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A NOTE ON ESSENTIALLY NORMAL OPERATORS

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ABSTRACT

Associated with every compact metric space X, there is a classifying structure $\mathrm{Ext}(X)$ for the equivalence classes of extensions of $C(X)$ by the C^*-algebra of compact operators. In this note we give a new proof that $\mathrm{Ext}(X)$ is a group, based on operator-theoretic techniques.

In a recent paper [3], Brown, Douglas and Fillmore have classified essentially normal operators up to compact perturbations. More generally, for each compact metric space X, they consider the class of all (separably acting) separable C^*-algebras $\mathscr{A}$, which contain the algebra $\mathscr{C}(\mathscr{H})$ of all compact operators on the underlying Hilbert space $\mathscr{H}$, and for which the quotient C^*-algebra $\mathscr{A}/\mathscr{C}(\mathscr{H})$ is isomorphic to $C(X)$. This family of algebras is associated with an abelian group $\mathrm{Ext}(X)$ which classifies the algebra to an appropriate equivalence. When X is a subset of the complex plane, this reduces to considering the class of essentially normal operators having X as their essential spectrum, with respect to the relation *unitary equivalence modulo compact perturbations*. In this case, the Fredholm index gives rise to a homomorphism of $\mathrm{Ext}(X)$ into the free abelian group having one generator for each hole of X, and the classification result alluded to in the first sentence asserts that this homomorphism is injective.

Now in the general case, it is not very hard to show that $\mathrm{Ext}(X)$ is a commutative semigroup with zero, and one of the main results of Brown-Douglas-Fillmore is that $\mathrm{Ext}(X)$ is in fact a group: i.e. has inverses. The proof is quite indirect, and it may be of interest to have an alternate proof which is independent of their machinery. The purpose of this note is to point out how such a proof can be based on a lifting theorem of Vesterstrøm [5] and Andersen [1] (see also Ando [2]), together with some elementary considerations from dilation theory.

For simplicity, we only consider the case where X is a compact subset of the complex plane $\mathbb{C}$, but the reader may easily see that everything goes through for arbitrary compact metric spaces.

Let $X \subseteq \mathbb{C}$ be compact. We define $EN(X)$ to be the class of all essentially normal operators A (i.e., $A^*A - AA^*$ is compact), such that A acts on a

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separable infinite dimensional Hilbert space $\mathscr{H}_A$ and has X as its essential spectrum (the essential spectrum of A is written $\sigma_e(A)$, and as usual is defined as the spectrum of the image of A in the Calkin algebra). Note that the Hilbert Space $\mathscr{H}_A$ is allowed to vary with A. If A and B are two elements of $EN(X)$, then so is the direct sum $A \oplus B$, as well as every compact perturbation $A + K$, $K \in \mathscr{C}(\mathscr{H}_A)$.

Two elements $A,B \in EN(X)$ are said to be *equivalent* ($A \sim B$) if there is a unitary operator $U: \mathscr{H}_A \to \mathscr{H}_B$ such that $UA - BU$ is compact (equivalently, $UAU^* - B \in \mathscr{C}(\mathscr{H}_B)$). Ext($X$) is defined as the set of all equivalence classes $EN(X)/\sim$. If $a,b \in$ Ext(X), one may define $a + b$, as the equivalence class of any operator of the form $A \oplus B$, where A and B are arbitrarily chosen elements of a and b respectively. It is easy to see that $+$ is well-defined, and is a commutative associative binary operation on Ext(X).

Now one of the preliminary results of [3] implies that if $A,N \in EN(X)$ and N is normal, then $A \oplus N \sim A$. By taking A to be normal we see that $N \sim N \oplus A \sim A \oplus N \sim A$, so that the normal elements of $EN(X)$ determine a single class in Ext(X); moreover, the preceding implies further that this class functions as a zero for the additive semigroup Ext(X). Thus, to show that Ext(X) is a group, it suffices to show that for every $a \in$ Ext(X), the equation $a + x = 0$ has a solution $x \in$ Ext(X). This is the content of the following:

Theorem A. *For every $A \in EN(X)$, there exists $B \in EN(X)$ such that $A \oplus B$ has the form $N + K$, where N is normal and K is compact.*

To prove Theorem A we shall make use of the following special case of the lifting theorem [1], [5] mentioned above (note, incidentally, that liftings with similar properties are discussed in [2]):

Theorem B. *Let $\mathfrak{A}$ be a separable C^*-algebra with unit, let $\mathfrak{M}$ be a closed two-sided ideal in $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{M}$ is commutative, and let π: $\mathfrak{A} \to \mathfrak{A}/\mathfrak{M}$ be the canonical projection. Then there is a positive linear map ϕ: $\mathfrak{A}/\mathfrak{M} \to \mathfrak{A}$, which preserves identities, such that $\pi \circ \phi$ is the identity map of $\mathfrak{A}/\mathfrak{M}$.*

Now let $A \in \mathscr{L}(\mathscr{H})$ be essentially normal with $\sigma_e(A) = X$. We will produce an operator B with similar properties such that $A \oplus B$ is normal plus compact. Let $\mathscr{A}$ be the C^*-algebra generated by A, 1 and $\mathscr{C}(\mathscr{H})$: $\mathscr{A} = C^*(A) + \mathscr{C}(\mathscr{H})$. Since the image $\hat{A}$ of A in $\mathscr{A}/\mathscr{C}(\mathscr{H})$ is normal, generates $\mathscr{A}/\mathscr{C}(\mathscr{H})$ (along with 1) as a C^*-algebra, and has X as its spectrum, it follows that $\mathscr{A}/\mathscr{C}(\mathscr{H})$ is isomorphic with $C(X)$ in such a way that $\hat{A}$ corresponds to the independent variable function $z \in C(X)$. Combining the positive map of Theorem B with the inverse of this isomorphism, we obtain a positive linear map ϕ: $C(X) \to \mathscr{A}$ having the properties

(i) $\phi(1) = 1$

(ii) $\phi(z) = A + K$, *where* $K \in \mathscr{C}(\mathscr{H})$

(iii) $\phi(fg) - \phi(f)\,\phi(g) \in \mathscr{C}(\mathscr{H})$, *for all* $f, g \in C(X)$

(iv) $\phi(f) \in \mathscr{C}(\mathscr{H})$ *iff* $f = 0$.

Now a familiar dilation theorem of Naimark [4] implies that there is a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ (which may be taken as separable because $C(X)$ is norm-separable and $\mathscr{H}$ is separable) and a *-representation $\pi: C(X) \to \mathscr{L}(\mathscr{K})$ such that

$$\phi(f) = P\pi(f) \mid_{\mathscr{H}}, \quad f \in C(X),$$

where P denotes the projection of $\mathscr{K}$ on $\mathscr{H}$. We claim: $PT - TP$ is compact for every T in the C^*-algebra $\pi(C(X))$. Indeed, condition (iii) above implies that $PSTP - PSPTP$ is compact for every $S,T \in \pi(C(X))$; taking $S = T^*$ we obtain $PT^*P^{\perp}TP \in \mathscr{C}(\mathscr{H})$, hence $P^{\perp}TP \in \mathscr{C}(\mathscr{H})$ and (replacing T with T^* and taking adjoints) $PTP^{\perp} \in \mathscr{C}(\mathscr{H})$. It follows that $PT - TP = PTP^{\perp} - P^{\perp}TP$ is compact.

This implies that, relative to the decomposition $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}^{\perp}$, every operator in $\pi(C(X))$ has a 2×2 block matrix of the form

$$\begin{bmatrix} X & * \\ * & Y \end{bmatrix}$$

where the off-diagonal terms are compact.

Now let N be the normal operator on $\mathscr{K}$ given by $N = \pi(z)$, and define $B \in \mathscr{L}(\mathscr{H}^{\perp})$ by $B = P^{\perp}N \mid_{\mathscr{H}^{\perp}}$. Condition (ii) and the preceding show that

$$N = \begin{bmatrix} A + K & L \\ M & B \end{bmatrix},$$

where K,L,M are all compact, and therefore $A \oplus B = N +$ compact.

Note that this last formula implies that $B^*B - BB^*$ is compact.

Finally, note that $\sigma_e(B) \subseteq X$. Indeed, $\sigma_e(A) \cup \sigma_e(B) = \sigma_e(A \oplus B) = \sigma_e(N) \subseteq X$, since $\sigma_e(N) \subseteq \sigma(N) = \sigma(\pi(z)) \subseteq \text{range}(z) = X$; and the assertion follows. We can arrange to have $\sigma_e(B) = X$ by replacing B with $B \oplus M$ where M is any normal operator having essential spectrum X. That completes the proof.

This proof suggests an equivalent definition of $\text{Ext}(X)$ for a general compact metric space X in terms of positive linear maps. Briefly, define $E(X)$ to be the class of all positive linear maps ϕ of $C(X)$ into $\mathscr{L}(\mathscr{H}_\phi)$ ($\mathscr{H}_\phi$ denoting a variable separable Hilbert space) satisfying

(i) $\phi(1) = 1$

(ii) $\phi(fg) - \phi(f)\,\phi(g) \in \mathscr{C}(\mathscr{H}_\phi)$, for all $f,g \in C(X)$

(iii) $\phi(f) \in \mathscr{C}(\mathscr{H}_\phi)$ *iff* $f = 0$.

For every such ϕ, the C^*-algebra $\mathscr{A} = \phi(C(X)) + \mathscr{C}(\mathscr{H}_\phi)$ defines an extension of $\mathscr{C}(\mathscr{H}_\phi)$ by $C(X)$; and by Theorem B it follows easily that every such extension arises in this way. The equivalence defined in [3] for extensions becomes the equivalence relation: $\phi \sim \psi$ *iff* there is a unitary operator $U: \mathscr{H}_\phi \to \mathscr{H}_\psi$ such that $U\phi(f) - \psi(f)\,U$ is compact for every $f \in C(X)$. $\text{Ext}(X)$ is then defined as the set of equivalence classes $E(X)/\sim$.

The direct sum operation,

$$(\phi \oplus \psi)(f) = \phi(f) \oplus \psi(f), f \in C(X),$$

induces the correct addition in Ext(X), making Ext(X) into an additive semigroup. The *representations* of $C(X)$ (satisfying (iii)) define a unique element of Ext(X), which functions as a zero, and finally the argument we have given shows that Ext(X) is a group.

This formulation suggests a generalisation in which $C(X)$ is replaced by a fixed noncommutative separable C^*-algebra with unit. Probably, one should consider only *completely* positive maps ϕ, since in that case Stinespring's theorem [4] provides an effective substitute for the dilation theorem used in proving Theorem A.

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