) if and only if $a \leqq b$ implies $\mu(a) \leqq \mu(b)$ for all $a, b \in L$. Given $\mu, \nu \in \text{val}(L)$ such that $\mu \geqq 0$ and $\nu \geqq 0$, we shall write $\mu \ll \nu$ if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every finite chain of the form $0 = a_1 \leqq b_1 \leqq a_2 \leqq b_2 \leqq \cdots \leqq b_n = 1$ for which $\sum_{k=1}^{n} (\nu(b_k) - \nu(a_k)) \leqq \delta$, one has $\sum_{k=1}^{n} (\mu(b_k) - \mu(a_k)) \leqq \varepsilon$. $L_1(v)$ will denote the space $\{\mu \in \text{val}(L): |\mu| \ll v\}$, where v denotes the distinguished valuation giving rise to the metric on L. It is easy to see that $L_1(v)$ is closed under the lattice operations, the vector space operations, and is closed in the norm of val (L), so that $L_1(v)$ is a *Banach lattice* (these routine verifications are left for the reader).

The second subspace of val (L) (which we write as L_*) is defined as the space of all elements $\mu \in \text{val}(L)$ whose modulus $|\mu|$ satisfies the condition $\lim_n |\mu|(a_n) = |\mu|(a)$ for every increasing sequence $a_n \in L$ with $\bigvee_n a_n = a$, and dually for decreasing sequences (i.e., $|\mu|$ is *normal* in the terminology of 3.1). Now since the map $\mu \mapsto |\mu|$ of L into itself is uniformly norm-continuous, it can easily be seen that L_* is closed in the norm of val (L); moreover, another series of simple arguments shows that L_* is also a Banach sublattice of val (L).

Now it is almost obvious that $L_1(v) \subseteqq L_*$; however, the inclusion is usually proper. We want to digress momentarily to discuss this significant point. Note first that, while L_* is defined purely in terms of the lattice-theoretic properties of L, $L_1(v)$ depends on the *pair* (L, v). Thus L_* is an *isomorphism* invariant of the lattice L, while $L_1(v)$ is not; in particular, while every lattice automorphism α of L onto itself must carry L_* onto itself, it can happen that $L_1(v \circ \alpha)$ is not contained in $L_1(v)$, so that $L_1(v \circ \alpha) \neq L_1(v)$. This phenomenon is illustrated by the following simple example.

Example. Let L be the complete chain consisting of the unit interval $[0, 1]$ with its usual ordering, and let v be the valuation $v(x) = x$, $0 \leqq x \leqq 1$. Let μ be any probability measure on $[0, 1]$ which is singular relative to Lebesgue measure m, is nonatomic, and assigns positive mass to every interval (a, b), $a < b$. Then the function $\alpha: L \to L$ defined by $\alpha(x) = \mu([0, x])$, $0 \leqq x \leqq 1$, is a continuous strictly increasing function mapping $[0, 1]$ onto itself. Thus, α is an automorphism of L. On the other hand, the condition $L_1(v \circ \alpha) = L_1(v)$ would imply μ is absolutely continuous relative to m, an absurdity; indeed, the singularity of μ is reflected by the fact that $L_1(v) \cap L_1(v \circ \alpha) = \{0\}$.

The existence of such "singular" automorphisms (and isomorphisms) in this class of distributive lattices is a phenomenon that does not occur in the special case of Boolean algebras. Indeed, one can show that $L_1(v) = L_*$ when L is a Boolean algebra (the proof of this can be based on the device used in the lemma on p. 253 of [8]). We now state the result of this section.

THEOREM 3.4.2. *Every separable distributive complete metric lattice is isomorphic to a complete sublattice of the measure algebra $M(X, m)$ of a standard finite measure space (X, m) which generates $M(X, m)$ as a complete Boolean algebra. Moreover, if v is the valuation associated with the metric of L, then the isomorphism implements an isometric lattice isomorphism $\mu \mapsto f_\mu$ of $L_1(v)$ onto $L^1(X, m)$, in which $\mu(a) = \int f_\mu \chi_A dm$, where $A \subseteqq X$ is the set corresponding to $a \in L$.*

The quickest way to a proof is via the following theorem of Kakutani ([18], [8]) concerning (L)-spaces (recall that an (L)-space is a real Banach lattice in which $\|x + y\| = \|x\| + \|y\|$ for all $x \geqq 0$, $y \geqq 0$).

THEOREM. *Let E be an (L)-space which contains an element e such that, for all nonnegative $x \in E$, $x \wedge e = 0$ implies $x = 0$. Then E is isomorphic (as a Banach lattice) to $L^1(X, m)$, where (X, m) is some probability space.*

To prove 3.4.2, we observe first that $L_1(v)$ is an (L)-space. For that, it suffices to show that the larger space val (L) is an (L)-space, i.e., satisfies $\|\mu + \nu\| = \|\mu\| + \|\nu\|$ for all $\mu, \nu \geqq 0$. But in this case, the definition of $\|\mu\|$ implies that $\|\mu\| = \mu(1)$, so that

$$\|\mu + \nu\| = (\mu + \nu)(1) = \mu(1) + \nu(1) = \|\mu\| + \|\nu\|$$

is obvious.

Next, note that the element $e = v$ satisfies the "weak unit" hypothesis of Kakutani's theorem. Indeed, let $\mu \in L_1(v)$, $\mu \geqq 0$, be such that $\mu \wedge v = 0$, and choose any positive number α. Then since $\mu \ll v$ one may choose δ, $0 < \delta < \alpha$, so that for every chain $a_1 \leqq b_1 \leqq a_2 \leqq b_2 \leqq \cdots \leqq b_n$ in L with $\sum_k (v(b_k) - v(a_k)) \leqq \delta$, one has $\sum_k (\mu(b_k) - \mu(a_k)) \leqq \alpha$. Now $\mu \wedge v(1) = 0$, so by definition of $\mu \wedge \nu$ there is a chain $0 = c_0 \leqq c_1 \leqq \cdots \leqq c_p = 1$ in L such that

$$\sum_{j=1}^{p} \min\left(\mu(c_j) - \mu(c_{j-1}),\ v(c_j) - v(c_{j-1})\right) \leqq \delta\ .$$

Divide $\{1, 2, \cdots, p\}$ into two disjoint subsets A, B such that $\mu(c_j) - \mu(c_{j-1}) \leqq v(c_j) - v(c_{j-1})$ if and only if $j \in A$. Then we have

$$\sum_{j \in A} (\mu(c_j) - \mu(c_{j-1})) + \sum_{j \in B} (v(c_j) - v(c_{j-1})) \leqq \delta\ ,$$

and in particular, each of these two sums is $\leqq \delta$. By the choice of δ,

$\sum_B (v(c_j) - v(c_{j-1})) \leqq \delta$ implies $\sum_B (\mu(c_j) - \mu(c_{j-1})) \leqq \alpha$. Adding this to $\sum_A \leqq \delta$ we obtain

$$\sum_{j=1}^{p} (\mu(c_j) - \mu(c_{j-1})) \leqq \delta + \alpha \leqq 2\alpha .$$

and since the series telescopes to $\mu(1) - \mu(0) = \mu(1)$, we conclude that $\| \mu \| = \mu(1) \leqq 2\alpha$. Since α was arbitrary, the conclusion $\mu = 0$ follows.

Before invoking Kakutani's theorem, we want to collect a few results concerning a natural action of L on val (L), defined for $a \in L$, $\mu \in$ val (L), by

$$a \cdot \mu(x) = \mu(a \wedge x) .$$

Because L is distributive and μ is a valuation, it is clear that $a \cdot \mu$ is a valuation. Moreover, if $\mu \geqq 0$ then for all $x \leqq y$ we have $0 \leqq \mu(a \wedge y) - \mu(a \wedge x) \leqq \mu(y) - \mu(x)$ ([8, p. 76]); therefore $0 \leqq a \cdot \mu \leqq \mu$. It follows from the Jordan decomposition ([8, pp. 83–84]) that $\| a \cdot \mu \| \leqq \| \mu \|$ for arbitrary $\mu \in$ val (L).

LEMMA. *For all $\mu, \nu \in$ val (L) with $\mu, \nu \geqq 0$, and for all $a, b \in L$, one has*

(i) $a \cdot \mu \wedge (\nu - a \cdot \nu) = 0$,

(ii) $a \cdot (\mu \wedge \nu) = (a \cdot \mu) \wedge \nu$,

(iii) $(a \wedge b) \cdot \mu = a \cdot \mu \wedge b \cdot \mu$, *and dually.*

Moreover, $a \cdot L_1(v) \subseteqq L_1(v)$, and $a \cdot \sigma = \lim_{n \to \infty} (na \cdot v) \wedge \sigma$ strongly, for every $\sigma \in L_1(v)$, $\sigma \geqq 0$.

Proof. (i) Since both $a \cdot \mu$ and $\nu - a \cdot \nu$ are isotone, according to the definition of $\wedge$ in val (L), $a \cdot \mu \wedge (\nu - a \cdot \nu) = 0$ will follow if we can exhibit a chain $0 = c_0 \leqq c_1 \leqq \cdots \leqq c_n = 1$ in L such that, for all $1 \leqq j \leqq n$, either $a \cdot \mu(c_j) - a \cdot \mu(c_{j-1}) = 0$ or $(\nu - a \cdot \nu)(c_j) - (\nu - a \cdot \nu)(c_{j-1}) = 0$. Now simply consider the chain $0 \leqq a \leqq 1$.

(ii) Since $a \cdot (\mu \wedge \nu) \leqq a \cdot \mu$ and $a \cdot (\mu \wedge \nu) \leqq \mu \wedge \nu \leqq \nu$, the inequality $a \cdot (\mu \wedge \nu) \leqq (a \cdot \mu) \wedge \nu$ is obvious. On the other hand, the linear map $\lambda \mapsto a \cdot \lambda$ of val (L) into itself is clearly idempotent, and therefore so is the map $P: \lambda \longrightarrow \lambda - a \cdot \lambda$. Moreover, the preceding remarks show that $\lambda \geqq 0$ implies $P\lambda \geqq 0$. Hence, $0 \leqq P(a \cdot \mu \wedge \nu) \leqq P(a \cdot \mu) = 0$, so that $a \cdot \mu \wedge \nu = a \cdot (a \cdot \mu \wedge \nu)$. Since the last term is dominated by $a \cdot (\mu \wedge \nu)$, the assertion follows.

(iii) Utilizing (ii), we can write $a \cdot \mu \wedge b \cdot \mu = a \cdot (\mu \wedge b \cdot \mu) = a \cdot (b \cdot \mu) = (a \wedge b) \cdot \mu$. The dual relation follows from this because, for each $x \in L$,

$$\begin{aligned}(a \vee b) \cdot \mu(x) &= \mu((a \vee b) \wedge x) = \mu(a \wedge x \vee b \wedge x) \\ &= \mu(a \wedge x) + \mu(b \wedge x) - \mu(a \wedge b \wedge x) .\end{aligned}$$

Hence, $(a \vee b)\cdot\mu = a\cdot\mu + b\cdot\mu - (a \wedge b)\cdot\mu = a\cdot\mu + b\cdot\mu - a\cdot\mu \wedge b\cdot\mu = a\cdot\mu \vee b\cdot\mu$, where the last equality uses the formula $\sigma \vee \tau + \sigma \wedge \tau = \sigma + \tau$, valid in any vector lattice.

Note that if $\mu \ll v$, then $a\cdot\mu \leqq \mu \ll v$ implies $a\cdot\mu \ll v$; i.e., $a\cdot L_1(\sigma) \subseteqq L_1(\sigma)$. Finally, we claim that $\lim_n (na\cdot v) \wedge \mu = a\cdot\mu$, whenever $\mu \in L_1(v)$. Indeed, since we have previously shown that v is a weak unit for $L_1(v)$ (i.e., $\lambda \wedge v = 0$ implies $\lambda = 0$, for all $\lambda \in L_1(v)$ with $\lambda \geqq 0$), the sequence $nv \wedge \mu$ must converge strongly to μ([18, Lemmas 3.3 and 3.9]). So from (ii) we conclude that $a\cdot\mu = a\cdot\lim_n (nv \wedge \mu) = \lim_n (na\cdot v) \wedge \mu$, proving the lemma.

Recall now from [18] that in any (L)-space E having a weak unit e (one assumes that $\|e\| = 1$), the set B of all elements x in E satisfying $x \geqq 0$ and $x \wedge (e - x) = 0$ is a complete Boolean algebra relative to the lattice operations inherited from L and the complementation $x \mapsto e - x$. Moreover, $w(x) = \|x\|$ defines a positive normal valuation on B. So assume $v(1) = 1$, and let B be the corresponding Boolean algebra in $L_1(v)$.

Now imbed L in B as follows: for $a \in L$, put $\hat{a} = a\cdot v$. Taking $\mu = \nu = v$ in part (i) of the lemma, we see that $\hat{a} \in B$, and (iii) asserts that $a \rightarrow \hat{a}$ is a lattice homomorphism of L into B, which carries unit to unit. Since $\|\hat{a}\| = \hat{a}(1) = v(a)$, we see that the map is an *isometry* of (L, v) into (B, w). Since both (L, v) and (B, w) are complete metric lattices, a standard argument shows that, in fact, $a \mapsto \hat{a}$ is an isomorphism of L onto a complete (therefore closed) sublattice of B.

Now Kakutani's theorem implies that there is a probability space (X, m) and an isometric lattice isomorphism of $L_1(v)$ onto $L^1(X, m)$ taking v to the function 1, which carries B onto the measure algebra $M(X, m)$ (regarded as the set of all characteristic functions in $L^1(X, m)$), such that the functional $\mu \mapsto \mu(1)$ carries over to integration against the measure m (see [18]). In particular, L becomes identified with a complete sublattice L' of $M(X, m)$.

Of course, $v(a) = \int \chi_A dm$, where $\chi_A \in L'$ is the image of $a \in L$. We now observe that this holds more generally; i.e., for every $\mu \in L_1(v)$ and $a \in L$, one has $\mu(a) = \int f\chi_A dm$, where $f \in L^1(X, m)$ is the image of μ. To see this, we may clearly assume $\mu \geqq 0$. Then by the lemma, we have

$$\mu(a) = (a\cdot\mu)(1) = \lim_n n\hat{a} \wedge \mu(1) = \lim_n \int (n\chi_A) \wedge f dm = \int \chi_A f dm ,$$

as asserted.

One can now deduce that $L^1(X, m)$ is the closed linear span of L';

equivalently, we assert that if $f \in L^{\infty}(X, m)$ is such that $\int f\chi_A dm = 0$ for all $\chi_A \in L'$, then $f = 0$ almost everywhere. Indeed, since f is integrable it corresponds to some element $\mu \in L_1(v)$. We have $\mu(a) = \int \chi_A f dm = 0$ for every $a \in L$ (using the preceding), so that $\mu = 0$ and hence $f = 0$ as an element of $L^1(X, m)$.

It follows easily from the preceding paragraph that L' generates the full measure algebra of (X, m) as a complete Boolean algebra.

Finally, we claim that $L^1(X, m)$ is norm-separable. For if $\{a_1, a_2, \cdots\}$ is a (metrically) dense subsequence of L, then $\{\chi_{A_1}, \chi_{A_2}, \cdots\}$ is metrically dense in L'. By the preceding paragraph, then, the (countable) Boolean algebra generated by $\{\chi_{A_1}, \chi_{A_2}, \cdots\}$ is dense in $M(X, m)$ in the usual metric of $M(X, m)$. This clearly implies that $L^1(X, m)$ is separable.

Because of this, we may replace (X, m) with an equivalent measure space (X', m') such that X' is a standard Borel space. That completes the proof of 3.4.2. □

Remark. By an obvious construction based on the device used in the proof of 1.2.2, we may also assume that there is a standard partial ordering $\leqq$ of X (which may be chosen to be *strict,* see 1.2.3), such that the image of L is precisely the lattice of (equivalence classes of) increasing Borel sets in X. Noting 1.2.1, we deduce the nontrivial half of the following lattice-theoretic characterization of commutative subspace lattices.

COROLLARY. *In order that a complete distributive lattice L be isomorphic to a separably acting commutative subspace lattice, it is necessary and sufficient that L be countably generated* (*as a complete lattice*) *and have a positive normal valuation.*

This result suggests a much broader, and undoubtedly more difficult, question.

Problem. Characterize the class of (countably-generated, complete) lattices whice are isomorphic to separably-acting subspace lattices.

What we have in mind here is something perhaps analogous to the characterization of von Neumann algebras as those C^*-algebras which are "complete" (i.e., closed under the operation of taking bounded monotone limits), and admit a faithful family of "normal" states. Note that the above corollary is analogous to this in the commutative case.

Non-distributive lattices may still admit isotone valuations (consider the dimension function on the projection lattice of a finite factor), but as this example suggests, valuations correspond to *traces* on von Neumann algebras, not states. Thus, a first step in attacking the above problem might be to introduce a lattice-theoretic counterpart of normal states on von Neumann algebras.

3.5. Numerical invariants

We now introduce a numerical invariant which is capable of distinguishing between the lattices $\mathfrak{L}(2^\infty, \leqq, m_p)$, $0 < p < 1$, of example 3.0.2. The reader may note some formal similarity between this quantity and the Kolmogorov-Sinai invariant of ergodic theory [7], in spite of the fact that the two are based on quite different principles.

Throughout this section, L will denote a distributive separable complete metric lattice; for convenience, we require L to be infinite. By a *state* of L we mean an isotone valuation μ which is normalized so that $\mu(0) = 0$, $\mu(1) = 1$; the convex set of all states is denoted by Σ. Σ_* will denote the subset of Σ consisting of all normal states; thus, $\Sigma_* = \{\mu \in L_*: \mu \geqq 0, \mu(1) = 1\}$. A generator G of L will be called *proper* if G_∞ is the trivial sublattice 2. Now given any state μ and any generator $G = \{a_1, a_2, \cdots\}$, we define the number

$$\alpha(G, \mu) = \limsup_n \mu(a_n) .$$

Note that $\alpha(G, \mu)$ does not depend on the particular enumeration of the elements of G (so long as $a_i \neq a_j$ for $i \neq j$). Finally, define

$$\alpha(L) = \inf_{G,\mu} \alpha(G, \mu) ,$$

where the infimum is extended over all proper generators G and all normal states $\mu \in \Sigma_*$ (if L has no proper generator, $\alpha(L)$ is taken to be 1).

Since the definition of Σ_* makes no reference to the distinguished valuation of L it follows that $\alpha(L)$ depends only on the structure of L as a (complete distributive) lattice, and thus *$\alpha(L)$ is an isomorphism invariant of L*. We now establish some results which will make the computation of $\alpha(L)$ very easy in certain cases.

THEOREM 3.5.1. *Let B be a basis for L and let G be any proper generator. Then $\alpha(B, \mu) \leqq \alpha(G, \mu)$ for every state μ.*

Proof. By definition of $\alpha(B, \mu)$ there is a sequence $p_n \in B$ such that $p_i \neq p_j$ if $i \neq j$ and $\lim_n \mu(p_n) = \alpha(B, \mu)$. Since each p_n is a prime we know from 3.1.2 that there exist elements $a_{n1}, a_{n2}, \cdots \in G$ such that $p_n = a_{n1} \wedge a_{n2} \wedge \cdots$, $n = 1, 2, \cdots$. We claim that there is a sequence $1 = n_1 < n_2 < \cdots$ of positive integers and a sequence $b_k \in G$ such that $p_{n_k} \leqq b_k$ for all k, and $b_i \neq b_j$ if $i \neq j$. Indeed, choose b_1 to be any element of

$\{a_{11}, a_{12}, \cdots\}$. Given $n_1 < \cdots < n_k$ and $b_1, \cdots, b_k$ as above, the set $\{a_{nj}: n > n_k, j \geqq 1\}$ must be infinite (since the complete lattice it generates contains $p_{n_k+1}, p_{n_k+2}, \cdots$), and therefore it contains an element a_{rj} which does not belong to $\{b_1, \cdots, b_k\}$. Now put $n_{k+1} = r$ and $b_{k+1} = a_{rj}$ to complete the inductive step.

We now have

$$\alpha(B, \mu) = \lim_k \mu(p_{n_k}) \leqq \limsup_k \mu(b_k) \leqq \alpha(G, \mu) ,$$

as required. □

THEOREM 3.5.2. *Let $\{M_n: n \geqq 1\}$ be a sequence of complete sublattices of L such that $M_n \supseteqq M_{n+1}$, and put $M_\infty = \bigcap_n M_n$.*

Then for every $\mu \in L_*$, the sequence $\sup\{|\mu(x)|: x \in M_n\}$ *decreases to* $\sup\{|\mu(x)|: x \in M_\infty\}$.

Proof. Let v be the distinguished valuation of L, and put $w = v + |\mu|$. Then w is a positive valuation of L, which is *normal* because $\mu \in L_*$. It follows that the pair (L, w) defines a complete metric lattice ([8, p. 80]). Note also that, since $|\mu| \leqq w$ implies $|\mu| \ll w$, we have $\mu \in L_1(w)$.

By 3.4.2 and the remark following it, we may assume that L is the subspace lattice $\mathfrak{L}(X, \leqq, m)$ associated with a standard partially ordered finite measure space $(X, \leqq, m)$, and that μ has the form $\mu(P_E) = \int \chi_E h dm$, where h belongs to $L^1(X, m)$. Writing h in the form $h(x) = f(x)\bar{g}(x)$ with $f, q \in L^2(X, m)$ we see that $\mu(P) = (Pf, g)$, for every projection $P \in L = \mathfrak{L}(X, \leqq, m)$.

For each $n \geqq 1$, we may find a standard partial ordering $\leqq_n$ of X such that $M_n = \mathfrak{L}(X, \leqq_n, m)$ (i.e., for n fixed, choose a sequence $E_1, E_2, \cdots$ in $L(X, \leqq)$ such that $\{P_{E_1}, P_{E_2}, \cdots\}$ generates M_n as a subspace lattice, define $x \leqq_n y$ to mean $\chi_{E_i}(x) \leqq \chi_{E_i}(y)$ for every $i = 1, 2, \cdots$, and now apply 1.2.2). To prove the theorem, let $\lambda_n = \sup\{|\mu(P)|: P \in M_n\}$ for $n \geqq 1$, and put $\lambda_\infty = \sup\{|\mu(P)|: P \in M_\infty\}$. It is clear that $\lambda_n \geqq \lambda_\infty$ for every n, and λ_n decreases to its limit $\lambda = \lim_n \lambda_n$.

To see that $\lambda \leqq \lambda_\infty$ choose, for each n, a projection $P_n \in M_n$ such that $\lambda_n - 1/n \leqq |\mu(P_n)| \leqq \lambda_n$. Let $\mathcal{K}_n$ denote the closure of $\{P_n, P_{n+1}, \cdots\}$ in the weak operator topology. Because the unit ball of $\mathfrak{L}(L^2(X, m))$ is compact in the weak operator topology and the sets $\mathcal{K}_n$ have the finite intersection property, we may find an operator A in $\bigcap_n \mathcal{K}_n$. Clearly $|\mu(A)| = \lambda$, and we claim that A belongs to the weakly closed convex hull of M_∞ (since the the linear functional $T \mapsto (Tf, g)$ is weakly continuous and coincides on $\mathfrak{L}(X, \leqq, m)$ with μ, it will follow that $\lambda = |\mu(A)| \leqq \sup\{|\mu(P)|: P \in M_\infty\} =$

λ_∞, as required). Let $k \in L^\infty(X, m)$ be such that $A = L_k$; since $0 \leqq A \leqq I$ we may also assume that $0 \leqq k(x) \leqq 1$ for all $x \in X$. For each t, $0 \leqq t \leqq 1$, let E_t be the projection corresponding to the Borel set $\{x: k(x) \geqq t\}$. Fix $n \geqq 1$. Then since $\mathcal{K}_n$ is contained in the weakly closed convex hull of $M_n = \mathcal{L}(X, \leqq_n, m)$, it follows from 2.1.3 that the function k is essentially increasing relative to the ordering $\leqq_n$. This implies that $\{x: f(x) \geqq t\}$ differs from a set in $L(X, \leqq_n)$ by a set of measure zero (1.1.11), and therefore $E_t \in \mathcal{L}(X, \leqq_n, m) = M_n$. Because n is arbitrary we have $E_t \in \bigcap_n M_n = M_\infty$. Now the argument in the second part of the proof of 2.1.3 shows that $A = \int_0^1 E_t dt$, from which it follows that A belongs to the weakly closed convex hull of $\{E_t: 0 \leqq t \leqq 1\} \subseteqq M_\infty$. The claim follows. □

It does not appear to be possible to prove 3.5.2 without utilizing the additional structure granted by 3.4.2. Our principal need for this result is through the following.

COROLLARY 1. *Let G be a proper generator of L. Then for any two normal states μ, ν of L, one has $\alpha(G, \mu) = \alpha(G, \nu)$.*

Proof. Enumerate the elements of G, say $a_1, a_2, \cdots$, where $a_i \neq a_j$ if $i \neq j$. It clearly suffices to show that $\lim_n |\mu(a_n) - \nu(a_n)| = 0$. Define $\sigma(b) = \mu(b) - \nu(b)$, $b \in L$. Clearly $\sigma \in \operatorname{val}(L)$. Moreover, since $|\sigma| \leqq \mu + \nu$ and both μ and ν are normal, it follows that $|\sigma|$ is normal, i.e., $\sigma \in L_*$. Now let M_n be the complete sublattice generated by $\{a_n, a_{n+1}, \cdots\}$. Since G is a proper generator, we have $\bigcap_n M_n = \{0, 1\}$; and because $\sigma(0) = \sigma(1) = 0$, we see from 3.5.2 that $\sup\{|\sigma(b)|: b \in M_n\}$ tends to 0 as $n \to \infty$. It follows in particular that $|\mu(a_n) - \nu(a_n)| = |\sigma(a_n)|$ tends to 0 as $n \to \infty$. □

COROLLARY 2. *Assume that L has a basis B. Then for every normal state μ,*

$$\alpha(L) = \alpha(B, \mu) .$$

Proof. Fix $\mu \in \Sigma_*$. We want to show that for every $\nu \in \Sigma_*$ and every proper generator G, $\alpha(B, \mu) \leqq \alpha(G, \nu)$. But 3.5.1 implies $\alpha(B, \mu) \leqq \alpha(G, \mu)$, and the preceding corollary asserts that $\alpha(G, \mu) = \alpha(G, \nu)$. □

The main application will be in terms of the following result. Recall that for a sublattice $M \subseteqq L$, $\mathcal{P}(M)$ denotes the set of primes of M.

THEOREM 3.5.3. *Assume that L admits a factorization $L = \bigotimes_n L_n$ into primary sublattices L_n. Then $B = \bigcup_n \mathcal{P}(L_n)$ is a basis for L, and $\alpha(L) = \alpha(B, \mu)$ for every $\mu \in \Sigma_*$.*

Proof. It suffices to show that $\bigcup_n \mathcal{P}(L_n)$ is a basis for L. It is clearly a generator (since $\mathcal{P}(L_n)$ generates L_n), and from property 3.2.3(iii) of factorizations it follows easily that $\bigcup_n \mathcal{P}(L_n)$ is proper. Finally, to see that each $p \in \mathcal{P}(L_n)$ is prime in L one simply uses 3.2.4, noting that $1 \in \mathcal{P}(L_k)$ for $k \neq n$ because L_k is primary. □

Discussion of Example 3.0.2. Now consider the lattices $L_p = \mathfrak{L}(2^\infty, \leqq, m_p)$, $0 < p < 1$, of Problem 3.0.2. We have already seen at the end of Section 3.2 that L_p has a factorization $\bigotimes_n L_{n,p}$ into three-element chains $L_{n,p} = \{0, a_n, 1\}$, and clearly $\mathcal{P}(L_{n,p}) = \{a_n, 1\}$. Taking $B = \{1, a_1, a_2, \cdots\}$ and the normal state v determined by the measure m_p, we see that $v(a_n) = p$ for every n, so by 3.5.3 we conclude that $\alpha(\mathfrak{L}(2^\infty, \leqq, m_p)) = p$, for every $0 < p < 1$. In particular, $\mathfrak{L}(2^\infty, \leqq, m_p)$ and $\mathfrak{L}(2^\infty, \leqq, m_q)$ are not isomorphic if $p \neq q$. Since these lattices are always reflexive, we conclude:

COROLLARY. *If $p \neq q$, then the operator algebras* alg $\mathfrak{L}(2^\infty, \leqq, m_p)$ *and* alg $\mathfrak{L}(2^\infty, \leqq, m_q)$ *are not similar.*

UNIVERSITY OF CALIFORNIA, BERKELEY

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