THE DEGREE OF POLYNOMIAL CURVES WITH A FRACTAL GEOMETRIC VIEW

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Abstract— Polynomial curves are multifaceted with respect to behaviour, function and application. These mathematical functions can be expressed in terms of the equation $z_n = z_{n-1}^i + c$, where 'i' is a natural number and 'z_n', 'z_{n-1}' and 'c' are all complex numbers. While analysing the analyticity properties of these curves, all the curves are found to exhibit a uniform relationship in the appearance of bulbs having the same period. Similarly, there is also a uniform relationship between the degree of the polynomial generating the curve and the number of period 2 bulbs. Also, the number of major points of attraction in each curve is found to be proportional to the degree of the polynomial generating the curve.

1. INTRODUCTION

Generally, most figures can be defined precisely in terms of mathematical equations. However, there are many more which are hard to define because of their chaotic nature. The simplest way to define chaos would be to say it has no definition and is all randomness. Nevertheless, the science of fractals tries to prove that there is some determinism in this randomness and chaos. In this paper, the determinism in some curves which appear chaotic has been shown. These curves have been referred as "Polynomial Curves".

Polynomial curves can be defined as the set of all complex numbers 'c' where the orbit of $z_0 = 0 + 0i$ in the equation

$$z_n = z_{n-1}^i + c \qquad (1)$$

where 'i' is a positive integer, is bounded. These can be generated (Stevens 1993) as fractal curves (Voss 1988) by representing the values of 'c' in the complex plane. For each pixel, the equation is solved for 'z_i' with 'z_0' set to 0. The equation is then iterated till it either blows up or reaches a maximum number of iterations without blowing up. The regions that don't blow up are then coloured black while others can be coloured according to the number of iterations needed to blow up. As the regions coloured black are the ones of our interest, the rest are all plotted white. This concept has been used by us to generate general polynomial curves and study their analyticity (Hoggar 1992) properties and their fractal geometry (Devaney 1995). It may be mentioned here that the curve generated by substituting 2 for 'i' in (1) is known more famously as the Mandelbrot set (Peitgen and Richter 1986).

2. Fractal Behaviour of the Polynomial Curves:

Polynomial curves have been generated by us using the equation (1) where 'z_n', 'z_{n-1}' and 'c' are complex numbers with 'i' being a natural number and 'z_0' set to 0 + 0i. Then these curves are the set of all c-values where the orbit of 'z_0' is bounded.

2.1 Generation of the Polynomial Curves:

Let's consider the curve having the equation

$$z_n = z_{n-1}^3 + c \qquad (2)$$

over the range which has been shown in Fig.1. Then the various bulbs are classified according to their respective periods.

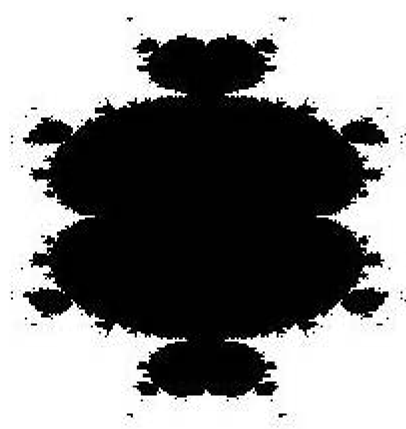

Figure 1

Now, let's consider the two primary bulbs in the above figure. These are the two largest bulbs in the figure. An arbitrary point (0, 0.5) inside one of these bulbs is taken which corresponds to a c-value of 0 +0.5i. This process is iterated by substituting this c-value in

Eq. (2) with 'z_0' as 0 + 0i. Iterating 'z' according to Eq. (2), the values generated are 0 + 0.38i, 0 + 0.45i, 0 + 0.41i, 0 + 0.43i, 0 + 0.42i, 0 + 0.43i, 0 + 0.42i, 0 + 0.42i, 0 + 0.42i and so on. From the above data, it can be concluded that z finally settles down to a constant value of 0 + 0.42i. Similarly, it can be shown that for any arbitrary point inside this bulb, z will finally settle down to a constant value. Thus, a point on this bulb remains constant or in other words, cycles with a period 1. Hence it's called a period 1 bulb or a primary bulb.

Now, let's consider the largest bulbs attached to these primary bulbs. An arbitrary point (0,1) inside one of these bulbs is taken which was equivalent to 0 + 1i. Taking this as c in Eq. (2) and iterating the z_i's, the values of successive z_i's are 0 + 0i, 0 + 1i, 0 +0i, 0 +1i, 0 + 0i, 0 + 1i and so on. Thus it can be concluded that the value of z in this case oscillates between 0 + 0i and 0 + 1i. In other words, the orbit of 'z_0' is found to be cycling with a period 2. These bulbs will be known as period 2 bulbs.

As before, it can be shown that any arbitrary point on any of the four large bulbs attached to the primary bulbs produces an orbit for z_0 with a period 2. Likewise, this can be extended to find bulbs of period 3 or more.

This process is then repeated with curves of degree 4 or higher. For each of the curves, the periods of various bulbs are determined and then they are classified according to their periods. This classification then helps in identifying certain similarities in the relationships between these bulbs irrespective of the power of these curves. The curves generated by substituting 4, 5 and 6 for i in Eq. (1), have been shown in Figs. 2-4 respectively.

Figure 2

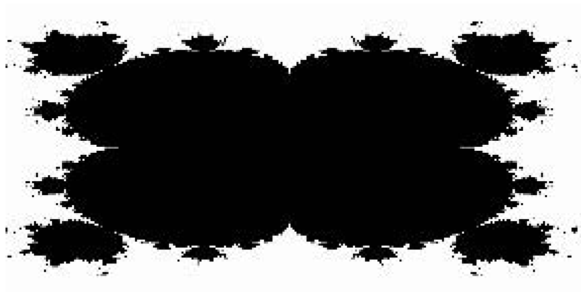

Figure 3

Figure 4

2.2. The mathematical relationship between the bulbs:

Let us now consider the curve given in Fig. 1 (i.e. the curve generated from Eq. (1)). It can be shown by the process in the previous section that the largest bulb between the bulbs of period 1 and those of period 2 have a period of 3. Similarly, the largest bulb between the bulbs of period 2 and those of period 3 are those of period 5, the largest between period 3 bulbs and period 5 bulbs are period 8 bulbs and so on.

Let 'm' and 'n' represent two consecutive periods of numbers, then the largest bulb between a period 'm' bulb and a period 'n' bulb is the bulb of period 'm+n', which reflects a Fibonacci property in the series.

Interestingly, the curves generated by taking polynomials of higher degree also show the same property of following the Fibonacci sequence in that the largest bulb between any two bulbs with periods of consecutive Fibonacci numbers is the bulb with a period of the next highest Fibonacci number.

Similarly, let's consider the largest bulb between the period 1 bulb and period 3 bulbs. The largest bulb between these two bulbs is the bulb of period 4. Proceeding likewise, it can be seen that the largest bulb between the period 1 bulb and the period 4 bulb is the period 5 bulb, largest between period 1 and period 5 bulbs is the period 6 bulb and so on. Thus the largest bulb between the period 1 bulb and a bulb of period m is the bulb of period m+1. This has been experimentally proved to be true for all curves generated from polynomials satisfying Eq. (1).

3. Observations on Curves

Detailed studies on the fractal behaviour of the polynomial curves generated by us are listed below.

3.1. Properties of the bulbs:

The period two bulbs for all curves are found to be exhibiting a sense of uniformity. To understand this better, let's consider the period 2 bulbs of the curve formed by iterating Eq. (2). It is found that the number of period two bulbs is twice the number of the primary bulbs. It may also be mentioned here that there are two primary bulbs for this polynomial curve. Similarly, let's consider the curve generated by iterating the equation

$$z_n = z_{n-1}^4 + c \qquad (3)$$

By referring to Fig. 2, it can be observed that there are three period 2 bulbs per primary bulb. Considering that there are three period 1 bulbs in the figure, the total number of period 2 bulbs for the entire figure is 9.

It can be shown that this trend will be true for all curves satisfying Eq. (1). Thus, if there are 'j' primary bulbs, then the number of period 2 bulbs 'k' is given by

$$k = j \times j = j^2 \qquad (4)$$

In our earlier work[3], we had shown that if a curve satisfies Eq. (1), then

$$j = i - 1 \qquad (5)$$

where 'i' is the degree of the polynomial generating the curve. Substituting this value of 'j' in Eq. (5), we get

$$k = (i - 1)^2 \qquad (6)$$

3.2. Attractors in the curves

The major points of attraction are defined as those points of attraction that lie either entirely on a primary bulb or at the points of intersection of the primary bulb with the period 2 bulbs. A sense of uniformity is observed in the number of major points of attraction and the degree of the curve. In order to discuss it in detail, let's consider the curve generated by iterating Eq. (2). Counting the major points of attraction, their number is found to be 6.

Similarly, let's consider the figure generated by iterating the equation

$$z_n = z_{n-1}^4 + c \qquad (7)$$

Counting the major points of attraction, their number is found to be 9. Similarly, exploring the figures generated by the substitution of higher values for 'i' in Eq. (1), the number of major points of attraction are found to be 12, 15 and 18 for figures having i-values of 5, 6 and 7 respectively. Iterating in this manner, it is found that for all curves satisfying Eq. (1), the number of major points of attraction 'mpa' is given by

$$mpa = 3 \times (i - 1) \qquad (8)$$

4. CONCLUSIONS

After detailed observations on the polynomial curves, the following points may be inferred.

1. The period of the largest bulb in between two bulbs having periods of successive Fibonacci numbers is the next highest Fibonacci number.
2. The period of the largest bulb in between the primary bulb and any other bulb is the natural number following the period of the other bulb.
3. The number of period 2 bulbs in a curve is the square of the natural number whose value is one less than the degree of the curve.
4. The number of major points of attraction is thrice the number which is one less than the degree of the curve.

The above-mentioned results can be used to make further studies on the attractors in the polynomial curves.

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