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Prediction Theory and Group Representations

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics

by

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To Sandy

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ABSTRACT OF THE DISSERTATION

Prediction Theory and Group Representations

by

William Barnes Arveson

Doctor of Philosophy in Mathematics

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Professor Henry A. Dye, Chairman

This study represents an attempt to lift some of the ideas and theorems of prediction theory to a noncommutative algebraically invariant setting. The definition of determinism, taken as fundamental in this regard, applies to the faithful weakly continuous representations of locally compact ordered groups. A decomposition similar to the Wold theorem for stochastic processes is valid and, when the underlying group is discrete, the idea of determinism is used to characterize the existence of a canonical finite trace on the von Neumann algebra generated by the image group.

These considerations lead naturally to operator algebras which are in many respects noncommutative analogs of the algebra H_∞ of bounded analytic functions in the unit disc $\{|z| < 1\}$. A factorization theorem is valid in this context, and this is used to study the relation between appropriate versions of Jensen's formula and the theorem of Szegö.

0. Introduction

The prediction theory of a stochastic process may be regarded as the analysis of a unitary representation of an ordered abelian group with respect to a fixed cyclic vector. For example, taking the additive abelian group of integers, one studies a sequence ξ_n, $n = 0, \pm 1, \pm 2, \ldots$, of vectors in a Hilbert space $\mathcal{H}$ with the properties that $\{\xi_n\}$ is fundamental in $\mathcal{H}$ and the inner product (ξ_{n+k}, ξ_n) is independent of n. These conditions are precisely what is needed to make the mapping $U\xi_n = \xi_{n+1}$ defined on $\{\xi_n\}$ extend uniquely to a unitary operator on $\mathcal{H}$. The formula $U^n\xi_o = \xi_n$ asserts that the original sequence is nothing other than the orbit of the cyclic vector ξ_o with respect to the representation $n \to U^n$ of the integers. It is an essential feature that, from the point of view of prediction theory, properties of this representation have no interest aside from the relation that they bear to the given sequence.

On the other hand, in the study of group representations per se, one is interested in global properties of the representation itself, i.e., algebraic or unitary invariants. In this study, we have attempted to lift to this context algebraically invariant analogs of some of the ideas and theorems of prediction theory. Among these, the idea of determinism we have taken as fundamental.

Section 1 contains a discussion of a related idea, called degeneracy, pertaining to the action of unitary groups in Hilbert space. When the group G is abelian, this property is used to characterize the condition that the spectrum of the C^* - algebra generated by G be homeomorphic in a natural way to the character group of the discrete group G.

The definition of determinism, as given in Section 2, applies to a faithful weakly continuous unitary representation of a locally compact group which, we emphasize, need not be abelian. Every such representation decomposes into the direct sum of a deterministic and a regular part, the regular part being in a sense maximally nondeterministic. A good deal more can be said if the underlying group is discrete and left - linear, where by definition a group G is left - linear if it admits a linear order $\leq$ such that $x \leq y$ implies $zx \leq zy$ for all $x,y,z \in G$. Here determinism relates in an essential way to the existence of a canonical finite trace on the von Neumann algebra generated by the image group of the representation: the trace exists if, and only if, the representation is not deterministic. This trace can be used to prove that in the above decomposition, the regular part is algebraically equivalent to the left regular representation of G in $\ell_2(G)$. Many questions remain open regarding the situation where G is left-linear and merely locally compact and unimodular.

In section 6, we begin a study of a class of operator algebras which seems to provide a natural noncommutative generalization of the algebra H_∞ of functions bounded and analytic in the unit disc $\{|z| < 1\}$. The setting is this: given a von Neumann algebra $\mathcal{B}$ with a distinguished faithful normal finite trace φ, one considers an ultraweakly closed Banach subalgebra $\mathcal{A}$ of $\mathcal{B}$ with the properties

(i) $I \in \mathcal{A}$

(ii) φ is multiplicative on $\mathcal{A}$,

(iii) $\mathcal{A} + \mathcal{A}^*$ contains a weakly dense self-adjoint subalgebra of $\mathcal{B}$.

Prototypes for these algebras can be constructed using the left regular representation of any left-linear discrete group. Though cast in a noncommutative situation, these conditions are not unlike the axioms for a Dirichlet function algebra, the principal difference being the preeminence of weak operator topologies in place of the uniform topology. For example, it is shown that every self-adjoint operator in $\mathcal{B}$ is the strong limit of a sequence $A_n + A_n^*$, where $A_n \in \mathcal{A}$. We are concerned primarily with the validity of an appropriate version of the Jensen formula for the trace φ and with the theorem of Szegö. It is proved that every positive invertible operator in $\mathcal{B}$ admits a factorization AA^* where both A and A^{-1} belong to $\mathcal{A}$, and this fact is used to deduce that Szegö's theorem and Jensen's formula are equivalent. It is emphasized, however, that at this time neither of these propositions has been established independently of the other in any noncommutative setting.

1. Preliminaries

The terminology and background material summarized here is standard and, for the most part, is drawn from the texts (2) (8) and (13).

It is to be understood in the following that subspaces of Hilbert spaces are always taken to be closed in the norm topology, and the term projection means self-adjoint projection.

By a C^*-algebra we mean a uniformly closed subalgebra of the ring $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on the Hilbert space $\mathcal{H}$, containing the identity and closed under the adjoint operation. A bounded linear functional ρ on the C^*- algebra $\mathcal{A}$ is called positive if $\rho(T^*T) \geq 0$ for every $T \in \mathcal{A}$. The collection $\Sigma(\mathcal{A})$ of all positive linear functionals of norm 1 is convex and compact in the relative weak* - topology, and is called the state space of $\mathcal{A}$; the elements of $\Sigma(\mathcal{A})$ are states. Functionals of the form $\omega_\xi(T) = (T\xi, \xi)$ where ξ is a unit vector in $\mathcal{H}$ are called canonical states. Every state ρ satisfies the Schwarz inequality in the sense that $|\rho(T^*S)|^2 \leq \rho(T^*T)\, \rho(S^*S)$ for all $S, T \in \mathcal{A}$.

A mapping of one C^*- algebra into another preserving the adjoint and the algebraic operations is called a $*$-homomorphism; such a map is a $*$-isomorphism if it is in addition 1-1 and onto. Every state ρ of a C^*- algebra $\mathcal{A}$ gives rise to a canonical $*$-homomorphism of $\mathcal{A}$ in the following way. Let $\mathcal{N}_\rho = \{T \in \mathcal{A}: \varphi(T^*T)=0\}$. $\mathcal{N}_\rho$ is a uniformly closed left ideal in $\mathcal{A}$, and the quotient space $\mathcal{A}/\mathcal{N}_\rho$ becomes a prehilbert space when endowed with the inner product

$$(A + \mathcal{N}_\rho, B + \mathcal{N}_\rho) = \rho(B^*A).$$

Let $\mathcal{H}_\rho$ be the Hilbert space completion of $\mathcal{A}/\mathcal{N}_\rho$. For every $T \in \mathcal{A}$, the mapping $L_T(A + \mathcal{N}_\rho) = TA + \mathcal{N}_\rho$ defined on $\mathcal{A}/\mathcal{N}_\rho$ extends uniquely to an operator in $\mathcal{L}(\mathcal{H}_\rho)$, which we denote by the same symbol L_T. Then $T \to L_T$ is a $*$-homomorphism of $\mathcal{A}$ into $\mathcal{L}(\mathcal{H}_\rho)$ and, though we shall not need this fact, the image of $\mathcal{A}$ is a C^*- subalgebra of $\mathcal{L}(\mathcal{H}_\rho)$.

If $\mathcal{A}$ is an abelian C^*- algebra, we may form the set $\sigma(\mathcal{A})$ of all algebra homomorphisms of $\mathcal{A}$ into the algebra $\mathbb{C}$ of complex numbers, which are not identically zero. Every $\omega \in \sigma(\mathcal{A})$ is already norm continuous and satisfies $\omega(A^*) = \bar{\omega}(A)$. $\sigma(\mathcal{A})$ is a compact Hausdorff space in the relative weak*-topology of the dual space of $\mathcal{A}$, and it is called the <u>maximal ideal space</u> or the <u>spectrum</u> of $\mathcal{A}$. The mapping $A \in \mathcal{A} \to \hat{A}$, where $\hat{A}$ is the function taking on the value $\omega(A)$ at the point $\omega \in \sigma(\mathcal{A})$, is an algebraic isomorphism of $\mathcal{A}$ onto the algebra of all continuous functions on $\sigma(\mathcal{A})$ such that $\widehat{A^*}$ is the conjugate of the function $\hat{A}$. This isomorphism we refer to as the <u>Gelfand transform</u>.

A <u>von Neumann algebra</u> is a C^*- algebra closed in the weak operator topology. In addition to the usual weak and strong operator topologies on von Neumann algebras, we shall make frequent use of the ultraweak (UW) and ultrastrong (US) topologies. The reason for this is, of course, that it is these topologies, and not the former, that are preserved under $*$-isomorphisms. Recall that the UW - topology (resp. US - topology) on the von Neumann algebra $\mathcal{A}$ is the topology having as a subbase (resp. base) at 0 all sets of the form

$\{T\in\mathcal{A}: |\sum_{k=1}^{\infty}(T\xi_k,\eta_k)| \leq 1\}$ (resp. $\{T\in\mathcal{A}: \sum_{k=1}^{\infty}\|T\xi_k\|^2 \leq 1\}$), where $\xi_k, \eta_k\in\mathcal{H}$, $\sum\|\xi_k\|^2 < \infty$, $\sum\|\eta_k\|^2 < \infty$. The UW and US - topologies have the same closed convex sets and the same continuous linear functionals, namely those of the form $F(T) = \sum_{k=1}^{\infty}(T\xi_k,\eta_k)$, where $\sum\|\xi_k\|^2 < \infty$, $\sum\|\eta_k\|^2 < \infty$.

Let $\{T_\alpha;\alpha\in A\}$ be a family of positive operators in $\mathcal{L}(\mathcal{H})$ such that $\sup\|T_\alpha\| < \infty$ and which is directed $\uparrow$ in the sense that for every α, $\beta\in A$ there exists $\gamma\in A$ such that T_γ dominates both T_α and T_β (all with respect to the usual operator order). Then $T = \text{LUB}\ T_\alpha$ exists in $\mathcal{L}(\mathcal{H})$, and in fact T belongs to the strong closure of $\{T_\alpha\}$. In particular, T belongs to any von Neumann algebra which contains all the T_α.

Let $\mathcal{A}^+$ denote the set of positive elements in the von Neumann algebra $\mathcal{A}$. A linear mapping Φ of $\mathcal{A}$ into another von Neumann algebra is said to be <u>positive</u> if $\Phi(T) \geq 0$ for every $T\in\mathcal{A}^+$, <u>normal</u> if it is positive and $\Phi(\text{LUB}\ T_\alpha) = \text{LUB}\ \Phi(T_\alpha)$ for every bounded directed $\uparrow$ family $\{T_\alpha\}\subseteq\mathcal{A}^+$. In order that a state ρ be normal, it is necessary and sufficient that it be UW-continuous; if this is the case, then the canonical $*$-homomorphism $T\in\mathcal{A}\to L_T\in\mathcal{L}(\mathcal{H}_\rho)$ associated with ρ is a normal mapping.

A function φ defined on $\mathcal{A}^+$ and taking values in $[0, +\infty]$ is called a <u>trace</u> if it has the properties

(i) $\varphi(A+B) = \varphi(A) + \varphi(B)$, for all $A, B \in \mathcal{A}^+$

(ii) $\varphi(UAU^{-1}) = \varphi(A)$ for every $A \in \mathcal{A}^+$ and every unitary $U \in \mathcal{A}$

(iii) $\varphi(\lambda A) = \lambda\varphi(A)$ for all $A \in \mathcal{A}^+$, $\lambda > 0$.

The trace φ is said to be faithful if $A \in \mathcal{A}^+$, $\varphi(A) = 0$ entails $A = 0$, normal if for every bounded directed $\uparrow$ family $A_\alpha \in \mathcal{A}^+$ one has $\varphi(\text{LUB } A_\alpha) = \sup \varphi(A_\alpha)$, semifinite if every $A \in \mathcal{A}^+$ such that $\varphi(A) > 0$ dominates a $B \in \mathcal{A}^+$ for which $0 < \varphi(B) < \infty$, and finite if $\varphi(A) < \infty$ for every $A \in \mathcal{A}^+$.

If φ is a trace on $\mathcal{A}^+$, put $\mathcal{J}_\varphi = \{T \in \mathcal{A}: \varphi(T^*T) < \infty\}$. $\mathcal{J}_\varphi$ is a two-sided ideal in $\mathcal{A}$, and we may form the second ideal

$$\mathfrak{J}_\varphi = \mathcal{J}_\varphi^2 = \left\{\sum_{k=1}^{n} S_k T_k : S_k, T_k \in \mathcal{J}_\varphi\right\};$$

$\mathfrak{J}_\varphi$ is called the ideal of definition of the trace φ. φ extends uniquely to a linear functional on $\mathfrak{J}_\varphi$ satisfying $\varphi(ST) = \varphi(TS)$ for every $S, T \in \mathcal{J}_\varphi$. When normal, φ is semifinite if, and only if, $\mathfrak{J}_\varphi$ is weakly dense in $\mathcal{A}$. For every $X \in \mathfrak{J}_\varphi$, the functional $\rho(A) = \varphi(XA)$ is a bounded linear functional on $\mathcal{A}$ which is positive when X is ; if φ is normal then ρ is UW-continuous.

A von Neumann algebra $\mathcal{A}$ is said to be finite if for every nonzero $A \in \mathcal{A}^+$ there exists a finite normal trace φ on $\mathcal{A}$ such that $\varphi(A) > 0$.

Let G be a locally compact Hausdorff topological group, and denote by $\mathcal{C}_{00}(G)$ the family of continuous functions on G having compact support. Let $I(f) = \int f(x)dx$ be a left Haar integral on $\mathcal{C}_{00}(G)$ in the sense of (8), normalized in the customary way when G is compact or discrete. By the <u>convolution</u> of f with g we mean the usual $(f*g)(x) = \int f(t)\, g(t^{-1}x)dt$. $\mathcal{C}_{oo}(G)$ is an algebra with respect to addition and convolution; and if Δ is the modular function of the group defined by the condition $\int f(x)dx = \int f(x^{-1})\Delta^{-1}(x)dx$ for $f \in \mathcal{C}_{oo}(G)$, then $\mathcal{C}_{oo}(G)$ admits the natural involution $f^*(x) = \bar{f}(x^{-1})\Delta^{-1}(x)$.

By a <u>unitary representation</u> of G we mean a homomorphism $x \in G \to U_x$ of G into the unitary group on some Hilbert space $\mathcal{H}$. Restricting the topologies of $\mathcal{H}$ to the unitary group leads to various notions of continuity for representations, and most of these coalesce since the weak, strong, ultraweak, and ultrastrong topologies all coincide on the unitary group. If $x \to U_x$ is a weakly continuous representation of G and $f \in \mathcal{C}_{oo}(G)$ then there exists a uniquely determined operator T_f in the von Neumann algebra $\mathcal{A}$ generated by $\{U_x\}$ satisfying

$$(T_f\xi, \eta) = \int f(x)\ (U_x\xi, \eta)dx$$

for every $\xi, \eta \in \mathcal{H}$. We will sometimes write $T_f = \int f(x)\, U_x\, dx$. An application of the dominated convergence theorem leads to the conclusion that

$$\rho(T_f) = \int f(x)\ \rho(U_x)dx$$

for every $f \in \mathcal{C}_{oo}(G)$ and every UW-continuous linear functional

$$\rho(T) = \sum_{k=1}^{\infty} (T\xi_k, \eta_k),\ \ \Sigma\|\xi_k\|^2 < \infty\ \ \Sigma\|\eta_k\|^2 < \infty,$$

on $\mathcal{A}$. The mapping $f \in \mathcal{C}_{oo}(G) \to T_f$ is a homomorphism of $\mathcal{C}_{oo}(G)$

into $\mathcal{A}$, preserving involution $T_{f^*} = T_f^*$. Not only is the image of $\mathcal{C}_{00}(G)$ weakly dense, but so is the $*$-subalgebra consisting of the operators

$$\sum_{k=1}^{n} T_{f_k} T_{g_k}, \text{ where } f_k, g_k \in \mathcal{C}_{00}(G).$$

A unitary representation $x \to U_x$ is said to be faithful if $U_x \neq I$ whenever $x \neq e$. For $f \in \mathcal{C}_{00}(G)$ and $x \in G$, define $\ell_x f \in \mathcal{C}_{00}(G)$ by $(\ell_x f)(t) = f(x^{-1}t)$, $t \in G$. ℓ_x extends uniquely to a unitary operator in $L_2(G, dx)$, and the map $x \to \ell_x$ is a weakly continuous faithful unitary representation of G called the left regular representation.

2. Action Characteristics of Abelian Unitary Groups

Let $\mathcal{H}$ be a Hilbert space and let U be a unitary operator on $\mathcal{H}$, generating an infinite cyclic group. The spectrum of U can be identified with a compact subset of the unit circle $\{|z| = 1\}$. Under what conditions on U will its spectrum fill out the unit circle?

We are going to consider this problem in the following more general setting. Given an abelian unitary group G on the Hilbert space $\mathcal{H}$, let $\mathcal{A}$ be the C^*- algebra generated by G and let $\sigma(\mathcal{A})$ denote the maximal ideal space of this algebra. Let Γ be the compact character group of the discrete group G. There is a natural mapping α of $\sigma(\mathcal{A})$ into Γ; α merely restricts the complex homomorphism $\omega \in \sigma(\mathcal{A})$ to the group G. α is automatically continuous by definition of the topologies involved, and it is 1-1 because a continuous linear functional on $\mathcal{A}$ is completely determined by its values on G, the latter being a fundamental set in $\mathcal{A}$. Our question becomes, what conditions on G will ensure that α map $\sigma(\mathcal{A})$ onto Γ? In theorem 1, this contingency is characterized in terms of the action of the group G.

Versions of this problem have been considered before. Some time ago, Kodira and Kakutani (12) showed essentially that α is onto Γ when G is the discrete unitary group determined by the regular representation of a locally compact abelian group in its own L_2 space. Their proof involves the Plancherel theorem and is not available in this context. Recently, A. Ionescu Tulcea (10) has proved that if U is the unitary operator induced in L_2 of a

σ-finite measure space by a nonperiodic measure preserving transformation, then the spectrum of U is the entire unit circle.

Now let G be an arbitrary unitary group in the Hilbert space $\mathcal{H}$. We say the action of G is <u>nondegenerate</u> if for every finite subset F of G, there exists a nonzero vector $\xi \in \mathcal{H}$ such that $U\xi \perp V\xi$ for every $U \neq V$ in F. Using the facts that G is a group and that unitary operators preserve orthogonality, it is easily seen that this condition is equivalent to the following: for every finite subset F of G such that $I \notin F$, there exists a nonzero vector $\xi \in \mathcal{H}$ such that $\xi \perp U\xi$ for every $U \in F$.

It can be seen that the unitary group in L_2 determined by the left regular representation of a locally compact group is nondegenerate. This situation is really a special case of the more general example from ergodic theory. As it is not our intention to enter a lengthy discussion of measure theoretic details for this example, we shall merely sketch results, all of which are known in one form or another. Let X be a locally compact Hausdorff space and let m be a regular Borel measure on the σ-algebra $\mathcal{B}$ of Borel sets in X (8). By a <u>measure preserving</u> transformation (MPT) we mean a mapping σ: $X \to X$ such that $\sigma^{-1}(B) = \{x \in X: \sigma(X) \in B\} \in \mathcal{B}$ and $m(\sigma^{-1}B) = m(B)$ for every $B \in \mathcal{B}$. The set of all MPT's of X form a semigroup S with identity under the multiplication $(\sigma\tau)(x) = \sigma(\tau x)$, σ, $\tau \in S$, $x \in X$. Let G be a subgroup of S whose identity is the identity of S. For

$\sigma \in G$, define the operator U_σ on $L_2(x,\mathcal{B},m)$ by $(U_\sigma f)(x) = f(\sigma^{-1}x)$, $f \in L_2$. Then $\sigma \to U_\sigma$ is a faithful unitary representation of G in $L_2(X,\mathcal{B},m)$. The group G is said to be freely-acting if, for every $\sigma \in G$ different from the identity and every $F \in \mathcal{B}$ such that $m(F) > 0$, there exists a Borel subset E of F such that $0 < m(E) < \infty$ and $m(E \cap \sigma^{-1}E) = 0$. This definition is essentially von Neumann's, and a discussion of it can be found in (4). If G is a freely-acting group of MPT's, then by taking intersections in the obvious way, we conclude that for every finite subset $\sigma_1,\ldots,\sigma_n$ in G all different from the identity, and every Borel set F such that $0 < m(F) < \infty$, there exists a nonnull Borel subset E of F such that $m(E) < \infty$ and $m(E \cap \sigma_k^{-1} E) = 0$, $1 \leq k \leq n$. By considering the characteristic function of E as an element of $L_2(X,\mathcal{S}, m)$ it follows that the unitary group $\{U_\sigma\colon \sigma \in G\}$ is nondegenerate.

Applying this to the above example, we need note merely that, with respect to left Haar measure on a locally compact group, the group of left translations constitutes a freely-acting group of MPT's.

As a second example, let $(X, \mathcal{S}, m)$ be a σ-finite measure space, and let τ be an invertible MPT which is nonperiodic in the sense that $\{x\colon \tau^n x \neq x\}$ is not a null set, for every $n \geq 1$. It is not difficult to show that for every $n \geq 1$, there exists a set $B_n \in \mathcal{S}$ such that $0 < m(B_n) < \infty$ and $m(B_n \cap \tau^{-k} B_n) = 0$ for $k = 1,2,\ldots,n$ (10). As in the last example, it follows that the cyclic unitary group induced in $L_2(X, \mathcal{S}, m)$ by τ is nondegenerate.

We turn now to the main result of this section.

Theorem 2.1: Let G be an abelian unitary group on $\mathcal{H}$, generating the C^*- algebra $\mathcal{A}$. In order that the image of $\sigma(\mathcal{A})$ under the natural mapping α be all of Γ, it is necessary and sufficient that the action of G be nondegenerate.

In this event, of course, α will be a homeomorphism. First, we prove sufficiency.

Lemma 2.1: Let G be any subset of the unitary group in an abelian C^*- algebra $\mathcal{A}$, and let γ be any complex-valued function defined on G. In order that there exist an $\omega \in \sigma(\mathcal{A})$ whose restriction to G is γ, it is necessary and sufficient that

$$\inf_{\|\xi\|=1} \sum_{k=1}^{n} \|U_k\xi - \gamma(U_k)\xi\| = 0,$$

for every finite subset $U_1,\ldots,U_n$ of G.

Proof: (Necessity) Let $\omega \in \sigma(\mathcal{A})$, $U_1,\ldots,U_n \in G$. Clearly it suffices to show that $\inf \sum_{k=1}^{n} \|U_k\xi - \omega(U_k)\xi\|^2 = 0$, $\|\xi\| = 1$.
Let $A = \sum_{k=1}^{n} (U_k - \omega(U_k)I)^*(U_k - \omega(U_k)I)$. Then A is a positive operator in $\mathcal{A}$. If $\inf_{\|\xi\|=1} (A\xi,\xi) = \epsilon > 0$, then $A - \epsilon I \geq 0$ which implies that A is regular. But by construction, the Gelfand transform of A has a zero at $\omega \in \sigma(\mathcal{A})$, a contradiction.

For sufficiency, note first that $|\gamma(U)| = 1$ for all $U \in G$. Indeed if $\|\xi\| = 1$, then

$\|U\xi - \gamma(U)\xi\| \geq |\, \|U\xi\| - |\gamma(U)| \cdot \|\xi\| \,| = |1 - |\gamma(U)||$; and by taking the infimum over $\|\xi\| = 1$, we get $|\, 1 - |\gamma(U)| \,| = 0$.

For every $U \in G$, define $K_U = \{\omega \in \sigma(\mathcal{A}) : \omega(U) = \gamma(U)\}$. We have to show that $\bigcap \{K_U : U \in G\} \neq \emptyset$. Since each K_U is a compact subset of $\sigma(\mathcal{A})$, it suffices to show that these sets have the finite intersection property. Fix $U_1, \ldots, U_n \in G$ and let $T = \frac{1}{n} \sum_{k=1}^{n} \bar{\gamma}(U_k) U_k \in \mathcal{A}$. Then for every $\xi \in \mathcal{H}$,

$$\|(A-I)\xi\| = \|\frac{1}{n} \sum_{k=1}^{n} (\bar{\gamma}(U_k)U_k - I)\xi\| \leq$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} \|\bar{\gamma}(U_k)U_k\, \xi - \xi\| = \frac{1}{n} \sum_{k=1}^{n} \|U_k\xi - \gamma(U_k)\xi\|.$$

So by hypothesis $\inf_{\|\xi\|=1} \|(A-I)\xi\| = 0$, implying that A-I is not regular. There exists, therefore, an element $\omega \in \sigma(\mathcal{A})$ such that

$$1 = \omega(I) = \omega(A) = \frac{1}{n} \sum_{k=1}^{n} \bar{\gamma}(U_k)\omega(U_k).$$

Since each summand has unit modulus and 1 is an extreme point of the unit disc, we have $\bar{\gamma}(U_k)\omega(U_k) = 1$ for $k = 1,2,\ldots,n$. Therefore $\omega \in \bigcap_{k=1}^{n} K_{U_k}$, completing the proof of lemma 1.

The author is indebted to Professor H. A. Dye for suggesting the following line of argument, thereby simplifying considerably the proof of sufficiency. The neat proof given for lemma 2.2 is his.

We shall write $|E|$ for the number of elements in the set E, and $E \setminus F$ for the set-theoretic difference consisting of those elements of E not in F.

<u>Lemma 2.2:</u> Let F be a finite subset of an abelian group H. Then for every $\epsilon > 0$ there exists a finite subset S of H such that $|FS \setminus S| \le \epsilon\, |S|$.

<u>Proof</u>: Say $F = \{x_1, x_2, \ldots, x_n\}$. For every $r \ge 1$, let $F_r = \{x_1^{r_1} x_2^{r_2}, \ldots, x_n^{r_n} : 1 \le r_k \le r\}$. The sequence F_r is increasing and $FF_r \subseteq F_{r+1}$. We claim $|F_{r+1}| \le (1+\epsilon)\, |F_r|$ for some r. Otherwise, $|F_{r+1}| > (1+\epsilon)|F_r|$ for all $r \ge 1$ and hence $|F_r| > (1+\epsilon)^{r-1}\, |F_1| = (1+\epsilon)^{r-1}$. This means that $(1+\epsilon)^{r-1} < r^n$ for every $r \ge 1$ since by construction $|F_r| \le r^n$, which is absurd.

Now choose such an r, and let $S = F_r$. Then

$$|FS \setminus S| \le |F_{r+1} \setminus F_r| = |F_{r+1}| - |F_r|$$

$$\le (1+\epsilon)|F_r| - |F_r| = \epsilon |S|,$$ proving lemma 2.2.

Now let F be a finite subset of G, and let $\gamma \in \Gamma$. By lemma 2.1, it suffices to show that for every $\epsilon > 0$, there exists $\xi \in \mathcal{H}$, $\|\xi\| = 1$, such that

$$\max_{U \in F} \|U\xi - \gamma(U)\xi\| = \max_{U \in F} \|\bar{\gamma}(U)U\xi - \xi\| \le 2\epsilon.$$

Let $F' = \{\bar{\gamma}(U)U : U \in F\}$ and let G' be the group $\{\bar{\gamma}(U)U : U \in G\}$. It is clear that G' is a nondegenerate subgroup of the unitary group in $\mathcal{A}$.

By lemma 2.2, there exists a finite subset $S \subseteq G'$ such that $|F'S \setminus S| \leq \epsilon |S|$. By nondegeneracy, choose a nonzero $\zeta \in \mathcal{H}$ such that $V\zeta \perp W\zeta$ for all $W \neq V$ in $S \cup F'S$.

Let $\xi = \sum_{V \in S} V\zeta$. Clearly $\|\xi\|^2 = |S| \cdot \|\zeta\|^2 > 0$. If $W \in G'$ then

$W\xi - \xi = \sum_{WS} V\zeta - \sum_{S} V\zeta = \sum_{WS \setminus S} V\zeta - \sum_{S \setminus WS} V\zeta$, since the summands cancel over $S \cap WS$. Now

$$|S \setminus WS| = |S| - |S \cap WS| = |WS| - |S \cap WS| = |WS \setminus S|;$$

so that if $W \in F'$, then by orthogonality

$$\begin{aligned} \|W\xi - \xi\|^2 &= (|WS \setminus S| + |S \setminus WS|)\ \|\zeta\|^2 \\ &= 2|WS \setminus S|\ \|\zeta\| \leq 2|F'S \setminus S| \cdot \|\zeta\|^2 \\ &\leq 2\epsilon |S| \cdot \|\zeta\|^2 = 2\epsilon \|\xi\|^2. \end{aligned}$$

The desired conclusion follows by normalizing ξ.

It remains to prove that the condition is necessary. Let F be a finite subset of G such that $I \notin F$. Assume first that F contains both self-adjoint and non self-adjoint elements, the distinct self-adjoint unitaries being $U_1, \ldots, U_m$. For each of the remaining elements V, V and V^{-1} are distinct: we discard one of them from F when (and only when) both are present. Let the distinct elements remaining be $V_1, \ldots, V_n$. Clearly the sets $\{U_1, \ldots, U_m\}$, $\{V_1, \ldots, V_n\}$ and $\{V_1^{-1}, \ldots, V_n^{-1}\}$ are disjoint, and if $F_o = \{U_1, \ldots, U_m, V_1, \ldots, V_n\}$ then $\xi \perp F\xi \Leftrightarrow \xi \perp F_o\, \xi$, for every $\xi \in \mathcal{H}$.

Now suppose $\xi \perp F_o\xi$ fails for every $\xi \neq 0$ in $\mathcal{H}$. Let $\mathcal{V}$ be the real vectorspace of all bounded self-adjoint ($\rho(T^*) = \bar{\rho}(T)$ for all $T \in \mathcal{A}$) linear functionals on $\mathcal{A}$, and let Ω be the subset of $\mathcal{V}$ consisting of all canonical states $\omega_\xi(T) = (T\xi,\xi)$ with $\|\xi\| = 1$. Observe that Ω in convex. For let $\xi,\eta \in \mathcal{H}$, $\|\xi\| = \|\eta\| = 1$ and take $\theta \in [0,1]$. Consider the functional $\rho(T) = \theta\omega_\xi(T) + (1-\theta)\omega_\eta(T)$ defined on the von Neumann algebra $\mathcal{B} = \bar{\mathcal{A}}^W$. As ρ is weakly continuous and $\mathcal{B}$ is abelian, ρ already has the form $\rho = \omega_\zeta$ for some $\zeta \in \mathcal{H}$ (see (2), p.233). We have $\|\zeta\|^2 = \rho(I) = \theta\|\xi\|^2 + (1-\theta)\|\eta\|^2 = 1$ and hence the restriction of ρ to $\mathcal{A}$ is in Ω.

Consider the linear mapping

$$\rho \in \mathcal{V} \to (\rho(U_1),\ldots,\rho(U_m),\ \rho(V_1),\ldots\rho(V_n))$$

of $\mathcal{V}$ into the m+2n-dimensional real vectorspace $\mathcal{R}^m \oplus \mathcal{C}^n = \{(x_1,\ldots,x_m,\ z_1,\ldots,z_n): x_i \in \mathcal{R},\ z_j \in \mathcal{C}\}$. The image K of Ω is a convex subset of $\mathcal{R}^m \times \mathcal{C}^n$, and by our assumption on F_o, K does not contain the origin. By a standard separation theorem, there exists a nontrivial real linear functional f on $\mathcal{R}^m \times \mathcal{C}^n$ such that $f(K) \geq 0$.

It is easily seen that f has the form $f(x_1,\ldots,x_m,z_1,\ldots,z_n) = \sum_{k=1}^{m} a_k x_k + \sum_{k=1}^{n} b_k z_k + \sum_{k=1}^{n} \bar{b}_k \bar{z}_k$, where $a_k \in \mathcal{R}$, $b_k \in \mathcal{C}$. Define the operator $T = \sum a_k U_k + \sum b_k V_k + \sum \bar{b}_k V_k^{-1} \in \mathcal{A}$. For every $\xi \in \mathcal{H}$, $\|\xi\| = 1$ we have

$$\omega_\xi(T) = \sum a_k \omega_\xi(U_k) + \sum b_k \omega_\xi(V_k) + \sum \bar{b}_k \overline{\omega_\xi(V_k)}$$
$$= f(\omega_\xi(U_1),\ldots,\omega_\xi(U_m),\ \omega_\xi(V_1),\ldots,\omega_\xi(V_n)) \geq 0.$$

Therefore T is positive. By hypothesis, we may identify $\sigma(\mathfrak{A})$ with Γ by virtue of the mapping α, the Gelfand transform $A\in\mathfrak{A}\to\hat{A}\in\mathcal{C}(\Gamma)$ taking G isomorphically into the character group of Γ. The continuous function

$$\hat{T} = \Sigma\, a_k\, \hat{U}_k + \Sigma\, b_k\, \hat{V}_k + \Sigma\, \bar{b}_k\, \hat{V}_k^{-1}$$

is nonnegative everywhere and its Haar integral is zero because the characters $\hat{U}_k$, $\hat{V}_k$, $\hat{V}_k^{-1}$ are all different from the function 1. Hence $\hat{T}$ vanishes identically. But by construction the characters on the right are distinct and therefore linearly independent, so that $a_1 = \dots = a_m = b_1 = \dots = b_n = 0$, contradicting our original choice of f.

A parallel argument applies if the original set F consists entirely of self-adjoint or non self-adjoint elements. One merely replaces $\mathcal{R}^m \oplus \mathcal{C}^n$ with $\mathcal{R}^m$ or $\mathcal{C}^n$ depending on which case occurs. This completes the proof of theorem 1.

We conclude this discussion with the remark that if G is any nondegenerate unitary group and $\mathfrak{A}$ is the generated C^*-algebra, then there exists a state φ of $\mathfrak{A}$ such that $\varphi(U) = 0$ for every $U\in G$ different from the identity. That is, the sets $K_U = \{\rho\in\Sigma(\mathfrak{A}):\ \rho(U)=0\}$ are weak* compact subsets of the state space of $\mathfrak{A}$, and for $U \neq I$ every finite intersection contains a canonical state. Thus $\bigcap_{U\neq I} K_U \neq \emptyset$. Of course the state φ is uniquely determined by this condition. If G is abelian then φ may be identified with the Haar integral over

$\Gamma = \sigma(\mathcal{A})$, and it is therefore a faithful state. In general, a simple continuity argument shows that $\varphi(AB) = \varphi(BA)$ for every $A, B \in \mathcal{A}$.

3. A Decomposition for Certain Unitary Representations

Let G be a locally compact Hausdorff topological group, and let $x \in G \to U_x$ be a faithful unitary representation of G on the Hilbert space $\mathcal{H}$. For every $f \in \mathcal{C}_{oo}(G)$, we may form

$$T_f = \int f(x)\, U_x\, dx,$$

dx denoting left-invariant Haar measure. One has

$$T_f\, T_g = T_{f*g} \quad \text{and}$$

$$T_f^* = T_{f*} \quad ,$$

where $(f*g)(x) = \int f(t) g(t^{-1}x) dt.$ and $f^*(x) = \bar{f}(x^{-1}) \Delta^{-1}(x)$, Δ being the modular function defined by $\int f(x)dx = \int f(x^{-1})\, \Delta^{-1}(x) dx$ for $f \in \mathcal{C}_{oo}(G)$.

Let $\mathcal{J}_\infty$ be the intersection of all subspaces of the form $\bar{\mathcal{M}}_C^{UW}$,

where

$$\mathcal{M}_C = \{\int f(x) U_x\, dx:\ f \in \mathcal{C}_{oo}(G),\ f(C) = \{0\}\},$$

C ranging over the family $\mathcal{C}$ of all compact subsets of G.

<u>Proposition 3.1</u>: $\mathcal{J}_\infty$ is an UW-closed two-sided ideal in $\mathcal{B}$.

<u>Proof</u>: Clearly $\mathcal{J}_\infty$ is an UW-closed subspace, and it suffices to show that $\mathcal{J}_\infty^* \subseteq \mathcal{J}_\infty$ and $\mathcal{B}\mathcal{J}_\infty \subseteq \mathcal{J}_\infty$.

Let $T \in \mathcal{J}_\infty$, and let $C \in \mathcal{C}$. Now C^{-1} is compact, and since $T \in \mathcal{J}_\infty$, we can find a net $f_n \in \mathcal{C}_{oo}$ (G) such that $f_n(C^{-1}) = 0$ and $T_{f_n} = \int f_n(x)\, U_x dx \to T$ ultraweakly. Then $T_{f_n^*} = T_{f_n}^* \to T^*$ UW, and

$f_n^*(x) = \bar{f}_n(x^{-1})\Delta^{-1}(x) = 0$ whenever $x \in C$. This proves $T^* \in \{T_f : f(C)=0\}^{-UW}$, and hence $T^* \in \mathcal{J}_\infty$ since C is arbitrary.

The set $\mathcal{S} = \{T \in \mathcal{B} : T\mathcal{J}_\infty \subseteq \mathcal{J}_\infty\}$ is an UW-closed subspace because $\mathcal{J}_\infty$ is; and since $\mathcal{B} = \overline{\text{span}}^{UW}\{U_x : x \in G\}$, it suffices to show that $\mathcal{S}$ contains every U_x, $x \in G$. Let $x \in G$, $C \in \mathcal{C}$, $T \in \mathcal{J}_\infty$. $x^{-1}C$ is compact, and there exists a net $f_n \in \mathcal{C}_{oo}(G)$ such that $f_n(x^{-1}C) = 0$ and $T_{f_n} \to T$ UW. Then $U_x T_{f_n} \to U_x T$ UW. But

$$U_x T_{f_n} = U_x \int f_n(t)\, U_t\, dt = \int f_n(x^{-1}t)\, U_t\, dt, \text{ and } f_n(x^{-1}t) = 0$$

whenever $t \in C$. Because C is arbitrary, we have $U_x T \in \mathcal{J}_\infty$, completing the proof.

The ideal $\mathcal{J}_\infty$ consists of those operators of $\mathcal{B}$ which are in a sense at ∞. We shall henceforth refer to $\mathcal{J}_\infty$ as the <u>tail ideal</u>. We say the representation is <u>deterministic</u> or <u>regular</u> according as the extreme case $\mathcal{J}_\infty = \mathcal{B}$ or $\mathcal{J}_\infty = 0$ occurs. As we shall see, the idea of determinism seems to be a fruitful one only when the underlying group G is linearly ordered. Nevertheless, it will be convenient to have the definition at our disposal now. We will see also that this notion of determinism (for linear groups) is closely related to the idea bearing the same name in the prediction theory of stochastic processes indexed by the integers or the reals.

It is well known that every UW-closed two-sided ideal in a von Neumann algebra is principal; indeed, there exists a unique projection $P_\infty \in \mathcal{B} \cap \mathcal{B}'$ such that $\mathcal{J}_\infty = P_\infty \mathcal{B} = \mathcal{B} P_\infty$ (2). Of course, P_∞ reduces the representation, and $\mathcal{B}$ splits into the direct sum

$\mathcal{J}_\infty \oplus \mathcal{B}(I - P_\infty)$. It is significant that the summands are pure types. That is:

<u>Theorem 3.1</u>: The subrepresentation $x \to U_x P_\infty$ on $P_\infty\mathcal{H}$ is deterministic, and the subrepresentation $x \to U_x(I - P_\infty)$ on $(I - P_\infty)\mathcal{H}$ is regular. $Q_\infty = I - P_\infty$ is characterized as the largest central projection $Q \in \mathcal{B} \cap \mathcal{B}'$ such that the subrepresentation $x \to U_x Q$ on $Q\mathcal{H}$ is regular.

<u>Proof:</u> Consider first the subrepresentation $x \to U_x P_\infty$ in $P_\infty \mathcal{H}$, and let $\mathcal{J}_\infty$ be the tail ideal. P_∞ is the identity here, and it suffices to show that $P_\infty \in \mathcal{J}_\infty$. Let $C \in \mathcal{C}$, and choose a net $f_n \in \mathcal{C}_{00}(G)$ such that $f_n(C) = 0$ and $T_{f_n} = \int f_n(x) U_x dx \to P_\infty$ ultraweakly on $\mathcal{H}$. Then $T_{f_n} P_\infty = \int f_n(x) (U_x P_\infty) dx \to P_\infty^2 = P_\infty$ UW on $\mathcal{H}$ and therefore on the subspace $P_\infty \mathcal{H}$. Hence $P_\infty \in \mathcal{J}_\infty$.

Now consider $x \to U_x(I - P_\infty)$ on $(I - P_\infty)\mathcal{H}$, and let $\mathcal{K}_\infty$ be the tail ideal. Suppose Q is a nonnull projection in $\mathcal{K}_\infty$. Let $C \in \mathcal{C}$, and take a net $f_n \in \mathcal{C}_{00}(G)$ such that $f_n(C) = 0$ and $Q_n = \int f_n(x)\, U_x(I - P_\infty) dx \to Q$ UW. Considered as an operator on $\mathcal{H}$, we have $Q_n = \int f_n(x)\, U_x\, dx - \int f_n(x)\, U_x\, P_\infty\, dx =$

$$= \int f_n(x)\, U_x\, dx - \left(\int f_n(x)\, U_x\, dx\right) P_\infty.$$

Now $\left(\int f_n(x)\, U_x dx\right) P_\infty \in \mathcal{J}_\infty \subseteq \{\int g(x) U_x dx : g(C) = 0\}^{UW}$, so that Q_n, and finally Q itself, is in $\{\int g(x) U_x\, dx ; g(C) = 0\}^{UW}$. Because C is arbitrary, $Q \in \mathcal{J}_\infty$. Hence $P_\infty + Q \in \mathcal{J}_\infty$. But since $Q \leq I-P$, $P_\infty + Q$ is a projection, and $P_\infty < P_\infty + Q$ because $Q \neq 0$. This contradicts the fact that P_∞ is the largest projection in $\mathcal{J}_\infty$.

To prove the last statement, let Q be any central projection such that $x \to U_x Q$ is regular. Fix $C \in \mathcal{C}$, and take $f_n \in \mathcal{C}_{00}(G)$ such that $f_n(C) = 0$, $\int f_n(x)\, U_x\, dx \to P_\infty\, UW$. Then $\int f_n(x)\, (U_x Q) dx = (\int f_n(x)\, U_x\, dx)\, Q \to P_\infty Q\, UW$. Again, since C is arbitrary, $P_\infty Q$ belongs to the tail ideal of $\{U_x Q: x \in G\}$. We conclude $P_\infty Q = 0$, or $Q \leq I - P_\infty$, as asserted.

In the remainder of this section, we discuss a relation between the tail ideal and a distinguished trace on $\mathcal{B}$, assuming that the latter exists.

Let $\mathcal{B}_o$ be the $*$-subalgebra of operators $T_f = \int f(x)\, U_x\, dx$, $f \in \mathcal{C}_{00}(G)$, and let $\mathcal{B}_o^2$ be the $*$-subalgebra consisting of all operators of the form $\sum_{k=1}^{n} T_{f_k} T_{g_k}$, where $f_k, g_k \in \mathcal{C}_{00}(G)$. It is well-known that $\mathcal{B}_o^2$ is weakly dense in $\mathcal{B}$.

Now suppose there is given a trace φ on $\mathcal{B}^+$ satisfying $\varphi(T_f^* T_f) = \int |f(x)|^2\, dx$ for every $f \in \mathcal{C}_{00}(G)$. Let $\mathcal{J} = \{T \in \mathcal{B}: \varphi(T^*T) < \infty\}$ and let $\mathfrak{J} = \left\{\sum_{k=1}^{n} A_k B_k: A_k, B_k \in \mathcal{J}\right\}$ be the ideal of definition of φ. Since every element of $\mathcal{B}_o^2$ is a linear combination of operators of the form $T_f^* T_f = T_{f^* * f}$ (viz., $f^* g =$

$$= 1/4[(f+g)^* * (f+g) - (f-g)^* * (f-g) - i(f+ig)^* * (f+ig) + i(f-ig)^* * (f-ig)]),$$

we have $\mathcal{B}_o^2 \subseteq \mathfrak{J}$; hence $\mathfrak{J}$ is weakly dense in $\mathcal{B}$, so that if φ is normal to begin with it is necessarily also semifinite. Now

$$\varphi(T_{f^*{*}f}) = \varphi(T_f^* T_f) = \int |f(x)|^2\, dx = \int |f(x^{-1})|^2\, \Delta^{-1}(x)\, dx =$$
$$= \int f^*(x)\, f(x^{-1})\, dx = (f^* {*} f)(e).$$

Again, since the operators $T_{f^*{*}f}$ span $\mathcal{B}_o^2$, it follows that for every h of the form $h = \sum_{k=1}^{n} f_k {*} g_k$, f_k, $g_k \in \mathcal{C}_{oo}(G)$, one has $\varphi(T_h) = h(e)$. Thus we may evaluate φ on the dense subalgebra $\mathcal{B}_o^2$.

Note here that such a group is already unimodular. Indeed since φ is a trace, we have

$$\int |f(x)|^2\, dx = (f^* {*} f)(e) = \varphi(T_f^*\, T_f) = \varphi(T_f T_f^*) =$$
$$= (f {*} f^*)(e) = \int |f(x)|^2 \Delta(x)\, dx.$$

Because this holds for every $f \in \mathcal{C}_{oo}(G)$, we have $\Delta \equiv 1$.

Theorem 3.2: Let φ be a normal trace on $\mathcal{B}^+$ satisfying $\varphi(T_f^*\, T_f) = \int |f(x)|^2 dx$ for every $f \in \mathcal{C}_{oo}(G)$. Then $\mathcal{J}_\infty \subseteq \{T \in \mathcal{B} : \varphi(T^* T) = 0\}$. Equality obtains if G is discrete.

First, we establish a lemma,

Lemma 3.1: Let $A \in \mathcal{J}_\infty$ and let $h = \sum_{k=1}^{n} f_k {*}\, g_k$, f_k, $g_k \in \mathcal{C}_{oo}(G)$. Then $\varphi(AT_h) = 0$.

Proof: Note first that T_h, and therefore AT_h belong to the ideal of definition of φ, so that it is legitimate to consider $\varphi(AT_h)$. Let C_h be a compact set such that $h = 0$ off C_h. If f is any function in $\mathcal{C}_{oo}(G)$ such that $f(C_h^{-1}) = 0$, then $f(x)h(x^{-1}) \equiv 0$ on G; consequently, $(f {*} h)(e) = \int f(x)h(x^{-1})dx = 0$. Hence $\varphi(T_f T_h) = \varphi(T_{f*h}) = (f {*} h)(e) = 0$ for every such f. This says that the functional

$\omega(T) = \varphi(TT_h)$ vanishes on the subspace $\mathcal{S} = \{T_f : f \in \mathcal{C}_{oo}(G),$ $f(\bar{C}_h^{-1}) = 0\}$. As $T_h \in \mathcal{J}$ and φ is normal, ω is bounded and UW-continuous. Hence ω vanishes on $\bar{\mathcal{S}}^{UW}$. But $\bar{C}_h^{-1}$ is compact, and $A \in \mathcal{J}_\infty \subseteq \bar{\mathcal{S}}^{UW}$. We conclude $\varphi(AT_h) = \omega(A) = 0$, as asserted.

Turning now to the proof of the theorem, fix $T \in \mathcal{J}_\infty$, and let $\mathcal{J}$ be the ideal of definition of φ, as above. $\mathcal{J}$ is a weakly dense two-sided ideal in $\mathcal{B}$, since φ is normal and semifinite. Hence, there exists a family $A_\alpha \in \mathcal{J}^+$, directed $\uparrow$ under the usual operator order such that LUB $A_\alpha = T^*T$ (see (2), p.45). Now $0 \leq A_\alpha \leq T^*T$, and T^*T belongs to the weakly closed ideal $\mathcal{J}_\infty$. Hence $A_\alpha \in \mathcal{J} \cap \mathcal{J}_\infty$, for every α.

Fix α, and define $\psi(X) = \varphi(A_\alpha X)$, for $X \in \mathcal{B}$. Because $A_\alpha \in \mathcal{J}$, ψ is a bounded UW-continuous linear functional on $\mathcal{B}$. Since $A_\alpha \in \mathcal{J}_\infty$, the lemma tells us that ψ vanishes on $\mathcal{B}_o^2$; and by UW-continuity, $\psi = 0$ on $(\mathcal{B}_o^2)^{-UW} = \mathcal{B}$. In particular, $\varphi(A_\alpha) = \psi(I) = 0$. Finally, we have $\varphi(T^*T) = \sup \varphi(A_\alpha) = 0$, by normality.

If G is discrete, $\mathcal{C}_{oo}(G)$ consists of the complex functions having finite support, and $\mathcal{J}_\infty = \bigcap \overline{\text{span}}^{UW} \{U_x : x \notin F\}$, F ranging over the finite subsets of G. Also, $\varphi(I) = \varphi(U_e) = \varphi(U_e^* U_e) = 1$, so that φ is in fact a normal state.

Let $T \in \mathcal{B}$, $\varphi(T^*T) = 0$. Choose a net $f_n \in \mathcal{C}_{oo}(G)$ such that

$T_{f_n} = \Sigma f_n(x) U_x \to T$ US. Then $T_{f_n}^* T_{f_n} \to T^*T$ UW. Since φ is UW-continuous, $\Sigma |f_n(x)|^2 = \varphi(T_{f_n}^* T_{f_n}) \to \varphi(T^*T) = 0$, as $n \to \infty$.

Let F be a finite set, and define $g_n(x) = f_n(x)$ for $x \notin F$, $g_n(x) = 0$ for $x \in F$. Let ψ be any UW-continuous linear functional on $\mathcal{B}$. Then

$$|\psi(T) - \psi(T_{g_n})| \leq |\psi(T-T_{f_n})| + |\psi(T_{f_n}-T_{g_n})|.$$

But $$|\psi(T_{f_n} - T_{g_n})| = |\sum_G (f_n(x) - g_n(x))\psi(U_x)| =$$

$$= |\sum_F f_n(x)\ \psi(U_x)| \leq \sum_F |f_n(x)|\ \|\psi\| \leq$$

$$\leq (|F| \sum_F |f_n(x)|^2)^{1/2} \|\psi\| \leq (|F| \sum_G |f_n(x)|^2)^{1/2} \|\psi\|,$$

where $|F|$ denotes the number of elements in F. It follows that $\lim_n |\psi(T) - \psi(T_{g_n})| = 0$ for every such ψ, and hence $T \in \overline{\text{span}}^{UW} \{U_x;\ x \notin F\}$. As F is arbitrary, $T \in \mathcal{J}_\infty$, completing the proof of the theorem.

It seems reasonable to expect that $\mathcal{J}_\infty = \{T \in \mathcal{B} : \varphi(T^*T) = 0\}$ for an arbitrary locally compact unimodular group G. We have not yet studied this conjecture sufficiently.

In any case, one has the following corollary.

Corollary: Let $x \to U_x$ be a faithful weakly continuous representation of the locally compact unimodular group G. If the representation is deterministic, then there is no normal trace on $\mathcal{B}^+$ such that $\varphi(T_f^* T_f) = \int |f(x)|^2 dx$, for every $f \in \mathcal{C}_{oo}(G)$.

Proof: For such a φ, we have $\{T \in \mathcal{B} : \varphi(T^*T) = 0\} \supseteq \mathcal{J}_\infty = \mathcal{B}$. Thus φ is identically 0, which is absurd.

In section 4 we consider the problem of the existence of such a trace, for the case where G belongs to a class of discrete groups.

4. Existence of the Canonical Trace

In this section we consider the faithful representations of a fairly extensive family of torsion-free discrete groups. The idea of determinism will be used to characterize the existence of a canonical trace on the generated von Neumann algebra; implicit in this is an algebraic characterization of the left regular representation. These algebras contain prototypes for the theory discussed in Section 6.

By a left-linear group we mean a pair $(G,\leq)$ consisting of a group G endowed with a linear order $\leq$ satisfying the left invariance rule $x \leq y$ implies $zx \leq zy$, for all $x, y, z \in G$. Right-linear groups are defined similarly, the order relation being invariant under right multiplication. For such a group, $x < y$ will mean $x \leq y$ and $x \neq y$.

The most elementary properties of left-linear groups are these.

(a) Every subgroup of a left-linear group is left-linear in the inherited order.

(b) If $(G,\leq)$ is left-linear, then the order $\precsim$ defined by $x \precsim y$ if and only if $y^{-1} \leq x^{-1}$ makes G into a right-linear group.

(c) If $(G,\leq)$ is left-linear and $S = \{x \in G: x \geq e\}$, then S is a subsemigroup of G satisfying $S \cap S^{-1} = \{e\}$, $S \cup S^{-1} = G$, and $x \leq y \Leftrightarrow x^{-1}y \in S$. G is simultaneously left- and right-linear with respect to $\leq$ if and only if $xSx^{-1} \subseteq S$, for every $x \in G$. Conversely, if S is a subsemigroup of G such that $S \cap S^{-1} = \{e\}$ and $S \cup S^{-1} = G$, then the relation $x \leq y \Leftrightarrow x^{-1}y \in S$ makes G into a left-linear group.

(d) If $\{G_\alpha : \alpha \in A\}$ is a family of left-linear groups, then the direct product ΠG_α, $\alpha \in A$, can be made into a left-linear group.

The statement (a) is obvious, and the proof of (b) and (c) is a routine verification. We indicate the proof of (d). Let $G = \Pi G_\alpha$, $\alpha \in A$. Well-order A. For every $x \neq e$ in G, write $x > e$ or $x < e$ according as the first $\alpha \in A$ such that $x_\alpha \neq e_\alpha$ (e_α being the identity of G_α) satisfies $x_\alpha > e_\alpha$ or $x_\alpha < e_\alpha$. It is easily seen that the set S consisting of the identity of G together with all $x \in G$ such that $x > e$ is a semigroup such that $S \cap S^{-1} = \{e\}$ and $S \cup S^{-1} = G$.

According to (c), an abstract group can be made into a left-linear group if, and only if, it contains a certain kind of subsemigroup. We now give another simple characterization of this property. Given a group G, let $\mathfrak{N}$ be the family of all normal subgroups N of G such that G/N can be made into a left-linear group. Clearly $G \in \mathfrak{N}$.

<u>Proposition 4.1:</u> $G/\cap\mathfrak{N}$ can be made into a left-linear group.

<u>Proof:</u> $N_0 = \cap\mathfrak{N}$ is a normal subgroup and, by (a) and (d), it suffices to show that G/N_0 can be mapped isomorphically into a direct product of left-linear groups.

Let $\hat{G}$ be the direct product of the groups G/N, $N \in \mathfrak{N}$. For $x \in G$, let $\hat{x}$ be the function on $\mathfrak{N}$ whose value at N is the coset $xN \in G/N$.

Then $x \to \hat{x}$ is a homomorphism of G into $\hat{G}$, and the kernel of this homomorphism is precisely $\cap \mathfrak{N} = N_o$. The induced isomorphism of the quotient group G/N_o is of the desired kind, and the proof is complete.

By a <u>left-linear</u> homomorphism of a group G, we mean a homomorphism of G into a left-linear group. The following corollary asserts that the left-linear groups are precisely those possessing sufficiently many left-linear homomorphisms.

<u>Corollary:</u> Let G be an abstract group. In order that there exist at least one left-linear ordering on G, it is necessary and sufficient that for every $x \neq e$ in G, there exists a left-linear homomorphism h of G such that h(x) is not the identity in h(G).

<u>Proof:</u> Necessity is obvious: the identity mapping taken for h works simultaneously for every $x \neq e$ in G. Conversely, if h is any homomorphism of G and if N is the kernel of h, then clearly $N \in \mathfrak{N}$ if and only if h is left-linear. Every $N \in \mathfrak{N}$ appears in this way, and $h(x) \neq$ identity in h(G) if and only if $x \notin N$. Thus the hypothesis implies that $\cap \mathfrak{N} = \{e\}$, and by the preceding theorem, $G \cong G/\cap \mathfrak{N}$ can be made into a left-linear group. QED.

Observe that a left-linear group G is torsion-free. Indeed, suppose $e < x$. Then $e < x = xe \leq x^2 = x^2 e \leq x^3$ ... which gives $e < x^n$ for every $n \geq 1$. Similarly if $x < e$, we have $x^n < e$ for every $n \geq 1$. Conversely, every <u>abelian</u> torsion-free group can be made into a left-linear group (1). Following are some examples of nonabelian left-linear groups.

1. G is the proper affine group consisting of the multiplicative group of matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with x and y real, $x > 0$.

   $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} x' & y' \\ 0 & 1 \end{pmatrix}$ means $x < x'$, or $x = x'$ and $y \leq y'$.

   $(G, \leq)$ is both left- and right-linear.

2. G is the group of matrices $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ with x, y, z real.

   $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \leq \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix}$ means $x < x'$,

   or $x = x'$ and $y < y'$, or $x = x'$, $y = y'$ and $z \leq z'$.

   Again, $(G, \leq)$ is both left- and right-linear.

3. G is the same group as in (a), but with the order

   $$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} x' & y' \\ 0 & 1 \end{pmatrix}$$

   meaning $y < y'$, or $y = y'$ and $x \leq x'$.

   Then $(G, \leq)$ is left-linear but not right-linear.

Examples 1 and 2 are standard (see (1)). We indicate the verification of 3. Let S be the subset of G consisting of all $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ satisfying $y > 0$, or $y = 0$ and $x \geq 1$. It is easily seen that $S \cup S^{-1} = G$ and $S \cap S^{-1} = \{\text{identity}\}$. Let $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x' & y' \\ 0 & 1 \end{pmatrix} \in S$. Then $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' & y' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} xx' & xy'+y \\ 0 & 1 \end{pmatrix}$. But $xy'+y$ is positive unless $y = y' = 0$. If $y = y' = 0$ then $x \geq 1$ and $x' \geq 1$, so that $xx' \geq 1$. Hence S is a semigroup, and by property (c) above, $(G, \leq)$ is a left-linear group.

Let $a > 1$. Then $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in S$. But for every $y > 0$, we have

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & (1-a)y \\ 0 & 1 \end{pmatrix} \notin S.$$

By property (c) again, $\leq$ is not a right-linear ordering, and example 3 is verified.

Examples 1 and 3 illustrate that among the linear orderings a given group admits, some may be both right and left-invariant while others are merely left invariant.

We proceed now to the main result of this section. Throughout, G will be a fixed left-linear discrete group. If $x \to U_x$ is a unitary representation of G on $\mathcal{H}$, we say a subspace $\mathcal{M}$ of $\mathcal{H}$ is <u>wandering</u> when $U_x\mathcal{M} \perp U_y\mathcal{M}$ for every $x \neq y$ in G. The notation $[\xi_\alpha : \alpha \in A]$ will denote the subspace generated by the set of vectors $\{\xi_\alpha : \alpha \in A\}$ in $\mathcal{H}$.

Lemma 4.1: Let $x \to U_x$ be any unitary representation of G on the Hilbert space $\mathcal{H}$, and let $\xi \in \mathcal{H}$ be such that $\xi \notin [U_x\xi : x > e]$. Then $[U_x\xi : x \geq e]$ contains a nontrivial wandering subspace.

Proof: Let $\mathcal{M} = [U_x\xi : x > e]$. Then $\mathcal{M}$ is a proper subspace of $[U_x\xi : x \geq e]$. Choose a unit vector $\zeta \in [U_x\xi : x \geq e] \ominus \mathcal{M}$. For $t > e$, $U_t\zeta \in [U_{tx}\xi : x \geq e] \subseteq \mathcal{M}$, since $e < t \leq tx$ for every $x \geq e$, by left-linearity. Hence $(U_t\zeta, \zeta) = 0$ for every $t > e$. If $t < e$, then $(U_t\zeta, \zeta) = \overline{(U_t^{-1}\zeta, \zeta)} = 0$. So $U_t[\zeta] \perp [\zeta]$ for every $t \neq e$, and it is clear from this that $[\zeta]$ is a wandering subspace.

Lemma 4.2: Let $\mathcal{B}$ be a von Neumann algebra, let $\mathcal{S}$ be any convex subset of $\mathcal{B}$, and take for $\mathcal{P}$ the collection of all normal states of $\mathcal{B}$. In order that an element T of $\mathcal{B}$ lie in the UW-closure of $\mathcal{S}$, it is necessary and sufficient that $\inf_{S \in \mathcal{S}} \varphi((T-S)^*(T-S)) = 0$ for every $\varphi \in \mathcal{P}$.

Proof; (Necessity) Because $\mathcal{S}$ is convex, the ultraweak and ultrastrong closures of $\mathcal{S}$ are the same (see (2), p. 41). Choose a net $T_n \in \mathcal{S}$ such that $T - T_n \to 0$ US. Then $(T-T_n)^*(T-T_n) \to 0$ UW, and for every $\varphi \in \mathcal{P}$, we have

$$\lim_n \varphi((T-T_n)^*(T-T_n)) = 0.$$

(Sufficiency) Suppose T satisfies the stated condition, and choose n arbitrary UW-continuous linear functionals $\rho_1, \ldots, \rho_n$ on $\mathcal{B}$. We have to show that $\max_k |\rho_k(T-S)|$ can be made arbitrarily small, with $S \in \mathcal{S}$.

Because each ρ_k is a linear combination of elements of $\mathcal{P}$ (by polarization), there exist $\varphi_1,\ldots,\varphi_N \in \mathcal{P}$ and $c_{kj} \in \mathcal{C}$, $1 \le k \le n$, $1 \le j \le N$, such that $\rho_k = \Sigma_j\, c_{kj}\varphi_j$, for every $k = 1,\ldots,n$. Now for every k and every $A \in \mathcal{B}$,

$$|\rho_k(A)|^2 \le [\Sigma_j |c_{kj}|\; |\varphi_j(A)|]^2 \le$$

$$\le \Sigma_j |c_{kj}|^2\, \Sigma_j |\,\varphi_j(A)|^2 \le \Sigma_j |c_{kj}|^2\, \Sigma_j \varphi_j(A^*A),$$

the last inequality by the Schwarz inequality for states in a C^*-algebra. Summing the right side on k, we obtain $\max_k |\rho_k(A)|^2 \le \Gamma\, \psi\,(A^*A)$, where $\Gamma = N\Sigma_{k,j}\, |c_{kj}|^2$ and $\psi = \frac{1}{N}\sum_{j=1}^{n} \varphi_j$. Since $\mathcal{P}$ is convex, $\psi \in \mathcal{P}$. Hence

$$\inf_{S\in\mathcal{S}}\; \max_k\; |\rho_k(T-S)| \le \Gamma^{1/2}[\inf_{S\in\mathcal{J}} \psi((T-S)^*(T-S))]^{1/2} = 0,$$

which was to be proved.

Let $x \to U_x$ be a faithful unitary representation of G on $\mathcal{H}$, and let $\mathcal{B}$ be the von Neumann algebra generated by $\{U_x\colon x \in G\}$. Note first that a normal trace on $\mathcal{B}^+$ such that $\varphi(T_f^*\, T_f) = \Sigma|f(x)|^2$ for all $f \in \mathcal{C}_{oo}(G)$ is nothing other than a normal state satisfying $\varphi(U_x) = 0$ for all $x \ne e$. Indeed if φ is such a trace, let $\delta(x) = 0$ for $x \ne e$ and $\delta(e) = 1$. Then $I = U_e = U_e^*\, U_e = T_\delta^*\, T_\delta$, so that $\varphi(I) = \Sigma|\delta(x)|^2 = 1$. Therefore φ is finite, the ideal of definition of φ is all of $\mathcal{B}$, and the extension of φ to $\mathcal{B}$ is in fact a normal state. As in the previous section, $\varphi(T_f T_g) = \varphi(T_{f*g}) = (f*g)(e)$ for every f, $g \in \mathcal{C}_{oo}(G)$, so that for every such f we have

$\varphi(T_f) = \varphi(T_{f*\delta}) = (f*\delta)(e) = f(e)$. From this it is clear that $\varphi(T_x) = \delta(x)$. Conversely, if φ is a normal state satisfying $\varphi(U_x) = \delta(x)$ for every $x \in G$, then $\varphi(T_f) = \Sigma f(x)\varphi(U_x) = f(e)$ for every $f \in \mathcal{C}_{oo}(G)$, and hence $\varphi(T_f^* T_f) = \varphi(T_{f^* * f})$ $= \Sigma|f(x)|^2$. Being a normal state, φ is UW-continuous, and we have $\varphi(T_f T_g) = \varphi(T_{f*g}) = (f*g)(e) = (g*f)(e) = \varphi(T_g T_f)$, for $f, g \in \mathcal{C}_{oo}(G)$. By an obvious continuity argument, it follows that $\varphi(AB) = \varphi(BA)$ for all $A, B \in \mathcal{B}$, showing that φ is indeed a trace.

<u>Theorem 4.1</u>: Let $x \to U_x$ be a faithful unitary representation of the left-linear discrete group G on the Hilbert space $\mathcal{H}$. Then the following are equivalent:

(a) There exists a normal state φ on $\mathcal{B}$ such that $\varphi(U_x) = \delta(x)$ for all $x \in G$.

(b) $\mathcal{J}_\infty \neq \mathcal{B}$ ($x \to U_x$ is not deterministic).

(c) $I \notin \overline{\text{span}}^{UW}\{U_x : x > e\}$.

If G is abelian, then (a) through (c) are equivalent to:

(d) There exists a nontrivial wandering subspace.

<u>Proof</u>: (a) $\Rightarrow$ (b): By theorem 3.2 and the preceding remarks, for such a φ one has $\mathcal{J}_\infty = \{T \in \mathcal{B} : \varphi(T^*T) = 0\}$. Hence $I \notin \mathcal{J}_\infty$.
(b) $\Rightarrow$ (c): Suppose $I \in \overline{\text{span}}^{UW}\{U_x : x > e\}$. We will show that $I \in \mathcal{J}_\infty$, contradicting (b). Let F be any finite subset of G, and let $P = \{x_1, x_2, \ldots, x_n\}$ be the nonnegative elements of F, $e \leq x_1 < x_2 < \ldots < x_n$. It suffices to show $I \in \overline{\text{span}}^{UW}\{U_x : x \geq e, x \notin P\}$.

Note that if $\mathcal{S}$ is any subset of $\mathcal{B}$ and T is any operator in $\mathcal{B}$, then $T(\overline{\mathcal{S}}^{UW}) \subseteq (T\mathcal{S})^{-UW}$, this being an immediate consequence of the fact that the mapping $B \in \mathcal{B} \to TB$ is UW-continuous. So by hypothesis, we have for every x, $U_x \in U_x \overline{\text{span}}^{UW} \{U_t : t > e\} \subseteq \overline{\text{span}}^{UW} \{U_{xt} : t > e\} = \overline{\text{span}}^{UW} \{U_t : t > x\}$, the last inequality reflecting left-linearity of the group G.

Now $U_{x_n} \in \overline{\text{span}}^{UW} \{U_t : t > x_n\}$. Hence, $\overline{\text{span}}^{UW} \{U_t : t > x_{n-1}\} = \overline{\text{span}}^{UW} \{U_t : t > x_{n-1},\ t \neq x_n\}$. But the left side contains $U_{x_{n-1}}$, so that $\overline{\text{span}}^{UW} \{U_t : t > x_{n-2}\} = \overline{\text{span}}^{UW} \{U_t : t > x_{n-2},\ t \neq x_n, x_{n-1}\}$. Continuing this in the obvious way, we obtain $\overline{\text{span}}^{UW} \{U_t : t \geq e\} = \overline{\text{span}}^{UW} \{U_t : t \notin P\}$, from which the conclusion follows.

(c) $\Rightarrow$ (a): Let $\mathcal{M}$ denote the subspace of all finite linear combinations of the U_x, $x > e$. By hypothesis and lemma 4.2, there exists a normal state ρ of $\mathcal{B}$ and $\epsilon > 0$ such that $\rho((I-T)^*(I-T)) \geq \epsilon$ for every $T \in \mathcal{M}$. Let $K_\rho = \{X \in \mathcal{B} : \rho(X^*X) = 0\}$. We pass to the canonical $*$-representation of $\mathcal{B}$ associated with the state ρ. That is, define the bilinear form $(\cdot, \cdot)$ on the vectorspace $\mathcal{B}/K_\rho$ by $(A + K_\rho, B + K_\rho) = \rho(B^*A)$. This converts $\mathcal{B}/K_\rho$ into a prehilbert space, whose completion we denote by $\mathcal{H}_\rho$. K_ρ is a left ideal in $\mathcal{B}$, and for every $T \in \mathcal{B}$ the mapping $L_T(X + K_\rho) = TX + K_\rho$ in $\mathcal{B}/K_\rho$ extends uniquely to a bounded operator on $\mathcal{H}_\rho$. The mapping $T \in \mathcal{B} \to L_T$ is a $*$-homomorphism of $\mathcal{B}$ into $\mathcal{L}(\mathcal{H}_\rho)$, and in fact it is a normal homomorphism because the state ρ is normal.

Let $\bar{m}$ denote the closure in $\mathcal{H}_\rho$ of the set $\{T + K_\rho : T \in m\} \subseteq \mathcal{B}/K_\rho$, and let $\xi = I + K_\rho$, considered as a unit vector in $\mathcal{H}_\rho$. Clearly $x \to L_{U_x}$ is a unitary representation of G on $\mathcal{H}_\rho$, and we have $\bar{m} = [L_{U_x}\xi : x > e]$. Moreover, by our choice of the state ρ, it follows that $(\xi - \eta, \xi - \eta) \geq \epsilon > 0$ for every $\eta \in \bar{m}$. Applying lemma 4.1 there exists a unit vector $\zeta \in \mathcal{H}_\rho$ such that $(L_{U_x}\zeta, \zeta) = \delta(x)$, for every $x \in G$. To complete the proof, set $\varphi(T) = (L_T\zeta, \zeta)$, $T \in \mathcal{B}$. Normality of φ is insured by the corresponding property of the homomorphism $T \to L_T$.

If G is abelian then so is $\mathcal{B}$, and it is known that every normal state of $\mathcal{B}$ has the form $\varphi(T) = (T\zeta, \zeta)$ for some unit vector $\zeta \in \mathcal{H}$ (see (2), p. 233). Obviously $[\zeta]$ is wondering if φ satisfies condition (a). On the other hand, if ζ is a unit vector contained in some wandering subspace, then $\varphi(T) = (T\zeta, \zeta)$ is a state satisfying (a). This completes the proof of theorem 4.1.

According to this theorem, the representation $x \to U_x$ is deterministic if, and only if, $I \in \overline{\text{span}}^{UW} \{U_x : x > e\}$.

Now take for G the additive group of integers in the usual ordering, and let U be any unitary operator on the Hilbert space $\mathcal{H}$ such that $U^n \neq I$ for every n. Then $n \in G \to U^n$ is a faithful representation of G which, by lemma 4.1 and theorem 4.1(d), is deterministic if and only if $\xi \in [U^n\xi : n \geq 1]$ for every $\xi \in \mathcal{H}$. This example makes clear the relation between our definition of determinism and the classical meaning of the word in prediction theory: in the latter, one studies a cyclic unitary representation of G with respect to a distinguished cyclic vector ξ_0, calling the sequence $\xi_n = U^n \xi_0$ deterministic when $\xi_0 \in [\xi_{-1}, \xi_{-2}, \xi_{-3}, \ldots] = [U^n \xi_0 : n \leq -1]$ ((3),p.564). Of course the reversal of order here is insignificant.

Next, we characterize regularity. If $x \to U_x$ and $x \to V_x$ are two unitary representations of the discrete group G on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, repectively, we say the representations are <u>algebraically equivalent</u> if there exists a $*$-isomorphism Φ of the von Neumann algebra generated by $\{U_x\}$ onto that generated by $\{V_x\}$ such that $\Phi(U_x) = V_x$ for all $x \in G$.

<u>Corollary:</u> Let G be a left-linear discrete group. In order that a faithful unitary representation of G be regular, it is necessary and sufficient that it be algebraically equivalent to the left regular representation of G on $\ell_2(G)$.

<u>Proof:</u> Let $x \to \ell_x$ be the left-regular representation of G in $\ell_2(G)$. That is, $(\ell_x f)(t) = f(x^{-1}t)$, for $f \in \ell_2(G)$, $x, t \in G$. It is easy to see that $x \to \ell_x$ has no tail ideal. For example, if $\mathcal{B}$ is the von Neumann algebra generated by the ℓ_x, then the functional $\varphi(\Sigma f(x)\ell_x) = f(e)$, for $f \in \mathcal{C}_{00}(G)$, extends uniquely to a faithful normal finite trace such that $\varphi(\ell_x) = \delta(x)$ ((12), p.301). Hence $\mathcal{J}_\infty = \{T \in \mathcal{B} : \varphi(T^*T) = 0\} = 0$. Moreover, $\mathcal{J}_\infty$ is defined in terms of the representatives of group elements, vectorspace operations, and the UW-topology, all of which are

preserved under an algebraic equivalence. Hence any representation algebraically equivalent to the left regular representation will have a trivial tail ideal.

Conversely, suppose $x \to U_x$ is a faithful representation of G such that $\mathcal{J}_\infty = 0$. By theorems 3.2 and 4.1, there exists a faithful normal finite trace φ on $\mathcal{B}$ such that $\varphi(U_x) = \delta(x)$. Now form the canonical Hilbert algebra. That is, endowed with the bilinear form $(A,B) = \varphi(AB^*)$, $\mathcal{B}$ becomes a prehilbert space, whose completion we denote by $\mathcal{H}_\varphi$. For $T \in \mathcal{B}$, the mapping $L_T A = T_A$ extends to a bounded linear transformation in $\mathcal{L}(\mathcal{H}_\varphi)$, and $T \to L_T$ is a $*$-isomorphism of $\mathcal{B}$ into $\mathcal{L}(\mathcal{B}_\varphi)$ (2). Let $\mathcal{L}$ denote the image of $\mathcal{B}$. Now define $W: \mathcal{H}_\varphi \to \ell_2(G)$ as follows; for $x \in G$, let WU_x be the characteristic function of the singleton $\{x\}$, in $\ell_2(G)$. It is easily seen that W extends uniquely to a unitary mapping of $\mathcal{H}_\varphi$ onto $\ell_2(G)$, such that $W L_{U_x} W^{-1} = \ell_x$. The conclusion follows.

We close this section with a few remarks about a distinguished subalgebra of $\mathcal{B}$. Let $x \to U_x$ be a regular faithful representation of the left-linear group G. Let $\mathcal{S}$ denote the linear span of the U_x for $x \geq e$. That is,

$$\mathcal{S} = \{T_f\colon f \in \mathcal{C}_{00}(G),\ f(x) = 0 \text{ for all } x < e\}.$$

By left-linearity $\mathcal{S}$, and therefore $\mathcal{a} = \mathcal{S}^{UW}$, is an algebra. Furthermore, if φ is the canonical trace and T_f, $T_g \in \mathcal{S}$, then we have

$$\varphi(T_f T_g) = \varphi(T_{f*g}) = (f*g)(e) = f(e)\, g(e) = \varphi(T_f)\varphi(T_g).$$

Since φ is UW-continuous, we can make an obvious continuity argument in each variable to conclude that $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A,B \in \mathcal{A}$. The fact that φ is multiplicative on $\mathcal{A}$ suggests that some of the theory of function algebras (i.e., Dirichlet or logmodular algebras) may generalize to this noncommutative context. Section 6 contains a discussion of this problem.

5. Examples

In the preceding section, we have seen that for a faithful representation of a left-linear discrete group, $\mathcal{J}_\infty = 0$ implies the existence of a canonical trace. It is reasonable to inquire whether or not the theorem remains true if one deletes the hypothesis that the group be left-linear. Our first example shows that this conjecture fails, even when the group is abelian.

Example 5.1: Let $\mathcal{H}$ be a Hilbert space having as an orthonormal base the vectors e_n, $n = 0, \pm 1, \pm 2, \ldots$. Let U be the unitary operator defined on $\mathcal{H}$ by $Ue_n = e_{n+1}$, $n = 0, \pm 1, \pm 2, \ldots$. Let Z be the additive group of integers, and let Λ be a finite nontrivial multiplicative subgroup of the unit circle. Let $G = \Lambda \times Z$, and for $x = (\lambda, n) \in G$, put $V_x = \lambda U^n$. It is clear that $x \to V_x$ is a faithful unitary representation of G on $\mathcal{H}$.

Let $\mathcal{B}$ be the von Neumann algebra generated by $\{V_x : x \in G\}$. Then of course, $\mathcal{B}$ is just the weak closure of span $\{U^n : n \in Z\}$. First, we claim $\mathcal{J}_\infty = 0$.

Let F be any finite subset of G, and put $E_F = \{n \in Z : \Lambda \times n \subseteq F\}$. Let $x = (\lambda, n) \in G$. If $x \notin F$ then $n \notin E_F$, so that $\lambda U^n \in \lambda\ \mathrm{span}\{U^k : k \notin E_F\}$ $= \mathrm{span}\ \{U^k : k \notin E_F\}$. Hence span $\{V_x : x \notin F\} \subseteq \mathrm{span}\ \{U^k : k \notin E_F\}$. As F ranges over the finite subsets of G, E_F ranges over the finite subsets of Z; and we have $\mathcal{J}_\infty \subseteq \bigcap \overline{\mathrm{span}}^{UW} \{U^k : k \notin E\}$, the intersection extended over all finite subsets E of Z. But the set on the right is just the tail ideal $\mathcal{J}_\infty$ of the representation $n \in Z \to U^n$ of the linear group Z in $\mathcal{H}$. It is clear by construction of U that this representation is unitarily equivalent to the left regular representation of Z in $\ell_2(Z)$. We conclude that J_∞, and therefore $\mathcal{J}_\infty$, is the null ideal.

We show now that there is no normal state φ on $\mathcal{B}$ such that $\varphi(V_x) = \delta(x)$ for all $x \in G$. Since $\mathcal{B}$ is abelian, such a state would have the form $\varphi(T) = (T\zeta,\zeta)$ where ζ is a unit vector in $\mathcal{H}$ ((2), p.233). Let $x = (\lambda,0) \in G$, $\lambda \neq 1$. Then $x \neq$ identity, and we have $\lambda = (\lambda\zeta,\zeta) = (V_x\zeta,\zeta) = \varphi(V_x) = 0$, an absurdity. This completes the discussion of example 1.

Let G be a discrete abelian linear group $x \in G \to U_x$ a faithful unitary representation. Suppose that the representation is nondeterministic (i.e., $\mathcal{J}_\infty \neq \mathcal{B}$). Then by theorem 4.1 there exists a nonzero vector ζ such that $\zeta \perp U_x\zeta$ for every $x \neq e$. So clearly the action of $\{U_x : x \in G\}$ is nondegenerate. It is not very surprising that the action of the image group may be nondegenerate when the representation itself is deterministic. We will indicate two different ways this can come about.

Example 5.2: Let $\mathcal{R}$ be the ordered group of real numbers in the usual Euclidean topology, and let $t \in \mathcal{R} \to \ell_t$ be the left regular representation of $\mathcal{R}$ in $L_2(\mathcal{R})$.

As we have seen in section 2, the group $\{\ell_t : t \in \mathcal{R}\}$ is nondegenerate. However, regarded as a representation of the discrete group $\mathcal{R}$, this representation is deterministic. Indeed, as $n \to \infty$, $\ell_{1/n} \to \ell_o = I$ ultraweakly. Hence

$$I \in \overline{\text{span}}^{UW} \{\ell_t : t > 0\}.$$

Example 3: Let K be the discrete circle group and let $\mathcal{H} = \ell_2(K)$. For $f \in \mathcal{H}$, define $(Uf)(\lambda) = \lambda f(\lambda)$, $\lambda \in K$. We will show that $n \to U^n$ is a faithful deterministic representation of the integers Z and that the action of the image group $\{U^n\}$ is nondegenerate.

It is clear that U is a unitary operator in $\mathcal{H}$. To show that the representation is faithful and nondegenerate, it suffices to show that for every prime integer $n \geq 2$, there exists $f_n \neq 0$ in $\mathcal{H}$ such that $f_n \perp U^k f_n$, $k = 1,2,\ldots,n-1$. Let ζ be a primitive n^{th} root of 1, and let f be the characteristic function of $\{1,\zeta,\zeta^2,\ldots,\zeta^{n-1}\}$. Then $(U^k f)(\zeta^j) = \zeta^{jk}$ for all $k \geq 1$, $0 \leq j \leq n-1$, and we have

$$(U^k f,f) = \sum_{j=o}^{n-1} (U^k f)(\zeta^j)\bar{f}(\zeta^j) = \sum_{j=o}^{n-1} \zeta^{kj}.$$

Now if $1 \leq k \leq n-1$, ζ^k is an n^{th} root of unity, and it is $\neq 1$ because n is prime. Therefore it satisfies the cyclotomic equation $1 + x + x^2 + \ldots + x^{n-1} = 0$. Hence $(U^k t,t) = \sum_{j=o}^{n-1} (\zeta^k)^j = 0$, as asserted.

Now suppose φ is a normal state of the von Neumann algebra generated by $\{U^n\}$, such that $\varphi(U^n) = 0$ for $n \neq 0$. As in example 1, φ has the form $\varphi(A) = (Af,f)$ for some $f \in \mathcal{H}$. This form persists, of course, if we restrict φ to the C^*-algebra $\mathcal{A}$ generated by $\{U^n\}$.

As the group $\{U^n\}$ is nondegenerate, we may take $\mathcal{A}$ as the algebra $\mathcal{C}(T)$ of continuous functions on the compact circle group T with the usual topology. In this correspondence, U^n becomes $g(z) = z^n$, and the condition $\varphi(U^n) = 0$ for $n \neq 0$, $\varphi(I) = 1$ identifies φ as the Haar integral over T. On the other hand, since $f \in \ell_2(K)$, the measure determined by the state $\omega_f(A) = (Af,f)$ on $\mathcal{C}(T)$ can be easily seen to have the form $\sum_{\lambda \in K} |f(\lambda)|^2 \delta_\lambda$ where δ_λ is the unit point mass concentrated at the point λ. But clearly this measure is singular with respect to Haar measure, and in particular this contradicts $\varphi = \omega_f$.

6. Analyticity in Operator Algebras

In this section, we begin a study of a class of operator algebras closely related to the Dirichlet and logmodular function algebras. In particular, we will discuss the possibility of generalizing to this noncommutative context the Jensen inequality and the classical theorem of Szegö, for analytic functions in the unit disc.

In the following, the concept of the determinant of an operator will play a central role. Fuglede and Kadison (5) have defined the determinant for operators in a II_1 factor. However, an examination of this paper shows immediately that their results (save those relating to uniqueness) are valid in the following more general situation. Let $\mathcal{B}$ be a von Neumann algebra, and let φ be a finite trace on $\mathcal{B}$ such that $\varphi(I) = 1$. Note first that log H is a self-adjoint operator in $\mathcal{B}$ whenever H is both positive and regular in $\mathcal{B}$. If $T \in \mathcal{B}$ is regular, so is $|T| = (T^*T)^{1/2}$ and the determinant of T is defined as $\Delta(T) = \exp \varphi(\log|T|)$. If T is singular, define

$$\Delta(T) = \overline{\lim_{\epsilon \downarrow 0}} \exp \varphi(\log(|T| + \epsilon I)).$$

Δ has the following properties:

1. $\Delta(\lambda T) = |\lambda|\, \Delta(T)$, for all $T \in \mathcal{B}$ and all complex λ.
2. $\Delta(ST) = \Delta(S)\Delta(T)$ for all $S, T \in \mathcal{B}$.
3. $\Delta(e^T) = |e^{\varphi(T)}| = \exp \varphi(\frac{1}{2}(T+T^*))$, for all $T \in \mathcal{B}$.

4. $\Delta(T) = \Delta(T^*) = (\Delta(T^*T))^{1/2}$, for all $T \in \mathcal{B}$.

5. $\Delta(H_1) \leq \Delta(H_2)$ when $0 \leq H_1 \leq H_2$.

6. $\Delta(T) \leq \lim_n \|T^n\|^{1/n}$.

7. Δ is uniformly continuous on regular elements.

8. if $0 \leq H \leq H_n$ and $H_n \to H$ uniformly then $\Delta(H_n) \to \Delta(H)$.

9. In general, $\overline{\lim_{n\to\infty}} \Delta(T_n) \leq \Delta(T)$, when $T_n \to T$ uniformly.

Recall first the well-known inequality of the arithmetic-geometric mean. Let $(X, \mathcal{S}, m)$ be a probability space, h a nonnegative integrable function. Then

$$\exp \int_X \log h(x)\, dm\,(x) \leq \int_X h(x)\, dm\,(x), \text{ the}$$

left side taken as 0 when $\int \log h(x)\, dm(x) = -\infty$. Moreover, equality obtains if, and only if, $h = \int_X h(x)\, dm(x)$ almost everywhere.

Proposition 6.1: Let $\mathcal{B}$ be a von Neumann algebra and let φ be a finite trace such that $\varphi(I) = 1$. Then for every $T \in \mathcal{B}$,

$$\Delta(T) \leq \varphi(|T|).$$

If T is invertible and φ faithful, the inequality is strict unless T is proportional to a unitary operator.

Proof: Let $\mathcal{B}_o$ be the von Neumann algebra generated by $|T|$. By spectral theory, there is an extremally disconnected compact Hausdorff space X and a $*$-isomorphism $A \in \mathcal{B}_o \to f_A \in \mathcal{C}(X)$ of $\mathcal{B}_o$ onto the algebra $\mathcal{C}(X)$ of all complex-valued continuous functions on X, mapping the projections in $\mathcal{B}_o$ onto the set of all characteristic functions of clopen sets in X.

Let m be the unique regular Borel probability measure on X satisfying

$$\int_X f_A(x)\,dm(x) = \varphi(A), \text{ for all } A \in \mathcal{B}_o.$$

If T is invertible, then we have

$$\Delta(T) = \exp \varphi(\log|T|) = \exp \int_X f_{\log|T|}(x)\,dx(x) =$$

$$= \exp \int_X \log f_{|T|}(x)\,dm(x) \leq \int_X f_{|T|}(x)\,dm(x) = \varphi(|T|),$$

by the ordinary arithmetic-geometric inequality. The conclusion follows in general by letting $\epsilon \downarrow 0$ in the inequality

$$\Delta(T) = \Delta(|T|) \leq \Delta(|T| + \epsilon I) \leq \varphi(|T|) + \epsilon.$$

If φ is faithful, then for every nonzero projection $P \in \mathcal{B}_o$, $0 < \varphi(P) = \int_X f_P(x)\,dm(x)$, so that $m(C) > 0$ for every nonnull clopen set $C \subseteq X$. Because the clopen sets are a base m is supported everywhere on X, and consequently, a continuous function which is constant a.e. (dm) is identically constant. If in addition T is invertible, then $\Delta(T) = \exp \int_X \log f_{|T|}(x)\,dm(x)$. So by the ordinary arithmetic-geometric mean inequality, $\Delta(T) = \varphi(|T|)$ implies $f_{|T|}(x) = \int f_{|T|}dm = \varphi(|T|)$ for all $x \in X$. Thus $(T^*T)^{1/2} = |T| = \varphi(|T|)I$, so that $\varphi^{-1}(|T|)T$ is a partial isometry in $\mathcal{B}$.

Of course $\mathcal{B}$ is a finite von Neumann algebra, and we conclude that $\varphi^{-1}(|T|)T$ is actually unitary. This completes the proof.

Additional properties of the determinant can be deduced from the following. Let $\mathcal{B}$ be a von Neumann algebra, φ a finite trace on $\mathcal{B}$ such that $\varphi(I) = 1$.

Proposition 6.2: If $H \in \mathcal{B}$ is positive, then $\Delta(H) = \inf \varphi(He^A)$, the infimum extended over all self-adjoint $A \in \mathcal{B}$ such that $\varphi(A) = 0$.

Proof: If A is self-adjoint and $\varphi(A) = 0$, then

$$\Delta(He^A) = \Delta(H)\Delta(e^A) = \Delta(H)e^{\varphi(A)} = \Delta(H).$$

Hence $\Delta(H) = \Delta(He^A) \leq \varphi(He^A)$, by the arithmetic-geometric mean inequality. This proves $\Delta(H) \leq \inf \varphi(He^A)$.

On the other hand, for every $\epsilon > 0$, $\inf \varphi(He^A) \leq \inf \varphi((H+\epsilon I)e^A)$. $H + \epsilon I$ is positive and regular, so that

$A = \varphi(\log(H + \epsilon I))I - \log(H + \epsilon I)$ is a bounded self-adjoint operator in $\mathcal{B}$, having zero trace. Moreover,

$e^A = \Delta(H + \epsilon I)(H + \epsilon I)^{-1}$, so that

$\varphi((H + \epsilon I)e^A) = \Delta(H + \epsilon I)\varphi(I) = \Delta(H + \epsilon I)$.

We have then $\inf \varphi(He^A) \leq \Delta(H + \epsilon I)$, and the opposite inequality follows by letting $\epsilon \downarrow 0$, using continuity of Δ under uniform decreasing limits on positive operators.

Corollary: Let $\mathcal{B}^+$ denote the set of positive elements of $\mathcal{B}$. Then for all $X, Y \in \mathcal{B}^+$, $\Delta(X+Y) \geq \Delta(X) + \Delta(Y)$ and in addition, Δ is upper semicontinuous on $\mathcal{B}^+$ with respect to the ultraweak topology. Δ is upper semicontinuous on $\mathcal{B}$ with respect to the ultrastrong topology.

Proof: The first statement follows from $\inf[\varphi(Xe^A)+\varphi(Ye^A)] \geq \inf \varphi(Xe^A) + \inf \varphi(Ye^A)$. The second statement comes from the fact that the infimum over a family of continuous functions uniformly bounded below is upper semicontinuous.

The last assertion follows similarly, using

$$\Delta(T) = [\Delta(T^*T)]^{1/2} = \inf\, [\varphi(T^*Te^A)]^{1/2},$$

and the fact that for every self-adjoint $A\in\mathcal{B}$ the mapping $T\in\mathcal{B}\to\varphi(T^*Te^A)$ is ultrastrongly continuous.

Of course, the corollary implies that Δ is a concave function on $\mathcal{B}^+$, since $\Delta(\lambda T) = \lambda\Delta(T)$ for every $\lambda \geq 0$, $T\in\mathcal{B}$.

The setting for the remainder of this paper will be this: we are given a von Neumann algebra $\mathcal{B}$, an UW-closed subalgebra $\mathcal{A}$ of $\mathcal{B}$, and a faithful normal finite trace φ on $\mathcal{B}$, satisfying

(i) $I\in\mathcal{A}$

(ii) $\mathcal{A}+\mathcal{A}^*$ contains a weakly dense $*$-subalgebra of $\mathcal{B}$

(III) φ is multiplicative on $\mathcal{A}$.

Examples are obtained by taking a left-linear discrete group G and a faithful regular unitary representation $x\in G\to U_x$. Let $\mathcal{B}$ be the von Neumann algebra generated by $\{U_x : x\in G\}$, $\mathcal{A} = \overline{\text{span}}^{US}\{U_x : x\geq e\}$, and take for φ the canonical trace satisfying $\varphi(U_x) = \delta(x)$. When G is the group of integers, $\mathcal{B}$ will be (isomorphic to) L_∞ of the unit circle, and $\mathcal{A}$ will consist of H_∞, the algebra of functions bounded and analytic in the interior of the disc.

We collect first the most elementary properties of the algebra $\mathcal{A}$. Define $\mathcal{A}_0 = \{T\in\mathcal{A}:\varphi(T) = 0\}$. Clearly $\mathcal{A}_0$ is an UW-closed subalgebra of $\mathcal{A}$.

Proposition 6.3:

(a) $\varphi(I) = 1$.

(b) $\mathcal{A}$ is antisymmetric; that is, $\mathcal{A}\cap\mathcal{A}^*$ consists solely of scalar multiples of the identity.

(c) $\mathcal{A} = \{T\in\mathcal{B}:\varphi(T\,\mathcal{A}_0) = \{0\}\}$.

(d) If $A\in\mathcal{B}$ is self-adjoint, there exists a sequence $T_n\in\mathcal{A}$ such that $T_n+T_n^*\to A$ strongly.

Proof: (a): is a trivial consequence of the fact that $\varphi^2(I) = \varphi(I^2) = \varphi(I)$.

(b): Let $A\in\mathcal{A}^*$, and put $B = A+A^*-\varphi(A+A^*)I$. Then B is self-adjoint, in $\mathcal{A}\cap\mathcal{A}^*\subseteq\mathcal{A}$, and has zero trace. Therefore $\varphi(B^2) = \varphi^2(B)=0$, giving $B = 0$ since φ is faithful. Similarly, one shows that $(A-A^*)/i = \varphi((A-A^*)/i)I$. It follows that $2A=(A+A^*)+i(A-A^*)/i = 2\varphi(A)I$

(c): Let $\mathcal{S} = \{T\in\mathcal{A}:\varphi(T\mathcal{A}_0) = \{0\}\}$. Clearly $\mathcal{A}\subseteq\mathcal{S}$, since φ is multiplicative on $\mathcal{A}$. Now by von Neumann's density theorem, $\mathcal{A}+\mathcal{A}^*$ is strongly dense in $\mathcal{B}$, and it is well-known that the strong topology on $\mathcal{B}$ coincides with the norm topology defined by the positive definite inner product $(A,B) = \varphi(AB^*)$, (see (2), p. 288).

Let $T\in\mathcal{S}$, and choose sequences A_n, $B_n\in\mathcal{A}$ such that $\|T - (A_n + B_n^*)\|_\varphi \to 0$. By putting $B_n' = B_n - \varphi(B_n)I$, $A_n' = A_n + \varphi(B_n^*)I$, we can even assume the B_n's are in $\mathcal{A}_o$. We have $(A_n,B_n^*) = \varphi(A_nB_n) = \varphi(A_n)\varphi(B_n) = 0$; and by definition of $\mathcal{S}$, $(T,B_n^*) = \varphi(TB_n) = 0$. So $T - A_n$ is orthogonal to B_n^*, for all n. Therefore

$$\|T-A_n\|_\varphi^2 + \| -B_n^*\|_\varphi^2 = \|T - (A_n+B_n^*)\|_\varphi^2 \to 0,$$

and in particular, $A_n \to T$. By the above remarks on the strong topology and the fact that strong and ultrastrong convergence is the same for sequences, we conclude that $T\in\bar{\mathcal{A}}^{US} = \bar{\mathcal{A}}^{UW} = \mathcal{A}$.

(d): Suppose $A\in\mathcal{A}$ is self-adjoint. By the proof of (c), there exist sequences $S_n, T_n\in\mathcal{A}$ such that $\|A - (S_n+ T_n^*)\|_\varphi \to 0$ as $n \to \infty$. Because φ is a trace,

$$\|A - (T_n+ S_n^*)\|_\varphi = \| [A - (S_n+ T_n^*)]^*\|_\varphi$$

$$= \|A - (S_n+ T_n^*)\|_\varphi \to 0.$$

Hence $\frac{1}{2}(S_n+ T_n) + \frac{1}{2}(S_n+ T_n)^* = \frac{1}{2}(S_n+ T_n^*) + \frac{1}{2}(T_n+ S_n^*)$ tends to A in the L_2 trace norm, and therefore strongly. This completes the proof of the proposition.

Now of course $\mathcal{A}= \mathcal{A}_o + \{\lambda I: \lambda\in\mathbb{C}\}$. So by (c) we have immediately that

$$\mathcal{A}_o= \{T\in\mathcal{B}: \varphi(T\mathcal{A}) = \{0\} \}.$$

If $\mathcal{S}$ is a subset of $\mathcal{B}$, $\mathcal{S}^{-1}$ will denote the set of all inverses of the regular elements of $\mathcal{S}$. Our interest centers on the following two statements:

1. $\Delta(T) = |\varphi(T)|$ for all $T \in \mathcal{Q} \cap \mathcal{Q}^{-1}$

2. For every positive $H \in \mathcal{B}$

$$\Delta(H) = \inf \varphi(H|I-B|^2), \text{ the inf taken over all } B \in \mathcal{Q}_0.$$

We shall refer to 1 and 2 as Jensen's formula and Szegö's theorem, respectively.

Helson and Lowdenslager (6) have given a measure-theoretic proof of Szegö's theorem which is valid in the case where $\mathcal{B}$ is abelian. The writer has been unable to generalize this proof to the noncommutative setting. Nevertheless, their key lemma carries over verbatim, and we digress now to establish this result.

Theorem 6.1: Let φ be a finite trace on a von Neumann algebra $\mathcal{B}$ such that $\varphi(I) = 1$, and let $\mathcal{B}_o$ be a weakly dense $*$-subalgebra of $\mathcal{B}$. Then for every positive $H \in \mathcal{B}$, one has

$$\inf \varphi(He^A) = \Delta(H),$$

the infimum extended over all self-adjoint $A \in \mathcal{B}_o$ with $\varphi(A) = 0$.

Proof: $\Delta(H) \leq \inf \varphi(He^A)$ follows from proposition 6.2. For the opposite inequality, suppose first that H is regular. The operator $Q = -\log H + \varphi(\log H)I$ is bounded and self-adjoint. By Kaplansky's density theorem (11), there exists a net $Q_n \in \mathcal{B}_o$ such that $Q_n = Q_n^*$, $\sup_n \|Q_n\| < \infty$, and $Q_n \to Q$ strongly. Since φ is strongly continuous on bounded sets, $\varphi(Q_n) \to \varphi(Q) = 0$, and we may even assume $\varphi(Q_n) = 0$ for every n.

Let $h(t)$, $t \in (-\infty, +\infty)$, be any real-valued continuous function vanishing at $\pm\infty$ and coinciding with e^t for $t \leq \sup_n \|Q_n\| + 1$.

Another lemma of Kaplansky (11) asserts that the mapping $T \to h(T)$ is strongly continuous on the self-adjoint operators in $\mathcal{B}$. Hence $e^{Q_n} = h(Q_n)$ tends strongly to $h(Q) = e^{Q} = H^{-1}\Delta(H)$. Now $\sup_n \|e^{Q_n}\| \leq \|h\|_\infty < \infty$, so that

$$\lim_n \varphi(He^{Q_n}) = \Delta(H)\varphi(H\,H^{-1}) = \Delta(H).$$

The desired inequality follows for regular H.

If H is now an arbitrary positive element in $\mathcal{B}$, then $H + \epsilon I$ is regular, for every $\epsilon > 0$, and $\varphi(He^A) \leq \varphi((H + \epsilon I)e^A)$ for every A in the prescribed set. Thus

$$\inf \varphi(He^A) \leq \Delta(H + \epsilon I),$$

and the proof can be completed by letting $\epsilon \downarrow 0$, using continuity of Δ under uniform decreasing limits on positive operators.

Now let $\mathcal{B}$, $\mathcal{A}$, $\mathcal{A}_o$, and φ be as defined above. The generalized Helson-Lowdenslager lemma reads as follows.

Corollary: For every positive operator $H \in \mathcal{B}$, one has

$$\Delta(H) = \inf_{T \in \mathcal{A}_o} \varphi(He^{T+T^*}).$$

Proof: The inequality $\leq$ follows, as before, from the arithmetic-geometric mean inequality. Let $\mathcal{C}$ be a $*$-subalgebra of $\mathcal{A} + \mathcal{A}^*$, weakly dense in $\mathcal{B}$. Theorem 6.1 asserts that $\Delta(H) = \inf \varphi(He^A)$, A ranging over the self-adjoint operators in $\mathcal{C}$ with $\varphi(A) = 0$. The opposite inequality will follow if we show that such an A has the form $T + T^*$, for some $T \in \mathcal{A}_o$.

Now $A = R + S^*$, $R,S \in \mathfrak{A}$. Since A self-adjoint, $R + S^* = R^* + S$, whence $R-S = (R-S)^*$ is a self-adjoint operator in $\mathfrak{A} \cap \mathfrak{A}^*$. By antisymmetry, $R-S = \lambda I$ for some real λ. So if we take $T = S + \frac{\lambda}{2} I \in \mathfrak{A}$, then $A = (S + \lambda I) + S^* = T + T^*$. Finally, putting $T_o = T - \varphi(T)I \in \mathfrak{A}_o$, we obtain the desired form

$$T_o + T_o^* = T+T^* - \varphi(T+T^*)I = A - \varphi(A)I = A.$$

Now suppose we are able to replace e^{T+T^*} with $e^{T^*}e^{T}$ in corollary 2. Then Szegö's theorem follows. Indeed, letting $A_T = I-e^T$, then clearly $A_T \in \mathfrak{A}$. And since $\varphi(A_T) = 1-e^{\varphi(T)} = 0$, we have $A_T \in \mathfrak{A}_o$. Thus

$$\inf_{T\in\mathfrak{A}_o} \varphi(H|I-T|^2) \leq \inf_{T\in\mathfrak{A}_o} \varphi(H|I-A_T|^2) = \Delta(H).$$

The opposite inequality follows from this one exactly as in the Helson-Lowdenslager proof, using the relation $\Delta(AA^*) = \Delta(A^*A)$. Although this alteration is trivial when $\mathcal{B}$ is abelian, it is not clear whether or not it can be accomplished in the general case.

Returning, now, to the main discussion, let $(\mathcal{B},\varphi)$ be the canonical Hilbert algebra consisting of the prehilbert space $\mathcal{B}$ endowed with the positive definite inner product $(A,B) = \varphi(AB^*)$. Let $\mathcal{L}$ (resp. $\mathcal{R}$) be the collection of all operators L_A (res. R_A) on the completion $\mathcal{H}_\varphi$ defined by $L_A T = AT$ (resp. $R_A T = TA$) for $T \in \mathcal{B}$. The mapping $A \to L_A$ (resp. $A \to R_A$) is a $*$-isomorphism (resp. $*$-antiisomorphism) of $\mathcal{B}$ onto $\mathcal{L}$ (resp. $\mathcal{R}$), and the fundamental commutation theorem asserts that $\mathcal{R}' = \mathcal{L}$ and $\mathcal{L}' = \mathcal{R}$ (2).

Lemma 6.1: Let ζ be any unit vector in the completion $\mathcal{H}_\varphi$ such that $\zeta \perp L_S\zeta$ for all $S \in \mathcal{A}_o$. Then ζ is bounded: in fact, ζ is a unitary operator in $\mathcal{B}$.

Proof: For $T \in \mathcal{B}$, define $A_\zeta T = L_T\zeta$. A_ζ is a linear transformation of the dense subspace $\mathcal{B}$ of $\mathcal{H}_\varphi$ into $\mathcal{H}_\varphi$.

Fix T, and choose a sequence $A_n \in \mathcal{A}$ such that $A_n + A_n^* \to T^*T$ strongly (proposition 6.3). Letting $B_n = A_n - \varphi(A_n)I$, then $B_n \in \mathcal{A}_o$ and $B_n + B_n^* + \varphi(A_n + A_n^*)I \to T^*T$ strongly. Since a *- isomorphism preserves strong convergence of sequences, we have

$$L_{B_n} + L_{B_n^*} + \varphi(A_n + A_n^*)\, L_I \to L_{T^*T} \text{ strongly.}$$

Now by hypothesis $(L_{B_n^*}\zeta, \zeta) = \overline{(L_{B_n}\zeta, \zeta)} = 0$ for all n. Thus

$$(A_\zeta T, A_\zeta T) = (L_T\zeta, L_T\zeta) = (L_{T^*T}\zeta, \zeta) =$$

$$= \lim_n [(L_{B_n}\zeta, \zeta) + (L_{B_n^*}\zeta, \zeta) + \varphi(A_n + A_n^*)]$$

$$= \lim_n \varphi(A_n + A_n^*) = \varphi(T^*T) = (T, T).$$

So A_ζ may be extended to an isometry acting in $\mathcal{H}_\varphi$.

If B, $T \in \mathcal{B}$, then $L_B A_\zeta T = L_B L_T \zeta = L_{BT}\zeta = A_\zeta BT = A_\zeta L_B T$. Hence $A_\zeta \in \mathcal{L}' = \mathcal{R}$, so there exists an isometry U in $\mathcal{B}$ such that $A_\zeta = R_U$. Because $\mathcal{B}$ is a finite von Neumann algebra U is unitary, and finally

$$\zeta = L_I\zeta = A_\zeta I = R_U I = U. \text{ QED}$$

<u>Theorem 6.2:</u> Let H be a positive regular element of $\mathcal{B}$, and let $d = [\inf_{T \in \mathcal{a}} \varphi(H|I-T|^2]^{1/2}$. Then $d^2 \geq \|H^{-1}\|^{-1} > 0$, and there exists $T \in \mathcal{a}_o$ with the following properties:

(a) $I-T \in \mathcal{a} \cap \mathcal{a}^{-1}$

(b) $H = d^2(I-T)^{-1}(I-T^*)^{-1}$.

<u>Proof:</u> Let K be the regular positive square root of H. Let $\mathcal{m}$ be the closed subspace of $\mathcal{H}_\varphi$ generated by $\mathcal{a}_o K$. Because K, and therefore R_K, is invertible we have $\mathcal{m} = R_K \mathcal{n}$, where $\mathcal{n}$ is the closed subspace generated by $\mathcal{a}_o$.

Now if $\xi \in \mathcal{n}$, then $I \perp \xi$, and

$$1 \leq \|I-\xi\|_\varphi = \|R_{K^{-1}}(K - R_K\xi)\|_\varphi \leq \|K^{-1}\| \ \|K-R_K\xi\|_\varphi.$$

It follows that $d^2 \geq \|K^{-1}\|^{-2} = \|H^{-1}\|^{-1}$.

Let η be the projection of K on $\mathcal{m}$, and let $\zeta = K-\eta$. By the above remarks, η has the form $R_K\xi$ for some $\xi \in \eta$, so that $\zeta = R_K(I-\xi)$. First, we show $\zeta = dU$, for some unitary U in $\mathcal{B}$.

If $A \in \mathcal{a}$, then $L_A\zeta = L_A R_K(I-\xi) = R_K L_A(I-\xi) = R_K(A-L_A\xi)$. Both A and $L_A\xi$ are in $\mathcal{n}$, hence $L_A\zeta \in R_K\mathcal{n} = \mathcal{m}$. But by definition of ζ, $\zeta \perp \mathcal{m}$. Therefore by the preceding lemma, there is a unitary operation $U \in \mathcal{B}$ such that $\zeta = \|\zeta\|U = dU$.

Now as an element of $\mathcal{H}_\varphi$, ξ is a norm limit of a sequence $A_n \in \mathcal{a}_o$. But $\xi = I - R_{K^{-1}}\zeta = I - dUK^{-1} \in \mathcal{B}$. Therefore $\varphi(|\xi-A_n|^2) \to 0$, and this implies that $\xi \in \mathcal{a}_o^{US} = \mathcal{a}_o$.

Consider the operator $D = (I-\xi)H(I-\xi^*) - d^2I$. For every $T\in\mathcal{B}$, we have

$$(D,T) = \varphi(L_{T^*}(I-\xi)K^2(I-\xi^*)) - d^2\,\varphi(T)$$

$$= (L_{T^*}\,\zeta,\zeta) - d^2\varphi(T).$$

Therefore $(D,T) = 0$ for all $T\in\mathcal{A}_0\cup\mathcal{A}_0^*\cup\{I\}$. Hence $D = 0$ since $\mathcal{A}_0\cup\mathcal{A}_0^*\cup\{I\}$ is fundamental in $\mathcal{H}_\varphi$, and we have $(I-\xi)H(*-\xi^*) = d^2I$. But in a finite von Neumann algebra, a product of operators is regular only when all the factors are, so that

$(I-\xi)^{-1}\in\mathcal{B}$, and $H = d^2(I-\xi)^{-1}\,(I-\xi^*)^{-1}$.

It remains only to show that $(I-\xi)^{-1}\in\mathcal{A}_0$, or what is the same, $\varphi(T(I-\xi)^{-1}) = 0$ for all $T\in\mathcal{A}$. For such a T, write $d^2\varphi(T(I-\xi)^{-1}) =$

$$= d^2\,\varphi(T(I-\xi)^{-1}(I-\xi^*)^{-1}(I-\xi^*)) = \varphi(TK^2(I-\xi^*))$$

$$= (R_KT,\ R_K(I-\xi)) = (R_KT,\zeta) = 0,\ \text{since } R_KT\in R_K\mathcal{N} = \mathcal{M}.$$

The theorem is proved.

Corollary: Let H be a regular positive operator in $\mathcal{B}$. Then H has a factorization $H = TT^*$ with $T\in\mathcal{A}\cap\mathcal{A}^{-1}$. T is uniquely determined up to a pmoportionality constant of modulus 1. The algebra $\mathcal{A}$ is logmodular in the sense that every self adjoint $A\in\mathcal{B}$ has the form

$$A = \log TT^*$$

where $T\in\mathcal{A}\cap\mathcal{A}^{-1}$.

Proof: The second statement follows from the first by taking $H = e^A$, and the existence of such a factorization for H is contained in the previous theorem.

Suppose S, $T \in \mathcal{A} \cap \bar{\mathcal{A}}^{-1}$ and $SS^* = TT^*$. Then $T^{-1}S = (S^{-1}T)^*$ giving $T^{-1}S \in \mathcal{A} \cap \mathcal{A}^*$. By antisymmetry, $T^{-1}S = \lambda I$, for some $\lambda \in \mathbb{C}$. It is clear that $|\lambda| = 1$, since

$$|\lambda|^2 I = T^{-1}SS^*T^{-1*} = T^{-1}TT^*T^{-1*} = I. \quad \text{QED.}$$

We turn now to Szegö's theorem and the Jensen formula.

Remark: Let $S \in \mathcal{A}$, $T \in \mathcal{A} \cap \bar{\mathcal{A}}^{-1}$, and suppose $SS^* \le TT^*$. Then $|\varphi(S)| \le |\varphi(T)|$.

Proof: By the Schwarz inequality for states in a C^*-algebra, we have

$$|\varphi(S)/\varphi(T)|^2 = |\varphi(T^{-1}S)|^2 \le \varphi(T^{-1}SS^*T^{-1*}) \le 1,$$

since $T^{-1}SS^*T^{-1*} = T^{-1}SS^*(T^*)^{-1} \le I$.

Theorem 6.3: The following propositions are equivalent.

(a) Jensen's formula on $\mathcal{A} \cap \bar{\mathcal{A}}^{-1}$.

(b) Jensen's inequality; i.e., $|\varphi(T)| \le \Delta(T)$ for all $T \in \mathcal{A}$.

(c) Szegö's theorem.

Proof: (c) $\Rightarrow$ (a) : Let $T \in \mathcal{A} \cap \bar{\mathcal{A}}^{-1}$. Then for every $S \in \mathcal{A}_0$, we have

$$\varphi((I-S)T) = (1 - \varphi(S))\,\varphi(T) = \varphi(T).$$

By the Schwarz inequality,

$$|\varphi(T)|^2 = |\varphi(I-S)T|^2 \le \varphi((I-S)TT^*(I-S^*))$$

for every $S \in \mathcal{A}_0$. By Szegö's theorem,

$$|\varphi(T)|^2 \le \inf_{S \in \mathcal{A}_0} \varphi(TT^*|I-S|^2) = \Delta(TT^*) = \Delta^2(T).$$

The same argument applies to T^{-1}, giving

$$|\varphi(T)|^{-2} = |\varphi(T^{-1})|^2 \le \Delta^2(T^{-1}) = \Delta^{-2}(T).$$

Therefore $|\varphi(T)| = \Delta(T)$.

(a) $\Rightarrow$ (b): Let $T \in \mathcal{A}$ and fix $\epsilon > 0$. Then $TT^* + \epsilon I$ is positive and regular. By the corollary to theorem 6.2, there exists $S \in \mathcal{A} \cap \mathcal{A}^{-1}$ such that $TT^* + \epsilon I = SS^*$; and by the remark preceding this theorem, $|\varphi(T)|^2 \leq |\varphi(S)|^2$. Applying the Jensen formula to the right side, we obtain $|\varphi(S)|^2 = \Delta^2(S) = \Delta(SS^*) = \Delta(TT^* + \epsilon I)$. But $\Delta(TT^* + \epsilon I) \downarrow \Delta(TT^*) = \Delta^2(T)$ as $\epsilon \downarrow 0$, and Jensen's inequality $|\varphi(T)| \leq \Delta(T)$ follows.

(b) $\Rightarrow$ (c): Let H be a positive operator in $\mathcal{B}$, and let $d = [\inf_{S \in \mathcal{A}_0} \varphi(H|I-S|^2)]^{1/2}$. We show first that $d^2 \leq \Delta(H)$.

Fix $\epsilon > 0$. Then $H + \epsilon I$ is positive and regular. Theorem 6.2 asserts that $H + \epsilon I$ can be put into the form

$$H + \epsilon I = d_\epsilon^2 (I-S)^{-1}(I-S^*)^{-1}, \text{ where}$$

$$d_\epsilon = [\inf_{S \in \mathcal{A}_0} \varphi((H+\epsilon I)|I-S|^2)]^{1/2}, \quad S \in \mathcal{A}_0 \text{ and } I-S \in \mathcal{A} \cap \mathcal{A}^{-1}.$$

Taking the determinant of both sides, there results

$$\Delta(H+\epsilon I) = d_\epsilon^2\, \Delta((I-S)^{-1}(I-S^*)^{-1}) = d_\epsilon^2\, \Delta^2((I-S)^{-1}).$$

Now by Jensen's inequality.

$$\Delta((I-S)^{-1}) \geq |\varphi((I-S)^{-1})| = |\varphi^{-1}(I-S)| = 1.$$

Hence $\Delta(H + \epsilon I) \geq d_\epsilon^2$. But clearly $d_\epsilon^2 \geq d^2$, and the desired inequality follows.

The opposite inequality is simple. Indeed, if $S \in \mathcal{A}_0$, then

$$\Delta(|I-S|^2) = \Delta^2(I-S) \geq |\varphi(I-S)|^2 = 1,$$

by Jensen's inequality. Hence,

$$\Delta(H) \le \Delta(H)\Delta(|I-S|^2) = \Delta(H|I-S|^2) \le \varphi(H)|I-S|^2),$$

by the arithmetic-geometric mean inequality. The conclusion follows by taking the inf over $S\in\mathfrak{A}_0$, and the proof of the theorem is complete.

Regarding the proof, we note that the second inequality $\Delta(H) \le d^2$ can be deduced from $d^2 \le \Delta(H)$ without invoking Jensen's inequality a second time. This can be seen in a number of ways. First, one can mimic the Helson-Lowdenslager proof of this assertion, using the fact that $\Delta(AA^*) = \Delta(A^*A)$. Instead, one can show that Jensen's inequality itself follows from $d^2 \le \Delta(H)$. Indeed, as in the proof of (c) $\Rightarrow$ (a), we have

$$|\varphi(T)|^2 = |\varphi((I-S)T)|^2 \le \varphi(TT^*|I-S|^2),$$

for all $T\in\mathfrak{A}$, $S\in\mathfrak{A}_0$; thus

$$|\varphi(T)|^2 \le \inf_{S\in\mathfrak{A}_0} \varphi(TT^*|I-S|^2) = d^2 \le \Delta(TT^*) = \Delta^2(T),$$

which is precisely Jensen's inequality.

Of course, theorem 6.3 is not decisive since we have been unable to establish any of its three assertions independently of the others, in any noncommutative context. We conclude with the following bit of information. Let $\exp(\mathfrak{A})$ denote the set of all e^T, $T\in\mathfrak{A}$.

<u>Proposition 6.4:</u> If the US-closure of $\exp(\mathfrak{A})$ contains $\mathfrak{A}\cap\mathfrak{A}^{-1}$, then Jensen's formula is valid.

<u>Proof:</u> Since φ is multiplicative on $\mathcal{a}$, Jensen's formula is always valid on exp $(\mathcal{a})$. Indeed if $T \in \mathcal{a}$, then

$$|\varphi(e^T)| = |e^{\varphi(T)}| = \Delta(e^T).$$

Take $T \in \mathcal{a} \cap \mathcal{a}^{-1}$, and choose a net e^{T_n}, $T_n \in \mathcal{a}$, tending to T ultrastrongly. Then

$$|\varphi(T)| = \lim_n |\varphi(e^{T_n})| = \lim_n |\Delta(e^{T_n})| \leq \Delta(T),$$

since the determinant is upper semicontinuous in the US-topology. The same argument applied to T^{-1} yields

$$|1/\varphi(T)| = |\varphi(T^{-1})| \leq \Delta(T^{-1}) = 1/\Delta(T).$$

Therefore $|\varphi(T)| = \Delta(T)$.

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