ANALYTICITY IN OPERATOR ALGEBRAS.

By William B. Arveson.[1]

Contents.

Received March 1, 1966.

[1] This research was supported, in part, by the U. S. Army Research Office (Durham).

1. Preliminaries.

1.1. Introduction. In the study of families of operators on Hilbert space, the self-adjoint algebras have occupied a preeminent position. Nevertheless, many problems in operator theory lead obstinately toward questions about algebras that are not necessarily self-adjoint. Indeed, the general theory of a single operator on a finite-dimensional space rests on an analysis of the polynomials in that operator, and has nothing at all to do with the *-operation. The first systematic investigation of non self-adjoint operator algebras was made by Kadison and Singer in their interesting and original paper on triangular operator algebras [11]. Earlier, there was the well-known work of Helson and Lowdenslager [6], and of Masani and Wiener, on matrix-valued analysic functions. This related primarily to function theory and to the least-squares prediction theory of stochastic processes. The second half of Helson's book (*Lectures on Invariant Subspaces,* Academic Press, 1964) gives a recent account of relevant parts of the subject.

In this paper, we present a theory of non self-adjoint operator algebras which we feel adds some unity to the very different perspectives of [6] and [11], and which is general enough to encompass a variety of examples. Roughly, a *subdiagonal algebra* is a pair $(\mathfrak{A}, \Phi)$ where $\mathfrak{A}$ is an algebra of operators on Hilbert space such that the closure of $\mathfrak{A} + \mathfrak{A}^*$ (in an appropriate topology) is a von Neumann algebra $\mathfrak{B}$, and where Φ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{A} \cap \mathfrak{A}^*$ which extends in a suitable way to $\mathfrak{B}$. It is not accidental that this description bears a resemblance to the definition of Dirichlet function algebras; we believe that the position of subdiagonal algebras in the class of general operator algebras is rather analogous to the position of Dirichlet algebras in the class of sup norm algebras. Some, but by no means all, maximal triangular algebras fall into this category. Significantly, this seems to depend more on the structure of the triangular algebra than on its reducibility properties (recall that the most incisive results about triangular algebras are available only when the algebra is hyperreducible). Our principal results apply to subdiagonal algebras which are *finite* in the sense that there is a faithful normal finite trace ϕ on $\mathfrak{B}$ such that $\phi \circ \Phi = \phi$, and which satisfy a maximality condition. The elements of $\mathfrak{B} \cap \mathfrak{B}^{-1}$ are related to $\mathfrak{A} \cap \mathfrak{A}^{-1}$ by a factorization theorem, and this is used to study the relation between generalizations of some properties of bounded analytic functions in the open unit disc. "Jensen's" inequality, for instance, is shown to be valid in some examples, both hyperreducible and irreducible, of triangular subalgebras of II_1 factors.

Chapter 2 contains the main definitions and some general results. In Chapter 3, we give a variety of examples of subdiagonal algebras. Chapter 4 is the core of this paper. It contains the factorization theorem and the discussion described in the preceding paragraph. We are unable, however, to establish Jensen's inequality in that general a setting, and this problem is taken up for a number of special cases in Chapter 5. The last chapter is an appendix on the conditional expectation mapping, a fundamental constituent of this theory.

A preliminary announcement of some of these results was made in the *Notices*, Amer. Math. Soc., February, 1965, p. 239.

1.2. On Terminology. We take our Hilbert spaces to be complex and separable or finite-dimensional. We assume the reader is familiar with the terminology and basic theory in [1], and we shall use terms like *faithful, normal, hyperfinite, state, trace* without further explanation. Regarding topologies, since the algebras we discuss are not self-adjoint, there can be a difference between, say, the weak and ultraweak closure of a particular one. Moreover, it is the "ultra" topologies, rather than the weak and strong topologies, that are algebraic invariants for von Neumann algebras. Therefore, most topological statements are in terms of ultraweak and ultrastrong. Our operators are bounded, and are denoted by upper case Roman letters, taken from both ends of the alphabet. Sets of operators are usually designated with Roman script letters, but there are a few exceptions to this. Subsets of Hilbert space are denoted by German letters. $\mathcal{S}'$ denotes the set of all operators that commute with each element of $\mathcal{S}$, $\mathcal{S}^*$ is the collection of all A^*, $A \in \mathcal{S}$, $\mathcal{S}^{-1}$ denotes the set of all inverses of the regular elements of $\mathcal{S}$, and $\mathcal{S}^+$ denotes the positive operators in $\mathcal{S}$. For an operator T, $|T|$ is the positive square root of T^*T.

2. Generalities.

2.1. Definitions. Let $\mathcal{B}$ be a von Neumann algebra, and let Φ be a faithful normal positive linear mapping of $\mathcal{B}$ into itself which is idempotent $(\Phi \circ \Phi = \Phi)$.

Definition 2.1.1. *A subalgebra $\mathcal{A}$ of $\mathcal{B}$ is said to be subdiagonal (with respect to Φ) if it has the following propedties*

(i) $\mathcal{A} + \mathcal{A}^*$ *is ultraweakly dense in* $\mathcal{B}$

(ii) $\Phi(AB) = \Phi(A)\Phi(B)$, $A, B \in \mathcal{A}$

(iii) $\Phi(\mathfrak{A}) \subseteq \mathfrak{A} \cap \mathfrak{A}^*$

(iv) $(\mathfrak{A} \cap \mathfrak{A}^*)^2$ *has trivial nullspace.*

Remark 2.1.2. It is significant that $\mathfrak{A}$ need not be closed in any of the operator topologies. Our purpose in this is mainly one of convenience for constructing examples. However, if $\mathfrak{A}^-$ is the ultraweak closure of $\mathfrak{A}$, then clearly $\mathfrak{A}^-$ is a subalgebra of $\mathfrak{B}$ satisfying (i) and (iv). (ii) and (iii) are also true for $\mathfrak{A}^-$, by a standard argument using ultraweak continuity of Φ. Therefore, the ultraweak closure of a subdiagonal algebra is subdiagonal.

Definition 2.1.3. *Let M be a von Neumann subalgebra of the von Neumann algebra $\mathfrak{B}$. By an expectation on M, we mean a positive linear mapping Φ of $\mathfrak{B}$ onto M which leaves the identity fixed, and which satisfies $\Phi(AX) = A\Phi(X)$, $A \in M$, $X \in \mathfrak{B}$.*

The term expectation calls attention to the analogy between this and the probabilistic mapping which associates with each bounded random variable of a probability space its conditional expectation relative to some fixed sub σ-field of the full σ-field. Properties of expectations are discussed at some length in the appendix. Some of the simplest and most important are these: if Φ is an expectation of $\mathfrak{B}$ on M, then M is precisely the set of fixed points of Φ, and for every $X \in \mathfrak{B}$,

$$\Phi(X)^*\Phi(X) \leqq \Phi(X^*X).$$

Hence, a normal expectation is continuous relative to either the ultraweak or ultrastrong topologies ([1], theorem 2, p. 56).

Now let $\mathfrak{A}$ be a subdiagonal subalgebra of $\mathfrak{B}$, with respect to Φ. The self-adjoint algebra $\mathfrak{D} = \mathfrak{A} \cap \mathfrak{A}^*$ is called the *diagonal* of $\mathfrak{A}$. Property 2.1.1 (iv) insures that the ultraweak closure $\mathfrak{D}^-$ of $\mathfrak{D}$ is a von Neumann algebra.

PROPOSITION 2.1.4. *Φ is an expectation on $\mathfrak{D}^-$.*

Proof. Let $\mathcal{S}$ be the set of $X \in \mathfrak{B}$ such that $\Phi(X) \in \mathfrak{D}^-$. $\mathcal{S}$ is an ultraweakly closed subspace of $\mathfrak{B}$, by continuity. By 2.1.1 (iii) $\mathcal{S}$ contains $\mathfrak{A}$. Since $\mathfrak{D}^-$ is self-adjoint and Φ preserves the *-operation, $\mathcal{S}$ contains $\mathfrak{A} + \mathfrak{A}^*$, and therefore $\mathfrak{B}$, by 2.1.1 (i). Hence $\Phi(\mathfrak{B}) \subseteq \mathfrak{D}^-$.

We claim $\Phi(A) = A$ for every $A \in \mathfrak{D}$. Since $\mathfrak{D}$ is spanned by its self-adjoint elements, it suffices to prove the formula for A a self-adjoint element of $\mathfrak{A}$. Using (i), one has

$$\Phi((A - \Phi(A))^2) = \Phi(A - \Phi(A))^2 = 0,$$

since

$$\Phi(A - \Phi(A)) = \Phi(A) - \Phi \circ \Phi(A) = 0.$$

As Φ is faithful, $(A - \Phi(A))^2 = 0$, and therefore $A = \Phi(A)$, as asserted. In particular, $\Phi(I) = I$, by continuity.

Now fix $A \in \mathfrak{D}$, and let $X \in \mathfrak{A} \cup \mathfrak{A}^*$. If $X \in \mathfrak{A}$, then $\Phi(AX) = \Phi(A)\Phi(X) = A\Phi(X)$; if $X \in \mathfrak{A}^*$, then

$$\Phi(AX) = \Phi^*(X^*A^*) = (\Phi(X^*)\Phi(A^*))^* = \Phi(A)\Phi(X) = A\Phi(X).$$

Thus, the set $\mathcal{S}_A$ of all $X \in \mathfrak{B}$ for which $\Phi(AX) = A\Phi(X)$ is an ultraweakly closed subspace which contains $\mathfrak{A} \cup \mathfrak{A}^*$. By (i) again, we conclude that $\Phi(AX) = A\Phi(X)$ for $A \in \mathfrak{D}$, $X \in \mathfrak{B}$. The formula extends to $A \in \mathfrak{D}^-$ immediately, by continuity, and the proof is complete.

COROLLARY 2.1.5. *If $\mathfrak{A}$ is a non-closed subdiagonal algebra with diagonal $\mathfrak{D}$, then the diagonal of $\mathfrak{A}^-$ is $\mathfrak{D}^-$. In particular, if $\mathfrak{A}$ is antisymmetric ($\mathfrak{D}$ = scalars), then so is $\mathfrak{A}^-$.*

Proof. By 2.1.4, Φ is an expectation on $\mathfrak{D}^-$. At the same time, Φ is an expectation on $\mathfrak{D}_1^- = \mathfrak{D}_1$, where $\mathfrak{D}_1 = \mathfrak{A}^- \cap \mathfrak{A}^{-*}$. Thus $\mathfrak{D}^- = \mathfrak{D}_1$ = set of fixed points of Φ.

2.2. Maximality of subdiagonal algebras. Throughout this section, $\mathfrak{A}$ will be a fixed subdiagonal subalgebra of $\mathfrak{B}$, with respect to Φ. $\mathfrak{A}$ is not necessarily closed. Let $\mathfrak{J} = \{T \in \mathfrak{A} : \Phi(T) = 0\}$, $\mathfrak{D} = \mathfrak{A} \cap \mathfrak{A}^*$. Clearly $\mathfrak{J}$ is an ideal in $\mathfrak{A}$, and $\mathfrak{A} = \mathfrak{D} + \mathfrak{J}$.

THEOREM 2.2.1. *Let $\mathfrak{A}_m$ be the set of all $X \in \mathfrak{B}$ for which $\Phi(\mathfrak{J}X\mathfrak{A}) = \Phi(\mathfrak{A}X\mathfrak{J}) = 0$. Then $\mathfrak{A}_m$ is a subdiagonal subalgebra of $\mathfrak{B}$ (with respect to Φ), $\mathfrak{A}_m \supseteq \mathfrak{A}$, and if $\mathfrak{A}_1$ is any other subdiagonal algebra with $\mathfrak{A}_1 \supseteq \mathfrak{A}$, then $\mathfrak{A}_m \supseteq \mathfrak{A}_1$.*

Proof. It is plain that $\mathfrak{A}_m$ is an ultraweakly closed subspace of $\mathfrak{B}$. If $\mathfrak{A}_1$ is any algebra $\supseteq \mathfrak{A}$ and if Φ is multiplicative on $\mathfrak{A}_1$, then for every $T \in \mathfrak{A}_1$, $A \in \mathfrak{A}$, $B \in \mathfrak{J}$, one has

$$\Phi(ATB) = \Phi(A)\Phi(T)\Phi(B) = 0,$$

and

$$\Phi(BTA) = \Phi(B)\Phi(T)\Phi(A) = 0.$$

Therefore $\mathfrak{A}_m \supseteq \mathfrak{A}_1$. In particular $\mathfrak{A}_m \supseteq \mathfrak{A}$. Thus, $\mathfrak{A}_m$ satisfies (i), (iii), and (iv) of 2.1.1. What remains to be proved, then, is that $\mathfrak{A}_m$ is an algebra and that Φ is multiplicative on $\mathfrak{A}_m$. Neither is obvious.

Since the space on which $\mathcal{B}$ acts is separable, there exists a faithful normal state σ on $\mathcal{B}$. For example, take $\sigma(T)=\sum_n \theta_n(T\xi_n,\xi_n)$, where $\xi_1,\xi_2,\cdots$ is an orthonormal base, and $\theta_1,\theta_2,\cdots$ is a sequence of positive numbers with sum 1. Let $\rho=\sigma\circ\Phi$. Then ρ is a faithful normal state of $\mathcal{B}$, and ρ preserves Φ in the sense that $\rho\circ\Phi=\rho$. Make $\mathcal{B}$ into a prehilbert space in the usual way, by setting $(X,Y)_\rho=\rho(Y^*X)$, and let $\mathfrak{H}_\rho$ be its completion. Let L_X denote (the extension of) left multiplication by $X\in\mathcal{B}$. Then $X\to L_X$ is a *-isomorphism of $\mathcal{B}$ on a von Neumann agebra acting on $\mathfrak{H}_\rho$ ([1], prop. 1, p. 57). Define the following closed subspaces of $\mathfrak{H}_\rho$:

$$\mathfrak{M}_\rho=[\mathcal{A}],\qquad \mathfrak{N}_\rho=[\mathcal{J}]$$
$$\mathfrak{M}_\rho{}^*=[\mathcal{A}^*],\qquad \mathfrak{N}_\rho{}^*=[\mathcal{J}^*].$$

Put $\mathcal{A}_M=\{X\in\mathcal{B}: L_X\mathfrak{M}_\rho\subseteq\mathfrak{M}_\rho,\ L_{X^*}\mathfrak{M}_\rho{}^*\subseteq\mathfrak{M}_\rho{}^*\}$. Clearly $\mathcal{A}_M$ is a subalgebra of $\mathcal{B}$. Thus it suffices to show that $\mathcal{A}_m=\mathcal{A}_M$, and Φ is multiplicative on $\mathcal{A}_M$.

First, we claim

$$\mathfrak{H}_\rho=\mathfrak{M}_\rho\oplus\mathfrak{N}_\rho{}^*=\mathfrak{M}_\rho{}^*\oplus\mathfrak{N}_\rho.$$

Indeed, if $A\in\mathcal{A}$ and $T\in\mathcal{J}$, then

$$(A,T^*)=\rho(TA)=\rho\circ\Phi(TA)=\rho(\Phi(T)\Phi(A))=0,$$
$$(A^*,T)=\rho(T^*A^*)=\bar\rho(AT)=\bar\rho\circ\Phi(AT)=0.$$

Thus $\mathfrak{M}_\rho\perp\mathfrak{N}_\rho{}^*$ and $\mathfrak{M}_\rho{}^*\perp\mathfrak{N}_\rho$. On the other hand, $\mathcal{A}+\mathcal{A}^*=\mathcal{A}+\mathcal{J}^*=\mathcal{A}^*+\mathcal{J}$ is ultraweakly (and therefore ultrastrongly) dense in $\mathcal{B}$, so for every $X\in\mathcal{B}$, there are nets $A_n,B_n\in\mathcal{A}$, $S_n,T_n\in\mathcal{J}$, such that $A_n+S_n{}^*\to X$ and $B_n{}^*+T_n\to X$ ultrastrongly. Thus both

$$|A_n+S_n{}^*-X|^2 \text{ and } |B_n{}^*+T_n-X|^2$$

tend to 0 ultraweakly. One has,

$$[\![A_n+S_n{}^*-X]\!]_\rho{}^2=\rho(|A_n+S_n{}^*-X|^2)\to 0$$

and

$$[\![B_n{}^*+T_n-X]\!]_\rho{}^2=\rho(|B_n{}^*+T_n-X|^2)\to 0.$$

This proves that each of $\mathfrak{M}_\rho\oplus\mathfrak{N}_\rho{}^*$ and $\mathfrak{M}_\rho{}^*\oplus\mathfrak{N}_\rho$ contains the $\mathfrak{H}_\rho$-closure of $\mathcal{B}$, which is $\mathfrak{H}_\rho$ itself.

We can now show that $\mathcal{A}_m\subseteq\mathcal{A}_M$. Let $X\in\mathcal{A}_m$. By the last paragraph, it suffices to show that $L_X\mathfrak{M}_\rho\perp\mathfrak{N}_\rho{}^*$ and $L_{X^*}\mathfrak{M}_\rho{}^*\perp\mathfrak{N}_\rho$. Take $A\in\mathcal{A}$, $T\in\mathcal{J}$. Then $(L_XA,T^*)_\rho=\rho(TXA)=\rho\circ\Phi(TXA)=0$, and

$$(L_{X^*}A^*, T)_\rho = \rho(T^*X^*A^*) = \bar{\rho}(AXT) = \bar{\rho}\circ\Phi(AXT) = 0,$$

by definition of $\mathfrak{A}_m$. Hence $L_X\mathfrak{A} \perp \mathfrak{J}^*$ and $L_{X^*}\mathfrak{A}^* \perp \mathfrak{J}$. The conclusion follows by closing the manifolds $\mathfrak{A}$, $\mathfrak{J}^*$, $\mathfrak{A}^*$, $\mathfrak{J}$ in $\mathfrak{H}_\rho$.

Now let $\mathfrak{D}^-$ be the von Neumann algebra generated by the diagonal $\mathfrak{D} = \mathfrak{A} \cap \mathfrak{A}^*$, and let P be the projection of $\mathfrak{H}_\rho$ onto the subspace of $\mathfrak{H}_\rho$ determined by $\mathfrak{D}^-$. It is easy to see that the restriction of P to elements of $\mathfrak{B}$ is Φ (cf., 2.1.4 and 6.1.1 (iv)). First, note that for every $A \in \mathfrak{A}$, $PL_{A^*} = L_{\Phi(A^*)}P$ on $\mathfrak{M}_\rho{}^*$. For if $B \in \mathfrak{A}$, then

$$PL_{A^*}B^* = P(A^*B^*) = \Phi(A^*B^*) = \Phi(A^*)\Phi(B^*)$$
$$= L_{\Phi(A^*)}P(B^*).$$

The formula now extends, by boundedness of PL_{A^*} and $L_{\Phi(A^*)}P$, to the closure $\mathfrak{M}_\rho{}^*$ of $\mathfrak{A}^*$. Second, note that $I \in \mathfrak{M}_\rho \cap \mathfrak{M}_\rho{}^*$; for I is an ultrastrong limit of operators $D_n \in \mathfrak{D} = \mathfrak{A} \cap \mathfrak{A}^* \subseteq \mathfrak{M}_\rho \cap \mathfrak{M}_\rho{}^*$, hence $|I - D_n|^2 \to 0$ ultraweakly, hence $\rho(|I - D_n|^2) \to 0$. Third, we claim $\Phi(XA) = \Phi(X)\Phi(A)$, if $A \in \mathfrak{A}$, $X \in \mathfrak{A}_M$. Indeed, $\Phi(XA)^* = \Phi(A^*X^*) = PL_{A^*}X^*$, and

$$(\Phi(X)\Phi(A))^* = \Phi(A^*)\Phi(X^*) = L_{\Phi(A^*)}PX^*.$$

But $X^* = L_{X^*}I \in L_{X^*}\mathfrak{M}_\rho{}^* \subseteq \mathfrak{M}_\rho{}^*$, because $I \in \mathfrak{M}_\rho{}^*$; so that by the preceding lines, $\Phi(XA)^*$ and $[\Phi(X)\Phi(A)]^*$ are the same element of $\mathfrak{H}_\rho$. The conclusion follows by taking the adjoint (in $\mathfrak{B}$). Therefore, for $X \in \mathfrak{A}_M$, $A \in \mathfrak{A}$, we have

$$PL_XA = \Phi(XA) = \Phi(X)\Phi(A) = L_{\Phi(X)}PA.$$

Hence, $PL_X = L_{\Phi(X)}P$ on $\mathfrak{M}_\rho$, by continuity. Now we can prove that Φ is multiplicative on $\mathfrak{A}_M$. If $X, Y \in \mathfrak{A}_M$, then $Y = L_YI \in L_Y\mathfrak{M}_\rho \in \mathfrak{M}_\rho$, so that

$$\Phi(XY) = PL_XY = L_{\Phi(X)}PY = L_{\Phi(X)}\Phi(Y) = \Phi(X)\Phi(Y).$$

Thus, $\mathfrak{A}_M$ is an algebra on which Φ is multiplicative, and which contains $\mathfrak{A}_m$. By the first part of the argument, $\mathfrak{A}_m = \mathfrak{A}_M$, and that completes the proof.

Definition 2.2.2. *A subalgebra of* $\mathfrak{B}$ *which is subdiagonal with respect to* Φ *is called maximal subdiagonal if it is contained properly in no larger subdiagonal algebra of* $\mathfrak{B}$ *(with respect to* Φ*).*

Remark 2.2.3. The preceding theorem shows that every subdiagonal algebra $\mathfrak{A}$ is contained in exactly one maximal subdiagonal algebra $\mathfrak{A}_m$. Of course, $\mathfrak{A}_m$ is ultraweakly closed. It is reasonable to ask whether $\mathfrak{A}_m$ is the ultraweak closure of $\mathfrak{A}$, or what is the same, is every ultraweakly closed subdiagonal algebra already maximal? For example, the proof of 2.2.1 shows that for every $X \in \mathfrak{A}_m$ and every normal Φ-preserving state ρ, one has $\inf \rho(|X - A|^2) = 0$, $A \in \mathfrak{A}$; were this true for all normal states, then one could easily show that $\mathfrak{A}_m = \mathfrak{A}^-$. We will prove that $\mathfrak{A}_m = \mathfrak{A}^-$ for a number of examples (cf., 3.1.3, 5.1.2, 5.3.2, 5.4.2(i), 5.5.5 (i)), however the general question is still open.

COROLLARY 2.2.4. *Suppose there is a faithful normal semifinite trace ϕ on $\mathfrak{B}^+$ such that $\phi \circ \Phi = \phi$. Then $\mathfrak{A}_m = \{X \in \mathfrak{B} : \Phi(X\mathfrak{I}) = 0\}$.*

Proof. It suffices to show that if $\Phi(X\mathfrak{I}) = 0$, then $X \in \mathfrak{A}_m$. Let $\mathfrak{J}^+ = \{S \in \mathfrak{B} : S \geqq 0, \phi(S) < +\infty\}$. Then $\mathfrak{J}^+$ is the positive part of a two-sided ideal $\mathfrak{J}$ in $\mathfrak{B}$, $\mathfrak{J}$ is ultraweakly dense because ϕ is normal and semifinite, and ϕ extends uniquely to a linear functional $\dot{\phi}$ on $\mathfrak{J}$ satisfying $\dot{\phi}(RS) = \dot{\phi}(SR)$, $S \in \mathfrak{B}$, $R \in \mathfrak{J}$ ([1], p. 80, prop. 1). Moreover, $\phi \circ \Phi = \phi$ on $\mathfrak{B}^+$ implies $\Phi(\mathfrak{J}^+) \subseteq \mathfrak{J}^+$, and hence, $\Phi(\mathfrak{J}) \subseteq \mathfrak{J}$. Finally, $\dot{\phi} \circ \Phi = \dot{\phi}$ on $\mathfrak{J}$.

We claim $\Phi(AXT) = 0$ for every $A \in \mathfrak{A}$, $T \in \mathfrak{I}$. Let $D \in \mathfrak{D} \cap \mathfrak{J}$. Then

$$\begin{aligned}\dot{\phi}(D\Phi(AXT)) &= \dot{\phi} \circ \Phi(DAXT) = \dot{\phi}(DAXT) \\ &= \dot{\phi}(XTDA) = \dot{\phi} \circ \Phi(XTDA) = 0,\end{aligned}$$

since $DA \in \mathfrak{J}$ and $TDA \in \mathfrak{I}$. The claim will follow, then, if we show that when $C \in \mathfrak{D}$ and $\phi(DC) = 0$ for every $D \in \mathfrak{D} \cap \mathfrak{J}$, then $C = 0$. Let $C = UC_1$ be the polar decomposition of C, with U a partial isometry, and $C_1 \geqq 0$ satisfying $U^*UC_1 = C_1$. We have $(\mathfrak{D} \cap \mathfrak{J})U^* \subseteq \mathfrak{D} \cap \mathfrak{J}U^* \subseteq \mathfrak{D} \cap \mathfrak{J}$, so that for every $D \in \mathfrak{D} \cap \mathfrak{J}$,

$$\dot{\phi}(DC_1) = \dot{\phi}(DU^*UC_1) = \dot{\phi}(DU^*C) = 0.$$

Since $\mathfrak{J}$ is ultraweakly dense in $\mathfrak{B}$, there is a bounded directed increasing net $H_\alpha \in \mathfrak{J}^+$ such that $\mathrm{LUB}\, H_\alpha = C_1$ ([1], p. 45, cor. 5). We have

$$\mathrm{LUB}\, \Phi(H_\alpha) = \Phi(\mathrm{LUB}\, H_\alpha) = \Phi(C_1) = C_1,$$

so we can assume $H_\alpha \in \mathfrak{D} \cap \mathfrak{J}^+$ for every α. Now

$$\begin{aligned}\phi(C_1^2) &= \phi(C_1^{\frac{1}{2}}C_1^{\frac{1}{2}}) = \mathrm{LUB}\, \phi(C_1^{\frac{1}{2}}H_\alpha C_1^{\frac{1}{2}}) \\ &= \mathrm{LUB}\, \dot{\phi}(H_\alpha C_1) = 0,\end{aligned}$$

and since ϕ is faithful, we have $C^*C = C_1^2 = 0$.

$\Phi(TXA) = 0$ can be proved by a very similar argument, and this shows that $X \in \mathfrak{A}_m$.

3. Examples of Subdiagonal Algebras.

3.1. Algebras based on invariant subspaces. Let $\mathcal{B}$ be a von Neumann algebra, and let $\mathcal{P}$ be a nonempty *abelian* family of projections in $\mathcal{B}$. Put

$$\mathcal{A} = \{X \in \mathcal{B} : EXE = XE \text{ for all } E \in \mathcal{P}\}$$
$$\mathcal{D} = \{X \in \mathcal{B} : EX = XE \text{ for all } E \in \mathcal{P}\}.$$

Clearly $\mathcal{A}$ is a weakly closed subalgebra of $\mathcal{B}$, and $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^* = \mathcal{P}' \cap \mathcal{B}$.

If E and F are two commuting $\mathcal{A}$-invariant projections, then $E \wedge F = EF$ is $\mathcal{A}$-invariant. There will be no effect on $\mathcal{A}$, therefore, if we replace $\mathcal{P}$ by the smallest family of projections which contains $\mathcal{P}$, O, I, and which is closed under the lattice operation $\wedge$. The new family is abelian; it is just the set of all finite products of elements of $\mathcal{P}$, together with O and I. We assume, in the sequel, that $\mathcal{P}$ has these additional properties.

It can happen that for certain combinations of $\mathcal{B}$ and $\mathcal{P}$, there is no faithful normal expectation of $\mathcal{B}$ on $\mathcal{D}$ (cf. remark 3.1.3). We shall confine attention to the other cases, where $\mathcal{D}$ is *compatible* with $\mathcal{B}$ in the sense that there is an expectation with these properties (definition 6.1.4).

THEOREM 3.1.1. *If $\mathcal{D}$ is compatible with $\mathcal{B}$, and if Φ is the unique expectation on $\mathcal{D}$ (cf. 6.2.2), then Φ is multiplicative on $\mathcal{A}$. If $T \in \mathcal{A}$ and $\Phi(T) = 0$, then there is a uniformly bounded net $T_n \in \mathcal{A}$ such that each T_n is nilpotent, $\Phi(T_n) = 0$, and $T_n \to T$ strongly.*

If $\mathcal{P}$ is linearly ordered, then every element of $\mathcal{B}$ is in the strong closure of a bounded subset of $\mathcal{A} + \mathcal{A}^*$. In particular, $\mathcal{A} + \mathcal{A}^*$ is ultraweakly dense in $\mathcal{B}$.

Proof. Let $X, Y \in \mathcal{A}$, $E \in \mathcal{P}$. Then

$$\Phi(XYE) = \Phi(XEYE) = \Phi(XEY)E = E\Phi(XEY)$$
$$= \Phi(EXEY) = \Phi(XEY).$$

Hence, $\Phi(XYA) = \Phi(XAY)$ holds whenever A is a linear combination of elements of $\mathcal{P}$. Since $\mathcal{P}$ is a self-adjoint semigroup, these operators form a $*$-algebra having the same commutant as $\mathcal{P}$. By von Neumann's density theorem, these A's are σ-weakly dense in $\mathcal{P}''$; and by continuity, the formula remains valid when A ranges over $\mathcal{P}''$. In particular, $\Phi(XYU) = \Phi(XUY)$ for every U in the unitary group G of $\mathcal{P}''$. The same argument applies when Y is replaced by YU^{-1}, and we have $\Phi(XY) = \Phi(XUYU^{-1})$ for every $U \in G$. Making use of weak continuity of Φ on bounded sets, it follows that $\Phi(XY) = \Phi(XS)$ for every S in $C(Y)$, the weakly closed convex hull of UYU^{-1}, $U \in G$. Because $\Phi(Y) \in C(Y)$ (6.2.1 and 6.2.2), we have

$$\Phi(XY) = \Phi(X\Phi(Y)) = \Phi(X)\Phi(Y).$$

For the second statement, let $T \in \mathcal{A}$, $\Phi(T) = 0$. Let $\mathcal{F} = \{E_1, \cdots, E_n\}$ be a finite subset of $\mathcal{P}$, containing I. It will be convenient to introduce some compact notation. Let S_n be the set of all n-tuples $\alpha = (\alpha_1, \cdots, \alpha_n)$ where $\alpha_i = +1$ or -1. For a projection P, let $P^{+1} = P$ and $P^{-1} = I - P$. For $\alpha \in S_n$, put $E^\alpha = E^{\alpha_1} E^{\alpha_2} \cdots E^{\alpha_n}$. As α ranges over S_n, E^α ranges over the atoms of the Boolean algebra generated by $\mathcal{F}$. Of course, it is possible for some of the E^α to be 0, but every nonzero atom appears exactly once as the image of some α. Thus $\sum_\alpha E^\alpha T E^\alpha$ is simply $T_{\mathcal{F}}$, of the remarks preceding 6.1.6, and in particular this sum is independent of the particular labeling of the elements of $\mathcal{F}$. Corollary 6.1.8 shows that with the finite subsets directed by inclusion, $T - T_{\mathcal{F}}$ tends boundedly and strongly to T. It is clear that $T - T_{\mathcal{F}}$ belongs to $\mathcal{A}$, and that $\Phi(T - T_{\mathcal{F}}) = -\Phi(T_{\mathcal{F}}) = -(\Phi(T))_{\mathcal{F}} = 0$. The proof will be completed by showing that $T - T_{\mathcal{F}}$ is nilpotent.

Give S_n a lexicographic order as follows. If $\alpha \neq \beta$, let i be the first index such that $\alpha_i \neq \beta_i$. If $\alpha_i = -1$ and $\beta_i = +1$, write $\alpha < \beta$; if $\alpha_i = +1$ and $\beta_i = -1$, write $\alpha > \beta$. Clearly $<$ totally orders S_n. Now

$$T - T_{\mathcal{F}} = \sum_{\alpha,\beta} E^\alpha T E^\beta - T_{\mathcal{F}} = \sum_{\alpha \neq \beta} E^\alpha T E^\beta$$
$$= \sum_{\alpha > \beta} E^\alpha T E^\beta + \sum_{\alpha < \beta} E^\alpha T E^\beta.$$

Note first that if $\alpha < \beta$ then $E^\alpha T E^\beta = 0$. Indeed, $\alpha_i = -1$ and $\beta_i = +1$ for some i, and $E^\alpha = E^\alpha(1 - E_i)$ and $E^\beta = E_i E^\beta$. Thus

$$E^\alpha T E^\beta = E^\alpha(1 - E_i) T E_i E^\beta = 0,$$

since T leaves E_i invariant. Therefore

$$T - T_{\mathcal{F}} = \sum_{\alpha > \beta} E^\alpha T E^\beta.$$

Let $m = 2^n$. Then $(T - T_{\mathcal{F}})^{m+1}$ is a sum of operators of the form

$$E^{\alpha^1} T E^{\beta^1} E^{\alpha^2} T E^{\beta^2} \cdots E^{\beta^m} E^{\alpha^{m+1}} T E^{\beta^{m+1}}$$

where $\alpha^1 > \beta^1, \alpha^2 > \beta^2, \cdots, \alpha^{m+1} > \beta^{m+1}$. Now $E^\beta E^\alpha = 0$ unless $\alpha = \beta$. Therefore if at least one of these products is nonzero, we must have

$$\alpha^1 > \beta^1 = \alpha^2 > \beta^2 = \alpha^3 > \cdots > \beta^m = \alpha^{m+1} > \beta^{m+1}.$$

In particular, $\alpha^1, \alpha^2, \cdots, \alpha^{m+1}$ are all distinct, contradicting the fact that S_n contains only m elements. Therefore $(T - T_{\mathcal{F}})^{m+1} = 0$.

Assume, now, that $\mathfrak{P}$ is linearly ordered. Fix $X \in \mathfrak{B}$, let $\mathfrak{F}$ and S_n be as in the preceding paragraphs, and take $\alpha > \beta$ in S_n. We claim $E^\alpha X E^\beta \in \mathfrak{A}$. For by definition of the order, $\alpha_i = +1$ and $\beta_i = -1$ for some i, $E^\alpha = E_i E^\alpha$, and $E^\beta(1 - E_i)$. Let F be a projection. If $F \leqq E_i$, then $E^\beta F = E^\beta(1 - E_i)F = 0$ and $E^\alpha X E^\beta F = F E^\alpha X E^\beta F = 0$. If $F \geqq E_i$, then $FE_i = E_i$, $FE^\alpha = E^\alpha$, and $FE^\alpha X E^\beta F = E^\alpha X E^\beta F$. In either case, then, $E^\alpha X E^\beta$ leaves F invariant. By hypothesis, every element of $\mathfrak{P}$ falls into one of these cases, so that $E^\alpha X E^\beta$ belongs to $\mathfrak{A}$.

Now write

$$X = \sum_{\alpha,\beta} E^\alpha X E^\beta = X_{\mathfrak{F}} + \sum_{\alpha > \beta} E^\alpha X E^\beta + \sum_{\alpha < \beta} E^\alpha X E^\beta$$
$$= (X_{\mathfrak{F}} - \Phi(X)) + \Phi(X) + \sum_{\alpha > \beta} E^\alpha X E^\beta + (\sum_{\alpha < \beta} E^\beta X^* E^\alpha)^*.$$

The sum of the last three terms belongs to $\mathfrak{D} + \mathfrak{A} + \mathfrak{A}^* = \mathfrak{A} + \mathfrak{A}^*$, its norm $\| X - X_{\mathfrak{F}} + \Phi(X) \| \leqq 3 \| X \|$, and it tends strongly to X because $X_{\mathfrak{F}} - \Phi(X)$ tends strongly to 0, as $\mathfrak{F} \to \mathfrak{P}$.

COROLLARY 3.1.2. *If $\mathfrak{P}$ is linearly ordered and $\mathfrak{D}$ is compatible with $\mathfrak{B}$, then $\mathfrak{A}$ is maximal subdiagonal, with respect to the expectation Φ.*

Proof. That $\mathfrak{A}$ is subdiagonal is immediate from 3.1.1. For maximality, it suffices to show that if $X \in \mathfrak{B}$, $\Phi(TXA) = 0$ whenever $T, A \in \mathfrak{A}$, $\Phi(T) = 0$, then $X \in \mathfrak{A}$ (2.2.1).

We have to show that $(I - E)XE = 0$ for every $E \in \mathfrak{P}$. Fix E, and consider the operator $T = EX^*(I - E)$. Clearly

$$\Phi(T) = E\Phi(X^*)(I - E) = \Phi(X^*)E(I - E) = 0.$$

Note also that $T \in \mathfrak{A}$. For if $F \in \mathfrak{P}$, and $F \leqq E$, then $(I - E)F = 0$, so that $FTF = 0 = TF$. If $E \leqq F$, then $FE = E$ so

$$FTF = FEX^*(I - E)F = EX^*(I - E)F = TF.$$

Thus $FTF = TF$ in either case.

We have

$$\Phi(TT^*) = \Phi(EX^*(I - E)XE) = \Phi(TXE) = 0$$

by hypothesis, since $E, T \in \mathfrak{A}$ and $\Phi(T) = 0$. Since Φ is faithful, $T^* = 0$, as required.

Remark 3.1.3. Let $\mathfrak{B}$ be a factor and let $\mathfrak{A}$ be a hyperreducible maximal triangular subalgebra of $\mathfrak{B}$ ([11]). That is, $\mathfrak{D} = \mathfrak{A} \cap \mathfrak{A}^*$ is a maximal abelian subalgebra of $\mathfrak{B}$, $\mathfrak{A}$ is a maximal subalgebra of $\mathfrak{B}$ with that property, and $\mathfrak{D}$ is generated by the hulls of $\mathfrak{A}$ (a hull is a projection of $\mathfrak{B}$ left invariant under $\mathfrak{A}$). It is shown in ([11], Theorem 3.1.1) that the set $\mathfrak{P}$ of hulls is linearly ordered, and

$$\mathfrak{A} = \{X \in \mathfrak{B}\,; EXE = XE \text{ for every } E \in \mathfrak{P}\}.$$

Of course, $\mathfrak{D} = \{X \in \mathfrak{B} : EX = XE, E \in \mathfrak{P}\}$; hence, the hypotheses of 3.1.1 are satisfied whenever $\mathfrak{D}$ is compatible with $\mathfrak{B}$. This always occurs when $\mathfrak{B}$ is finite (remark 6.1.5), and it occurs for some combinations of $\mathfrak{P}$ and $\mathfrak{B}$ in all other types I_∞, II_∞, or III. For example, if $\mathfrak{B}$ is the ring of all bounded operators on an infinite dimensional Hilbert space, then $\mathfrak{D}$ is compatible with $\mathfrak{B}$ if $\mathfrak{D}$ is totally atomic: or what is the same, if $\mathfrak{D}$ is generated by its 1-dimensional projections. If $\mathfrak{D}$ is nonatomic, the situation is quite different. It is still true that $\mathfrak{A} + \mathfrak{A}^*$ is ultraweakly dense (this can be read out of Theorem 3.3.1 of [11], for instance), but there is no normal expectation on $\mathfrak{D}$ (prop. 6.2.4), and the theory developed below does not apply.

3.2. Algebras based on ordered groups. Let G be a group, and let S be a subsemigroup of G with the properties:

$$S \cap S^{-1} = \{e\}$$
$$S \cup S^{-1} = G.$$

Define the relation $\leqq$ in G by $x \leqq y$ iff $x^{-1}y \in S$. This relation is transitive because S is closed under multiplication, and the other conditions on S guarantee that it is in fact a linear order on the group elements. Moreover, $x \leqq y$ implies $zx \leqq zy$ for every $z \in G$. This order will be invariant under right multiplication if (and only if) S is normal in the sense that $zSz^{-1} \subseteq S$, for every $z \in G$. Clearly S is the set of all x in G for which $x \geqq e$. Conversely, given a left-invariant linear order on G, the set $S = \{x \in G : x \geqq e\}$ is a semigroup having the two properties listed above.

Definition 3.2.1. *An ordered group is a pair* $(G, \leqq)$ *consisting of a group* G *and a linear order* $\leqq$ *on* G *which is invariant under left multiplication.*

Note that we do *not* require the order to be right-invariant. An ordered group must be torsion-free. For if $x \geqq e$, $x \neq e$, and $n \geqq 2$, then

$$x^n = x^{n-1}x \geqq x^{n-1} \geqq \cdots \geqq x;$$

so that $x^n = e$ implies $e = x^n \geqq x \geqq e$, hence $x = e$. The same argument works if $x^{-1} \geqq e$. If G is abelian, this condition is also sufficient: every torsion-free abelian group can be made into an ordered group.[2]

Considerable attention has been given to the harmonic analysis and function theory based on ordered discrete abelian groups. Here, we shall concentrate on the nonabelian case. Typical examples of countable ordered groups are the following:

Example 1. Let G be the multiplicative group of 2×2 matrices (a_{ij}), where $a_{21} = 0$, $a_{22} = 1$, a_{11} and a_{12} are rational real numbers with $a_{11} > 0$. Say $(a_{ij}) \leqq (b_{ij})$ if $a_{11} < b_{11}$, or $a_{11} = b_{11}$ and $a_{12} \leqq b_{12}$. This defines a linear order which is both right- and left-invariant.

Example 2. Let G be the same group, with the new order $(a_{ij}) \leqq (b_{ij})$ meaning $a_{12} < b_{12}$, or $a_{12} = b_{12}$ and $a_{11} \leqq b_{11}$. This order is left-invariant but not right-invariant.

Example 3. Let G be the free group on two generators. It is known that there is a linear order on G which is both left- and right-invariant.[3]

Let $(G, \leqq)$ be a countable ordered discrete group, and let $\mathcal{B}$ be the von Neumann algebra generated by the left regular representation $x \to U_x$ of G on the separable Hilbert space $l^2(G)$; here, for $f \in l^2(G)$, $U_x f(y) = f(x^{-1}y)$. If f_0 is the characteristic function of the identity of G, then $\phi(T) = (Tf_0, f_0)$ is a faithful normal trace on $\mathcal{B}$ such that $\phi(I) = 1$. Every operator in $\mathcal{B}$ has an expansion $T = \sum_x \lambda_x U_x$, $\lambda_x = \phi(TU_x^*)$, in the sense that the net of finite sums $\lambda_{x_1} U_{x_1} + \cdots + \lambda_{x_n} U_{x_n}$ converges to T in the trace norm $[X] = \phi(X^*X)^{\frac{1}{2}}$. Significantly, convergence need not occur in the strong, or even the weak, operator topology (cf. remark 3.2.3 below). The expansions of sums, products, and adjoints are obtained by carrying out the usual formal operations with the infinite sums, and one has $\phi(\sum \lambda_x U_x) = \lambda_e$.

Let $\mathcal{A}_0$ be the set of all finite linear combinations of the U_x with $x \geqq e$. $\mathcal{A}_0 + \mathcal{A}_0^*$ is the *-subalgebra of $\mathcal{B}$ consisting of finite sums of the U_x, and is therefore σ-weakly dense in $\mathcal{B}$. If $A = \sum \lambda_x U_x$ and $B = \sum \mu_y U_y$ belong to $\mathcal{A}_0$, then $AB = \sum \nu_z U_z$, where $\nu_z = \sum_x \lambda_x \mu_{x^{-1}z}$ (here, all sums are actually finite). Thus $\phi(AB) = \nu_e = \sum \lambda_x \mu_{x^{-1}}$. Now $\lambda_x = 0$ whenever $x < e$, and $\mu_{x^{-1}} = 0$ whenever $x > e$. So $\phi(AB) = \lambda_e \mu_e = \phi(A)\phi(B)$. We have proved that the

[2] For example, see Birkhoff, "Lattice Theory," *American Mathematical Society Colloquium Publications*, vol. 25 (1948), p. 224, Theorem 14.

[3] L. Fuchs, *Partially Ordered Algebraic Systems*, Addison-Wesley, 1963, p. 49, Theorem 9.

mapping $\Phi(X) = \phi(X)I$ is a faithful normal positive idempotent ϕ-preserving map of $\mathcal{B}$ onto $\{\lambda I\}$ with respect to which $\mathfrak{A}_0$ is subdiagonal. Let $\mathfrak{A}$ be the maximal subdiagonal algebra determined by $\mathfrak{A}_0$. By Theorem 2.2.1, the elements A of $\mathfrak{A}$ are defined by the condition

$$\phi(U_x A U_y) = \phi(U_y A U_x) = 0, \text{ for every } x \geqq e \text{ and } y > e.$$

This is clearly equivalent to $\phi(AU_z) = 0$ for every $z > e$. We conclude that $\mathfrak{A}$ is precisely the set of all operators in $\mathcal{B}$ having a representation $\sum \lambda_x U_x$, with $\lambda_x = 0$ for $x < e$.

Remark 3.2.3. We have noted that the net of partial sums $\lambda_{x_1}U_{x_1} + \cdots + \lambda_{x_n}U_{x_n}$ of $T \in \mathcal{B}$ need not converge to T even in the weak operator topology. This anomaly is closely associated with the well-known fact that the partial sums of the Fourier series of a bounded measurable function f on the unit circle need not converge to f in the weak* topology of L^∞. Indeed, using this very fact from the unit circle, one can prove that if G is a countable discrete group having at least one element x of infinite order, then there exists an operator T in the sub von Neumann algebra generated by the powers of U_x whose net of finite partial sums does not converge weakly to T. Going back to the algebras $\mathfrak{A}_0$ and $\mathfrak{A}$ of the last paragraph, we see that, even though every $A \in \mathfrak{A}$ can be approximated in the norm $[\![\ ]\!]$ with its finite partial sums (each of which belongs to $\mathfrak{A}_0$), it does not follow from this that $\mathfrak{A}$ is contained in the weak closure of $\mathfrak{A}_0$; the answer to the question raised in remark 2.2.3 is not clear even in this special case of algebras based on ordered groups.

On the unit circle, the above difficulty can be repaired by replacing the partial sums with the sequence of Cesaro means ([9], p. 20); the latter sequence does converge weak* to f. In this noncommutative context, however, there is no obvious analog of the Cesaro means. Nevertheless, by making use of special properties of the group in example 1, we can show that for this case, and for some of the examples to follow, $\mathfrak{A}$ is the ultraweak closure of $\mathfrak{A}_0$ (cf. Theorem 5.5.5 (i)).

Observe that examples 1, 2 and 3, $\mathcal{B}$ is a II_1 factor ([1], p. 302, prop. 5). In case 3, $\mathcal{B}$ is non-hyperfinite ([1], p. 304). It is known[4] that for case 1, $\mathcal{B}$ is hyperfinite. We will sketch a proof of this, however, since we lack a specific reference. It suffices to produce a maximal abelian von Neumann subalgebra M of $\mathcal{B}$ and an *abelian* group $\mathcal{G}$ of unitary operators of $\mathcal{B}$ such that

[4] The author is indebted to Prof. H. A. Dye for calling this to his attention.

(i) $UMU^{-1} = M$, for every $U \in \mathfrak{G}$

3.2.4 (ii) $M \cup \mathfrak{G}$ generates $\mathfrak{B}$

(iii) for every $U \in \mathfrak{G}$ not the identity, and every nonnull projection P of M, there is a nonnull subprojection $P_0 \leqq P$ in M such that $UP_0U^{-1} \neq P_0$ (free action).

([3], p. 576 and pp. 569-570). Let H (resp. K) be the subgroup consisting of all $(a_{ij}) \in G$ for which $a_{11} = 1$ (resp. $a_{12} = 0$). These are abelian groups, and H is normal. Let M be the von Neumann algebra generated by $\{U_h : h \in H\}$, and let $\mathfrak{G} = \{U_k : k \in K\}$. We have $U_k M U_k^{-1} = M$ for every $k \in K$, because H is normal; $M \cup \mathfrak{G}$ generates $\mathfrak{B}$ because $H \cup K$ generates G, and M is maximal abelian because the set $\{hgh^{-1} : h \in H\}$ is infinite for every g not in H ([1], p. 307, ex. 12). To prove (iii), it suffices to show that for every $k \in K$, $k \neq e$, U_k acts ergodically in M (cf. section 3.3). Let $A \in M$, and suppose $U_k A U_k^{-1} = A$. Then $U_{k^n} A U_{k^{-n}} = A$ for every $n = 0, \pm 1, \pm 2, \cdots$. If $A = \sum \lambda_h U_h$, the sum extended over H, then the last condition implies $\lambda_h = \lambda_{k^n h k^{-n}}$, $n = 0, \pm 1, \pm 2, \cdots$, for every $h \in H$. A simple computation shows, however, that for every $h \in H$, $h \neq e$, $\{k^n h k^{-n} : n = 0, \pm 1, \pm 2, \cdots\}$ is infinite. This, together with the fact that the sequence λ_h is square summable, forces $\lambda_h = 0$ for every $h \neq e$. Thus $A = \lambda_e I$, proving ergodicity.

Of course, $\mathfrak{B}$ does not depend on the order on G, and what we have said applies to example 2 as well.

The scond class of examples relating to ordered groups is a generalization of the matrix-valued analytic functions studied by Helson and Lowdenslager, Wiener and Masani, et al. These algebras are interesting, and they are not without unexpected difficulties (cf. question 5.3.6). However, our main purpose in introducing them is because of the essential role that they play in the proof of Jensen's inequality for example 1 of this section and for example 6 of 3.3.

Example 4. Let M_0 be a II_1 factor acting on a separable Hilbert space $\mathfrak{H}_0$, and let G be a countable discrete abelian ordered group, written multiplicatively. We need not assume, at this point, that M_0 is hyperfinite. Let $\mathfrak{H}$ be the direct sum of $\aleph_0$ copies of $\mathfrak{H}_0$, realized as the set $l^2(G, \mathfrak{H}_0)$ of all $\mathfrak{H}_0$-valued functions F on G such that $\sum_x \| F(x) \|^2$ is finite. Let W_x be the operator "translation by x," defined by $W_x F(y) = F(x^{-1}y)$, and let P_x be the projection on the x-th coordinate space, defined by $(P_x F)(y) = \delta_{x,y} F(y)$. One has $W_x P_y = P_{xy} W_x$. Every $T_0 \in M_0$ determines an operator T on $\mathfrak{H}$ by way of $(TF)(x) = T_0 F(x)$, $x \in G$. The map $T_0 \rightarrow T$ is well known to be a

*-isomorphism of M_0 on a von Neumann algebra M acting on $\mathfrak{H}$. Each W_x commutes with the operators in M.

Let $\mathcal{B}$ be the von Neumann algebra generated by M and the unitary group $\{W_x: x \in G\}$. One may think of $\mathcal{B}$ as the tensor product of M with the von Neumann algebra generated by $\{W_x\}$, or alternately, as the set of all bounded operators on $\mathfrak{H}$ admitting a matrix representation $(T_{x,y})$, $x, y \in G$, with $T_{x,y} \in M_0$ and $T_{x,y} = T_{s,t}$ whenever $xy^{-1} = st^{-1}$. In addition, as we will see in Section 5, $\mathcal{B}$ is *-isomorphic to the algebra of (equivalence classes) of bounded M_0-valued measurable functions on the character group of G, with respect to Haar measure. Though we shall have to refer to the measure-theoretic model later, the construction given is most convenient for our present purposes.

We now introduce the operation in $L(\mathfrak{H})$ that replaces a matrix with its "diagonal" part. For every bounded operator X, let

$$\Phi(X) = \sum_{x \in G} P_x X P_x.$$

Φ is a faithful normal expectation of $L(\mathfrak{H})$ onto $\{P_x: x \in G\}'$ (6.1.3, example 2). For every $y \in G$, $y \neq e$, one has

$$\Phi(W_y) = \sum P_x W_y P_x = \sum W_y P_{y^{-1}x} P_x = 0,$$

by orthogonality. We claim that the restriction of Φ to $\mathcal{B}$ is an expectation on M. Indeed, $\Phi(TX) = T\Phi(X)$ for $T \in M$, $X \in \mathcal{B}$, because M commutes with every P_x, and $\Phi(I) = I$ by definition. It suffices, therefore, to show that $\Phi(\mathcal{B}) \subseteq M$. If X has the form $T_1 W_{x_1} + \cdots + T_n W_{x_n}$, $T_i \in M$, $x_i \in G$, then $\Phi(X) = T_1 \Phi(W_{x_1}) + \cdots + T_n \Phi(W_{x_n}) = T_1 \delta_{x_1,e} + \cdots + T_n \delta_{x_n,e} \in M$. Such X's form an ultraweakly dense *-subalgebra of $\mathcal{B}$, and the claim follows by continuity of Φ.

Let ϕ_1 be the canonical trace on M, normalized so that $\phi_1(I) = 1$. For $X \in \mathcal{B}$, put $\phi(X) = \phi_1 \circ \Phi(X)$. Then ϕ is a normal state of $\mathcal{B}$ which preserves Φ. ϕ is faithful because both ϕ_1 and Φ are. If $S, T \in M$ and $x, y \in G$, then

$$\phi(SW_x TW_y) = \phi(STW_x W_y) = \phi_1(ST\Phi(W_{xy})),$$

which is 0 or $\phi_1(ST) = \phi_1(TS)$, according as $y \neq x^{-1}$ or $y = x^{-1}$. But

$$\phi(TW_y SW_x) = \phi(TSW_y W_x) = \phi_1(TS\Phi(W_y W_x)),$$

and it follows that $\phi(SW_x TW_y) = \phi(TW_y SW_x)$. By linearity and continuity, ϕ is a trace on $\mathcal{B}$.

LEMMA 3.2.5. *Every operator in* $\mathfrak{B}$ *has a unique "Fourier" expansion* $T = \sum T_x W_x$, $x \in G$, *with* $T_x \in M$, *the finite sums converging to* T *in the trace norm. Here*, $T_x = \Phi(TW_x^*)$.

Proof. Let $\mathfrak{H}'$ be the Hilbert space completion of $\mathfrak{B}$ with respect to the trace norm $[X] = \phi(X^*X)^{\frac{1}{2}}$. Let $\mathfrak{H}_x'$ be the closure of the submanifold MW_x. If $R, S \in M$ and x, y are distinct elements of G, then

$$(RW_x, SW_y) = \phi(W_y^{-1}S^*RW_x) = \phi(S^*RW_{xy^{-1}})$$
$$= \phi \circ \Phi(S^*RW_{xy^{-1}}) = \phi(S^*R\Phi(W_{xy^{-1}})) = 0.$$

Hence, $\mathfrak{H}_x' \perp \mathfrak{H}_y'$. The $\mathfrak{H}_x'$ subspaces span $\mathfrak{H}'$ because the linear manifold generated by MW_x, $x \in G$, is ultrastrongly dense, and therefore $[\ \ ]$-dense, in $\mathfrak{B}$. Let Q_x be the orthogonal projection of $\mathfrak{H}'$ on $\mathfrak{H}_x'$. We claim that for $X \in \mathfrak{B}$, $Q_x(X) = \Phi(XW_x^*)W_x$. Indeed, the mapping $X \to \Phi(XW_x^*)W_x$ is an idempotent linear transformation of $\mathfrak{B}$ onto MW_x, and

$$[\Phi(XW_x^*)W_x]^2 = \phi(W_x^*\Phi^*(XW_x^*)\Phi(XW_x^*)W_x) = \phi(\Phi^*(XW_x^*)\Phi(XW_x^*))$$
$$\leqq \phi \circ \Phi((XW_x^*)^*XW_x^*) = \phi(W_xX^*XW_x^*) = \phi(X^*X) = [X]^2.$$

This map has, therefore, a unique norm-depressing extension of $\mathfrak{H}'$ onto $\mathfrak{H}_x'$, and these properties identify the extensions as Q_x.

Now for every $\xi \in \mathfrak{H}'$, the net of finite sums $\sum Q_x\xi$ converges to ξ in norm. Taking $\xi = T \in \mathfrak{B}$, the convergence statement follows. Uniqueness is immediate from the uniqueness of components in an orthogonal expansion, proving the lemma.

As in examples (1), (2) and (3), the finite sums $\sum T_xW_x$ need not converge to T in the weak operator topology. The expected formal rules are valid, provided one interprets convergence in the sense of the metric $[\ \ ]$.

Let $\mathfrak{A}_0$ be the algebra consisting of all finite sums $\sum T_xW_x$, where $T_x \in M$ and $T_x = 0$ except for finitely many $x \geqq e$. One has $\Phi(\sum T_xW_x) = T_e$, and it follows immediately that Φ is multiplicative on $\mathfrak{A}_0$. Clearly $\mathfrak{A}_0 \cap \mathfrak{A}_0^* = M$, by uniqueness of "Fourier" coefficients, and $\mathfrak{A}_0 + \mathfrak{A}_0^*$ is an ultraweakly dense *-subalgebra of $\mathfrak{B}$. Let $\mathfrak{A}$ be the maximal subdiagonal algebra determined by $\mathfrak{A}_0$. If $T = \sum T_xW_x \in \mathfrak{A}$ and $y > e$, then

$$0 = \Phi(TW_y) = \sum T_x\Phi(W_{xy}) = T_{y^{-1}}.$$

Therefore $T_x = 0$ for all $x < e$. If, conversely, $T = \sum T_xW_x$ is an element of $\mathfrak{B}$ for which $T_x = 0$ whenever $x < e$, then a straightforward application of 3.2.4 show that $T \in \mathfrak{A}$. Thus $\mathfrak{A}$ is the collection of all operators in $\mathfrak{B}$

admitting an expansion $\sum T_x W_x$, with $T_x = 0$ for $x < e$. We will see later that $\mathcal{A}$ is the ultraweak closure of $\mathcal{A}_0$ (Theorem 5.3.2).

Remark 3.2.6. It is possible to carry out this construction starting with an *arbitrary* von Neumann algebra M_0 which acts on a separable Hilbert space. One defines $\mathfrak{H}$, M, the group $\{W_x : x \in G\}$, the projections $\{P_x\}$, and Φ in exactly the same way. M no longer comes equipped with a trace, but this can be obviated as follows. Let ρ_1 be a faithful normal state of M (e.g., let $\rho_1(X) = \sum (2^n \| \xi_n \|^2)^{-1} (X\xi_n, \xi_n)$, where $\xi_1, \xi_2, \cdots$ is a sequence of vectors dense in $\mathfrak{H}$). Define ρ on $\mathcal{B}$ by $\rho = \rho_1 \circ \Phi$. Then ρ is a faithful normal Φ-preserving state. For $X \in \mathcal{B}$, let $[\![X]\!]_\rho = \rho(X^*X)^{\frac{1}{2}}$. The $*$-operation is no longer isometric with respect to $[\![\ ]\!]_\rho$, but as it turns out, this feature is not an essential one. Indeed, this proof of Lemma 3.2.5 is valid as stated if we replace $[\![\ ]\!]$ with $[\![\ ]\!]_\rho$. If one defines $\mathcal{A}_0$ and $\mathcal{A}$ as above, the same argument shows that $\mathcal{A}$ is the set of all $\sum T_x W_x \in \mathcal{B}$ for which $T_x = 0$ for $x < e$. This remark applies, in particular, when M_0 is a type III factor or the ring of all bounded linear operators, on a separable space.

3.3. Algebras based on groups of measurable transformations. Let $(S, \mathcal{S}, m)$ be a separable, nonatomic, σ-finite measure space, and let G be a countable ordered group which acts on S in the sense that, for each $x \in G$, $x: s \rightarrow xs$ is a measurable (i.e., Borel structure-preserving) transformation of S onto itself, $x(ys) = (xy)s$, and $es = s$, e being the identity in G. Suppose, in addition, that G has the following properties:

(i) (quasi-invariance) $m(xE) = 0$ iff $m(E) = 0$, for every $E \in \mathcal{S}$, $x \in G$.

3.3.0 (ii) (free action) For every $x \in G$, $x \neq e$, and every $E \in \mathcal{S}$ of positive measure, there exists a subset $E_0 \subseteq E$ in $\mathcal{S}$ such that $xE_0 \cap E_0 = \emptyset$ and $m(E_0) > 0$.

(iii) If $E \in \mathcal{S}$ and $xE \subseteq E$ for every $x \geqq e$ in G, then E or its complement has measure zero.

Remark 3.3.1. Condition (iii) implies at once that the action of G is ergodic. The converse may fail, however, even in very familiar situations. For example, if $(S, \mathcal{S}, m)$ is the real line with respect to Lebesgue measure and G is the group of rational translations, endowed with the usual order, then the semi-infinite interval $[0, +\infty)$ is invariant under positive translations and neither it nor its complement has measure zero. At the same time, G is well known to be an ergodic group of measure-preserving transformations.

Ergodicity is equivalent to (iii) in the two important cases where G is singly generated or it leaves invariant some finite measure equivalent to m. First, suppose n is a finite measure equivalent to m and invariant under G, and G is ergodic. If $E \in \mathcal{S}$ and $xE \subseteq E$ for all $x \geqq e$, then

$$n(E) = n(E - xE) + n(xE).$$

Since $n(E) = n(xE) < \infty$, we have $n(E - xE) = 0$, $x \geqq e$. Let $E_0 = \bigcup_{x \in G} xE$. Then $E \subseteq E_0$, and $E_0 - E = \bigcup_{x \geqq e} (x^{-1}E - E)$. But $n(x^{-1}E - E) = n(E - xE)$ $= 0$ for every $x \geqq e$, so that $n(E_0 - E) = 0$. Therefore $E_0 - E$ has m-measure zero. But clearly $xE_0 = E_0$ for every $x \in G$, and by ergodicity, E_0 or its complement has measure zero. The same statement applies to E, so that (iii) is satisfied. If $G = \{x^k : k = 0, \pm 1, \pm 2, \cdots\}$ is singly generated, then by interchanging x and x^{-1} if necessary, we can assume $x^k \leqq x^l$ iff $k \leqq l$. Suppose G is ergodic and $x^kE \subseteq E$, $k \geqq 0$, for some $E \in \mathcal{S}$. It suffices to show that $m(E - x^kE) = 0$, $k = 1, 2, \cdots$. For if this is the case, and $E_0 = \bigcup_{-\infty}^{\infty} x^kE$, then an argument similar to that used above shows that $m(E_0 - E) = 0$ and E_0 is left fixed under G. Thus E or its complement has measure zero. Now suppose to the contrary, that $E - x^kE$ has positive measure for some $k \geqq 1$, then

$$F \cap x^kF = (E - xE) \cap (x^kE - x^{k+1}E) \subseteq (E - xE) \cap xE = \emptyset.$$

It follows that the sets x^kF, $k = 0, \pm 1, \pm 2, \cdots$ are disjoint. Since m is σ-finite and nonatomic, there exists a subset F_0 of F such that $0 < m(F_0) < m(F)$. Therefore $\bigcup_{-\infty}^{\infty} x^kF_0$ and $\bigcup_{-\infty}^{\infty} x^k(F - F_0)$ are two disjoint subsets of S, each having positive measure, each left invariant under G. This contradicts ergodicity.

As a final note, we show that when G is singly-generated, ergodicity also implies (iii). Let $G = \{x^k ; k = 0, \pm 1, \pm 2, \cdots\}$, fix $x^k \neq e$, and take $E \in \mathcal{S}$, $m(E) > 0$. By replacing x by x^{-1}, if necessary, we can assume $k \geqq 1$. First, we claim that $m(E_0 - x^kE_0) > 0$ for some $E_0 \subseteq E$. If not, let a be a positive (finite) number smaller than $m(E)$. Choose, by σ-finiteness and nonatomicity, an $E_0 \subseteq E$ such that

$$0 < \max_{0 \leqq j \leqq k-1} m(x^jE_0) \leqq k^{-1}a,$$

and let

$$F_0 = E_0 \cup x^kE_0 \cup x^{2k}E_0 \cup x^{3k}E_0 \cup \cdots$$

$$= E_0 \cup (x^k E_0 - E_0) \cup (x^{2k} E_0 - x^k E_0) \cup \cdots$$
$$= E_0 \cup (x^k E_0 - E_0) \cup x^k (x^k E_0 - E_0) \cup \cdots$$

then $E_0 \subseteq F_0$, and $F_0 - E_0$ is a union of sets of the form $x^{jk}(x^k E_0 - E_0)$, $j = 1, 2, \cdots$. Thus $m(F_0 - E_0) = 0$, and $m(E_0) = m(F_0)$. Similarly, one has $m(x^j E_0) = m(x^j F_0)$, $1 \leqq j \leqq k-1$. Of course, $x^k F_0 \subseteq F_0$. Now the set $F = F_0 \cup xF_0 \cup \cdots \cup x^{k-1} F_0$ is invariant under x and has positive measure. As ergodicity implies 3.3.0 (iii) for singly-generated groups, it follows that the complement of F has measure zero. But this is impossible, because

$$m(F) \leqq k \max_{0 \leqq j \leqq k-1} m(x^j(F_0) = k \max_{0 \leqq j \leqq k-1} m(x^j E_0) \leqq a,$$

and $a < m(E)$. Therefore, there is an $E_0 \subseteq E$ such that $m(E_0 - x^k E_0) > 0$. Let H be any subset of $E_0 - x^k E_0$ having finite positive measure. Then $H \cap x^k H = \emptyset$, proving 3.3.0 (ii).

Example 5. Take $(S, \mathcal{S}, m)$ to be the unit interval with Lebesgue measure. It is known ([5], pp. 78-79) that there exists an invertible measurable transformation σ of S onto itself which satisfies (i) and is ergodic with respect to m, but which preserves no σ-finite measure equivalent to m. Let G be the group of integers with the usual order, acting on S by $ns = \sigma^n s$. By remark 3.3.1, G satisfies (ii) and (iii).

Example 6. Let Λ be the character group of the discrete additive group Q of rational real numbers, with normalized Haar measure m. Let G be the discrete multiplicative group of positive rationals, with the usual order. Each element x of G acts as an automorphism $r \in Q \rightarrow xr$ (multiplication by x) of Q. This action of G can be induced to give an action $\lambda \in \Lambda \rightarrow x\lambda$ of Λ in the usual way: $[x\lambda, r] = [\lambda, xr]$, $x \in G$, $\lambda \in \Lambda$, $r \in Q$. Each x is a continuous automorphism of Λ, and therefore preserves Haar measure. We claim that *every* $x \neq e$ acts ergodically in Λ. If f is a bounded measurable function in Λ invariant under x, and

$$f(\lambda) = \sum_{r \in Q} c_r [\lambda, r]$$

is the Fourier series of f, then $f(\lambda) = f(x\lambda) = f(x^2\lambda) = \cdots$ a.e. implies that $c_{x^n r} = c_r$ for every $r \in Q$, $n = 0, \pm 1, \pm 2, \cdots$. But if $r \neq 0$ the orbit $\{x^n r\}$ is infinite. Since $\sum_r |c_r|^2 < \infty$, we must have $c_r = 0$ whenever $r \neq 0$, so that f is constant. By remark 3.3.1, G satisfies conditions (ii) and (iii).

Given $(S, \mathcal{S}, m)$ and G satisfying 3.3.0 (i), (ii), and (iii), there is a well-known procedure for constructing factors of type II or III. By the

Radon-Nikodym theorem, there exists, for each $x \in G$, a positive function w_x such that

$$m(xE) = \int_E w_x(s)\,dm(s)$$

for every $E \in \mathcal{S}$. Form the Hilbert space $\mathfrak{H}$ of all complex functions F on $G \times S$ which are measurable in the second variable, and for which

$$\| F \|^2 = \sum_{x \in G} \int_S | F(x,s) |^2\, dm(s)$$

is finite. For $x \in G$, put

$$(U_x F)(y,s) = w_{x^{-1}}(s)^{-\frac{1}{2}} F(x^{-1}y, x^{-1}s)\,;$$

and for $f \in L^\infty(S, \mathcal{S}, m)$, let

$$(L_f F)(x,s) = f(s) F(x,s).$$

$x \to U_x$ (resp. $f \to L_f$) is a faithful unitary (resp. self-adjoint) representation of the group G (resp. the B^*-algebra $L^\infty(S, \mathcal{S}, m)$), and one has

$$U_x L_f U_x^* = L_{f_x}, \quad \text{where } f_x(s) = f(x^{-1}s).$$

Finite sums of operators of the form $L_f U_x$ form a *-algebra with identity, whose weak closure we denote by $\mathcal{B}$. $\mathcal{B}$ is a factor, whose type will be II_1, II_∞ or III according as G preserves a finite, σ-finite or no σ-finite measure on $\mathcal{S}$ equivalent to m. Thus the $\mathcal{B}$ arising from example 5 is type III, and example 6 leads to a II_1. In all cases, the set $\mathfrak{D}$ of all L_f, $f \in L^\infty(S, \mathcal{S}, m)$, is a maximal abelian von Neumann algebra in $\mathcal{B}$. In the case where m is an invariant measure with $m(S) = 1$, the canonical normalized trace Tr on $\mathcal{B}$ has the property that $\mathrm{Tr}(L_f U_x) = 0$ or $\int f(s)\,dm(s)$, according as $x \neq e$ or $x = e$.

Making use of the order on G, we can construct a subdiagonal subalgebra of $\mathcal{B}$ which is at the same time an irreducible triangular algebra in the sense of [11]. For $x \in G$, let P_x be the projection on the x-th coordinate space, defined by $(P_x F)(y,s) = 0$ if $y \neq x$ and $F(x,s)$ if $y = x$. A simple calculation shows that $U_x P_y = P_{xy} U_x$, $x, y \in G$. As in 3.2, the mapping

$$\Phi(X) = \sum_{x \in G} P_x X P_x$$

is a faithful normal expectation of $L(\mathfrak{H})$ onto the algebra of all bounded operators which commute with each P_x. We have

$$\Phi(L_f U_y) = \sum P_x L_f U_y P_x = \sum L_f P_x U_y P_x = \sum L_f P_x P_{yx} U_y,$$

and therefore $\Phi(L_fU_y)=0$ or L_f according as $y\neq e$ or $y=e$. It follows immediately that the restriction of Φ to $\mathfrak{B}$ is a faithful normal expectation of $\mathfrak{B}$ on $\mathfrak{D}$. Let $\mathfrak{A}_0$ be the set of all finite sums $L_{f_1}U_{x_1}+\cdots+L_{f_n}U_{x_n}$, with $x_i\geqq e$. Because $L_fU_xL_gU_y=L_{fg_x}U_{xy}$, and $vy\geqq e$ when $x\geqq e$ and $y\geqq e$, $\mathfrak{A}_0$ is an algebra. Moreover, from $\Phi(L_fU_xL_gU_y)=\Phi(L_{fg_x}U_{xy})=L_{fg_x}\Phi(U_{xy})$ and $\Phi(L_fU_x)\Phi(L_gU_y)=L_f\Phi(U_x)L_g\Phi(U_y)=L_{fg}\Phi(U_x)\Phi(U_y)$, and the fact that for $x,y\geqq e$, $\Phi(U_x)\Phi(U_y)=\Phi(U_{xy})=0$ unless $x=y=e$, it follows that Φ is multiplicative on $\mathfrak{A}_0$. Since $\mathfrak{A}_0+\mathfrak{A}_0{}^*$ is a weakly dense *-subalgebra of $\mathfrak{B}$, $\mathfrak{A}_0$ is subdiagonal. Let $\mathfrak{A}$ be the maximal subdiagonal subalgebra of $\mathfrak{B}$ determined by $\mathfrak{A}_0$.

PROPOSITION 3.3.2. (i) $\Phi(U_xTU_x^{-1})=U_x\Phi(T)U_x^{-1}$, $x\in G$, $T\in L(\mathfrak{H})$.

(ii) *$\mathfrak{A}$ is the set of all $T\in\mathfrak{B}$ for which*

$$\Phi(TU_x)=0,\quad x>e.$$

Proof. (i)

$$\Phi(U_xTU_x^{-1})=\sum_{y\in G}P_yU_xTU_x^{-1}P_y$$

$$=\sum_y U_xP_{x^{-1}y}TP_{x^{-1}y}U_x^{-1}=U_x(\sum_y P_{x^{-1}y}TP_{x^{-1}y})U_x^{-1}=U_x\Phi(T)U_x^{-1}.$$

(ii) By Theorem 2.2.1, $T\in\mathfrak{A}$ if and only if $T\in\mathfrak{B}$ and

$$\Phi(L_fU_xTL_gU_y)=\Phi(L_gU_yTL_fU_x)=0,$$

for all $x\geqq e$, $y\geqq e$, $f,g\in L^\infty(S,\mathcal{S},m)$. Such an operator satisfies condition (ii).

If, conversely, T satisfies (ii), and f, g, x, y are as above, then

$$\Phi(L_fU_xTL_gU_y)=L_f\Phi(U_xTU_y)L_{g_{y^{-1}}}=L_fU_x\Phi(TU_yU_x)U_x^{-1}L_{g_{y^{-1}}}$$

$$=L_fU_x\Phi(TU_{yx})U_x^{-1}L_{g_{y^{-1}}}=0,$$

since $yx>e$. Similarly,

$$\Phi(L_gU_yTL_fU_x)=L_gU_y\Phi(TU_{xy})U_y^{-1}L_{f_{x^{-1}}}=0$$

because $xy>e$. That completes the proof.

In a sense, then, $\mathfrak{A}$ is the collection of operators in $\mathfrak{B}$ whose negative "Fourier coefficients" are all zero. Finally, we claim that $\mathfrak{A}$ is an irreducible triangular subalgebra of $\mathfrak{B}$ [11]. Indeed, $\mathfrak{A}\cap\mathfrak{A}^*$ is a maximal abelian subalgebra of $\mathfrak{B}$ (namely $\mathfrak{D}$, the range of Φ), so that $\mathfrak{A}$ is triangular. Let P be therefore $P\in\mathfrak{D}$ by maximality. Say $P=L_f$ where f is the characteristic func-
a projection of $\mathfrak{B}$ which is invariant under $\mathfrak{A}$. P reduces $\mathfrak{D}=\mathfrak{A}\cap\mathfrak{A}^*$, and

tion of a measurable subset E of S. Then $PU_xP = U_xP$, $x \geqq e$, is equivalent to $L_f L_{f_x} = L_{f_x}$, $x \geqq e$. Therefore $xE - E$ has measure zero, for every $x \geqq e$. Put $E_1 = \bigcup_{x \geqq e} xE$. E_1 is a measurable set containing E, clearly $m(E_1 - E) = 0$, and E_1 is invariant under the action of every $x \geqq e$. Therefore E_1 or its complement has measure zero, by 3.3.0 (i); thus E or its complement has measure zero. This means that P is O or I, proving irreducibility.

4. Finite Subdiagonal Algebras.

4.1. Some remarks.

Definition 4.1.0. *Let Φ be a linear mapping of a von Neumann algebra $\mathcal{B}$ into itself. A positive linear functional ρ is said to preserve Φ if $\rho \circ \Phi(X) = \rho(X)$, for every $X \in \mathcal{B}$. A subdiagonal subalgebra of $\mathcal{B}$ is called finite if some faithful normal finite trace of $\mathcal{B}$ preserves the expectation associated with the subalgebra.*

Remark 4.1.1. In a finite von Neumann algebra, there may well exist faithful normal expectations which are preserved by no nonzero finite trace. Indeed, let ρ be any faithful normal state of $\mathcal{B}$ which is not a trace, and put $\Phi(X) = \rho(X)I$, $X \in \mathcal{B}$. Then Φ is an expectation with the stated properties, and the only linear functionals that preserve Φ are proportional to ρ. The question we wish to raise here is whether a subdiagonal subalgebra of a finite von Neumann algebra is automatically finite? In view of the above remarks about expectation in general, it is quite conceivable that the answer could be sometimes no. If $\Phi(\mathcal{B})$ has the form $N' \cap \mathcal{B}$ with N an abelian self-adjoint subalgebra of $\mathcal{B}$, however, then one can easily show, using 6.2.1 and 6.2.2, that every finite normal trace preserves Φ.

This section centers around Theorem 4.2.2, which says that when $\mathfrak{A}$ is a finite maximal subdiagonal subalgebra of $\mathcal{B}$, then every regular element of $\mathcal{B}$ is the product of a unitary operator and an operator of $\mathfrak{A}$ whose inverse also belongs to $\mathfrak{A}$. Phrased in this way for technical convenience, the result has a number of equivalent formulations which underscore the considerable structural similarity between these (noncommutative) operator algebras and certain algebras of analytic functions. Some of these are:

(i) Every regular positive operator in $\mathcal{B}$ has the form A^*A, for some $A \in \mathfrak{A} \cap \mathfrak{A}^{-1}$.

(ii) Every self-adjoint operator in $\mathcal{B}$ has the form $\log|A|$, for some $A \in \mathfrak{A} \cap \mathfrak{A}^{-1}$.

(i) and (ii) specialize to familiar statements about the algebra H^∞ of functions bounded and analytic in the unit disc. (i) resembles a factorization of the kind necessary in the theory of linear least-square prediction ([6], p. 208). (ii) is the statement that every real-valued bounded measurable function u on the unit circle is the set of boundary values of a function $\log|F(z)|$, where F and $1/F$ are bounded and analytic. Here, F is unique up to a constant factor, and one can take

$$F(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u(e^{i\theta})\, d\theta, \quad |z|<1$$

([9], p. 182). Specializing (ii) in a different direction gives the amusing result that every self-adjoint $n \times n$ matrix (over the complex numbers) can be written $\log|T|$, where T is a nonsingular $n \times n$ matrix with zeros above the main diagonal.

The remainder of this section deals with the generalization of Jensen's inequality, for analytic functions in the unit disc, to finite subdiagonal algebras. In these considerations, the determinant function occupies an essential position. In 4.3 we collect a number of its properties we shall have to use later, most of which are not discussed in [4]. In 4.4, we show that in maximal subdiagonal algebras, Jensen's inequality is equivalent to certain other propositions that are analogs of important results from the H^p theory of the unit disc. It is not known if these statements are generally valid for maximal finite subdiagonal algebras. They are established for a variety of special cases in 4.4.6 and in Section 5.

4.2. A factorization theorem.

We come now to a main result.

THEOREM 4.2.1. *Let $\mathfrak{A}$ be a finite maximal subdiagonal subalgebra of $\mathfrak{B}$. Then every regular operator in $\mathfrak{B}$ admits a factorization UA, where U is unitary and $A \in \mathfrak{A} \cap \mathfrak{A}^{-1}$. If UA and VB are two such representations for the same operator, then there is a unitary W in the diagonal of $\mathfrak{A}$ such that $V = UW^{-1}$ and $B = WA$.*

Proof. For uniqueness, suppose $UA = VB$ as above. Then

$$W = V^{-1}U = BA^{-1}$$

is unitary, belongs to $\mathfrak{A}$, and $W^* = W^{-1} = AB^{-1} \in \mathfrak{A}$. Thus $W \in \mathfrak{A} \cap \mathfrak{A}^*$, as asserted.

For existence, let ϕ be a faithful normal finite trace which preserves the expectation Φ associated with $\mathcal{A}$, and let $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$ be the diagonal of $\mathcal{A}$. Let $\mathfrak{H}_\phi$ be the canonical Hilbert space associated with ϕ, and let $\mathcal{L}$ (resp. $\mathcal{R}$) denote the von Neumann algebra of left (resp. right) multiplications by elements of $\mathcal{B}$, acting on $\mathfrak{H}_\phi$. One has $\mathcal{R}' = \mathcal{L}$ and $\mathcal{L}' = \mathcal{R}$ ([1], pp. 57, 86). For $\mathcal{S}$ a subset of $\mathcal{B}$, $[\mathcal{S}]$ will denote the closed subspace of $\mathfrak{H}_\phi$ generated by $\mathcal{S}$.

If P is the orthogonal projection of $\mathfrak{H}_\phi$ on $[\mathcal{D}]$, then the restriction of P to bounded elements of $\mathfrak{H}_\phi$ is simply Φ (for example, see 6.1.1 (iv)). An obvious argument, using multiplicativity of Φ, now shows that $PR_A\xi = R_{\Phi(A)}P\xi$ and $PL_A\xi = L_{\Phi(A)}P\xi$, for every $\xi \in [\mathcal{A}]$, R_T and L_T denoting right and left multiplication by T.

A vector $\zeta \in \mathfrak{H}_\phi$ is called *right-wandering* if $\zeta \perp R_T\zeta$ for every $T \in \mathcal{J} = \{A \in \mathcal{A} : \Phi(A) = 0\}$. Our first task is to identify the right-wandering vectors.

LEMMA 4.2.2. *If ζ is right-wandering, then there exists a partial isometry U in $\mathcal{B}$ such that $L_U\zeta \in [\mathcal{D}]$, and L_{U^*U} is the projection on $[\mathcal{R}\zeta]$.*

Proof. The subspace $[\mathcal{D}]$ is invariant under the von Neumann algebra $\mathcal{R}_0$ of right multiplications by elements of $\mathcal{D}$, and $[\mathcal{D}]$ contains a separating vector for $\mathcal{R}_0$ (the identity operator). Thus every normal state of $\mathcal{R}_0$ is the vector state defined by some vector in $[\mathcal{D}]$ ([1], p. 233, theoreme 4).

There exists, therefore, a $\zeta_0 \in [\mathcal{D}]$ such that $(R_D\zeta, \zeta) = (R_D\zeta_0, \zeta_0)$ for every $D \in \mathcal{D}$. If $T \in \mathcal{J}$, then $(R_T\zeta, \zeta) = 0$ because ζ is right-wandering, and $(R_T\zeta_0, \zeta_0) = (R_T\zeta_0, P\zeta_0) = (PR_T\zeta_0, \zeta_0) = (R_{\Phi(T)}P\zeta_0, \zeta_0) = 0$. The same is true if $T \in \mathcal{J}^*$. Thus $(R_X\zeta, \zeta) = (R_X\zeta_0, \zeta_0)$ for every X in $\mathcal{D} + \mathcal{J} + \mathcal{J}^* = \mathcal{A} + \mathcal{A}^*$, and since $\mathcal{A} + \mathcal{A}^*$ is ultraweakly dense in $\mathcal{B}$ and $X \to R_X$ is ultraweakly continuous, it follows that $\omega_\zeta = \omega_{\zeta_0}$ on $\mathcal{R}$. Now define the mapping

$$W : R_X\zeta \to R_X\zeta_0.$$

We have

$$\| WR_X\zeta \|^2 = (R_X\zeta_0, R_X\zeta_0) = (R_{XX^*}\zeta_0, \zeta_0)$$
$$= (R_{XX^*}\zeta, \zeta) = \| R_X\zeta \|^2$$

so that W may be uniquely extended to a partial isometry whose initial space is $[\mathcal{R}\zeta]$. A standard argument shows that $W \in \mathcal{R}' = \mathcal{L}$, so there exists an operator $U \in \mathcal{B}$ such that $W = L_U$. U is a partial isometry because W is, and by definition of W we have $L_U\zeta = WR_I\zeta = R_I\zeta_0 = \zeta_0 \in [\mathcal{D}]$, proving the lemma.

Now let X be a regular element of $\mathcal{B}$. $L_X[\mathcal{J}]$ is a closed subspace of $\mathfrak{H}_\phi$, because X is invertible, so there exists a vector $\xi \in [\mathcal{J}]$ such that $L_X\xi$ is the projection of $X = L_X I$ on $L_X[\mathcal{J}]$. Note first that $\zeta = L_X(I - \xi)$ is right-wandering. Indeed, for $T \in \mathcal{J}$,

$$(R_T\zeta, \zeta) = (R_T L_X(I - \xi), \zeta) = (L_X R_T(I - \xi), \zeta) = 0,$$

since $L_X R_T(I - \xi) \in L_X[\mathcal{J}]$ and $\zeta = X - L_X\xi$ is orthogonal to $L_X[\mathcal{J}]$. By Lemma 4.2.2, there exists a partial isometry $U \in \mathcal{B}$ such that $L_{UX}(I - \xi) \in [\mathcal{D}]$ and L_{U^*U} is the projection on $[\mathcal{R}\zeta]$.

We wish to show now that U is unitary. If $Y \in \mathcal{A} + \mathcal{A}^*$, say

$$Y = S + D + T^*,\ S, T \in \mathcal{J},\ D \in \mathcal{D},$$

then

$$\begin{aligned}(R_Y\zeta, \zeta) &= (R_S\zeta, \zeta) + (R_D\zeta, \zeta) + (\zeta, R_T\zeta) \\ &= (R_D\zeta, \zeta) = (R_{\Phi(Y)}\zeta, \zeta),\end{aligned}$$

since ζ is right-wandering. By ultraweak continuity it follows that $(R_Y\zeta, \zeta) = (R_{\Phi(Y)}\zeta, \zeta)$ for every $Y \in \mathcal{B}$. Now suppose $R_Y\zeta = 0$ for some $Y \in \mathcal{B}$. Putting $H = \Phi(YY^*)^{\frac{1}{2}}$, we have

$$(R_H\zeta, R_H\zeta) = (R_{\Phi(YY^*)}\zeta, \zeta) = (R_{YY^*}\zeta, \zeta) = \| R_Y\zeta \|^2 = 0.$$

Hence $R_H\zeta = 0$. But $R_H\zeta = R_H L_X(I - \xi) = L_X R_H(I - \xi)$, and because L_X has trivial nullspace, we have $H - R_H\zeta = R_H(I - \xi) = 0$. The conditions $H \in \mathcal{D} \subseteq [\mathcal{D}], R_H\xi \in [\mathcal{J}] \subseteq [\mathcal{D}]^\perp$, and $H = R_H\xi$, imply that, as an element of $\mathfrak{H}_\phi$, H is orthogonal to itself. Hence, $\phi(YY^*) = \phi(H^2) = (H, H) = 0$, so that $Y = 0$. Thus ζ is a separator for $\mathcal{R}$. By a general result on separating vectors for cyclic von Neumann algebras ([1], p. 232, prop. 4), the projection on $[\mathcal{R}\zeta]$ is equivalent to I in $\mathcal{R}' = \mathcal{L}$. Since $\mathcal{L}$ is finite, we have $[\mathcal{R}\zeta] = \mathfrak{H}_\phi$. Thus L_U is an isometry in a finite von Neumann algebra; we conclude that L_U, and finally U itself, is unitary.

Let $A = UX$. It is clear that A is an invertible operator in $\mathcal{B}$. We will complete the proof by showing that $A \in \mathcal{A}$ and $A^{-1} \in \mathcal{A}$.

For the first conclusion, it suffices to show that $\Phi(A\mathcal{J}) = 0$ (cor. 3.2.4). In terms of $\mathfrak{H}_\phi$, this is equivalent to the statement that $L_A[\mathcal{J}]$ is orthogonal to $[\mathcal{D}]$, since Φ is the restriction of P to bounded elements. This is what we shall prove. Let η have the form $R_T(I - \xi) = T - R_T\xi$, $T \in \mathcal{J}$. It is clear that $\eta \in [\mathcal{J}]$, since $\xi \in [\mathcal{J}]$ and $R_T[\mathcal{J}] \subseteq [\mathcal{J}T] \subseteq [\mathcal{J}]$, and we have $L_A\eta = L_A R_T(I - \xi) = R_T L_U L_X(I - \xi)$. By the choice of U, $L_U L_X(I - \xi)$ belongs to $[\mathcal{D}]$; hence $L_A\eta \in R_T[\mathcal{D}] \subseteq [\mathcal{J}]$. Since $[\mathcal{J}] \perp [\mathcal{D}]$, it follows that $L_A\eta$ is othogonal to $[\mathfrak{D}]$ whenever η has this particular form. The desired result will follow, therefore, if we show that $\{R_T(I-\xi) : T \in \mathfrak{J}\}$ is dense in $[\mathfrak{J}]$. Since X is regular, this will be true if we prove that the vectors of the form

$$L_X R_T(I-\xi) = R_T L_X(I-\xi) = R_T\zeta, \quad T \in \mathfrak{J},$$

are dense in $L_X[\mathfrak{J}]$. For this, note first that

$$\mathfrak{H}_\phi = [R_T\zeta : T \in \mathfrak{J}] \oplus [R_{A^*}\zeta : A \in \mathfrak{A}].$$

Indeed, the sum on the right is direct, because ζ is right-wandering. On the other hand, since $\mathfrak{J} + \mathfrak{A}^* = \mathfrak{A} + \mathfrak{A}^*$ is ultrastrongly dense in $\mathfrak{B}$, the elements of the form $R_T\zeta + R_{A^*}\zeta$, $T \in \mathfrak{J}$, $A \in \mathfrak{A}$, are dense in $[\mathfrak{R}\zeta]$ in the $\mathfrak{H}_\phi$-metric. But we have seen above that $[\mathfrak{R}\zeta] = \mathfrak{H}_\phi$, and this establishes the direct sum decomposition of $\mathfrak{H}_\phi$. We can now write

$$L_X[\mathfrak{J}] \ominus [R_T\zeta : T \in \mathfrak{J}] = L_X[\mathfrak{J}] \cap [R_{A^*}\zeta : A \in \mathfrak{A}].$$

But the two subspaces on the right are orthogonal, for if $\omega \in [\mathfrak{J}]$ and $A \in \mathfrak{A}$, then $(L_X\omega, R_{A^*}\zeta) = (R_A L_X\omega, \zeta) = (L_X R_A\omega, \zeta)$. The last inner product has to be zero, because $L_X R_A\omega \in L_X R_A[\mathfrak{J}] \subseteq L_X[\mathfrak{J}]$, and ζ is, by definition, that part of X which is orthogonal to $L_X[\mathfrak{J}]$. That proves $L_X[\mathfrak{J}] = [R_T\zeta : T \in \mathfrak{J}]$, and hence, $A \in \mathfrak{A}$.

The proof that $A^{-1} \in \mathfrak{A}$ proceeds along different lines. As we pointed out above, U has been chosen so that $L_A(I-\xi) = L_U L_X(I-\xi)$ belongs to $[\mathfrak{D}]$. Observe first, that $L_A(I-\xi)$ is actually in $\mathfrak{D} \cdots$ i.e., is bounded. For with P the projection on $[\mathfrak{D}]$, then

$$P(I-\xi) = \Phi(I) - P\xi = I, \text{ since } \xi \in [\mathfrak{J}] \subseteq [\mathfrak{D}]^\perp,$$

and, using the fact that $A \in \mathfrak{A}$, one has

$$L_A(I-\xi) = PL_A(I-\xi) = L_{\Phi(A)}P(I-\xi) = L_{\Phi(A)}I = \Phi(A) \in \mathfrak{D}.$$

It is immediate from this that $I - \xi = L_{A^{-1}}\Phi(A) = A^{-1}\Phi(A)$ is also bounded, and, restating the last sentence, we have

$$A(I-\xi) = \Phi(A).$$

We claim now that $\Phi(A)$ is regular. Choose $\epsilon > 0$ so that $X^*X \geqq \epsilon I$, X^*X being a regular positive operator. Then

$$\Phi(A)^*\Phi(A) = (I-\xi)^*A^*A(I-\xi) {}^*X^*X(I-\xi) \geqq \epsilon(I-\xi)^*(I-\xi).$$

Applying Φ to both sides and using the Schwarz inequality (6.1.1 (i)), there results

$$\Phi(A)^*\Phi(A) \geqq \epsilon\Phi(I-\xi)^*\Phi(I-\xi) = \epsilon I,$$

because $\Phi(I-\xi) = I - \Phi(\xi) = I$. Thus $\Phi(A)^*\Phi(A)$ is regular. But in a finite von Neumann algebra a product of bounded operators is regular only when all the factors are (for example, see [4], p. 528; it is plain that their result, stated for II_1 factors, is valid in any finite von Neumann algebra), and the regularity of $\Phi(A)$ is established. Of course, $\Phi(A)^{-1}$ will belong to $\mathfrak{D}$. Now we can write $A^{-1} = (I-\xi)\Phi(A)^{-1} = R_{\Phi(A)^{-1}}(I-\xi)$. As an element of $\mathfrak{H}_\phi$, the right side belongs to $R_{\Phi(A)^{-1}}[\mathfrak{A}] \subseteq [\mathfrak{A}]$; so that for every $T \in \mathfrak{J}$, $\Phi(A^{-1}T) = PR_T(A^{-1}) = R_{\Phi(T)}P(A^{-1}) = 0$. Therefore, $A^{-1} \in \mathfrak{A}$, by 3.2.4, and the proof of the theorem is complete.

Remark 4.2.3. The starting point for this proof is a simple projection argument first used by Wold in his study of stationary stochastic processes (Wold, H., *A Study in the Analysis of Stationary Times Series*, Uppsala, 1938). Later, a similar argument was conspicuously exploited by Helson and Lowdenslager in their papers on prediction theory [6]. The argument has since been applied to obtain a number of important results for a class of function algebras [8], [9].

It is clear that if, in the factorization $X = UA$, X itself belongs to A, then so does U. Under the hypotheses of 4.2.1, we have the following:

COROLLARY 4.2.4.

(i) *Let $X \in \mathfrak{B}$. If there exists a regular $S \in \mathfrak{B}$ such that $SXS^{-1} \in \mathfrak{A}$, then S can be taken to be unitary.*

(ii) *Every regular positive operator in $\mathfrak{B}$ has the form A^*A, with $A \in \mathfrak{A} \cap \mathfrak{A}^{-1}$. If $A^*A = A_1^*A_1$ are two such factorizations, then $A_1 = UA$ for some unitary U in $\mathfrak{A} \cap \mathfrak{A}^*$.*

(iii) *Every self-adjoint operator in $\mathfrak{B}$ has the form*

$$\log|A|, \text{ for some } A \in \mathfrak{A} \cap \mathfrak{A}^{-1}.$$

Proof. (i) Write $S^{-1} = UA$, as in 4.2.1. Then $SXS^{-1} = A^{-1}U^{-1}XUA$, and this belongs to $\mathfrak{A}$ if, and only if, $U^{-1}XU$ belongs to $A\mathfrak{A}A^{-1} = \mathfrak{A}$.

(ii) Let H be positive and regular. Then so is the positive square root of H, and we have $H^{\frac{1}{2}} = UA$, with U unitary and $A \in \mathfrak{A} \cap \mathfrak{A}^{-1}$. Hence, $H = H^{\frac{1}{2}*}H^{\frac{1}{2}} = A^*U^{-1}UA = A^*A$. For uniqueness if $A^*A = A_1^*A_1$, then $(A_1A^{-1})^* = (A_1A^{-1})^{-1} \in \mathfrak{A} \cap \mathfrak{A}^{-1}$, so that A_1A^{-1} is a unitary operator in $\mathfrak{A} \cap \mathfrak{A}^*$.

(iii) If X is self-adjoint, then by (ii), we have $e^{2X} = A^*A$ for some $A \in \mathcal{A} \cap \mathcal{A}^{-1}$. Hence, $X = \log(e^{2X})^{\frac{1}{2}} = \log(A^*A)^{\frac{1}{2}} = \log|A|$.

4. 3. Determinants. In this section, ϕ will be a distinguished trace on the von Neumann algebra $\mathcal{B}$ such that $\phi(I) = 1$. While ϕ is necessarily continuous in the norm topology, we shall require it to be neither faithful nor normal. For a regular operator $X \in \mathcal{B}$, the *determinant* of X is defined by

$$\Delta(X) = \exp \phi(\log|X|).$$

If X is singular, set

$$\Delta(X) = \inf \Delta(|X| + \epsilon I),$$

the inf taken over all positive numbers ϵ.

The determinant was introduced in II_1 factors by Fuglede and Kadison ([4], esp. see § 5), who established the following basic properties

(i) $\Delta(XY) = \Delta(X)\Delta(Y)$, $X, Y \in \mathcal{B}$

(ii) $\Delta(e^T) = |\exp \phi(T)|$, $T \in \mathcal{B}$

(iii) $\Delta(X^*) = \Delta(X)$, $\Delta(\lambda X) = |\lambda| \Delta(X)$, $X \in \mathcal{B}$, λ a complex number

4. 3. 1. (iv) Δ is norm-continuous on the set of regular elements of $\mathcal{B}$

(v) $\Delta(H_1) \leqq \Delta(H_2)$ if $0 \leqq H_1 \leqq H_2$, and $\Delta(H_2) \to \Delta(H_1)$ if H_2 tends uniformly to H_1 from above.

(vi) $\Delta(T) \leqq \lim_n \| T^n \|^{1/n}$.

Their proof of (i)-(vi), stated for ϕ the canonical trace on a II_1 factor, is valid without change in the more general situation described here. If $P_1, \cdots, P_n$ are mutually orthogonal projections with sum I and if $X = \lambda_1 P_1 + \cdots + \lambda_n P_n$, $\lambda_i \neq 0$, then a straightforward application of the definition gives $\Delta(X) = \prod_j |\lambda_j|^{\theta_j}$, where $\theta_j = \phi(P_j)$. If P is a single projection, then

$$\Delta(P) = \inf_{\epsilon > 0} \Delta(P + \epsilon I) = \inf_{\epsilon > 0} (1 + \epsilon)^{\phi(P)} \epsilon^{\phi(I-P)},$$

since $P + \epsilon I = (1 + \epsilon)P + \epsilon(I - P)$. Therefore, $\Delta(P)$ is 0 or 1 according as $\phi(P) < 1$ or $\phi(P) = 1$. In particular, if ϕ is faithful, the determinant of every proper projection is zero. More generally, if T is any operator having a nullspace, say $T = TP$ with P a proper projection, then $\Delta(T) = \Delta(T)\Delta(P) = 0$. If ϕ is faithful and normal and if the spectral representation of the positive operator H is given by $\int \lambda \, dE_\lambda$, then

$$\Delta(H) = \exp \int \log \lambda \phi(dE_\lambda),$$

the right side taken as 0 when $\int \log \lambda \phi(dE_\lambda) = -\infty$ [4]. Thus for H a singular operator with no nullspace, one has $\Delta(H) = 0$ if, and only if, $\int \log \lambda \phi(dE_\lambda) = -\infty$.

Here, we shall require some additional properties of Δ. Unless stated otherwise, the following results do not depend on faithfulness or normality of the trace ϕ.

PROPOSITION 4.3.2. *For every positive H in $\mathfrak{B}$, one has*

$$\Delta(H) = \inf \phi(HK),$$

the infimum taken over all regular positive K for which $\Delta(K) \geqq 1$. In fact, the inf *can be restricted to those K belonging to the von Neumann algebra generated by H, and having the stated properties.*

Proof. Let L be a regular positive operator in $\mathfrak{B}$. We claim $\Delta(L) \leqq \phi(L)$. For $\log L = \phi(\log L)I$ is a self-adjoint operator having zero trace. The inequality $1 + t \leqq e^t$, valid for all real numbers t, implies, via the operational calculus, that

$$I + \log L - \phi(\log L)I \leqq \exp(\log L - \phi(\log L)I) = L/\Delta(L).$$

Now apply ϕ to both sides to obtain $1 \leqq \phi(L)/\Delta(L)$.

Turning now to the proof of the proposition, assume first that H is regular, positive, satisfying $\Delta(K) \geqq 1$. Then

$$\phi(HK) = \phi(K^{\frac{1}{2}}HK^{\frac{1}{2}}) \geqq \Delta(K^{\frac{1}{2}}HK^{\frac{1}{2}}) = \Delta^2(K^{\frac{1}{2}})\Delta(H)$$
$$= \Delta(K)\Delta(H) \geqq \Delta(H).$$

Hence, $\inf \phi(HK) \geqq \Delta(H)$. To see that equality is achieved, put $K = \Delta(H)H^{-1}$. Observe that K is regular, positive, and belongs to the von Neumann algebra generated by H. Moreover, by 4.3.1 (i) and (iii), $\Delta(K) = \Delta(H)\Delta(H^{-1}) = \Delta(HH^{-1}) = 1$, and we have $\phi(HK) = \Delta(H)\phi(I) = \Delta(H)$.

In case H is singular, we have

$$\Delta(H) = \inf_{\epsilon>0} \Delta(H + \epsilon I) = \inf_{\epsilon} \inf_{K} \phi((H + \epsilon I)K)$$
$$= \inf_{K} \inf_{\epsilon} \phi((H + \epsilon I)K) = \inf_{K} \phi(HK),$$

as required. The second statement is also valid here because H and $H + \epsilon I$ generate the same von Neumann algebra.

COROLLARY 4.3.3.

(i) $\Delta(H_1+H_2) \geqq \Delta(H_1)+\Delta(H_2),\ H_i \geqq 0.$

(ii) Δ *is upper semicontinuous (u.s.c.) in the norm topology.*

Assuming ϕ *to be normal, one has*

(iii) Δ *is u.s.c. in the ultrastrong topology.*

(iv) Δ *is u.s.c. on positive operators in the ultraweak topology.*

Proof. (i) is immediate from 4.3.2 because the inf of a sum is not less than the sum of the infima.

(ii). For every $X \in \mathfrak{B}$, one has $\Delta(X) = \Delta(|X|) = \inf \phi(|X|K)$. For K fixed, the map $X \rightarrow \phi(|X|K)$ is norm-continuous, and so (ii) follows.

(iv). Suppose ϕ is normal, and therefore ultraweakly continuous. For every X, we have

$$\Delta(X) = \Delta(X^*X)^{\frac{1}{2}} = [\inf_K \phi(X^*XK)]^{\frac{1}{2}} = \inf_K \phi(X^*XK)^{\frac{1}{2}}$$

If X_α is a net tending ultrastrongly to X_0, then $X_\alpha{}^*X_\alpha \rightarrow X_0{}^*X_0$ ultraweakly. Hence, each function $X \rightarrow \phi(X^*XK)^{\frac{1}{2}}$ (K fixed) is ultrastrongly continuous. The conclusion follows as in (ii).

(iii) is immediate from 4.3.2, which expresses the restriction of Δ to positive operators explicitly as an infimum of ultraweakly continuous linear functionals.

COROLLARY 4.3.4 (*generalized Hadamard inequality*). *Let* Φ *be a* ϕ-*preserving expectation on a von Neumann subalgebra* M *of* $\mathfrak{B}$. *Then*

$$\Delta(H) \leqq \Delta(\Phi(H))$$

for every positive H *in* $\mathfrak{B}$.

Proof. Let $\mathcal{S}$ be the set of regular positive K in $\mathfrak{B}$ such that $\Delta(K) \geqq 1$. Then

$$\Delta(H) = \inf_{\mathcal{S}} \phi(HK) \leqq \inf_{\mathcal{S} \cap M} \phi(HK).$$

But if $K \in M$, then $\phi(HK) = \phi \circ \Phi(HK) = \phi(\Phi(H)K)$. Since $\Phi(H) \in M$, the second statement of 4.3.2 says that $\inf \phi(\Phi(H)K)$, $K \in \mathcal{S} \cap M$, is simply $\Delta(\Phi(H))$, completing the proof.

Remark 4.3.5. Taking $\Phi(H) = \phi(H)I$ in 4.3.4, we obtain $\Delta(H) \leqq \phi(H)$ for every positive H in $\mathfrak{B}$, extending the remark in the proof of 4.3.2. 4.3.3 generalizes Hadamard's determinant inequality in the following way. Take for $\mathfrak{B}$ the full algebra of $n \times n$ matrices over the complex numbers, and let ϕ be the canonical trace, normalized so that $\phi(I) = 1$. It is easily checked that for every matrix X, $\Delta(X) = |\det X|^{1/n}$, where det is the usual complex-valued determinant. Let Φ be the mapping which replaces all off-diagonal entries of X with 0 and leaves the diagonal entries fixed. Then Φ is a ϕ-preserving expectation of $\mathfrak{B}$ onto the algebra of diagonal matrices (6.1.3 example 2, et seq.). By 4.3.4, we have

$$\Delta^2(X) = \Delta(XX^*) \leqq \Delta(\Phi(XX^*)),$$

or

$$|\det X| \leqq [\det \Phi(XX^*)]^{\frac{1}{2}}.$$

If $X = (a_{ij})$, then $\Phi(XX^*)$ is the diagonal matrix whose i-th term is $l_i^2 = |a_{i1}|^2 + |a_{i2}|^2 + \cdots + |a_{in}|^2$. Therefore, the preceding inequality is

$$|\det(a_{ij})| \leqq l_1 l_2 \cdots l_n,$$

which is precisely the classical Hadamard inequality.

One more property of Δ will be needed in the sequel.

Lemma 4.3.6. *For every regular operator X,*

$$\Delta(X) = \lim_{t\to 0} \phi(|X|^t)^{1/t}.$$

Proof. Taking $S = \log |X|$, it will suffice to prove that for every self-adjoint S,

$$\phi(S) = \lim_{t\to 0} t^{-1} \log \phi(e^{tS});$$

the desired relation follows by exponentiating both sides.

Let $F(t) = \log \phi(e^{tS}) = \log(1 + t\phi(S) + t^2/2\, \phi(S^2) + \cdots)$. F is a differentiable function on $-\infty < t < +\infty$, vanishing at $t = 0$, and one has $F'(t) = \phi(e^{tS})^{-1}\phi(Se^{tS})$. The required limit is simply,

$$\lim_{t\to 0} t^{-1}F(t) = F'(0) = \phi(S),$$

proving the lemma.

Let $H \geqq 0$ belong to $\mathfrak{B}$, and set $\rho(X) = \phi(HX)$, $X \in \mathfrak{B}$. Then ρ is a positive linear functional on $\mathfrak{B}$. By 4.3.2, $\Delta(H) = \inf \rho(K)$, so that $\Delta(H)$ is completely determined by ρ. Thus we can regard Δ as a function defined on *certain* positive linear functionals on $\mathfrak{B}$. Using the above formula, we can now extend the domain of definition of Δ to all positive functionals on $\mathfrak{B}$.

Definition 4.3.7. *If ρ is a positive linear functional on $\mathfrak{B}$, define*

$\Delta(\rho) = \inf \rho(K)$, *$K$ ranging over the regular positive operators in* $\mathcal{B}$ *satisfying* $\exp \phi(\log K) \geqq 1$.

It is clear that, by definition, Δ is upper semicontinuous in the weak* topology of the dual space of $\mathcal{B}$, and by the preceding remarks, if $\rho(X) = \phi(HX)$, $H \geqq 0$, then $\Delta(\rho) = \Delta(H)$. This new determinant function has a number of properties resembling those of the original one, but we shall not dwell on these here. In the sequel, we shall need only the definition.

4. 4. Jensen's inequality. If f is an H^∞ function in the unit disc (i. e., f is bounded and analytic for $|z| < 1$), it is well known that

$$f(e^{i\theta}) = \lim_{r\uparrow 1} f(re^{i\theta})$$

exists almost everywhere on the unit circle ([9], p. 38). Fix $|z| < 1$. Then one version of Jensen's inequality states that

$$|f(z)| \leqq \exp \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| P_z(\theta)\, d\theta,$$

where $P_z(\theta) = \operatorname{Re}(\dfrac{1+ze^{i\theta}}{1-ze^{-i\theta}})$ is the Poisson kernel representing evaluation at the point z. In this section we are going to study an analogous inequality for subdiagonal operator algebras. Throughout, $\mathfrak{A}$ will be a finite maximal subdiagonal subalgebra of $\mathcal{B}$, Φ and ϕ will be, respectively, the associated expectation of $\mathcal{B}$ on the diagonal $\mathfrak{D}$ of $\mathfrak{A}$, and the (normalized) trace which preserves it.

By Jensen's inequality, we mean the following statement:

$$(\alpha) \qquad \Delta(\Phi(A)) \leqq \Delta(A), \text{ for every } A \in \mathfrak{A}.$$

Notice that if $\mathfrak{A}$ is antisymmetric, then necessarily $\Phi(X) = \phi(X)I$ for every $X \in \mathcal{B}$, and (α) becomes simply $|\phi(A)| \leqq \Delta(A)$, $A \in \mathfrak{A}$, an exact analog of Jensen's inequality on H^∞.

Question 4. 4. 1. The most exasperating of the open questions about finite subdiagonal algebras is whether Jensen's inequality is universally valid. Although the statement (α) makes sense for $\mathfrak{A}$ an arbitrary finite subdiagonal algebra, one may confine attention to maximal finite subalgebras. For every subdiagonal algebra is contained in a maximal one, and the maximal algebra will be finite whenever the original one is. This is evidently more than a trivial reduction because one can draw on the factorization theorem in the maximal case. In Chapter 5 we shall prove Jensen's inequality for a number of examples. Also, see Cor. 4. 4. 6.

In addition to Jensen's inequality, we shall consider the following two relations:

(β) $\Delta(\Phi(A)) = \Delta(A)$, for every $A \in \mathfrak{A} \cap \mathfrak{A}^{-1}$.

(γ) For every normal positive linear functional ρ on $\mathfrak{B}$,

$$\inf \rho(|D + T|^2) = \Delta(\rho),$$

the infimum taken over all $D \in \mathfrak{D}$ and $T \in \mathfrak{A}$ such that $\Delta(D) \geqq 1$ and $\Phi(T) = 0$.

We shall refer to (β) and (γ) as Jensen's formula and Szegö's theorem, respectively.

Remark 4.4.2. Note that when $\mathfrak{A}$ is antisymmetric, (γ) becomes $\inf \rho(|\lambda I + T|^2) = \Delta(\rho)$, λ ranging over the scalars of modulus $\geqq 1$, and T as before. Thus, the equation simplifies to

$$\inf \rho(|I + T|^2) = \Delta(\rho), \quad T \in \mathfrak{A}, \phi(T) = 0.$$

The reader should compare this with the classical theorem of Szegö, one version of which runs like this. Let w be a positive function on the unit circle which is integrable with respect to $d\sigma = (2\pi)^{-1} d\theta$. Then

$$\inf_f \int |1 + f|^2 w \, d\sigma = \Delta(w),$$

where f ranges over all H^∞ functions such that $\int f \, d\sigma = 0$, and $\Delta(w)$ is the geometric mean of w, defined as $\exp \int \log w \, d\sigma$ if $\log w$ is integrable and 0 otherwise. For the analogy to be complete, one needs two additional facts. First, it is well known that the absolutely continuous measures $d\mu = w \, d\sigma$, with w positive and integrable, correspond exactly to the positive normal linear functionals on the von Neumann algebra of multiplications by $L^\infty(d\sigma)$ functions, acting on $L^2(d\sigma)$ (the underlying space is the unit circle, of course). This hinges on the known fact that every normal state of an abelian von Neumann algebra is a vector state. In this correspondence, $w \, d\sigma$ determines a functional whose value at the operator "multiplication by f" ($f \in L^\infty(d\sigma)$) is simply $\int f(x) w(x) d\sigma(x)$. Second, one needs the fact that for such a w, $\Delta(w)$, as defined in this paragraph, is given by $\Delta(w) = \inf_h \int h(x) w(x) d\sigma(x)$, h ranging over all positive measurable functions such that both h and $1/h$ are bounded and for which $\exp \int \log h \, d\sigma \geqq 1$. The proof of that statement follows along measure-theoretic lines which are formally very close to the proof of 4.3.2; we omit these details.

THEOREM 4.4.3. *(α), (β), and (γ) are equivalent.*

Proof. $(\alpha)\Rightarrow(\gamma)$: Let

$$\mathcal{S}_1=\{H\in\mathcal{B}\cap\mathcal{B}^{-1}: H\geqq 0, \Delta(H)\geqq 1\},$$
$$\mathcal{S}_2=\{A^*A: A\in\mathcal{A}, \Delta(\Phi(A))\geqq 1\},$$

and let ρ be any positive linear functional on $\mathcal{B}$, not necessarily normal. The statement (γ) is that $\inf\rho(H_1)$, $H_1\in\mathcal{S}_1$ coincides with $\inf\rho(H_2)$, $H_2\in\mathcal{S}_2$. As ρ is norm-continuous, it suffices to show that $\mathcal{S}_1\subseteq\mathcal{S}_2\subseteq\mathcal{S}_1^-$, $\mathcal{S}_1^-$ denoting the norm closure of $\mathcal{S}_1$. Let $H\in\mathcal{S}_1$. By cor. 4.2.4 (ii), $H=A^*A$ for some $A\in\mathcal{A}\cap\mathcal{A}^{-1}$. By (α), we have $\Delta(\Phi(A))^{-1}=\Delta(\Phi(A^{-1}))\leqq\Delta(A^{-1})=\Delta(A)^{-1}$, so that $\Delta(\Phi(A))^2\geqq\Delta(A)^2=\Delta(H)\geqq 1$. Hence $H=A^*A\in\mathcal{S}_2$, so $\mathcal{S}_1\subseteq\mathcal{S}_2$.

Let $B^*B\in\mathcal{S}_2$, $B\in\mathcal{A}$, $\Delta(\Phi(B))\geqq 1$, and take $\epsilon>0$. By cor. 4.2.4 (ii), one has $B^*B+\epsilon I=C^*C$, with $C\in\mathcal{A}\cap\mathcal{A}^{-1}$. We claim

$$\Phi(B)^*\Phi(B)\leqq\Phi(C)^*\Phi(C).$$

By the Schwarz inequality 6.1.1 (i),

$$\begin{aligned}\Phi(C^{-1})^*\Phi(B)^*\Phi(B)\Phi(C^{-1})&=\Phi(BC^{-1})^*\Phi(BC^{-1})\\ &\leqq\Phi(C^{*-1}B^*BC^{-1})\leqq\Phi(C^{*-1}(B^*B+\epsilon I)C^{-1})=\Phi(I)=I.\end{aligned}$$

The inequality persists after multiplying on the left and right by $\Phi(C)^*$ and $\Phi(C)$, resp., and the desired inequality follows. Now for every $\epsilon>0$,

$$\Delta(B^*B+\epsilon I)=\Delta^2(C)\geqq\Delta^2(\Phi(C))\geqq\Delta^2(\Phi(B))\geqq 1,$$

by the preceding lines, so that $B^*B+\epsilon I\in\mathcal{S}_1$. Letting $\epsilon\to 0$, we have $B^*B\in\mathcal{S}_1^-$.

$(\gamma)\Rightarrow(\beta)$. Let $A\in\mathcal{A}\cap\mathcal{A}^{-1}$, and put $\rho(X)=\phi(A^*XA)$. Then by (γ),

$$\Delta^2(A)=\Delta(\rho)=\inf_D\inf_T\rho(|D+T|^2),$$

D ranging over the diagonal elements of determinant $\geqq 1$ and T ranging over $\mathcal{J}$, the set of $T\in\mathcal{A}$ such that $\Phi(T)=0$. Write

$$\begin{aligned}\rho(|D+T|^2)&=\phi(|DA+TA|^2)\\ &=\phi(|D\Phi(A)+D(A-\Phi(A))+TA|^2)\\ &=\phi(|D\Phi(A)|^2)+\phi(|D(A-\Phi(A))+TA|^2),\end{aligned}$$

since $\mathcal{D}$ and $\mathcal{J}$ are orthogonal with respect to the bilinear form $(X,Y)=\phi(Y^*X)$ on $\mathcal{B}$. Therefore

$$\inf_T \rho(|D+T|^2) = \phi(|D\Phi(A)|^2) + \inf_T \phi(|D(A-\Phi(A)) + TA|^2)$$
$$= \phi(|D\Phi(A)|^2),$$

this by taking $T = -D(A-\Phi(A))A^{-1} \in \mathcal{J}$. The right side is

$$\phi(\Phi(A)\Phi(A)^*D^*D),$$

and we have $\Delta^2(A) = \inf_D \phi(\Phi(A)\Phi(A)^*D^*D)$, $D \in \mathfrak{D}$, $\Delta(D) \geqq 1$. By 4.3.2, this is not less than $\Delta(\Phi(A)\Phi(A)^*) = \Delta^2(\Phi(A))$, and this proves $\Delta(A) \geqq \Delta(\Phi(A))$. The same argument applies to A^{-1}, giving

$$\Delta(A)^{-1} = \Delta(A^{-1}) \geqq \Delta(\Phi(A^{-1})) = \Delta(\Phi(A))^{-1}.$$

Therefore $\Delta(A) = \Delta(\Phi(A))$.

$(\beta) \Rightarrow (\alpha)$. Let $A \in \mathfrak{A}$, and take $\epsilon > 0$. By 4.2.4 (ii), $A^*A + \epsilon I$ has the form B^*B, $B \in \mathfrak{A} \cap \mathfrak{A}^{-1}$. As in the proof of $(\alpha) \Rightarrow (\gamma)$, we have $\Phi(A)^*\Phi(A) \leqq \Phi(B)^*\Phi(B)$, so that, using (β), we obtain

$$\Delta(\Phi(A))^2 \leqq \Delta(\Phi(B))^2 = \Delta^2(B) = \Delta(B^*B) = \Delta(A^*A + \epsilon I).$$

Letting $\epsilon \downarrow 0$, it follows that $\Delta(\Phi(A))^2 \leqq \Delta(A^*A) = \Delta(A)^2$. The proof is now complete.

PROPOSITION 4.4.4. *Each of the following conditions is necessary and sufficient for the validity of Jensen's inequality.*

(i) $\Delta(I+T) \geqq 1$, *for every* $T \in \mathfrak{A}$, $\Phi(T) = 0$.

(ii) $A, B \in \mathfrak{A} \cap \mathfrak{A}^{-1}$, $A^*A = BB^*$ *implies*

$$\Delta(\Phi(A)) = \Delta(\Phi(B)).$$

Proof. The necessity of both conditions is clear. Suppose (i) is true. Let $A \in \mathfrak{A} \cap \mathfrak{A}^{-1}$. Write $A = \Phi(A)(I+T)$, where $T = \Phi(A^{-1})A - I$. Clearly $T \in \mathfrak{A}$ and $\Phi(T) = \Phi(A^{-1})\Phi(A) - I = 0$. Hence

$$\Delta(A) = \Delta(\Phi(A))\Delta(I+T) \geqq \Delta(\Phi(A)).$$

Replacing A by A^{-1}, and using $\Phi(A^{-1}) = \Phi(A)^{-1}$, shows that the inequality is actually equality, and condition (β) follows.

Assume (ii) is true, and let $A \in \mathfrak{A} \cap \mathfrak{A}^{-1}$. Using 4.2.4 (ii), we can find a sequence $B_n \in \mathfrak{A} \cap \mathfrak{A}^{-1}$ such that $B_0 = A$, $(B_n^*B_n)^{\frac{1}{2}} = B_{n+1}^*B_{n+1}$, $n \geqq 0$. From $B_n^*B_n = B_{n+1}^*B_{n+1}B_{n+1}^*B_{n+1}$, one has $(B_nB_{n+1}^{-1})^*B_nB_{n+1}^{-1} = B_{n+1}B_{n+1}^*$. By hypothesis,

$$\Delta(\Phi(B_n))\Delta(\Phi(B_{n+1}))^{-1} = \Delta(\Phi(B_nB_{n+1}^{-1})) = \Delta(\Phi(B_{n+1})),$$

hence $\Delta(\Phi(B_n)) = \Delta^2(\Phi(B_{n+1})) = \Delta^2(\Phi(B_{n+1})^*)$. Iterating this n times gives $\Delta(\Phi(A)) = \Delta(\Phi(B_n))^{2^n} = \Delta(\Phi(B_n)^*)^{2^n}$. Using the Schwarz inequality for Φ and the fact that $\Delta \leqq \phi$ on positive operators, we have

$$\begin{aligned}\Delta(\Phi(A))^2 &= \Delta(\Phi(B_n)^*\Phi(B_n))^{2^n} \leqq \Delta(\Phi(B_n^*B_n))^{2^n} \\ &\leqq \phi \circ \Phi(B_n^*B_n)^{2^n} = \phi(B_n^*B_n)^{2^n} \\ &= \phi((A^*A)^{2^{-n}})^{2^n}.\end{aligned}$$

By Lemma 4.3.6, the right side tends to $\Delta(A^*A) = \Delta(A)^2$, as $n \to \infty$, and this proves $\Delta(\Phi(A)) \leqq \Delta(A)$. As in the proof involving condition (i), the opposite inequality follows from this one, and therefore (β) is true.

Remark 4.4.5. Let f and $1/f$ belong to H^∞ of the unit disc, let z be a point in the interior of the disc, and let g be any other H^∞ function such that $|g| \leqq |f|$ almost everywhere on the unit circle. Then $|g(z)| \leqq |f(z)|$ ([9], p. 62). This property of analytic functions, equivalent to the maximum modulus principle, has an exact analog in this setting, and it has an interesting relation to 4.4.4 (ii). Specifically, let $A \in \mathcal{A} \cap \mathcal{A}^{-1}$ and $B \in \mathcal{A}$ be such that $B^*B \leqq A^*A$. We claim $\Phi(B)^*\Phi(B) \leqq \Phi(A)^*\Phi(A)$. For by the Schwarz inequality (6.1.1 (i)),

$$\begin{aligned}\Phi(A)^{*-1}\Phi(B)^*\Phi(B)\Phi(A)^{-1} &= \Phi(BA^{-1})^*\Phi(BA^{-1}) \\ &\leqq \Phi((BA^{-1})^*BA^{-1}) = \Phi(A^{*-1}B^*BA^{-1}) \leqq \Phi(I) = I,\end{aligned}$$

and the inequality follows by multiplying on the left and right by $\Phi(A)^*$ and $\Phi(A)$ (note that this simple argument works as well in non-finite cases). In particular, if A and B both belong to $\mathcal{A} \cap \mathcal{A}^{-1}$, and if $A^*A = B^*B$, then by symmetry, $\Phi(A)^*\Phi(A) = \Phi(B)^*\Phi(B)$.

This fact might lead one to conjecture as follows: $A, B \in \mathcal{A} \cap \mathcal{A}^{-1}$ and $A^*A = BB^*$ implies $\Phi(A)^*\Phi(A) = \Phi(B)\Phi(B)^*$. Were this true, condition (ii) of 4.4.4 would of course be satisfied, and (α), (β) and (γ) would follow. The following example shows that this conjecture fails, even when the diagonal is abelian. Let $\mathcal{A}$ be the algebra of complex 2×2 matrices (a_{ij}), with $a_{21} = 0$. Here $\Phi(b_{ij}) = (c_{ij})$, where $c_{11} = b_{11}$, $c_{22} = b_{22}$, $c_{12} = c_{21} = 0$, and $\mathcal{B}$ is the full 2×2 matrix algebra. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} \\ 0 & 2^{\frac{1}{2}} \end{bmatrix}.$$

Then A and B belong to $\mathcal{A} \cap \mathcal{A}^{-1}$, $A^*A = BB^*$, but $\Phi(A)^*\Phi(A) = I$ and

$$\Phi(B)\Phi(B)^* = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}.$$

If B_1 is any other element of $\mathfrak{A} \cap \mathfrak{A}^{-1}$ such that $B_1B_1^* = BB^* = A^*A$, then by uniqueness of the factorization (applied to the maximal subdiagonal algebra $\mathfrak{A}^*$), $B_1^* = UB^*$ for some diagonal unitary U. Hence

$$\Phi(B_1)^*\Phi(B_1) = U\Phi(B)^*\Phi(B)U^{-1} = \Phi(B)^*\Phi(B).$$

Observe, however, that one still has $\Delta(\Phi(A)) = |\det\Phi(A)|^{\frac{1}{2}} = 1$ and $\Delta(\Phi(B)) = |\det\Phi(B)|^{\frac{1}{2}} = 1 = \Delta(\Phi(A))$.

COROLLARY 4.4.6. *If $\mathfrak{B}$ is abelian or if $\mathfrak{A}$ arises from a linearly ordered family of projections as in* 3.1, *then Jensen's inequality is valid.*

In particular, if $\mathfrak{B}$ is a finite factor and $\mathfrak{A}$ is a hyperreducible maximal triangular subalgebra of $\mathfrak{B}$, the same conclusion holds.

Proof. Suppose first that $\mathfrak{B}$ is abelian, and take $A, B \in \mathfrak{A} \cap \mathfrak{A}^{-1}$ such that $A^*A = BB^* = B^*B$. By remark 4.4.5, $\Phi(A)^*\Phi(A) = \Phi(B)^*\Phi(B)$; in particular, $\Delta(\Phi(A))^2 = \Delta(\Phi(B))^2$, and the result follows by 4.4.4 (ii).

Assume $\mathfrak{A}$ arises from an abelian linearly ordered family of projections as in 3.1. By 4.4.4 (i), it suffices to show that $\Delta(I+T) \geqq 1$, for every T in $\mathfrak{J} = \{A \in \mathfrak{A} : \Phi(A) = 0\}$. Suppose, first, that T is quasi nilpotent. Put

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} T^n.$$

The series converges uniformly because $\| T^n \|^{1/n} \to 0$, and we have $S \in \mathfrak{A}$ and $\Phi(S) = 0$. If

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n, \quad |z| < 1,$$

then $e^{f(z)} = 1 + z$, so by standard operational calculus, $e^S = I + T$. Therefore, $\Delta(I+T) = \Delta(e^S) = |\exp \phi(S)| = |\exp \phi \circ \Phi(S)| = 1$. Let $\mathcal{S}$ be the set of all $T \in \mathfrak{J}$ such that $\Delta(I+T) \geqq 1$. $\mathcal{S}$ is ultrastrongly closed, by 4.3.3 (iii), and we have just seen that $\mathcal{S}$ contains all quasi nilpotent elements of $\mathfrak{J}$. But the nilpotent elements of $\mathfrak{J}$ are ultrastrongly dense in $\mathfrak{J}$, by 3.1.1, so that $\mathcal{S} = \mathfrak{J}$, as required.

By remark 3.1.3, the last situation includes the case where $\mathfrak{B}$ is a finite factor and $\mathfrak{A}$ is a hyperreducible maximal triangular subalgebra of $\mathfrak{B}$.

5. Maximality and Jensen's inequality for special cases based on ordered groups. This chapter contains proofs of Jensen's inequality and the maximality conjecture (remark 2.2.3) for examples 1, 4, and 6. The reader should be warned that, while these results are important for the theory, our methods for establishing them are somewhat elliptical. To sum up, we first consider example 4, starting with a *hyperfinite* factor M and an ordered abelian group G, and represent that subdiagonal algebra as an algebra of M-valued functions on the character group of G. Making use of hyperfiniteness, we obtain the desired results as a relatively straightforward generalization of known theorems of Helson-Lowdenslager-Wiener on matrix-valued analytic functions. Significantly, the proof of Jensen's inequality, in the matrix-value case, makes an essential reference to the *complex* determinant function det for $n \times n$ matrices. The results are obtained for examples 1 and 6 by producing an algebraic (non-spatial) imbedding of their respective subdiagonal algebras into the above algebra of M-valued functions, in such a way that the desired conclusions follow from the analysis of example 4.

Everything depends, therefore, on an argument involving the function det; for this reason, there is little hope that the arguments here will cast light on the problem of Jensen's inequality in general finite subdiagonal algebras (see [4], Theorem 4).

5.1. A lemma. Let $\mathcal{B}$ be a von Neumann algebra, $\mathcal{D}$ a von Neumann subalgebra, $x \in G \to U_x \in \mathcal{B}$ a faithful unitary representation of the ordered group G in $\mathcal{B}$ satisfying $U_x \mathcal{D} U_x^{-1} = \mathcal{D}$ for all x, and let Φ be a faithful normal expectation of $\mathcal{B}$ on $\mathcal{D}$ such that $\Phi(U_x) = 0$, $x \neq e$. Assume, further, that $\mathcal{D} \cup \{U_x \colon x \in G\}$ generates $\mathcal{B}$.

Definition 5.1.1. *By a near identity we mean a net* ψ_n *of linear mappings of* $\mathcal{B}$ *into itself such that, for every* $T \in \mathcal{B}$,

(i) $\sup_n \| \psi_n(T) \| < \infty$

(ii) $\psi_n(T)$ *is a finite linear combination of elements of the form* $\Phi(TU_x^*)U_x$, $x \in G$

(iii) $\psi_n(T) \to T$ *weakly, as* $n \to \infty$.

LEMMA 5.1.2. *Suppose a near identity exists, and let* $\mathcal{A}_0$ *be the algebra generated by* $\mathcal{D}$ *and* $\{U_x \colon x \geqq e\}$. *Then* $\mathcal{A}_0$ *is subdiagonal with respect to* Φ, *and the ultrastrong closure of* $\mathcal{A}_0$ *is maximal subdiagonal.*

Proof. $\mathcal{A}_0$ is the set of finite sums $A_1 U_{x_1} + \cdots + A_n X_{x_n}$, $A_i \in \mathcal{D}$, $x_i \geqq e$ in G, and it is clear that $\mathcal{A}_0 + \mathcal{A}_0^*$ is a *-subalgebra of $\mathcal{B}$ (all finite sums $\sum A_x U_x$ with no restrictions on x) which is weakly, and therefore ultraweakly dense in $\mathcal{B}$. From $\Phi(AU_x) = A\Phi(U_x) = A$ or 0 according as $x = e$ or $x \neq e$, it follows easily that Φ is multiplicative on $\mathcal{A}_0$. Thus $\mathcal{A}_0$ is a subdiagonal algebra.

Let $\mathfrak{J}$ be the ideal of all elements of $\mathfrak{A}_0$ on which Φ vanishes. By 2.2.1, the algebra $\mathfrak{A}$ of all $X \in \mathfrak{B}$ for which $\Phi(\mathfrak{A}_0 X \mathfrak{J}) = \Phi(\mathfrak{J} X \mathfrak{A}_0) = 0$ is maximal subdiagonal, and contains $\mathfrak{A}_0^-$. We will prove that $\mathfrak{A} \subseteq \mathfrak{A}_0^-$. Let $X \in \mathfrak{A}$. Choose a near identity ψ_n. For each n, $\psi_n(X)$ is a linear combination of operators of the form $\Phi(XU_x^*)U_x$, $x \in G$. If $x < e$, $U_x^* = U_{x^{-1}} \in \mathfrak{J}$, and therefore $\Phi(XU_x^*)U_x = 0$, by definition of $\mathfrak{A}$. This proves that $\psi_n(X) \in \mathfrak{A}_0$ for each n. Thus X is a weak limit of a bounded subset of $\mathfrak{A}_0$, hence X belongs to the ultraweak closure of $\mathfrak{A}_0$. But $\mathfrak{A}_0$ is convex, so that its ultraweak and ultrastrong closures are the same, and the proof is complete.

5.2. Preliminary constructions. Let $\mathfrak{H}_0$ be a separable Hilbert space and let Γ be a compact abelian group with normalized Haar measure dm. A function $F\colon \Gamma \rightarrow \mathfrak{H}_0$ is called *measurable* if $(F(\gamma), \eta)$ is Borel-measurable for every $\eta \in \mathfrak{H}_0$. The measurable functions form a vector space with respect to the pointwise operations, and for every $\gamma \in \Gamma$, one has $\| F(\gamma) \| = \sup |(F(\gamma), \eta)|$, η ranging over a denumerable dense subset of the unit ball in $\mathfrak{H}_0$. Hence $\gamma \rightarrow \| F(\gamma) \|$ is measurable, and by polarization, $\gamma \rightarrow (F(\gamma), G(\gamma))$ is measurable whenever F and G are. $L^2(\Gamma, \mathfrak{H}_0)$ is the set of all measurable F for which

$$\| F \|^2 = \int \| F(\gamma) \|^2 \, dm(\gamma) < +\infty .$$

We make the traditional convention of identifying functions that coincide almost everywhere. $L^2(\Gamma, \mathfrak{H}_0)$ is well known to be a Hilbert space with respect to the above norm, and one has $(F, G) = \int (F(\gamma), G(\gamma)) \, dm(\gamma)$.

Let M be a von Neumann algebra acting on $\mathfrak{H}_0$. A function T from Γ to M is called *measurable* if $(T(\gamma)\xi, \eta)$ is measurable for all $\xi, \eta \in \mathfrak{H}_0$. If T is measurable, it follows that $\gamma \rightarrow (T(\gamma)F(\gamma), G(\gamma))$ is measurable for every $F, G \in L^2(\Gamma, \mathfrak{H}_0)$, and the M-valued measurable functions form an algebra with involution with respect to the usual pointwise operations. Another application of separability shows that $\| \Gamma(\gamma) \|$ is a real-valued measurable function when T is an M-valued measurable function. $L^\infty(\Gamma, M)$ will denote the set of all M-valued measurable functions for which $\| T \| = \operatorname{essup} \| \Gamma(\gamma) \|$ is finite (again, identifying functions agreeing almost everywhere dm). $L^\infty(\Gamma, M)$ is a B^*-algebra under the pointwise operations. In fact, putting

$$(L_T F)(\gamma) = T(\gamma) F(\gamma), T \in L^\infty(\Gamma, M), F \in L^2(\Gamma, \mathfrak{H}_0),$$

one has that $T \rightarrow L_T$ is an isometric $*$-isomorphism of $L^\infty(\Gamma, M)$ onto a von Neumann algebra $\mathfrak{B}$ acting on $L^2(\Gamma, \mathfrak{H}_0)$ ([1], p. 160, prop. 2, and p. 180, prop. 1).

Let G be the discrete group having Γ as its character group. For $x \in G$, $\gamma \in \Gamma$, write $[x,\gamma] = \gamma(x)$, as usual. One may define "Fourier" coefficients of an element $F \in L^2(\Gamma, \mathfrak{H}_0)$ as follows. For x fixed in G, the map

$$\xi \in \mathfrak{H}_0 \rightarrow \int (F(\gamma), \xi) \overline{[x,\gamma]}\, dm(\gamma)$$

is a bounded linear functional on $\mathfrak{H}_0$. There exists, therefore, an element F_x of $\mathfrak{H}_0$ such that

$$(F_x, \xi) = \int (F(\gamma), \xi) \overline{[x,\gamma]} dm(\gamma), \qquad \xi \in \mathfrak{H}_0.$$

The mapping $Q_x \colon F \rightarrow [x, \cdot] F_x$ is easily seen to be a self-adjoint projection in $L^2(\Gamma, \mathfrak{H}_0)$, and by a standard argument, $Q_x \perp Q_y$ when $x \neq y$ and $\sum Q_x = I$.

In a similar way, we can define Fourier coefficients of an element $T \in L^\infty(\Gamma, M)$. For $x \in G$,

$$\langle \xi, \eta \rangle = \int (T(\gamma)\xi, \eta) \overline{[x,\gamma]} dm(\gamma)$$

defines a bilinear form on $\mathfrak{H}_0$ such that $|\langle \xi, \eta \rangle| \leqq \| T \| \cdot \| \xi \| \cdot \| \eta \|$. By the well-known Riesz lema, there is an operator T_x on $\mathfrak{H}_0$, of norm not exceeding $\| T \|$, for which $\langle \xi, \eta \rangle = (T_x \xi, \eta)$. Occasionally, we shall write

$$T_x = \int T(\gamma) \overline{[x,\gamma]} dm(\gamma).$$

A standard calculation shows that T_x commutes with everything that commutes with $\{T(\gamma) : \gamma \in \Gamma\}$; therefore $T_x \in M$.

Recall that an approximate identity on Γ is a net f_n of continuous functions such that

(i) $f_n \geqq 0$

5.2.1. (ii) $\int f_n \, dm = 1$

(iii) $\lim_{n \rightarrow \infty} \sup_{\gamma \notin U} | f_n(\gamma) | = 0$, for every neighborhood U of the identity.

Approximate identities always exist. In fact, because every continuous function can be uniformly approximated by trigonometric polynomials, we may choose an approximate identity consisting entirely of these.

For $\omega \in \Gamma$, $F \in L^2(\Gamma, \mathfrak{H}_0)$, let

$$F_\omega(\gamma) = F(\gamma\omega).$$

LEMMA 5.2.2. (i) *Finite linear combinations of functions of the form* $[x, \cdot]\xi$, $x \in G$, $\xi \in \mathfrak{H}_0$, *are dense in* $L^2(\Gamma, \mathfrak{H}_0)$.

(ii) $F \to F_\omega$ *defines a weakly (therefore strongly) continuous unitary representation of* Γ *in* $L^2(\Gamma, \mathfrak{H}_0)$.

(iii) *Finite linear combinations of operators of the form* $L_{[x,\cdot]T}$, $x \in G$, $T \in M$, *are weakly dense in* $\mathfrak{B}$.

(iv) *For every* $T \in L^\infty(\Gamma, M)$, $\sum Q_x L_T Q_x$ *is multiplication by the constant operator function* $\int T(\gamma)\, dm(\gamma)$.

Proof. (i) and (ii) are well known; briefly, if $F \in L^2(\Gamma, \mathfrak{H}_0)$ is orthogonal to the linear span of the $[x, \cdot]\xi$, then all Fourier coefficients F_x of F are 0, therefore $(F(\gamma), \xi) \equiv 0$ for every $\xi \in \mathfrak{H}_0$, therefore $F = 0$. For (ii), $(F_{\omega_1})_{\omega_2} = F_{\omega_1\omega_2}$ and $\| F_\omega \| = \| F \|$ are obvious. By (i), continuity will follow if $(F_\omega, G) \to (F, G)$, as $\omega \to e$, whenever F has the form $[x, \cdot]\xi$; and for this F the assertion is clear.

(iii): Let $T \in L^\infty(\Gamma, M)$. We shall exhibit a net T_n, where each T_n is a linear combination of things of the form $[x, \cdot]T_x$, and such that $L_{T_n} \to L_T$ weakly. Choose an approximate identity p_n consisting of trigonometric polynomials on Γ: say

$$p_n(\gamma) = \sum_x a_n(x)\,[x, \gamma]$$

each a_n being a complex-valued function on G with finite support. Let $T_n(\gamma) = \sum a_n(x)\,[x,\gamma]\,T_x$.

If $F, G \in L^2(\Gamma, \mathfrak{H}_0)$, then

$$\begin{aligned}
(L_{T_n}F, G) &= \int \sum a_n(x)\,[x,\gamma]\,(T_x F(\gamma), G(\gamma))\, dm(\gamma) \\
&= \int\int \sum a_n(x)\,[x,\gamma]\,\overline{[x,\omega]}\,(T(\omega)F(\gamma), G(\gamma))\, dm(\omega)\, dm(\gamma) \\
&= \int\int p_n(\omega^{-1}\gamma)\,(T(\omega)F(\gamma), G(\gamma))\, dm(\omega)\, dm(\gamma) \\
&= \int\int p_n(\omega)\,(T(\omega^{-1}\gamma)F(\gamma), G(\gamma))\, dm(\gamma)\, dm(\omega) \\
&= \int\int p_n(\omega)\,(T(\gamma)F_\omega(\gamma), G_\omega(\gamma))\, dm(\gamma)\, dm(\omega) \\
&= \int p_n(\omega)\,(L_T F_\omega, G_\omega)\, dm(\omega),
\end{aligned}$$

using the Fubini theorem. Hence,

$$\begin{aligned}
|(L_{T_n - T}F, G)| &= \left| \int p_n(\omega)\,[(L_T F_\omega, G_\omega) - (L_T F, G)]\, dm(\omega) \right| \\
&\leqq \sup_{\omega \in U} |(L_T F_\omega, G_\omega) - (L_T F, G)| \\
&\quad + \sup_{\omega \notin U} p_n(\omega) \cdot 2 \cdot \| T \| \cdot \| F \| \cdot \| G \|,
\end{aligned}$$

for every neighborhood U of the identity. For an appropriate U, the first term will be small, because $\| F_\omega - F \|$ and $\| G_\omega - G \|$ both tend to 0 as ω

tends to the identity. For that U, the second term goes to 0 as $n\to\infty$, by 5.2.1 (iii). This proves that $L_{T_n}\to L_T$ weakly.

(iv) : If $F\in L^2(\Gamma,\mathfrak{H}_0)$ and $T\in L^\infty(\Gamma,M)$, then $Q_xF=[x,\cdot]F_x$. Hence,

$$(Q_xL_TQ_xF)(\gamma)=[x,\gamma]\int T(\omega)(Q_xF)(\omega)\overline{[x,\omega]}dm(\omega)$$
$$=[x,\gamma]\int T(\omega)F_xdm(\omega)=\int T(\omega)dm(\omega)\cdot(Q_xF)(\gamma).$$

So if T_e is the constant operator function $\int T(\omega)dm(\omega)$, we have $\sum Q_xL_TQ_x =\sum L_{T_e}Q_x=L_{T_e}$, as required. The lemmas is proved.

5.3. Discussion of Example 4. In Section 3.2, we described a procedure for constructing a von Neumann algebra $\mathcal{B}$, starting with a II_1 factor M_0 (acting on $\mathfrak{H}_0$) and an ordered abelian group G. The notation of that section is now in force. Let $\Gamma=\hat{G}$. We will show that the $\mathcal{B}$ of that section is unitarily equivalent to the algebra of multiplications by $L^\infty(\Gamma,M_0)$ functions, acting in $L^2(\Gamma,\mathfrak{H}_0)$.

Let F be an $\mathfrak{H}_0$-valued function on G such that $F(x)=0$ except for finitely many x. Define the function $UF\in L^2(\Gamma,\mathfrak{H}_0)$ by

$$(UF)(\gamma)=\sum_x[x,\gamma]F(x).$$

We have

$$\|UF\|^2=\sum_{x,y}(F(x),F(y))\int[x,\gamma]\overline{[y,\gamma]}dm(\gamma)$$
$$=\sum_x(F(x),F(x))=\|F\|^2.$$

Thus U is an isometric linear mapping of a dense subspace of $l_2(G,\mathfrak{H}_0)$ on a dense subspace of $L^2(\Gamma,\mathfrak{H}_0)$ (5.2.2 (i)). We denote its unitary extension by the same letter U. Let $T_0\in M_0$, $x\in G$. For F as above, we have

$$UW_xT_0F=\sum_s[s,\cdot](W_xT_0F)(s)=\sum_s[s,\cdot]T_0F(x^{-1}s)$$
$$=\sum_t[xt,\cdot]T_0F(t)=[x,\cdot]\sum_t[t,\cdot]F(t)$$
$$=L_{[x,\cdot]T_0}UF.$$

Therefore, U implements an isomorphism of $\mathcal{B}$ on the von Neumann algebra generated by finite linear combinations of operators of the form $L_{[x,\cdot]T_0}$, $x\in G$, $T_0\in M_0$. By Lemma 5.2.2 (iii), the latter is precisely the algebra of multiplications by $L^\infty(\Gamma,M_0)$ functions. This proves most of the following.

LEMMA 5.3.1. *The algebra $\mathcal{B}$ of example 4 is unitarily equivalent to the algebra of multiplications by $L^\infty(\Gamma,M_0)$ functions acting in $L^2(\Gamma,\mathfrak{H}_0)$.*

This correspondence associates W_x *with multiplication by the character* $[x,\cdot]$, *the image* M *of* M_0 (*in* $\mathcal{B}$) *with the algebra of multiplications by constant* M_0*-valued functions, the expectation* $\Phi(X)=\sum_x P_xXP_x$ *with the operation taking* L_T *onto multiplication by the constant function* $\int T(\gamma)dm(\gamma)$, *and the trace* ϕ *with the functional* $L_T\to\int \mathrm{Tr}(T(\gamma))dm(\gamma)$, Tr *being the canonical trace on* M_0.

Proof. Everything but the last two phrases has been proved or is obvious. By 5.2.2 (iv), the statement about Φ will follow if we show that $UP_xU^{-1}=Q_x$. Let F be an $\mathfrak{H}_0$-valued function on G having finite support. Then $(P_x)(y)=\delta_{x,y}F(y)$ has finite support, and

$$UP_xF=\sum_y [y,\cdot]\delta_{x,y}F(y)=[x,\cdot]F(x)$$

$$=[x,\cdot]\int \overline{[x,\gamma]}(UF)(\gamma)dm(\gamma)=Q_xUF,$$

by orthogonality of the characters. Hence $UP_x=Q_xU$, by 5.2.2 (i). The proof of the statement about traces is similar.

We now make the identifications described in Lemma 5.3.1: $\mathcal{B}$ is the algebra of $L^\infty(\Gamma,M_0)$ multiplications,

$$\Phi(L_T)=\int T(\gamma)dm(\gamma),\quad \phi(L_T)=\int \mathrm{Tr}(T(\gamma))dm(\gamma),$$

and $\mathfrak{A}_0$ is the algebra generated by the operators $W_x=L_{[x,\cdot]I}$, $x\geqq e$, and L_T, $T\in M_0$. $\mathfrak{A}$ will be the corresponding maximal subdiagonal algebra.

Our first results concern maximality of the closure of $\mathfrak{A}_0$.

THEOREM 5.3.2. *$\mathfrak{A}$ is the ultrastrong closure of $\mathfrak{A}_0$.*

Proof. The quadruple $(\mathcal{B},\mathfrak{A}_0,\Phi,\{W_x\})$ satisfies all the conditions preceding Lemma 5.1.2, so it suffices to produce a near identity ψ_n. Let p_n be an approximate identity on Γ consisting of trigonometric polynomials. If

$$p_n(\gamma)=\sum_x a_n(x)[x,\gamma],$$

each a_n being a complex function on G having finite support, set

$$\psi_n(L_T)=\sum a_n(x)L_{[x,\cdot]T_x},$$

where $T_x=\int T(\gamma)\overline{[x,\gamma]}dm(\gamma)$. In the proof of Lemma 5.2.2 (iii), it was shown that $\psi_n(T)\to T$ weakly, as $n\to\infty$. So we need only check conditions (i) and (ii) of definition 5.1.1.

Now $\Phi(L_T W_x^*) = \Phi(L_{T[\overline{x,\cdot}]}) = L_{T_x}$, so that

$$L_{[x,\cdot]T_x} = L_{[x,\cdot]} L_{T_x} = W_x \Phi(L_T W_x^*).$$

Thus ψ_n satisfies condition (ii). For (i), let $F, G \in L^2(\Gamma, \mathfrak{H}_0)$. Then

$$\begin{aligned}(\psi_n(L_T)F, G) &= \int \sum_x a_n(x)[x,\gamma](T_x(\gamma)F(\gamma), G(\gamma))\, dm(\gamma) \\ &= \int\int \sum_x a_n(x)[x,\gamma](T(\omega)[x,\omega]F(\gamma), G(\gamma))\, dm(\omega)\, dm(\gamma) \\ &= \int\int p_n(\omega^{-1}\gamma)(T(\omega)F(\gamma), G(\gamma))\, dm(\omega)\, dm(\gamma).\end{aligned}$$

For every $\gamma \in \Gamma$,

$$\begin{aligned}&\left| \int p_n(\omega^{-1}\gamma)(T(\omega)F(\gamma), G(\gamma))\, dm(\omega) \right| \\ &\quad \leqq \operatorname*{esssup}_{\omega} |(T(\omega)F(\gamma), G(\gamma))| \cdot \int |p_n| \leqq \|T\| \cdot \|F(\gamma)\| \cdot \|G(\gamma)\|.\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}|(\psi_n(L_T)F, G)| &\leqq \|T\| \int \|F(\gamma)\| \cdot \|G(\gamma)\|\, dm(\gamma) \\ &\leqq \|T\| \cdot \|F\| \cdot \|G\|.\end{aligned}$$

Hence, $\|\psi_n(L_T)\| \leqq \|T\|$ for every n. This completes the proof.

We turn, now, to the proof of Jensen's inequality. Let us suppress the subscript on M_0 and write it as M; the nought has long since lost its usefulness. We shall first assume M to be of type I_n, $n < \infty$, so that $\mathfrak{B}$ is an algebra of bounded measurable matrix-valued functions on Γ. This result is contained in the work of Helson and Lowdenslager ([6], pp. 191-192; also, see [15]). For completeness, and because on this case rests everything that is to follow, we represent the proof.

Realize M as the ring of all $n \times n$ matrices over the complex numbers, and let Tr be the trace on M normalized so that it is 1 at the identity. Elements of $L^\infty(\Gamma, M)$ take the form

$$T(\gamma) = (f_{ij}(\gamma)),$$

where each f_{ij} is a bounded complex-valued measurable function on Γ. For $(a_{ij}) \in M$, let $\det(a_{ij})$ be the usual (complex-valued) algebraic determinant. We shall require the following two facts.

LEMMA 5.3.3 (i) *For every $T \in M$,*

$$\operatorname{Tr}(\log |T|) = \log |\det T|^{1/n},$$

the left side taken as $-\infty$ if T is singular.

(ii) *For any trigonometric polynomial of the form*

$$p(\gamma) = a_1[x_1, \gamma] + \cdots + a_n[x_n, \gamma], \; x_i \geqq e,$$

a_i *complex numbers, one has*

$$\int \log |p(\gamma)| \, dm(\gamma) \geqq \log \left| \int p(\gamma) \, dm(\gamma) \right|,$$

$\log 0$ *being taken as* $-\infty$.

Proof. (i) If T is singular, the formula clearly holds. If T is nonsingular, say $T = UH$, where U is unitary and H is positive and nonsingular, then $|\det T| = |\det U \det H| = \det H$. So it suffices to prove (i) for T regular and positive. Let $\tau_1, \cdots, \tau_n$ be the eigenvalues of T, repeated according to multiplicities. Then $\tau_i > 0$ and $\log \tau_1, \cdots, \log \tau_n$ is the set of eigenvalues of the self-adjoint operator $\log T$. Hence

$$\begin{aligned} \log |\det T|^{1/n} &= \log |\lambda_1 \lambda_2 \cdots \lambda_n|^{1/n} \\ &= n^{-1}(\log \lambda_1 + \cdots + \log \lambda_n) = \mathrm{Tr}(\log T). \end{aligned}$$

(ii) is a known result ([6], Theorem 13, p. 199). It can be deduced from the results of Section 4 by causing the bounded measurable (complex-valued) functions on Γ to act in $L^2(\Gamma)$. Define $\mathfrak{A}_0$ in the obvious way to be the multiplications by trigonometric polynomials of the form in (ii), imbed $\mathfrak{A}_0$ in a maximal subdiagonal algebra with respect to the state defined by the Haar integral, and apply cor. 4.4.6.

LEMMA 5.3.4 (Helson-Lowderslager-Wiener). *If M is a factor of type I_n, $n < \infty$, then Jensen's inequality is valid.*

Proof. By prop. 4.4.4 (i), it suffices to show that $\Delta(L_T) \geqq 1$ for every $L_T \in \mathfrak{A}$ such that $\Phi(L_T) = I$. Suppose, first, that T has the form

$$T(\gamma) = I + \sum_{x > e} T_x[x, \gamma],$$

where all but finitely many T_x are 0.

The map $S \in L^\infty(\Gamma, M) \to L_S \in \mathfrak{B}$ is an isometric * isomorphisms, which preserves the operational calculus. Thus, $\log L_H = L_{\log H}$, for every positive $H \in L^\infty(\Gamma, M)$ such that $\operatorname{esssup}_\gamma \|H^{-1}(\gamma)\| < \infty$. Hence,

$$\begin{aligned} \log \Delta(L_T) &= \inf_{x>0} \log \Delta(|L_T| + \epsilon I) \\ &= \inf_\epsilon \log \Delta(L|_{T+\epsilon I}) = \inf_\epsilon \int \mathrm{Tr}[\log(|T(\gamma)| + \epsilon I)] \, dm(\gamma) \\ &\geqq \int \mathrm{Tr}(\log |T(\gamma)|) \, dm(\gamma), \end{aligned}$$

taking $\operatorname{Tr}(\log|T(\gamma)| = -\infty$ when $T(\gamma)$ is singular. By 5.3.3 (i),

$$\operatorname{Tr}(\log|T(\gamma)|) = n^{-1}\log|\det T(\gamma)|,$$

for every γ. Write $T(\gamma) = (f_{ij}(\gamma))$, where each entry has the form

$$(*) \qquad f_{ij}(\gamma) = \sum_{x} c_{ij}(x)\,[x,\gamma], \qquad 1 \leqq i, j \leqq n,$$

each c_{ij} being 0 except for finitely many $x \geqq e$. Now for any matrix (a_{ij}), $\det(a_{ij}) = p(f_{11}, a_{12}, \cdots, a_{nn})$, p being the familiar polynomial in n^2 variables. Hence,

$$\det T(\gamma) = p(a_{11}(\gamma), f_{12}(\gamma), \cdots, f_{nn}(\gamma)),$$

is itself a trigonometric polynomial of the form $b_1[x_1,\gamma] + \cdots + b_N[x_N,\gamma]$, $x_i \geqq e$. So by 5.3.3 (ii),

$$\int \operatorname{Tr}(\log|T(\gamma)|)\,dm(\gamma) = n^{-1}\int \log|\det T(\gamma)|\,dm(\gamma)$$
$$\geqq n^{-1}\log\left|\int \det T(\gamma)\,dm(\gamma)\right|.$$

Now, using the fact that the Haar integral is multiplicative on the algebra of trigonometric polynomials like (*), we have

$$\int \det T(\gamma)\,dm(\gamma) = \int p(f_{11}(\gamma), \cdots, f_{nn}(\gamma))\,dm(\gamma)$$
$$= p\left(\int f_{11}dm, \cdots, \int f_{nn}dm\right)$$
$$= \det\left(\int T(\gamma)\,dm(\gamma)\right) = \det I = 1.$$

This proves that $\log \Delta(L_T) \geqq 0$.

Now suppose L_T is an arbitrary element of $\mathfrak{A}$ for which $\Phi(L_T) = I$. By the proof of Theorem 5.3.2, there exists a net L_{T_n} in $\mathfrak{A}_0$ such that $L_{T_n} \to L_T$ ultrastrongly. By continuity of Φ, we can assume $\Phi(L_T) = I$ for every n. But the above paragraph shows that $\Delta(L_{T_n}) \geqq 1$ for every n; moreover, the set of all $L_S \in \mathfrak{A}$ for which $\Delta(L_S) \geqq 1$ is ultrastrongly closed (Corollary 4.3.3 (iii)). Therefore $\Delta(L_T) \geqq 1$, completing the proof of the lemma.

We are now ready to prove Jensen's inequality in the case of a hyperfinite M.

THEOREM 5.3.5. *If M is the hyperfinite factor, then Jensen's inequality is valid on $\mathfrak{A}$.*

Proof. By arguing exactly as in the preceding lemma, it suffices by 5.3.2, to show that $\Delta(L_T) \geqq 1$ for every $T \in L^\infty(\Gamma, M)$ of the form

$$T(\gamma) = I + \sum_{x} T_x[x,\gamma],$$

where $T_x \in M$ and $T_x = 0$ except for finitely many $x > e$.

By hyperfiniteness, there exists a sequence M_n of subfactors of M, each of type I_n, $n=1,2,\cdots$, for which $M_n \subseteq M_{n+1}$ and $\cup M_n$ is strongly dense in M. For each n, let $\mathcal{B}_n$ be the set of all multiplications L_S such that $S(\gamma) \in M_n$ for every $\gamma \in \Gamma$. Each $\mathcal{B}_n$ is a von Neumann subalgebra of $\mathcal{B}$, which we can identify with $L^\infty(\Gamma, M_n)$ in the obvious way, and $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$ because of the ordering on the M_n. Note that $\cup \mathcal{B}_n$ is an ultrastrongly (and therefore strongly) dense *-subalgebra of $\mathcal{B}$. For $\cup \mathcal{B}_n$ contains linear combinations of operators of the form $L_A L_{[x,\cdot]}$, with A a constant function in $\cup M_n$ and $x \in G$. Because $\cup M_n$ is ultrastrongly dense in M and the map $A \in M \to L_A$ is ultrastrongly continuous (cf. first paragraph in the dimension of example 4, and 5.3.1), it follows that the ultrastrong slosure of $\cup \mathcal{B}_n$ contains linear combinations of $L_A L_{[x,\cdot]}$, $A \in M$, $x \in G$, and therefore it contains $\mathcal{B}$.

Let E_n be the natural expectation of $\mathcal{B}$ on $\mathcal{B}_n$ (6.1.3, example 3). By the martingale theorem (6.1.9), $E_n(L_T) \to L_T$ boundedly and strongly, as $n \to \infty$. Since Δ is upper semicontinuous in the ultrastrong topology, it suffices to show that $\Delta(E_n(L_T)) \geqq 1$ for every n. Writing $L_T = I + \sum L_{T_x} L_{[x,\cdot]}$, we have

$$E_n(L_T) = I + \sum_{x>e} E_n(L_{T_x} L_{[x,\cdot]})$$

$$= I + \sum_{x>e} E_n(L_{T_x}) L_{[x,\cdot]},$$

since $L_{[x,\cdot]} \in \mathcal{B}_n$ for every n. By Lemma 5.3.4, we will be done if we show that each $E_n(L_{T_x})$ has the form L_R, for some constant function R taking values in M_n. Consider the "Fourier" coefficients of $E_n(L_{T_x})$. For every $y \in G$, $y \neq e$, we have

$$\phi(E_n(L_{T_x}) L^*_{[y,\cdot]}) = \phi(E_n(L_{T_x \overline{[y,\cdot]}}))$$

$$= \phi(L_{T_x \overline{[y,\cdot]}}) = \int \operatorname{Tr}(T_x \overline{[y,\gamma]})\, dm(\gamma)$$

$$= \operatorname{Tr}(T_x) \int \overline{[y,\gamma]}\, dm(\gamma) = 0.$$

Therefore $E_n(L_{T_x}) = L_S$, where $S(\gamma) = S$ is constant a.e. (dm). By definition of $\mathcal{B}_n$, S has to belong to M_n, and that completes the proof.

Question 5.3.6. The proof of this theorem can probably be modified along the obvious lines to prove Jensen's inequality when M is simply a hyperfinite von Neumann algebra. A more interesting question is whether the inequality is valid when M is a finite non-hyperfinite factor.

5.4. Discussion of Example 6. Let N be a II_1 factor acting on a separable Hilbert space $\mathfrak{H}_0$. Let G be a countable discrete abelian group, and suppose there is a unitary representation $x \in G \to U_x \in \mathcal{B}$ of G in N, and a maximal abelian subalgebra M of N, which satisfy the conditions 3.2.4. Let Tr be the canonical normalized trace on N.

Let $\Gamma = \hat{G}$, and let $\mathcal{B}$ be the von Neumann algebra of multiplications by $L^\infty(\Gamma, N)$ functions acting on $L^2(\Gamma, \mathfrak{H}_0)$. Define the trace ϕ and the expectation Φ on $\mathcal{B}$ as before:

$$\phi(L_T) = \int \mathrm{Tr}(T(\gamma))\, dm(\gamma),$$
$$\Phi(L_T) = L_{T_0}, \quad T_0 = \int T(\gamma)\, dm(\gamma),$$

m being Haar measure on Γ. We shall describe a way of algebraically imbedding N in $\mathcal{B}$.

Remark 5.4.1. Let $\mathcal{B}$ and $\mathcal{C}$ be von Neumann algebras, and suppose ϕ (resp. ψ) is a faithful normal finite trace on $\mathcal{B}$ (resp. $\mathcal{C}$). Let $\mathcal{B}_0$ (resp. $\mathcal{C}_0$) be a strongly dense *-subalgebra of $\mathcal{B}$ (resp. $\mathcal{C}$), and suppose θ is a *-isomorphism of $\mathcal{B}_0$ on $\mathcal{C}_0$ which is trace-preserving in the sense that $\psi \circ \theta(X) = \phi(X)$, $X \in \mathcal{B}^*$. Then θ extends uniquely to a trace-preserving *-isomorphism of $\mathcal{B}$ on $\mathcal{C}$. We shall sketch a proof of this known result, since we lack a specific reference. From the Hilbert algebra $(\mathcal{B}_0, \phi)$, that is, the *-algebra $\mathcal{B}_0$ endowed with the bilinear form $[A, B] = \phi(B^*A)$, and let $\mathfrak{H}$ be its completion. The set of (left-) bounded elements of $\mathfrak{H}$ can be identified with $\mathcal{B}$, using ([1], p. 71, Prop. 3 and p. 288, Lemma 1). Therefore, the map taking $B \in \mathcal{B}$ into left multiplication by B, acting on $\mathfrak{H}$, is onto the left ring. It is clear that this mapping is 1-1 and preserves all the algebraic operations; so that $\mathcal{B}$ is *-isomorphic to the left ring of $(\mathcal{B}_0, \phi)$. For $C, D \in \mathcal{C}$, put $\langle C, D\rangle = \psi(D^*C)$. This makes $(\mathcal{C}_0, \psi)$ into a Hilbert algebra, and for every $A, B \in \mathcal{B}_0$,

$$\langle\theta(A), \theta(B)\rangle = \psi(\theta(B)^*\theta(A)) = \psi \circ \theta(B^*A)$$
$$= \phi(B^*A) = [A, B].$$

Thus θ extends uniquely to a unitary mapping V of $\mathfrak{H}$ onto the completion $\mathfrak{K}$ of $(\mathcal{C}_0, \psi)$. V clearly implements a *-isomorphism of the respective left rings, and we have

$$VL_BV^* = L_{\theta(B)}, \quad B \in \mathcal{B}_0.$$

By the above remarks, $V \cdot V^*$ may be transferred down to give the required extension of θ. This extension preserves the trace, and it is clearly uniquely determined by θ.

Now let N_0 be the strongly dense *-subalgebra of N consisting of all sums $\sum_x A(x)U_x$, with $A(x) \in M$ and $A(x) = 0$ except for finitely many $x \in G$. Set $\theta(\sum A(x)U_x) = L_T$, where $T(\gamma) = \sum_x [x, \gamma]A(x)U_x$. Then θ is a *-monomorphism of N_0 into $\mathcal{B}$. Moreover,

$$\phi \circ \theta(\sum A(x)U_x) = \int \mathrm{Tr}(A(x)U_x)[x,\gamma]dm(\gamma) = \mathrm{Tr}(A(e)).$$

On the other hand, since $U_x \cdot U_x^*$ is a freely-acting automorphism of the subalgebra M $(x \neq e)$, one has

$$\mathrm{Tr}(A(x)U_x) = 0, \quad x \neq e$$

([3], p. 572). Therefore $\phi \circ \theta(\sum A(x)U_x) = \mathrm{Tr}(\sum A(x)U_x)$. If $\mathcal{B}_1$ denotes the strong closure of $\theta(N_0)$, then by remark 5.4.1, θ extends uniquely to a trace-preserving *-isomorphism of N onto $\mathcal{B}_1$.

Specializing, now, to the setting of example 6, let G be the discrete ordered multiplicative group of positive rationals. The construction described in 3.3 gives rise to a II_1 factor N, a maximal abelian subalgebra M of N, and a faithful unitary representation $x \in G \to U_x \in M$ all of which satisfy the conditions 3.2.4. It was pointed out in 3.2 that these conditions force N to be hyperfinite ([3], p. 576 and pp. 569-570). Let $\Gamma = \hat{G}$. By the above paragraphs, N is isomorphic to a certain subalgebra $\mathcal{B}_1$ of the algebra of multiplications by $L^\infty(\Gamma, N)$ functions acting in $L^2(\Gamma, \mathfrak{H}_0)$, $\mathfrak{H}_0 = l^2(G) \otimes L^2(\Gamma)$ being the space on which N acts. $\mathcal{B}_1$ is simply the von Neumann algebra generated by multiplications of the form

$$T(\gamma) = \sum_x A(x)U_x[x,\gamma]$$

$A(x) \in M$ and $A(x) = 0$ off a finite subset of G. If E_M denotes the natural expectation of N onto M (6.1.3 example 3), then $E_M(\sum A(x)U_x) = A(e)$, and

$$\Phi \circ \theta(\sum A(x)U_x) = \int \sum A(x)U_x[x,\gamma]dm(\gamma)$$
$$= \sum A(x)U_x \int [x,\gamma]dm(\gamma) = A(e).$$

By continuity, then, $\theta \circ E_M = \Phi \circ \theta$ holds on N. Thus E_M correspond to the restriction Φ_1 of Φ to $\mathcal{B}_1$. In particular, $\Phi(\mathcal{B}_1) = \Phi_1(\mathcal{B}_1) = \theta(M)$ is the algebra of multiplications by constant functions taking values in the abelian algebra M.

The subdiagonal algebra concerning us is easily described as follows. For $x \in G$, let V_x be multiplication by the function $T(\gamma) = U_x[x, \gamma]$; that is, $V_x = \theta(U_x)$. Let $\mathfrak{A}_{10}$ be the algebra consisting of the operators $\sum A(x) V_x$, $A(x) \in M$, $A(x) = 0$ except for a finite number of $x \geqq e$. $\mathfrak{A}_{10}$ is the image under θ of the subdiagonal algebra of example 6; therefore $\mathfrak{A}_{10}$ is a subdiagonal subalgebra of $\mathfrak{B}_1$, with respect to Φ_1, and $\mathfrak{A}_{10} \cap \mathfrak{A}_{10}^* = \theta(M)$. The maximal subdiagonal algebra $\mathfrak{A}_1$ determined by $\mathfrak{A}_{10}$ is, by 3. 3. 2 (ii), simply the set of $X \in \mathfrak{B}_1$ for which $\Phi(XV_x) = 0$, $x > e$.

THEOREM 5. 4. 2. (i) *$\mathfrak{A}_1$ is the ultraweak closure of $\mathfrak{A}_{10}$.* (ii) *Jensen's inequality is valid on $\mathfrak{A}_1$.*

Proof. (i). First, note that

$$V_x\theta(M) V_x^{-1} = \theta(U_x)\theta(M)\theta(U_x^{-1}) = \theta(U_x M U_x^{-1}) = \theta(M).$$

Thus $\mathfrak{A}_{10}$ will satisfy the hypotheses of Lemma 5. 1. 2 if we show that a near identity exists for $\mathfrak{B}_1$. Let ψ_n be a near identity for $\mathfrak{B}$; that is, ψ_n is a net satisfying conditions (i) and (iii) of Definition 5. 1. 1 together with the requirement that $\psi_n(X)$ be a finite linear combination of operators of the form $\Phi(XL_{[x,\cdot]}^*)L_{[x,\cdot]}$. We exhibited such a net in the proof of 5. 3. 2. Let ψ_{1n} be the restriction of ψ_n to $\mathfrak{B}_1$. For $X \in \mathfrak{B}_1$, $x \in G$, we have

$$\begin{aligned} \Phi(XL_{[x,\cdot]}^*) &= \Phi(XL_{[x,\cdot]}^* U_x^* U_x) \\ &= \Phi(XV_x^*) U_x = \Phi_1(XV_x^*) U_x, \end{aligned}$$

since $XV_x^* \in \mathfrak{B}_1$. Therefore, $\Phi(XL_{[x,\cdot]}^*)L_{[x,\cdot]} = \Phi)XV_x^*) V_x$. This proves that ψ_{1n} satisfies condition (ii) of 5. 1. 1 with respect to $\theta(M)$ and the group $\{V_x\}$. Conditions (i) and (iii) are, of course, as true for the restrictions as they are for the original net. Thus ψ_{1n} is a near identity for $\mathfrak{B}_1$, as required.

(ii). For this, there is almost nothing to prove. For we have $\mathfrak{A}_1$ imbedded as a subalgebra of the maximal subdiagonal algebra of Section 5. 3. ϕ_1 and Φ_1 behave in the correct way, that is, they are the restrictions of the corresponding objects on $\mathfrak{B}$. As we pointed out in the paragraphs preceding 5. 4. 2, N is hyperfinite. Thus Jensen's inequality follows from Theorem 5. 3. 5.

5. 5. Discussion of Example 1. Let G be the ordered discrete matrix group of example 1.

Remark 5.5.1. Let H (resp. K) be the subgroup of all $(a_{ij}) \in G$ for which $a_{11} = 1$ (resp. $a_{12} = 0$). Then H and K are abelian subgroups, H is normal, and every element of G can be written uniquely as a product hk, $h \in H$, $k \in K$; that is, G is the semidirect product $H \circledS K$.

Let $\mathcal{B}$ be the von Neumann algebra generated by the left regular representation $x \to l_x$, $(l_x f)(y) = f(x^{-1}y)$, $f \in l^2(G)$. As we have seen (3.2.4 et seq.), $\mathcal{B}$ is a hyperfinite factor, and the trace ϕ on $\mathcal{B}$ satisfies $\phi(l_x) = 0$, $x \neq e$. Normalize ϕ so that $\phi(I) = 1$. We shall first show that $\mathcal{B}$ is isomorphic to the factor arising in example 6 (the author is indebted to Prof. H. A. Dye for pointing this out to him). The desired results will then be easy consequences of this identification.

Let $\mathcal{B}_0$ (resp. M_0) be the *-algebra of linear combinations of l_x, with $x \in G$ (resp. $x \in H$). By remark 5.5.1, the finite sum appearing in $\mathcal{B}_0$ can be rearranged so that they have the form

$$\sum_{k \in K} A(k) l_k, \quad A(k) \in M_0,$$

and $A(k) = 0$ except for finitely many $k \in K$. This representation is plainly unique. Recall that, in example 6, one constructs the von Neumann algebra generated by certain multiplications by bounded measurable functions on $\hat{H}$, together with certain unitary operators U_k, $k \in K$, all of which act on the Hilbert space $L^2(K \times \hat{H}) = l^2(K) \otimes L^2(\hat{H})$. We map $\mathcal{B}_0$ into this algebra as follows. For

$$A = \sum_{h \in H} a(h) l_h \in M_0,$$

$a(h)$ being a complex function on H of finite support, let $\theta(A)$ denote the operator L_f, where

$$f(\gamma) = \sum_h a(h)[h, \gamma], \qquad \gamma \in \hat{H}.$$

Extend θ to $\mathcal{B}_0$ by the formula

$$\theta(\sum_{k \in K} A(k) l_k) = \sum_k \theta(A(k)) U_k.$$

θ is a *-homomorphism of $\mathcal{B}_0$ onto a strongly dense *-subalgebra of the von Neumann algebra of example 6. Moreover, if Tr denotes the canonical (normalized) trace on the latter, and if f is a bounded measurable function on $\hat{H}$, then $\mathrm{Tr}(L_f U_x) = 0$ or $\int f\, dm$ according as $x \neq e$ or $x = e$, dm denoting Haar measure on $\hat{H}$ (cf. discussion of example 6). Thus $\mathrm{Tr} \circ \theta(\sum A(k) l_k) = \mathrm{Tr} \circ \theta(A(e))$, so that if $A(e) = \sum_h a(h) l_h$ $(h \in H)$, then

$$\mathrm{Tr}\circ\theta(A(e)) = \int_H \sum a(h)[h,\gamma]dm(\gamma) = a(e)$$
$$= \phi(\sum_{k\in K} A(k)l_k).$$

Thus $\mathrm{Tr}\circ\theta = \phi$. By remark 5.4.1, θ extends uniquely to a *-isomorphism of $\mathcal{B}$ onto the von Neumann algebra of example 6.

Remark 5.5.2. θ has the important property that $\theta(l_k) = U_k$, $k\in K$. Thus, the subdiagonal algebra $\mathfrak{A}_{10}$ studied in example 6 can be realized in $\mathcal{B}$ as the algebra consisting of all finite sums $\sum A(k)l_k$, $A(k)\in M$ and $A(k)=0$ except for finitely many $k\in K$. The mapping Φ is here simply the natural expectation E_M of $\mathcal{B}$ on M, and the maximal subdiagonal algebra $\mathfrak{A}_1$ determined by $\mathfrak{A}_{10}$ is the set of all operators $X\in\mathcal{B}$ for which $E_M(Xl_k)=0$, $k\in K$, $k>e$.

The algebra at the center of this discussion is the set $\mathfrak{A}_0$ of finite sums of l_x, $x\geqq e$ in G. $\mathfrak{A}_0$ is subdiagonal with respect to ϕ (more precisely, with respect to the expectation on the scalars, $X\in\mathcal{B}\to\phi(X)I$), and the maximal subdiagonal algebra $\mathfrak{A}$ determined by $\mathfrak{A}_0$ is the set of all $x\in\mathcal{B}$ for which $\phi(Xl_x)=0$, $x\in G$, $x>e$.

Remark 5.5.3. It is clear that $\mathfrak{A}_0\subseteq\mathfrak{A}_{10}$. We claim that $\mathfrak{A}\subseteq\mathfrak{A}_1$. For this, we have to show that if $A\in\mathcal{B}$ and $\phi(Al_x)=0$, $x\in G$, $x>e$, then $E_M(Al_k)=0$, $k\in K$, $k>e$. For that, it will suffice to prove, for every $T\in M$, that $\phi(E_M(Al_k)T)=\phi\circ E_M(Al_kT)=\phi(Al_kT)=0$ (i.e., simply take $T=E_M(Al_k)^*$). First, suppose $T=l_h$, $h\in H$, say

$$k=\begin{bmatrix} a & 0\\ 0 & 1\end{bmatrix},\qquad h=\begin{bmatrix} 1 & s\\ 0 & 1\end{bmatrix},$$

a and s rational, $a>1$. Then

$$kh=\begin{bmatrix} a & as\\ 0 & 1\end{bmatrix},$$

so that $kh>e$ by definition of the order on G. Hence, $\phi(Al_kT)=\phi(Al_{kh})=0$. By taking finite linear combinations of l_h, we have $\phi(Al_kT)=0$ for $T\in M_0$, and since M_0 is ultraweakly dense in M, we have $\phi(Al_kM)=0$, as required.

LEMMA 5.5.4. (i) *$\mathfrak{A}_0\cap M$ is a subdiagonal subalgebra of M, with respect to ϕ.*

(ii) *If $\mathfrak{A}_M$ is the corresponding maximal subdiagonal subalgebra of M, then $\mathfrak{A}_M$, $\mathfrak{A}\cap M$, and the ultrastrong closure of $\mathfrak{A}_0\cap M$ all coincide.*

(iii) *E_M is a homomorphism of $\mathfrak{A}$ on $\mathfrak{A} \cap M$.*

(iv) *Jensen's inequality is valid on $\mathfrak{A} \cap M$.*

Proof. (i). ϕ is obviously multiplicative on $\mathfrak{A}_0 \cap M$, and $\mathfrak{A}_0 \cap M + (\mathfrak{A}_0 \cap M)^*$ contains M_0, an ultraweakly dense *-subalgebra of M.

(ii). ϕ is multiplicative on the ultrastrongly closed algebra $\mathfrak{A} \cap M$, so the inclusions $(\mathfrak{A}_0 \cap M)^- \subseteq \mathfrak{A} \cap M \subseteq \mathfrak{A}_M$ are obvious. We will show that $\mathfrak{A}_M \subseteq (\mathfrak{A}_0 \cap M)^-$. $\mathfrak{A}_0 \cap M$ being a convex set, its ultrastrong and ultraweak closures coincide; therefore $(\mathfrak{A}_0 \cap M)^-$ contains the ultraweak closure of all finite linear combinations of the l_h, $h \geqq e$ in H. Let $T \in \mathfrak{A}_M$. Then for every $h \in H$, $h > e$, $\phi(T_{l_h}) = \phi(T)\phi(l_h) = 0$. (ii) will follow, then, if we show that every operator S in M can be ultraweakly approximated by finite linear combinations of things of the form $\phi(Sl_h^{-1})l_h$, $h \in H$.

Because H is abelian, we can construct a near identity in M along the same lines as in the proof of 5.3.2. Very briefly, one first maps M_0 into the von Neumann algebra $\mathfrak{L}$ generated by the regular representation of H in $l^2(H)$; this map θ takes l_h into translation by h^{-1}. θ is a *-homomorphism of M_0 onto a weakly dense *-subalgebra of $\mathfrak{L}$, and if $\xi \in l^2(H)$ is the characteristic function of the identity in H, then $(\theta(T)\xi, \xi) = \phi(T)$, $T \in M_0$. By Remark 5.4.1, θ extends to a *-isomorphism of M on $\mathfrak{L}$. So we can identify M with $\mathfrak{L}$, and l_h with translation by h^{-1}. Going now to the compact abelian group $\hat{H}$, choose an approximate identity p_n consisting of trigonometric polynomials. Making use of the Plancherel theorem, the net p_n determines a near identity ψ_n in $\mathfrak{L}$, and therefore in M, and the result follows from the fact that $\psi_n(T) \to T$ ultraweakly.

(iii) E_M is multiplicative on $\mathfrak{A}_1$, and, by Remark 5.5.3, $\mathfrak{A} \subseteq \mathfrak{A}_1$. Therefore, E_M is multiplicative on $\mathfrak{A}$. Moreover, E_M leaves the elements of $\mathfrak{A} \cap M \subseteq M$ fixed. So we need only prove that $E_M(\mathfrak{A}) \subseteq \mathfrak{A}$. Let $A \in \mathfrak{A}$, and take $T \in \mathfrak{A} \cap M$ such that $\phi(T) = 0$. Then

$$\phi(TE_M(A)) = \phi \circ E_M(TA) = \phi(TA) = \phi(T)\phi(A) =)0.$$

Since $\mathfrak{A} \cap M$ is maximal subdiagonal (in M), we have $E_M(A) \in \mathfrak{A} \cap M \subseteq \mathfrak{A}$.

(iv) is immediate from Corollary 4.4.6.

THEOREM 5.5.5. (i). *$\mathfrak{A}$ is the ultrastrong closure of $\mathfrak{A}_0$.* (ii). *Jensen's inequality is valid on $\mathfrak{A}$.*

Proof. (i) It suffices to show that $\mathfrak{A} \subseteq \mathfrak{A}_0^-$. Take $A \in \mathfrak{A}$. By Remark 5.5.2 and the proof of 5.4.2 (i), A belongs to the ultrastrong closure of finite linear combinations of operators $E_M(Al_k^{-1})l_k$, with $k \in K$, $k \geqq e$. If $k = e$, then

$$E_M(Al_k^{-1})l_k = E_M(A) \in A \cap M \subseteq (\mathfrak{A}_0 \cap M)^- \subseteq \mathfrak{A}_0^-.$$

by 5.5.4 (ii) and (iii). So we need only prove that $Ml_k \subseteq \mathfrak{A}_0^-$, for every $k \in K$, $k > e$. If $h \in H$ and $k \in K$, $k > e$, then $hk = k(k^{-1}hk) > e$, as in Remark 5.5.3. Thus $l_h l_k = l_{hk} \in \mathfrak{A}_0$. Taking finite sums of $l_h l_k$ with h ranging over H, we obtain $M_0 l_k \subseteq \mathfrak{A}_0$. Since M_0 is ultrastrongly dense in M and since the map $T \in M \to Tl_k = l_k(l_k^{-1}Tl_k)$ is ultrastrongly continuous, the result follows.

(ii). Let $A \in \mathfrak{A}$. We have to prove that $\Delta(A) \geqq \Delta(\phi(A)I) = |\phi(A)|$. By Remark 5.5.2 and Theorem 5.4.2 (ii), one has $\Delta(A) \geqq \Delta(E_M(A))$. By Lemma 5.5.4 (iii) and (iv),

$$\Delta(E_M(A)) \geqq \Delta(\phi \circ E_M(A)I) = \Delta(\phi(A)I) = |\phi(A)|,$$

and the theorem is proved.

6. Appendix: Expectations. The conditional expectation mapping has been an indispensable technical device in probability theory since the fundamental work of Kolmogorov in 1933. A von Neumann algebra having a faithful normal finite trace, though non-commutative, is known to share many common features with the algebra of (equivalence classes of) bounded random variables of a probability space. In particular, one can define the conditional expectation on a given von Neumann subalgebra in a most natural way; when the subalgebra is the center, for example, this construction yields the center-valued trace ([13], [14]). For a different purpose, von Neumann ([12]) introduced "diagonal processes" relative to a maximal abelian subalgebra, and their properties were later exploited in some non-finite cases by Kadison and Singer ([9]), and by J. Schwartz (Non-Isomorphism of a Pair of Factors of Type III, Communications on Pure and Applied Mathematics, vol. XVI, 1963, pp. 111-120).

To our knowledge, however, there is no published study of conditional expectation in an infinite von Neumann algebra which is appropriate for our purposes. We shall give a brief account of those parts of the subject that bear on the results of this paper. The reader will note some contact between this appendix and [10]; e.g., compare 6.1.8 below with Lemma 2 of [10]. Nevertheless, our methods and emphasis are rather different.

6.1. Some properties of expectations. Let $\mathfrak{B}$ be a von Neumann algebra, and let M be a von Neumann subalgebra. An expectation on M is a positive linear map Φ of $\mathcal{B}$ into M such that $\Phi(I)=I$ and $\Phi(AX)=A\Phi(X)$, $A\in M$, $X\in\mathcal{B}$. (2.1.3). By taking adjoints, it follows immediately that $\Phi(XA)=\Phi(X)A$, $A\in M$, $X\in\mathcal{B}$.

PROPOSITION 6.1.1. *Let Φ be an expectation on M. Then*

(i) $\Phi(X)^*\Phi(X)\leqq\Phi(X^*X)$, $X\in\mathcal{B}$ (Schwarz inequality).

(ii) *M is the set of fixed points of Φ.*

(iii) $\Phi\circ\Phi=\Phi$.

(iv) *Let ρ be any state which preserves Φ $(\rho\circ\Phi=\rho)$, and let $[T]_\rho=\rho(T^*T)^{\frac{1}{2}}$, $T\in\mathcal{B}$. Then for every $X\in\mathcal{B}$,*

$$[X-\Phi(X)]_\rho=\inf[X-T]_\rho,\quad T\in M.$$

In particular, when ρ is faithful, $\Phi(X)$ is that element of M which best approximates X in the norm $[\ ]_\rho$.

(v) *Let ρ be any state which preserves Φ. If ρ is faithful then so is Φ; if ρ is faithful and normal, then Φ is normal.*

Proof. (i) follows by expanding the right side of the inequality

$$0\leqq\Phi[(X-\Phi(X))^*(X-\Phi(X))]$$

to obtain

$$0\leqq\Phi(X^*X)-\Phi(X)^*\Phi(X).$$

Here, one uses $\Phi(\Phi(X)^*X)=\Phi(X)^*\Phi(X)=\Phi(X^*\Phi(X))$ and

$$\Phi[\Phi(X)^*\Phi(X)]=\Phi(X)^*\Phi(X)\Phi(I)=\Phi(X)^*\Phi(X).$$

(ii) If $X=\Phi(X)$, then $X\in\Phi(\mathcal{B})$ so each fixed point belongs to M. If $A\in M$, then $\Phi(A)=\Phi(AI)=A\Phi(I)=A$.

(iii) is immediate from (ii).

(iv) The inequality $[X-\Phi(X)]_\rho\geqq\inf[X-T]_\rho$ is clear, because $\Phi(X)\in M$. If $S\in M$, then

$$\begin{aligned}\rho(S^*(X-\Phi(X)) &= \rho\circ\Phi(S^*)X-\Phi(X)))\\ &= \rho(S^*\Phi(X-\Phi(X)))=0.\end{aligned}$$

So for every $T\in M$,

$$\begin{aligned}[X-T]_\rho{}^2 &= [X-\Phi(X)+\Phi(X)-T]_\rho{}^2\\ &= [X-\Phi(X)]_\rho{}^2+2\operatorname{Re}\rho((X-\Phi(X))^*(\Phi(X)-T))+[\Phi(X)-T]_\rho{}^2\\ &= [X-\Phi(X)]_\rho{}^2+[\Phi(X)-T]_\rho{}^2\geqq[X-\Phi(X)]_\rho{}^2.\end{aligned}$$

The opposite inequality is now immediate.

(v) Suppose ρ is faithful, and $H \in \mathcal{B}$, $H \geqq 0$. If $\Phi(H) = 0$, then $\rho(H) = \rho \circ \Phi(H) = 0$; hence $H = 0$.

If ρ is also normal, let H_α be a bounded set of positive operators in $\mathcal{B}$, directed increasing with respect to the usual operator order. Then $\Phi(H_\alpha)$ is directed increasing, and

$$\text{LUB}\,\Phi(H_\alpha) \leqq \Phi(\text{LUB}\,H_\alpha),$$

because Φ is positive. One has

$$\begin{aligned}\rho(\text{LUB}\,\Phi(H_\alpha)) &= \text{LUB}\,\rho \circ \Phi(H_\alpha) = \text{LUB}\,\rho(H_\alpha)\\ &= \rho(\text{LUB}\,H_\alpha) = \rho \circ \Phi(\text{LUB}\,H_\alpha).\end{aligned}$$

Thus $\rho(\Phi(\text{LUB}\,H_\alpha) - \text{LUB}\,\Phi(H_\alpha)) = 0$, so that $\Phi(\text{LUB}\,H_\alpha) - \text{LUB}\,\Phi(H_\alpha) = 0$, completing the proof.

According to the following result of Tomiyama [13], an expectation on M can be characterized as an idempotent linear mapping of $\mathcal{B}$ on M having norm 1, and which leaves the identity fixed.

THEOREM 6.1.2 (Tomiyama). *Let Φ be a linear map of the von Neumann algebra $\mathcal{B}$ onto a von Neumann subalgebra M such that $\Phi \circ \Phi = \Phi$ and $\| \Phi(X) \| \leqq \| X \|$, $X \in \mathcal{B}$. Then Φ is positive, and $\Phi(AX) = A\Phi(X)$, $A \in M$, $X \in \mathcal{B}$.*

6. 1. 3. Examples.

(1) Let ρ be any state of $\mathcal{B}$, and put $\Phi(X) = \rho(X)I$. Then Φ is an expectation on the algebra of scalar multiplies of the identity. Φ will be faithful or normal if ρ has these properties.

(2) Let $P_1, P_2, \cdots$ be a sequence of mutually orthogonal projections in $\mathcal{B}$ having sum I. Put

$$\Phi(X) = \sum_n P_n X P_n,$$

the sum taken in the strong topology. We claim that Φ is a faithful normal expectation of $\mathcal{B}$ onto $M = \mathcal{B} \cap \{P_n\}'$. It is clear that Φ leaves the identity fixed and is a positive linear mapping of $\mathcal{B}$ into itself. By othogonality, we have $P_m\Phi(X) = P_m{}^2XP_m = P_mXP_m{}^2 = \Phi(X)P_m$, thus $\Phi(X) \in M$. If $A \in M$, then $P_nAXP_n = AP_nXP_n$ for every n, so that $\Phi(AX) = A\Phi(X)$. Hence Φ is an expectation on M. If $\Phi(X^*X) = 0$, then for $n = 1, 2, \cdots$, one has

$$(XP_n)^*XP_n = P_nX^*XP_n = P_n(\sum_m P_mX^*XP_m)P_n$$
$$= P_n\Phi(X^*X)P_n = 0,$$

so that $XP_1 = XP_2 = \cdots = 0$. Hence X vanishes on $\sum P_n$, proving that $X = 0$. For normality, let ξ be a unit vector in the underlying Hilbert space. Then

$$(\Phi(X)\xi, \xi) = \sum_n (P_nXP_n\xi, \xi) = \sum_n (X\xi_n, \xi_n),$$

where $\xi_n = P_n\xi$ and $\sum_n \| \xi_n \|^2 = \| \xi \|^2 < \infty$. Thus $\rho(X) = (\Phi(X), \xi, \xi)$ is a normal state, and normality of Φ is immediate from this.

Let $e_1, e_2, \cdots, e_n$ be an orthonormal base for the n-dimensional Hilbert space $\mathfrak{H}$, $n < \infty$, and let P_j be the projection on the subspace determined by e_j. $L(\mathfrak{H})$ can be realized as the ring of complex $n \times n$ matrices, with respect to this basis, and the expectation $\Phi(X) = \sum P_jXP_j$ is the mapping that replaces a matrix with its "diagonal" part: $\Phi(a_{ij}) = (a_{ij}')$, where $a_{ij}' = 0$ or a_{ij} according as $i \neq j$ or $i = j$.

(3) Suppose $\mathcal{B}$ is finite, and let ϕ be a faithful normal finite trace on $\mathcal{B}$. Let M be an arbitrary von Neumann subalgebra of $\mathcal{B}$. By a known Radon-Nikodym theorem, there exists, for every $X \in \mathcal{B}$, a uniquely determined element $\Phi(X) \in M$ such that

$$\phi(XA) = \phi(\Phi(X)A)$$

for every $A \in M$. It is not hard to show that Φ is a faithful normal expectation on M for which $\phi \circ \Phi = \phi$. For details, see [14]. We shall call Φ the *natural* expectation on M defined by ϕ.

(4) Let ϕ be a faithful normal semifinite trace on $\mathcal{B}$, and let M be a von Neumann subalgebra such that the restriction of ϕ to M is again semifinite. By a standard Hilbert algebra argument, one can show that there exists, for every $X \in \mathcal{B}$, a unique element $\Phi(X) \in M$ for which

$$\phi(XA) = \phi(\Phi(X)A)$$

for every $A \in M$ such that $\phi(|A|) < \infty$. Φ is a faithful normal expectation on M, and $\phi \circ \Phi = \phi$ holds on the ideal of definition of ϕ. We omit the details for this example, since it is not essential for the rest of the paper.

(5) In 6.2 we give other examples of expectations, which are neither faithful nor normal.

Definition 6.1.4. *Let $\mathfrak{B}$ be a von Neumann algebra. A von Neumann subalgebra M is said to be compatible with $\mathfrak{B}$ if there exists a faithful normal expectation of $\mathfrak{B}$ on M.*

Remark 6.1.5. Every von Neumann subalgebra of a finite $\mathfrak{B}$ is compatible with $\mathfrak{B}$. More generally, if $\mathfrak{B}$ is semifinite and if the restriction of some faithful normal semifinite trace to M is semifinite, then M is compatible with $\mathfrak{B}$. Example (2) shows that if $\mathfrak{P}$ is a sequence of mutually orthogonal projections in an arbitrary $\mathfrak{B}$, then $\mathfrak{P}' \cap \mathfrak{B}$ is compatible with $\mathfrak{B}$. On the other hand, we will see in 6.2 that a nonatomic maximal abelian (self-adjoint) algebra is not compatible with the ring of all bounded linear operators.

Let $\mathfrak{P}$ be an abelian family of projections in $\mathfrak{B}$ which contains O and I. For every finite subset $\mathfrak{F}$ of $\mathfrak{P}$ (containing O and I), and every $X \in \mathfrak{B}$, let

$$X_{\mathfrak{F}} = \sum AXA,$$

the sum extended over all atoms A in the finite Boolean algebra generated by $\mathfrak{F}$. The map $X \rightarrow X_{\mathfrak{F}}$ is an expectation of $\mathfrak{B}$ on $\mathfrak{F}' \cap \mathfrak{B}$, by example (2). Direct the finite subsets of $\mathfrak{P}$ in the increasing sense by set inclusion. Then the subalgebras $\mathfrak{F}' \cap \mathfrak{B}$ are directed decreasing, and their intersection is $\mathfrak{P}' \cap \mathfrak{B}$. It is natural to ask whether the net $\{X_{\mathfrak{F}}\}$ converges in some sense to the image of X under an expectation on $\mathfrak{P}' \cap \mathfrak{B}$. This need not occur. The following definition provides the general setting for this question.

Definition 6.1.6. *Let D be an increasing directed set. A net $\{\Phi_\alpha : \alpha \in D\}$ of expectations in $\mathfrak{B}$ is called a decreasing martigale if $\alpha \leqq \beta$ in D implies $\Phi_\alpha(\mathfrak{B}) \supseteqq \Phi_\beta(\mathfrak{B})$ and $\Phi_\beta \circ \Phi_\alpha = \Phi_\beta$.*

Increasing martingales can be defined by replacing $\leqq$ with $\geqq$ in 6.1.6.

THEOREM 6.1.7. *Let Φ_α, $\alpha \in D$, be a decreasing martingale in $\mathfrak{B}$. Suppose there exists a faithful normal expectation Φ on $\bigcap_\alpha \Phi_\alpha(\mathfrak{B})$ such that $\Phi \circ \Phi_\alpha = \Phi$ for every α.*

Then $\lim_\alpha \Phi_\alpha(X) = \Phi(X)$ strongly, for every $X \in \mathfrak{B}$.

Proof. Fix $X_1 \in \mathfrak{B}$, and put $X = X_1 - \Phi(X_1)$. Then $\Phi(X) = 0$ and $\Phi_\alpha(X) = \Phi_\alpha(X_1) - \Phi_\alpha \cdot \Phi(X_1) = \Phi_\alpha(X_1) - \Phi(X_1)$, since Φ_α leaves fixed every element of $\Phi_\alpha(\mathfrak{B}) \supseteqq \Phi(\mathfrak{B})$. Thus it suffices to assume $\Phi(X) = 0$.

Set $\mathcal{S}_\alpha = \{\Phi_\beta(X) : \beta \geqq \alpha\}$. Each $\mathcal{S}_\alpha$ is a nonempty subset of the ball of radius $\| X \|$, and one has

$$\mathcal{S}_{\alpha_1} \cap \mathcal{S}_{\alpha_2} \cap \cdots \cap \mathcal{S}_{\alpha_n} \supseteqq \mathcal{S}_\beta$$

where β is any element of D larger than $\alpha_1, \cdots, \alpha_n$. By compactness, there is at least one element T in the weak closure of every $\mathcal{S}_\alpha$. Since $\mathcal{S}_\alpha^- \subseteq \Phi_\alpha(\mathcal{B})$, we have $T \in \bigcap_\alpha \Phi_\alpha(\mathcal{B})$ and hence $T = \Phi(T)$. But Φ vanishes identically on each $\mathcal{S}_\alpha$, since $\Phi \circ \Phi_\beta(X) = \Phi(X) = 0$, and by weak continuity of Φ on bounded sets, we have $\Phi(T) = 0$. This proves that $T = 0$ is a weak cluster point of the set $\{\Phi_\alpha(X) : \alpha \in D\}$.

Since $\mathcal{B}$ acts on a separable Hilbert space, there is a faithful normal state ρ on $\mathcal{B}$. Put $\sigma = \rho \circ \Phi$. Then σ is faithful and normal because both ρ and Φ have these properties, and moreover $\sigma \circ \Phi = \sigma$. If $\beta \geqq \alpha$, then

$$\sigma(\Phi_\beta(X)^*\Phi_\beta(X)) = \sigma([\Phi_\beta \circ \Phi_\alpha(X)]^*\Phi_\beta \circ \Phi_\alpha(X))$$
$$\leqq \sigma(\Phi_\alpha(X)^*\Phi_\alpha(X)) = \sigma \circ \Phi(\Phi_\alpha(X)^*\Phi_\alpha(X)).$$

Also,

$$\Phi(\Phi_\alpha(X)^*\Phi_\alpha(X)) = \Phi \circ \Phi_\alpha(\Phi_\alpha(X)^*X)$$
$$= \Phi(\Phi_\alpha(X)^*X).$$

Therefore, using $\sigma \circ \Phi = \sigma$ once again, we have

$$\sigma(\Phi_\beta(X)^*\Phi_\beta(X)) \leqq \sigma(\Phi_\alpha(X)^*X)$$

whenever $\beta \geqq \alpha$. Now $\sigma(\Phi_\alpha(X)^*X)$ can be made small with an appropriate choice of α, by weaq continuity of the map $\tau(S) = \sigma(S^*X)$ on bounded sets and the fact that $\{\Phi_\alpha(X)\}$ clusters weakly at 0. Thus, the last inequality implies that

$$\lim_\alpha \sigma(\Phi_\alpha^*(X)\Phi_\alpha(X) = 0.$$

It follows that $\Phi_\alpha(X)$ tends strongly to 0 ([1], p. 62, prop. 4).

COROLLARY 6.1.8. *Let $\mathcal{P}$ be an abelian family of projections in $\mathcal{B}$ such that $\mathcal{P}' \cap \mathcal{B}$ is compatible with $\mathcal{B}$.*

Then for every $X \in \mathcal{B}$, $X_{\mathcal{F}} \to \Phi(X)$ strongly, where Φ is the faithful normal expectation on $\mathcal{P}' \cap \mathcal{B}$ (Φ is unique, by 6.2.2).

Proof. We have only to show that the mappings $X \to X_{\mathcal{F}}$ form a decreasing martingale and $\Phi(X_{\mathcal{F}}) = \Phi(X)$, $X \in \mathcal{B}$. The first statement is an obvious consequence of the definition of $X_{\mathcal{F}}$. Let $\mathcal{F}$ be a finite subset of $\mathcal{P}$. For every atom A in the Boolean algebra generated by $\mathcal{P}$, we have $\Phi(AXA) = A\Phi(X)A$, by the expectation property and the fact that

$A \in \mathfrak{P}' \cap \mathcal{B}$. On the other hand, $A \in \mathfrak{P}''$, and so it commutes with $\Phi(X)$. Hence $\Phi(AXA) = \Phi(X)A^2 = \Phi(X)A$, and one has

$$\sum_A \Phi(AXA) = \Phi(X) \sum_A A = \Phi(X),$$

proving the corollary.

We could, at this point, prove a convergence theorem for increasing martingales. All we shall require, however, is the following simple result in a finite von Neumann algebra. This is a variation of a known result of Umegaki.

PROPOSITION 6.1.9. *Let D be a directed set and let M_α, $\alpha \in D$, be a family of von Neumann subalgebras of $\mathcal{B}$ such that $\bigcup_\alpha M_\alpha$ is strongly dense in $\mathcal{B}$ and $\alpha \leqq \beta$ implies $M_\alpha \subseteq M_\beta$.*

Let ϕ be a faithful normal finite trace on $\mathcal{B}$, and let Φ_α be the expectation on M_α satisfying $\phi \circ \Phi_\alpha = \phi$. Then $\Phi_\alpha(X) \to X$ strongly, for every $X \in \mathcal{B}$.

Proof. Fix $X \in \mathcal{B}$. It suffices to show that $[X - \Phi_\alpha(X)] \to 0$, where $[T]$ is the trace norm $\phi^{\frac{1}{2}}(T^*T)$ ([1], p. 62, prop. 4). If $d_\alpha = \inf [X - T_\alpha]$, $T_\alpha \in M_\alpha$, then $\alpha \leqq \beta$ implies $d_\beta \leqq d_\alpha$, so $\lim_\alpha d_\alpha$ exists. This limit must be 0 because every element of $\mathcal{B}$ can be approximated in $[\ ]$ by elements of the strongly dense *-algebra $\bigcup_\alpha M_\alpha$ ([1], p. 288, Lemma 1). But by 6.1.1 (iv), $d_\alpha = [X - \Phi_\alpha(X)]$, and that completes the proof.

6.2. Some existence and uniqueness questions. This section concerns existence and uniqueness of faithful normal expectations on subalgebras of a specific form, including maximal abelian subalgebras. Throughout, $\mathcal{B}$ is a von Neumann algebra acting on the Hilbert space $\mathfrak{H}$, and M is a fixed subalgebra of the form $\mathcal{B} \cap N'$, where N is an abelian von Neumann subalgebra of $\mathcal{B}$.

Let G be a discrete abelian group. Let Λ be a linear functional on the Banach space of bounded complex-valued functions on G; $\Lambda_x f(x)$ denotes the value of Λ at f. Λ is called a *mean* if $0 \leqq f(x) \leqq 1$ implies

$$0 \leqq \Lambda_x f(x) \leqq 1, \quad \| \Lambda \| = 1,$$

and $\Lambda_x f(yx) = \Lambda_x f(x)$ for every $y \in G$. Every discrete abelian group admits a mean ([7], p. 231, thm. 17.5), and means are unique only when G is finite.

In this section, G will denote the abelian group of unitary operators in N. For $X \in \mathcal{B}$, $C(X)$ is the weakly closed convex hull of the set

$$\{UXU^{-1}; U \in G\}.$$

THEOREM 6.2.1. *Let* Λ *be a mean on* G. *Then there exists an expectation* Φ_Λ *of* $\mathfrak{B}$ *on* M *such that*

$$(\Phi_\Lambda(X)\xi, \eta) = \underset{U}{\Lambda}\,(UXU^{-1}\xi, \eta), \qquad X \in \mathfrak{B}, \xi, \eta \in \mathfrak{H}.$$

One has $\Phi_\Lambda(X) \in C(X) \cap M$.

Proof. Fix $X \in \mathfrak{B}$. For every pair $\xi, \eta \in \mathfrak{H}$, the mapping $U \to (UXU^{-1}\xi, \eta)$ is a function on G of modulus $\leqq \| X \| \| \xi \| \| \eta \|$. Thus $[\xi, \eta] = \underset{U}{\Lambda}(UXU^{-1}\xi, \eta)$ defines a bounded bilinear form on $\mathfrak{H} \times \mathfrak{H}$. By a well-known lemma of Riesz, there is a uniquely determined operator $\Phi_\Lambda(X)$ such that

$$(\Phi_\Lambda(X)\xi, \eta) = \underset{U}{\Lambda}\,(UXU^{-1}\xi, \eta).$$

It is clear that $X \to \Phi_\Lambda(X)$ defines a positive linear mapping which leaves I fixed, and by a standard separation theorem, $\Phi_\Lambda(X) \in C(X)$.

If $V \in G$, then for $\xi, \eta \in \mathfrak{H}$,

$$\begin{aligned}(V\Phi_\Lambda(X)V^{-1}\xi, \eta) &= (\Phi_\Lambda(X)V^{-1}\xi, V^{-1}\eta) = \underset{U}{\Lambda}(UXU^{-1}V^{-1}\xi, V^{-1}\eta) \\ &= \underset{U}{\Lambda}\,(VUX(VU)^{-1}\xi, \eta) = (\Phi_\Lambda(X)\xi, \eta).\end{aligned}$$

Thus $\Phi_\Lambda(X)$ commutes with every unitary operator in N, so that $\Phi_\Lambda(X) \in M$. If $A \in M$ and $X \in \mathfrak{B}$, then $UAXU^{-1} = AUXU^{-1}$ for every $U \in G$, and a similar string of identities shows that $\Phi_\Lambda(AX) = A\Phi_\Lambda(X)$. Hence Φ_Λ has all the stated properties.

It is quite possible that different means may give rise to different expectations, and that a given Φ_Λ may be neither faithful nor normal (cf. 6.2.4). In the desirable cases, however, these anomalies do not occur.

THEOREM 6.2.2. *Suppose that* M *is compatible with* $\mathfrak{B}$. *Then there is exactly one faithful normal expectation* Φ *on* M, *and* $\Phi_\Lambda = \Phi$ *for every mean* Λ.

Proof. Let Φ be any faithful normal expectation on M, and let $X \in \mathfrak{B}$. By 6.2.1, $C(X) \cap M$ contains $\Phi_\Lambda(X)$ for every mean Λ; thus it suffices to show that this intersection consists of the single point $\Phi(X)$. Since $G \subseteq N' \cap \mathfrak{B} = M$, we have $\Phi(UXU^{-1}) = U\Phi(X)U^{-1}$, for every $U \in G$. At the same time, everything in M commutes with G, so that $U\Phi(X)U^{-1} = \Phi(X)$.

Thus $\Phi(T) = \Phi(X)$ whenever T is a convex linear combination of operators UXU^{-1}, $U \in G$. Being normal, Φ is weakly continuous on bounded sets, so the last equality persists when T ranges over $C(X)$. In particular, if $T \in C(X) \cap M$, then $T = \Phi(T) = \Phi(X)$, and the proof is complete.

THEOREM 6.2.3. *In order that M be compatible with $\mathcal{B}$, it is necessary and sufficient that there exist a faithful normal state ρ of $\mathcal{B}$ such that*

$$\rho(UXU^{-1}) = \rho(X), \qquad U \in G, X \in \mathcal{B}.$$

Proof. Necessity. Let Φ be a faithful normal expectation on M. Since $\mathcal{B}$ acts on a separable Hilbert space, there is a faithful normal state σ on $\mathcal{B}$. Put $\rho = \sigma \circ \Phi$. Then ρ is faithful and normal because both σ and Φ are. By the proof of 6.2.2, $\Phi(UXU^{-1}) = \Phi(X)$ for $X \in \mathcal{B}$, $U \in G$. Thus

$$\rho(UXU^{-1}) = \sigma \circ \Phi(UXU^{-1}) = \sigma \circ \Phi(X) = \rho(X).$$

Sufficiency. Let Λ be a mean, and construct the expectation Φ_Λ. Let $X \in \mathcal{B}$, and let ρ be a state with the given properties. Therefore $\rho(T) = \rho(X)$ whenever $T \in C(X)$, by weak continuity of ρ on bounded sets. In particular, $\rho \circ \Phi_\Lambda(X) = \rho(X)$. By 6.1.1 (v), Φ_Λ is faithful and normal, and the proof is complete.

Remark 6.2.4. Take $\mathcal{B}$ to be the ring of all bounded linear operators on a separable $\mathfrak{H}$, and let M be a maximal abelian (self-adjoint) subalgebra of $\mathcal{B}$ which is totally atomic in the sense that every nonnull projection in M dominates a minimal projection in M. If the minimal projections are $P_1, P_2, \cdots$, then $\Phi(X) = \sum P_n X P_n$ is the (unique) faithful normal expectation on M (example 2 and 6.2.2).

If M is nonatomic (no minimal projections) the picture is quite different. Let Tr be the canonical semifinite trace on $\mathcal{B}^+$, and let HS denote the set of all $X \in \mathcal{B}$ such that $\mathrm{Tr}(X^*X) < +\infty$. HS is well known to be an ultraweakly dense two-sided ideal in $\mathcal{B}$, and every operator in HS is compact. Let G be the unitary group of M.

PROPOSITION 6.2.4. *If M is a (self-adjoint) nonatomic maximal abelian subalgebra of $\mathcal{B}$, the ring of all bounded operators on $\mathfrak{H}$, then $C(X) \cap M = \{0\}$, for every $X \in HS$. M is not compatible with $\mathcal{B}$; moreover, for every mean Λ, Φ_Λ is neither faithful nor normal.*

Proof. By a lemma of Dixmier ([1], Lemma 1, p. 127), $C(X) \cap M$ is a (nonvoid) subset of HS. Every operator T in this intersection is therefore compact. The eigenspace corresponding to any nonzero eigenvector of T is finite dimensional and its projection belongs to M because T does. But every nonnull projection in M is infinite dimensional, by nonatomicity. Thus $\sigma(T)=\{0\}$. Since T is normal, one has $T=0$.

Let ρ be a normal state such that $\rho(UXU^{-1})=\rho(X)$, $U\in G$, $X\in\mathfrak{B}$. Then by continuity, ρ is constant on $C(X)$. If $X\in HS$, then by the preceding paragraph, $0=\rho(0)=\rho(X)$. Thus ρ vanishes on HS. By continuity again, $\rho=0$, and this contradicts $\rho(I)=1$. By 6.2.3, M is not compatible with $\mathfrak{B}$.

Let Λ be a mean. Then $\Phi_\Lambda(X)\in C(X)\cap M$ for every X (6.2.1). Thus Φ_Λ vanishes on HS, and HS certainly contains nontrivial positive operators. Hence Φ_Λ is not faithful. If Φ_Λ were normal, it would have to vanish identically because HS is ultraweakly dense, and this would contradict $\Phi_\Lambda(I)=I$.

HARVARD UNIVERSITY.

REFERENCES.

[1] Jacques Dixmier, *Les Algebres d'operateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1957.

[2] H. A. Dye, "On groups of measure-preserving transformations I," *American Journal of Mathematics*, vol. 81 (1959), pp. 119-159.

[3] ———, "On groups of measure-preserving transformations II," *American Journal of Mathematics*, vol. 85 (1963), pp. 551-576.

[4] B. Fuglede and R. V. Kadison, "Determinant theory in finite factors," *Annals of Mathematics*, vol. 55 (1952), pp. 520-530.

[→ P. R. Halmos. "Recent progress in ergodic theory," *Bulletin of the American Mathematical Society*, vol. 67 (1961), pp. 78-79.

[→ H. Helson and D. Lowdenslager, "Prediction theory and Fourier series in several variables," *Acta Mathematica*, vol. 99 (1958), pp. 165-202.

[7] E. Hewitt and K. Ross, *Abstract Harmonic Analysis I*, Academic Press, New York, 1963.

[→ K. Hoffman, "Analytic functions and logmodular Banach algebras," *Acta Mathematica*, vol. 108 (1962), pp. 271-317.

[9] ———, *Banach Spaces of Analytic Functions, Prentice-Hall*, Englewood Cliffs, N. J., 1962.

[10] R. V. Kadison and I. M. Singer, "Extension of pure states," *American Journal of Mathematics*, vol. 81 (1959), pp. 383-400.

[11]———, "Triangular operator algebras," *American Journal of Mathematics*, vol. 82 (1960), pp. 227-259.

[12] J. von Neumann, "On rings of operators III," *Annals of Mathematics*, vol. 41 (1940), pp. 94-161.

[] Tomiyama, "On the projection of norm 1 in W^*-algebra," *Proceedings of the Japan Academy*, vol. 33 (1957), pp. 608-612.

[] H. Umegaki, "Conditional expectation in an operator algebra," *Tohoku Journal of Mathematics*, vol. 2 (1954), pp. 177-181.

[] N. Wiener, "On the factorization of matrices," *Commentarii Mathematici Helvetici*, vol. 29 (1955), pp. 97-11.