ON SUBALGEBRAS OF C*-ALGEBRAS

BY WILLIAM B. ARVESON¹

Communicated by Murray Gerstenhaber, January 14, 1969

In this note we announce some new methods and results in the theory of nonnormal Hilbert space operators and nonselfadjoint operator algebras. A main difficulty in the subject has been the apparent absence of relations between, say, a nonselfadjoint algebra of operators and its generated C^* -algebra. For example, given full information about the norm-closed algebra P(T) generated by all polynomials in a given (nonnormal) operator T, what can one say about the C^* -algebra $C^*(T)$ generated by T and the identity? While one cannot expect much of an answer in general, we will describe here a class of operators and operator algebras for which these relations are as simple as one could hope for.

All C^* -algebras are assumed to contain an identity (written as e), L(H) denotes the algebra of all bounded operators on a Hilbert space H, and $C^*(S)$ stands for the C^* -algebra generated by S and the identity where S is either an operator or a subset of a C^* -algebra. An operator is irreducible if it commutes with no nontrivial projections.

1. An extension theorem. Let S be a linear subspace of a C^* algebra B, such that S contains the identity of B. A linear map ϕ of S into another C*-algebra is *positive* if $\phi(x) \ge 0$ for every positive element x of S (note, however, that S may contain no positive elements other than scalars). A familiar theorem of M. Krein implies that if $S = S^*$, then every scalar-valued positive linear map of S has a positive extension to B. We first describe a generalization of Krein's theorem to operator-valued maps which is basic for virtually all of the sequel. If M_n is the algebra of all complex $n \times n$ matrices, then $B \otimes M_n$ is the *-algebra of all $n \times n$ matrices over B. There is a unique C*-algebra norm on $B \otimes M_n$, and $S \otimes M_n$ is a linear subspace of this C^{*}-algebra. A linear map ϕ of S into a C*-algebra B' induces, for every $n \ge 1$, a linear map $\phi_n: S \otimes M_n \rightarrow B' \otimes M_n$ by applying ϕ element by element to each matrix over S. ϕ is completely contractive or completely isometric according as each ϕ_n is contractive $(\|\phi_n\| \leq 1)$ or isometric. ϕ is completely positive if each ϕ_n is a positive linear map.

 $^{^{1}}$ Research supported by NSF grant GP-5585 and the U.S. Army Research Office, Durham.

THEOREM 1. Let S be a closed selfadjoint linear subspace of a C^* algebra B, which contains the identity of B, and let H be a Hilbert space. Then every completely positive linear map of S into L(H) has a completely positive extension to B.

It is easily shown that a scalar-valued positive linear map of S is already completely positive; thus Theorem 1 generalizes Krein's theorem. We remark also that Theorem 1 is false if one deletes the adverb "completely," even when B is commutative.

2. Boundary representations. Let S be a closed subspace of a C^* -algebra B, such that e belongs to S. S need not be selfadjoint.

DEFINITION. An irreducible representation π of B is called a boundary representation for S if the only completely positive linear extension of $\pi|_s$ to B is π itself.

When B = C(X) (for X a compact Hausdorff space), boundary representations correspond to points in the Choquet boundary of X relative to S. In the noncommutative cases, boundary representations occur in a variety of ways, as the following two examples show.

EXAMPLE 1. Let T be an operator on a Hilbert space. sp(T) and W(T) will denote the spectrum of T and the numerical range of T, respectively. A *character* of a C*-algebra is a one-dimensional representation (i.e., a complex homomorphism), necessarily irreducible. The following example seems to have independent interest.

THEOREM 2. For every point λ in $\operatorname{sp}(T) \cap \partial W(T)$, there is a character π of $C^*(T)$ such that $\pi(T) = \lambda$. π is a boundary representation for P(T).

COROLLARY. If $C^*(T)$ has no characters, then the spectral radius of T is less than ||T||.

EXAMPLE 2. Let ψ be an inner function in H^{∞} (of the unit disc), and let Z_{ψ} be the "zero set" of $\psi: Z_{\psi}$ consists of all zeros of ψ in the interior of the unit disc along with all boundary points λ such that ψ cannot be analytically continued from the interior to λ . S_{ψ} denotes the projection of the bilateral shift onto $H^2 \ominus \psi H^2$. S_{ψ} is an irreducible operator, so that the identity representation of $C^*(S_{\psi})$ is irreducible.

THEOREM 3. If $Z_{\psi} \cap \{ |z| = 1 \}$ has (linear) Lebesgue measure zero, then the identity representation is a boundary representation for $P(S_{\psi})$. If, on the other hand, Z_{ψ} contains the entire unit circle, then the identity representation is not a boundary representation for $P(S_{\psi})$.

We remark that the proof of the first half of Theorem 3 utilizes machinery we have not discussed, and the result is a principal one in this theory. The following theorem gives the key property of boundary representations.

THEOREM 4. Let S_i be a linear subspace of a C^* -algebra B_i , i = 1, 2, each containing the respective identity e_i . Assume $B_i = C^*(S_i)$, and let ϕ be a linear completely isometric map of S_1 on S_2 such that $\phi(e_1) = e_2$. Then for every boundary representation π of B_1 for S_1 there is a (unique) boundary representation π' of B_2 for S_2 such that $\pi' \odot \phi = \pi$ on S_1 .

Theorem 4 has a number of general consequences, relating to "Silov boundaries," of which we mention only the following.

COROLLARY. Let $S_i \subseteq B_i$, i = 1, 2, be as above. Assume that the intersection of the kernels of all boundary representations of B_i for S_i is 0.

Then every completely isometric linear map of S_1 on S_2 , which preserves the identity, is implemented by a *-isomorphism of B_1 on B_2 .

This corollary implies that certain subspaces (and subalgebras) of C^* -algebras completely determine the structure of the C^* -algebra they generate. It can be shown that the hypotheses are satisfied if there are enough unitary elements in the closure of $S_i + S_i^*$ to generate B_i as a C^* -algebra, i=1, 2.

3. Some applications. We describe three applications of the preceding in the theory of nonnormal operators on Hilbert space. Let Tbe a nonscalar simple algebraic operator (i.e., $C^*(T)$ is simple and Tsatisfies a polynomial equation p(T) = 0). Assume ||T|| = 1, and let $p(z) = (z-a_1)^{n_1}(z-a_2)^{n_2} \cdots (z-a_k)^{n_k}$ be the minimum polynomial of T. The corollary of Theorem 2 implies that $|a_i| < 1$, $1 \le i \le k$, and so the Blaschke product B_T which has p as its numerator is in H^{∞} . Tis called maximal if ||B(T)|| = 1 for every proper Blaschke divisor Bof B_T . Thus, if $p(z) = z^n$, then T is maximal iff $||T|| = \cdots = ||T^{n-1}||$ = 1. In general, simple algebraic operators can generate complicated C^* -algebras; for example, there are operators T having cube zero for which $C^*(T)$ is an (antiliminal) UHF algebra. Nevertheless, we have

THEOREM 5. If T is a maximal simple algebraic operator, then $C^*(T)$ is *-isomorphic with M_n , n being the order of the minimum polynomial of T. An irreducible maximal simple algebraic operator is finite-dimensional, and two such are unitarily equivalent if, and only if, they have the same minimum polynomial.

The second application is similar to the preceding, for operators having an "infinite" minimum polynomial. Let T be an arbitrary contraction, and let A be the norm-closure (in C(T), T being the unit

circle) of the polynomials in $e^{i\theta}$. Because the unit disc is a spectral set for T, we have $||p(T)|| \leq ||p||$ for every polynomial $p \in A$, and thus there is a unique contractive homomorphism ϕ of A such that $\phi(p) = p(T)$ for every polynomial p. Assume that ϕ has nonzero kernel. Then ker ϕ has the form $\psi \cdot A_K$, where K is a closed set in T of Lebesgue measure zero, ψ is an inner function for which $Z_{\psi} \cap T \subseteq K$, and A_K denotes all functions in A which vanish on K. ψ is called the *minimum* function for T. The minimum function is undefined if ker $\phi = 0$.

For each $n \ge 1$, ϕ induces a (contractive) homomorphism ϕ_n of the Banach algebra $A \otimes M_n (\subseteq C(T) \otimes M_n)$, whose kernel is $\psi \cdot A_K \otimes M_n$: *T* is called *maximal* if the canonical homomorphism of $A \otimes M_n/\psi \cdot A_K$ $\otimes M_n$ induced by ϕ_n is isometric, for every $n \ge 1$. It can be seen that this usage of the term is in harmony with the foregoing.

Let ψ be an inner function such that $Z_{\psi} \cap T$ is of measure zero. Then S_{ψ} is an example of a maximal operator which has ψ as its minimum function. Moreover, S_{ψ} is irreducible and $S_{\psi}^*S_{\psi} - S_{\psi}S_{\psi}^*$ has finite rank.

THEOREM 6. Let T_1 and T_2 be irreducible operators such that both commutators $T_i^*T_i - T_iT_i^*$ are compact. Assume that both operators possess minimum functions and are maximal. Then T_1 and T_2 are unitarily equivalent if, and only if, their minimum functions are proportional.

Our third application is to the Volterra operator V, defined on $L^2(0, 1)$ by $Vf(x) = \int_0^x f(t)dt$, $f \in L^2(0, 1)$. V is well known to be compact and irreducible. If C_0, C_1, \dots, C_k are $n \times n$ matrices, then $p(z) = C_0 + C_1 z + \dots + C_k z^k$ defines an M_n -valued polynomial; we define p(T) for an operator $T \in L(H)$ by $p(T) = C_0 \otimes I + C_1 \otimes T + \dots + C_k \otimes T^k$, regarded as an operator on $\mathbb{C}^n \otimes H$.

THEOREM 7. Let T be an irreducible operator for which $T^*T - TT^*$ is compact. Suppose ||p(T)|| = ||p(V)|| for every matrix-valued polynomial p. Then T is unitarily equivalent to V.

The norm condition ||p(T)|| = ||p(V)|| is assumed to hold for every M_n -valued polynomial p, and every $n \ge 1$. It can be shown that the condition for n = 1 already implies $||p(T)|| \le ||p(V)||$ for n > 1, but we do not know if the opposite inequality is also redundant for n > 1.

Two counterexamples are noteworthy. First, there exist (threedimensional) operators S and T for which ||p(S)|| = ||p(T)|| holds for all scalar-valued polynomials p, but not for all matrix-valued polynomials; put differently, not every isometric representation of a subalgebra of a C*-algebra is completely contractive, even when the

1969]

W. B. ARVESON

subalgebra is singly-generated. Second, while a characterization like Theorem 7 holds for operators other than V, it is of limited generality. For example, if ψ_1 and ψ_2 are inner functions such that Z_{ψ_1} and Z_{ψ_2} both contain the entire unit circle, then S_{ψ_i} is irreducible and $S_{\psi_i}^* S_{\psi_i} - S_{\psi_i} S_{\psi_i}^*$ is compact, $||p(S_{\psi_1})|| = ||p(S_{\psi_2})||$ is valid for every matrix-valued polynomial p, but these two operators are not unitarily equivalent if ψ_1 and ψ_2 are not proportional. The second statement of Theorem 3 explains why the proof of Theorem 7 breaks down for these examples.

Finally, we remark that Theorems 6 and 7 remain valid when the hypothesis $T^*T - TT^*$ compact is replaced with the weaker condition: the commutator ideal in $C^*(T)$ is a minimal (closed, two-sided) ideal. Full details and further developments will appear elsewhere.

UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720