Operators with Compact Imaginary Part

W. B. ARVESON

Let B be a C^*-algebra with identity e and let S be a linear subspace of B such that e belongs to S. An irreducible *-representation π of B on a Hilbert space $\mathfrak{H}$ is called a *boundary representation* for S if the restricted map $\pi|_S$: $S \to \mathcal{L}(\mathfrak{H})$ has a *unique* completely positive linear extension to B (namely π). Boundary representations were introduced in [1]; their usefulness for studying non self-adjoint families of operators derives from the following

Implementation theorem *Let S_i be a linear subspace of a C^*-algebra B_i , $i = 1, 2$, such that the identity of B_i belongs to S_i , and such that S_i generates B_i as a C^*-algebra. Assume that the intersection of the kernels of all boundary representations of B_i for S_i is $\{0\}$, $i = 1, 2$. Then every completely isometric linear map of S_1 on S_2 , which preserves identities, is implemented by a *-isomorphism of B_1 onto B_2 .*

Here is one way that kind of situation can occur in operator theory. Suppose we are given a finite set $\{T_1 , \cdots , T_n\}$ of operators on a Hilbert space $\mathfrak{H}$ which is irreducible in the sense that the only subspaces of $\mathfrak{H}$ invariant under the set $\{T_1 , \cdots , T_n , T_1^* , \cdots , T_n^*\}$ are $\{0\}$ and $\mathfrak{H}$. Let $\mathcal{S}$ be the (finite dimensional) linear span of $\{I, T_1 , \cdots , T_n\}$ and let $\mathcal{B}$ be the C^*-algebra generated by $\mathcal{S}$. Then the identity representation of $\mathcal{B}$ is irreducible and of course has trivial kernel, so the hypotheses of the implementation theorem will be satisfied when the identity representation is a boundary representation for $\mathcal{S}$. Unfortunately this need not be so in general, even in the "nice" situation where $\mathcal{B}$ is a type I C^*-algebra (in the case at hand this would imply that $\mathcal{B}$ contains the compact operators), see 3.5.4 of [1] for an example. Therefore it is of interest to know what additional conditions on $\{T_1 , \cdots , T_n\}$ will lead to the desired conclusion. The following theorem implies that it is enough to know that some T_i has compact imaginary part.

Theorem 1. *Let $\mathcal{S}$ be an irreducible set of operators on $\mathfrak{H}$ such that the norm-closed linear span of $\mathcal{S} \cup \mathcal{S}^*$ contains a nonzero compact operator. Let φ be a completely positive linear map of the C^*-algebra generated by $\mathcal{S}$ into $\mathcal{L}(\mathfrak{H})$ such that $||\varphi|| \leqq 1$ and $\varphi(T) = T$ for all T in $\mathcal{S}$. Then φ is the identity map.*

The earlier version of this theorem [2] required more, namely that every operator in $\mathcal{S}$ be compact. This somewhat more general form was obtained by C. A. Akemann and the author. This result, together with the Implementation

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Theorem, allows us to conclude that if S and T are irreducible operators acting on spaces $\mathfrak{H}$ and $\mathfrak{K}$, both of which have (nonzero) compact imaginary part, and if the map $\varphi : aI_{\mathfrak{H}} + bS \rightarrow aI_{\mathfrak{K}} + bT$ $(a, b \in \mathbf{C})$ is completely isometric, then there is a *-isomorphism between the respective C^*-algebras which implements φ. Moreover, the presence of compact operators allows us to conclude much more, namely that the *-isomorphism is itself implemented by a unitary operator from $\mathfrak{H}$ to $\mathfrak{K}$. Conclusion: if $aI_{\mathfrak{H}} + bS \rightarrow aI_{\mathfrak{K}} + bT$ defines a completely isometric linear map, then S and T are unitarily equivalent.

Given two specific operators S and T, it is generally hard to see if the above linear map is completely isometric. In the following discussion we will introduce a somewhat more manageable invariant. Let T be an operator on $\mathfrak{H}$ and let n be a positive integer. We define $\mathcal{W}_n(T)$ to be the set of all elements in M_n, the C^*-algebra of all complex $n \times n$ matrices, which have the form $\varphi(T)$ where φ runs over all completely positive linear maps of $\mathfrak{L}(\mathfrak{H})$ into M_n which preserve identities. The sequence $\{\mathcal{W}_1(T), \mathcal{W}_2(T), \cdots\}$ is called the Matrix Range of T. The term derives from the fact that $\mathcal{W}_1(T)$ can easily be shown to be the closure of the numerical range of T.

Theorem 2. *Let S and T be Hilbert space operators. Then the map $aI + bS \rightarrow aI + bT$ is completely isometric if, and only if, $\mathcal{W}_n(S) = \mathcal{W}_n(T)$ for $n = 1, 2, \cdots$.*

Theorem 2 and the preceding supply the nontrivial half of the following classification theorem.

Theorem 3. *Let S and T be two irreducible Hilbert space operators which have compact imaginary parts. Then S and T are unitarily equivalent if, and only if, they have the same matrix range.*

We remark that $\mathcal{W}_n(T)$ is completely determined by the "matrix valued" *affine* functions of T, *i.e.*, operators of the form $A \otimes I + B \otimes T$, where A and B are $n \times n$ matrices. That is to say, one does not have to consider higher order matrix valued polynomials in T (for example $A \otimes I + B \otimes T + C \otimes T^2$, where A, B, and C are $n \times n$ matrices), much less functions of both T and T^*. $\mathcal{W}_n(T)$ is therefore a *first order* unitary invariant for T. Thus, Theorem 3 implies that irreducible compact operators are completely determined by their first order properties. It goes without saying that this is not so in general for irreducible operators which are not compact, even when the operators generate a type I C^*-algebra. One can easily see, in fact, that there are many such operators which have the same matrix range but which are not unitarily equivalent ([1], 3.5.4). The full details of this material will appear in a sequel to [1].

REFERENCES

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University of California,
Berkeley

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