Estimation Of Integrated Covolatility For Asynchronous Assets In The Presence Of Microstructure Noise

by

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ABSTRACT

The use of high-frequency return data has led to dramatic improvements in both theoretical and applied finance research. Estimators of covariance among multiple processes have been proposed, such as realized variance and Hayashi-Yoshida estimator. We are introducing our new estimator, the random lead-lag estimator(RLLE), which coincides with the Hayashi-Yoshida estimator at very high frequency. We studied the performance of RLLE both with and without microstructure noise for non-synchronous data and obtained the optimal estimator with good bias-variance trade-off. Our result is confirmed by simulation. We also applied our method to real data application.

The dissertation is organized as follows: In Chapter 1, I introduce the background of the study. A literature review is included. Also included is the set up of the financial model for returns and existing estimators for integrated covariance. Chapter 2 introduces the new estimator: Random Lead and Lag estimator, as well as theoretical support for the estimator. The simulation results are presented in Chapter 3; real data application, in Chapter 4.

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Chapter 1

Chapter 1 – Introduction and Background

1.1 Introduction

Many of us examine the equity market and bond market during our morning coffee, using the old fashion of newspaper listings, or radio and TV, or online media. Yet, most of us do not think it is that revelent to our daily life, even with all these big slumps in the financial market these days. No matter how well or ill functioning our market is right now, it is a fact that the whole financial system is a marvelously complex thing which goes round and round like a huge machine with millions of gears. As humanity goes, this machine develops, breaks, and evolves; it may take other formats, but it is hard to imagine that it would totally stop or disappear. Whether we like or hate money, we know the financial world is a mirror of our society, and it

is a mixture of many things, with rigorous structures at some points, but with also with vague and abstract structures at other points. It is a blend of logic, wisdom, dreams, human anxiety and fear. All these uncertainties build up the broad stage for statistics to perform on, yet all the human interactions in these markets make lots of tasks rough and even distressing sometimes. Challenges are researchers' enemies and friends, and that is my reason for thinking that financial statistics is such an amazing field. There is no doubt we are facing a crisis in our financial system, and this means we will need more new thoughts and research efforts in the field. It might be difficult for researchers, but it also opens many doors to exciting discoveries.

The use of high frequency return data has led to dramatic improvements in financial research, especially in the area of volatility estimation. This is motivated by the recent availability of an almost continuous tick-by-tick record of quotes and transaction prices for a variety of financial assets. There is a rich literature of volatility estimation in finance triggered by the work of Andersen and Bollerslev (1998). A vast amount of research followed, such as Andersen et al. (2001a and b), Barndorff-Nielsen and Shephard (2001); Areal and Taylor (2002), Thomakos and Wong (2003), Martens et al. (2004), Pong et al. (2004) and Koopman et al. (2005).

Under microstructure noise, the process is 'contaminated' due to discretization or bid-ask spreads, and recently, extensive work has been done under this situation. Some references are Corsi et al(2001), At-Sahalia et al. (2005), Zhang, et al. (2005),

Bandi and Russell (2006), Barndorff-Nielsen et al(2006), Hansen and Lunde (2006) and Oomen (2006).

There is, however, a lot of scope for the study of the variance-covariance matrix for multiple stochastic processes, which is of great interest in most finance applications, such as portfolio selection and risk management. Especially when the observations are non-synchronous and microstructure noise is present in the processes, the estimation gets more complicated. There are quite a few estimators constructed to estimate this quantity under various situations and assumptions. The naive estimator is the *realized covariance*(RC) as in Jacod and Shiryaev (1987) or Karatzas and Shreve (1991), which is the analogue of realized variance for a single process. This estimator is well studied for synchronous data without microstructure noise, see Jacod and Protter (1998), Barndorff-Nielsen and Zhang (2001), Shephard (2002), and Mykland and Zhang (2000). RC is a consistent estimator for $< P^1, P^2 >$, with mixed normal distributed estimation error. Under non-synchronous trading, this estimator becomes highly biased due to the Epps effect (Epps, 1979). Other estimators have been proposed to correct the bias under the situation of no microstructure noise. Scholes and Williams (1977) add one lead and lag to the realized covariance, and Dimson (1979) and Cohen et al. (1983) generalized the number of leads and lags to k. See also de Jong and Nijiman (1997). Hayashi and Yoshida (2004, 2005) propose their *cumulative covariance* (CC) estimator for non-synchronous data, which is unbiased without microstructure noise. Martens (2004) studied some of these estimators and carried out

a comparison of different methods. See also De Pooter, Martens and Van Dijk (2006), for other important empirical studies of the properties of different estimators. When microstructure noise plays a role in the market, the CC estimator becomes biased and inconsistent, too. Most recently, Griffin and Oomen (2006) study the CC estimators under an i.i.d. noise scheme. Voev and Lunde (2006) have bias-corrected the CC estimator and examine the sub-sampling version of that estimator. Zhang (2006) examines the last-tick interpolation-based realized covariance under microstructure noise for non-synchronous data. Sheppard (2005), Bandi and Russell (2005b) have studied sample frequency and bias corrections for synchronous data.

While we were weighing on the merits and the weakness of the CC estimator, beyond the bias present under microstructure noise, we found that also the variance is big and it is computationally demanding. This motivated us to propose a new estimator, which we call the random lead-lag (RLL) estimator. Not like the Cohen estimator, where the leads and lags are fixed for all the moving averages, the RLL estimator allows them to vary randomly following two independent exponential distributions. In our simulation, we found that when no microstructure noise is present, the bias and variance of the RLL estimator both decrease while the frequency increases, which implies that the Hayashi-Yoshida estimator is still the best in the sense of small bias and variance. But it is computationally demanding as the frequency gets higher. As the frequency hits one minute, the change in bias and variance of the RLL estimator is negligible, hence we choose one minute as the optimal frequency. When

microstructure is present, the variance of the RLL estimator actually increases with increasing frequency, which opens up the scope for choosing an optimal frequency using bias variance trade-off.

It can be inferred from the above literature survey that although RC is consistent for integrated covariance in the ideal situation, it is highly biased for non-synchronous data at high frequencies, a situation which is required for the asymptotics to be valid. On the other hand, CC is unbiased in the absence of noise, but has very high variance when noise is present. The contribution of this paper is in proposing a discrete-time version of CC which converges to CC at high frequency and is hence asymptotically unbiased even for nonsynchronous data. On the other hand, our estimator can be calculated at lower frequencies, so that we can choose a lower frequency to keep the variance in control in the presence of microstructure noise. Our estimator is also a generalization of the class of fixed lead-lag estimators. It thus serves as a bridge between interval-based estimators like RC and tick-based estimators like CC.

1.2 Medeling Financial Returns and Realized Covariance

1.2.1 Basic formulation

In 1973, Fischer Black and Myron Scholes proposed the celebrated Black-Scholes model for log stock price. This model became the root of a lot of models and tech-

niques applied by today's financial analysts. The model looks like this:

$$dp(t,\omega) = \mu dt + \sigma dW(t,\omega), \quad t \geq 0 \tag{1.2.1}$$

Where $p(t,\omega)$ denotes the continuous price process of a security on the log scale at time t and under scenario ω, W is a standard Wiener process, wherein $\sigma > 0$ is called the volatility , and μ is the drift.

In most financial applications, not only the volatility, but also the variance-covariance matrix of multiple securities is of interest. Examples include application in portfolio selection or risk management, etc. Generally, the volatilities are time-varying. Hence, an extension to multiple asset prices and time-varying volatility is necessary:

$$dp^{(i)}(t,\omega) = \mu_i dt + \sigma_i(t) dW^{(i)}(t,\omega), \quad i = 1,\ldots,k \quad t \geq 0$$

where the quadratic variation between the two Wiener processes is $d < W^{(i)}, W^{(j)} >_t = \rho^{i,j}(t)$. From now on, we shall refer to ρ_t as the correlation. k denotes the number of securities and the superscript (i) denotes the i-th security.
$\{p^{(i)}(t,\omega)\}$ is the price process for security i in logarithmic scale. $\sigma_i(t) > 0, i = 1,\ldots,k$, are the volatilities , and $\mu_i(t), i = 1,\ldots,k$, are the drift terms. All of them generally will be (continuous) stochastic processes.

The quantity of interest is the integrated covariance

$$< p^{(i)}, p^{(j)} >_T = \int_0^T \sigma_i(t)\sigma_j(t)\rho^{i,j}(t)dt \tag{1.2.2}$$

For high frequency data, consider a fixed period of time of length $T > 0$. For concreteness, in simulation, we typically consider $T = 23,400$ seconds in a six and a half hour (NYSE) trading day. Traditionally daily returns are computed as

$$r_i = p(iT) - p((i-1)T), i = 1, 2, ...$$

where i indexes the day. However, for high frequency data, we have additional intra-day observations during each time period T. These observations usually are not observed at equal intervals of time. Most existing estimation methods require converting them artificially to regularly spaced data. Let us denote the common interval by h (seconds). $h \in \{1800, 600, 300, 60, 30, 10, 1\}$ are used throughout our simulation study. Let m be the number of intra-period intervals. Then $h = T/m$. The j-th intra-period return for the i-th period is defined as:

$$r_{j,i} = p((i-1)T + jh) - p((i-1)T + (j-1)h), j = 1, ..., m.$$

To convert the irregularly spaced observations to equal spaced intervals, two common schemes of data interpolation are used: 1. Previous tick interpolation, which uses the most recent values for each equal spaced sampling point. 2. Linear interpolation, which uses observations bracketing the desired time.

1.2.2 Non-synchronicity

In real data, the observations happen at random times corresponding to when trades take place in the market. These times do not coincide for multiple securities. Eg,

Asset A may record trades at 9:47 am, 9:52am, 9:56am etc whereas asset B records trades at 9:48am, 9:57am etc. This phenomenon is termed non-synchronicity. The following figure shows the trading times for Intel (INTC) and Amgen (AMGN) on Nasdaq for the first 200 seconds after market opened on Jan 2nd, 1990. The data are from the real data example in Chapter 4.

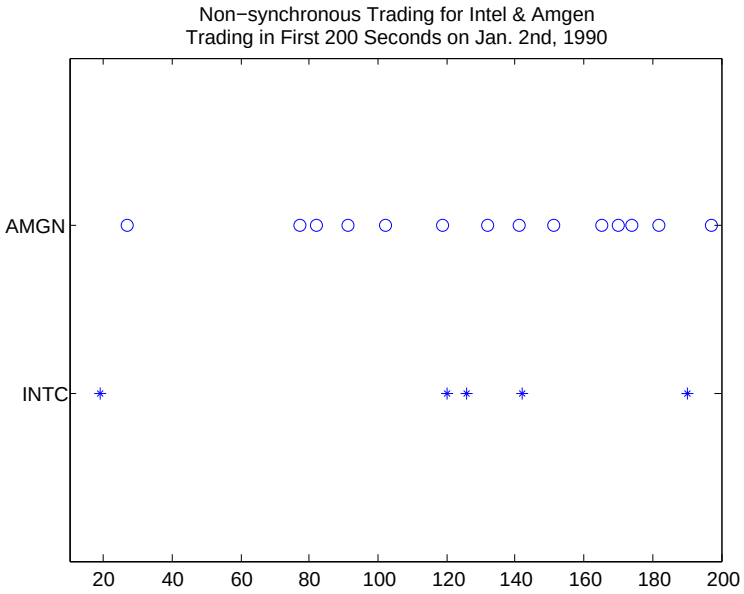


Figure 1.1: Non-synchrinicity

For multivariate problems, non-synchronous trading becomes a real concern. When data on two securities are observed non-synchronously, the Fisher effect (Fisher 1966) occurs irrespective of the underlying correlation structure. This effect is where returns sampled at equal spaced time will correlate with previous and successive returns on other assets. Also, the covariation measure becomes smaller as we increase the

sampling frequency. This was observed empirically by Epps(1979 JASA). Table 1.1 gives an example of how the realized covariance diminishes while the frequency gets higher.Hayashi and Yoshida (2005) show that even under very mild assumptions realized covariance is biased and inconsistent. In fact, under certain assumptions, it converges to zero.

Table 1.1: Realized covariance estimates based on Previous Tick and Linear Interpolation for various sampling intervals of Continental Airlines and Tidewater Inc.

h(second)	Realized-PT	Realized-LI
60	0.1516	0.0856
300	0.1350	0.1028
6000	0.0909	0.0598
1200	0.3574	0.2306
2340	0.3662	0.2775
4680	0.8595	0.8109

Some references for covariance estimation with non-synchronous high-frequency data are de Jong and Nijiman (1997); Lundin, Dacorogna and Mller (1999); Muthuswamy, Sarkar, Low, Terry(2001); Malliavin and Mancino (2002); Reno (2003); Yahashi and Yoshida (2004a and b.).

1.2.3 Microstructure noise and jumps

For the purpose of volatility estimation, Equation 1.2.4 is far from true in practice. One possible explanation is microstructure noise, such as discreteness of prices, bid

- ask spread, etc. The presence of noise does not pose any particular problem if data is observed less frequently–say, daily. Modern technology, however, permits almost continuous observation of data. When transaction prices or bid-ask quotes are contaminated with microstructure noise, the efficient price is not observed, so the observed prices lack martingale property. From now on, we suppose the observe log transaction or bid-ask quote price is P^*, and the underlying efficient log-price is P. For two assets, we will set their observed prices as $P_t^{(1)*}$ and $P_t^{(2)*}$, their efficient prices as $P^{(1)}$ and $P^{(2)}$ respectively. We write $P^{(i)*}$ as

$$P_t^{(i)*} = P_t^{(i)} + u_t^{(i)} \qquad i = 1, 2. \tag{1.2.3}$$

. $u_t^{(i)}$ is the noise component due to imperfection of the trading process, and it is a summary of variety of microstructure noise, which can be roughly grouped into the following categories: frictions inherent in the trading process, informational effects, and measurement or data recording errors. See At-Sahalia, Mykland and Zhang (2005). (Note: they have more explicit expressions for these three kinds of microstructure noise.)

We assume $u_t^{(i)}$ is independent of the efficient log-price $P_t^{(i)}$, which follows the SDE 1.2.1. For high frequency data, the length of time intervals h is really small, usually measured in seconds. The drift term is irrelevant under this circumstance, hence we will set $\mu_i(t) = 0, i = 1, 2$, throughout the study.

Recently, impressive work has been done studying the properties and performance

of the realized variance estimate of integrated volatility under the presence of microstructure noise. It is a fact that the estimator depends only on the noise up to first order approximation. The following are some references in the area: Zhang, Mykland and At-Sahalia (2005); At-Sahalia, Mykland and Zhang (2005); Barndorff-Nielsen and Shephard (2003, 2004a,b);Bandi and Russell (2004a,b);Hansen and Lunde (2006). Some suggested solutions to the problems caused by the microstructure noise include applying variance reduction techniques like sub-sampling, or using alternative estimators like bipolar variance.

Another suggested reason for failure of Equation 1.2.4 in real data is the presence of jumps in the stock price process. See Fan and Wang (2005). In our real data example in Chapter 4, we will present examples of price jump due to stock split for Microsoft (MSFT) and Amgen (AMGN). See Figure 4.2. They advocate the use of wavelets to separate jumps from the continuous process.

1.2.4 Realized variance/covariance estimator

The 'Previous tick' method is used throughout this paper. At each sampling point, the most recently observed price for each asset is recorded, i.e. $P_t^{(j)} = p_{N_j(t)}^{(j)}$, where $N_j(t) = sup_n\{n|t_n^{(j)} \leq t\}$. Follow the setup in section 1.2.1. Suppose we have two assets with prices $(P^{(i)}(t_k), P^{(j)}(t_k))_{k=0,1,\ldots,m}$ observed at times $\pi(m) = \{0 = t_0 < t_1 < \cdots < t_m = T\}$. These correspond to times at which trades take place. Then the

Realized covariance can be formulated as the following:

$$RC_{i,j}(\pi(m)) := \sum_{k=1}^{m}(P^{(i)}(t_k) - P^{(i)}(t_{k-1}))(P^{(j)}(t_k) - P^{(j)}(t_{k-1})).$$

While $i = j$, the realized covariance becomes the realized variance in the univariate case. While the observations are synchronous and there is no microstructure noise present, the estimator is consistent for the integrated covariance as defined in equation 1.2.1 (Jacod and Shiryaev (1987), Karatzas and Shreve (1991)), with a mixed Gaussian distribution (Jacod and Protter (1998), Zhang (2001), Barndorff-Nielsen and Shephard (2002), and Mykland and Zhang (200), et al.). That is:

$$RC_{i,j}(\pi(m)) \xrightarrow{P} < p^{(i)}, p^{(j)} >_T \text{ as } \pi(m) := \max \mid t_k - t_{k-1} \mid \to 0 \tag{1.2.4}$$

Barndorff-Nielsen and Shephard (BN-S)(2004) derive the asymptotic distribution for the realized covariance under certain general assumptions: The price is a continuous stochastic volatility semimartingale ($SVSM^C$) vector. They found that the drift term does not affect the result. They also derive the asymptotic distribution of realized correlation and realized beta. These two statistics are transformations of the realized covariance matrix. They are defined as follows:

Realized correlation:

$$\frac{\sum_{k=1}^{m}(P^{(i)}(t_k) - P^{(i)}(t_{k-1}))(P^{(j)}(t_k) - P^{(j)}(t_{k-1}))}{\sqrt{\sum_{k=1}^{m}(P^{(i)}(t_k) - P^{(i)}(t_{k-1}))^2}\sqrt{\sum_{k=1}^{m}(P^{(j)}(t_k) - P^{(j)}(t_{k-1}))^2}}$$

Realized beta:

$$\frac{\sum_{k=1}^{m}(P^{(i)}(t_k) - P^{(i)}(t_{k-1}))(P^{(j)}(t_k) - P^{(j)}(t_{k-1}))}{\sum_{k=1}^{m}(P^{(i)}(t_k) - P^{(i)}(t_{k-1}))^2}$$

Under certain conditions, both statistics have asymptotic normal distribution. Given the broad use of integrated covariance, correlation, beta in Derivative Pricing, Man-

agement, Portfolio Optimization and other applications, the estimation of these quantities are of great interest. The asymptotic distribution theories give a great theoretical guide to estimation methods.

For some background on Realized Variance, refer to Andersen, Bollerslev, Diebold and Ebens (2001); Andersen, Bollerslev, Diebold and Labys (2001); Barndorff-Nielsen and Shephard (2002, 2004).

1.2.5 Hayashi-Yoshida Estimator

Recently, Hayashi and Yoshida (2004,2005) propose an alternative estimator that takes care of the non-synchronicity problem using tick-by-tick data. We called it the Cumulative Covariance (CC) estimator, as it is called in Hayashi and yoshida (2005).

Similar to the notation in the subsection of Realized Covariance, suppose that we have two assets with prices: $p^{(1)*}(t^{(1)}_{k_1})$, where $k_1 = 0, 1, \ldots, m_1$, and $p^{(2)*}(t^{(2)}_{k_2})$, where $k_2 = 0, 1, \ldots, m_2$, respectively. $p^{(l)*}(t^{(l)}_{k_l})$, $l = 1, 2$, follow equation 1.2.3, and their observations are observed at times $\pi^{(1)}(m_1) = \{0 \leqslant t^{(1)}_0 < t^{(1)}_1 < \cdots < t^{(1)}_{m_1} \leqslant T\}$, and, $\pi^{(2)}(m_2) = \{0 \leqslant t^{(2)}_0 < t^{(2)}_1 < \cdots < t^{(2)}_{m_2} \leqslant T\}$, they usually correspond to the occurrence of transactions or quote-revisions. We assume these arrival times follow Poisson processes independent of the price process with constant arrival intensity over time, but they can vary across assets (see e.g. Hayashi and Yoshida, 2005).The Hayashi-Yoshida CC estimator accumulates the cross product of all overlapping (fully

and partially) returns. It is specified as follows:

$$\mathrm{CC} = \sum_{i=1}^{m_1} \sum_{j \in A_i} r_i^{(1)} r_j^{(2)} \tag{1.2.5}$$

Where $r_i^{(l)} = p_i^{(l)*} - p_{i-1}^{(l)*}, l = 1, 2$ and $A_i = \{j | (t_{i-1}^{(1)}, t_i^{(1)}) \cap (t_{j-1}^{(2)}, t_j^{(2)}) \neq \emptyset\}$.

They propose two versions of cumulative correlation estimators(we call them CCOR1 and CCOR2):

- First, if the volatilities of the two processes σ_1 and σ_2 are known, then the estimator is:

$$\mathrm{CCOR1} = \frac{1}{T} \sum_{i=1}^{m_1} \sum_{j \in A_i} \frac{r_i^{(1)} r_j^{(2)}}{\sigma_1 \sigma_2} \tag{1.2.6}$$

  where $r_i^{(j)}, j = 1, 2$ and A_i are the same as those in the expression of Cumulative Covariance estimator.

- Second, no matter if the volatilities σ_1 and σ_2 are known or not, the estimator is:

$$\mathrm{CCOR2} = \frac{\sum_{i=1}^{m_1} \sum_{j \in A_i} r_i^{(1)} r_j^{(2)}}{(\sum_{i=1}^{m_1} r_i^{(1)})^{1/2} (\sum_{i=1}^{m_2} r_i^{(2)})^{1/2}} \tag{1.2.7}$$

  where $r_i^{(j)}, j = 1, 2$, and A_i are the same as those in the expression of Cumulative Covariance estimator.

While microstructure noise is not present, these estimators of covariance and correlation are unbiased, consistent and asymptotically normal under fairly general assumptions. Hayashi and Yoshida prove consistency when the price process is continuous semi-martingale and the arrival processes are sequences of stopping times.

However, while microstructure noise is present, the performance of the estimators change. While the noise process is i.i.d. and independent of the efficient price process, and the two processes have independent poison arrival time, the CC estimator is still unbiased, but the noise makes it inconsistent. It is consistent while the arrival intensities λ_1 and λ_2 go to infinity, but then this is reduced to the Realized Covariance case(Griffin and Oomen (2006)). Under other noise specification, the estimators are not only inconsistent, but can also be biased(Voev, Lunde (2006)). Also, computation of the estimate takes very long. Another downside of the estimators are that they require the exact timing of transactions. Thus one may still be forced to rely on a realized covariance kind of estimator if they are not available (Griffin and Oomen (2006)).

Chapter 2

Chapter 2 – Random Lead-and-Lag Estimator

2.1 Interpretation

After assessing the upside and downside of the two major different estimators – Realized Covariance Estimator and Cumulative Covariance estimator–we found the following: under i.i.d. noise, the Realized Covariance estimator gets bigger bias while the frequency gets higher, and this is not surprising due to the Epps effect. The variance is bell shaped, so it gets bigger, but after a certain point, it starts to decrease. The Cumulative Covariance estimator is unbiased, but it has big variance, and it is computationally demanding. This motivates us to propose our new estimator–the Random Lead-Lag (RLL) estimator. One can think that the RLL is kind of between the Realized Covariance estimator and the Cumulative Covariance estimator. We also

propose optimal sampling frequency, and gain a good bias-variance trade-off for the new estimator. The computation of RLL is very fast. Following the setting of section 1.2.1, let the total time T(eg one trading day) be divided into m intervals of length $h = T/m$. We scale $T = 1(h = 1/m)$ to get more general results.Let $P_i^{(l)*}$ denote the observed log-price of stock l at the end of the i-th interval by the 'previous tick' method. Replace $P_i^{(l)*}$ by the efficient log-price of stock l by $P_i^{(l)}$ while microstructure noise is not present.

We define the random lead-lag estimator as:

$$\text{RLL} = \sum_{i=1}^{m} \sum_{j=K_{i1}}^{K_{i2}} (P_i^{(1)*} - P_{i-1}^{(1)*})(P_j^{(2)*} - P_{j-1}^{(2)*}) \tag{2.1.1}$$

where $K_{i1} = \max\{k \leq i-1 :$ at least one event for process 1 in the time interval $((k-1)h, kh]\}+1$ and $K_{i2} = \min\{k \geq i :$ at least one event for process2 in the time interval $((k-1)h, kh]\}$ are random variables.

This estimator can be interpreted as follows: wait till there is a trade for either stock. If there is a trade for both stocks, take the product of log returns. If there is a trade for stock 1 and not 2, impute the last non-zero log return for stock 2 and take product. And vice versa.

To show our main results, we summarize some general assumptions mentioned in the earlier text.

Assumption 2.1.1. *For high frequency data, since the intervals are usually very*

small, the drift term is irrelevant in this case. We assume the drift terms for both processes are zeros.

Assumption 2.1.2. *The observation price $P_t^{(i)*}, i = 1, 2$ follow the equation 1.2.3, and the efficient price $P_t^{(i)}, i = 1, 2$ are correlated Brownian motions, i. e. $P^{(i)} = \sigma_i W^{(i)}$, $i = 1, 2$, and $dW^{(1)}dW^{(2)} = \rho dt$, where ρ is a shorthand for $\rho^{1,2}(t)$, and it is deterministic and constant over time.*

Assumption 2.1.3. *The arrival times are two independent Poisson processes with arrival intensities λ_1 and λ_2. Hence the increments are independent i.i.d. exponential distributed.*

Assumption 2.1.4. *The market microstructure noise $u_t^{(1)}$ and $u_t^{(2)}$ are independent of each other, and they are independent of the efficient price processes. $u_t^{(i)} \sim (0, \phi_i^2)$,$i = 1, 2$.*

Theorem 2.1.5. *The random lead-lag estimator coincides with the previous-tick realized covariance estimator if there is at least 1 trade in each interval for either asset (High trading rate).*

Proof. When there is at least 1 trade in each interval for process 1, then for each fixed i, for any $k \leq i - 1$,

$$K_{i1} = i - 1 + 1 = i.$$

likewise, if there is at least 1 trade in each interval for process 2, then for fixed i, for any $k \geq i$,

$$K_{i2} = i.$$

Combine these two sides, we have:

$$\begin{aligned} \text{RLL} &= \sum_{i=1}^{m} \sum_{j=K_{i1}}^{K_{i2}} (P_i^{(1)*} - P_{i-1}^{(1)*})(P_j^{(2)*} - P_{j-1}^{(2)*}) \\ &= \sum_{i=1}^{m} (P_i^{(1)*} - P_{i-1}^{(1)*})(P_i^{(2)*} - P_{i-1}^{(2)*}) \\ &= \text{RC} \end{aligned}$$

□

Assumption 2.1.6. *The probability of two observations happening at the same time for different processes is zero. Hence, if the frequency is sufficiently high, intervals will be small enough that in each interval, there is at most one observation for both assets, i.e., the two assets are not observed in the same interval, and for each one, it is not observed more than once in an interval.*

Theorem 2.1.7. *Under the general assumptions and assumption 3.6,the random Lead-lag estimator coincides with the Hayashi-Yoshida estimator if there is at most 1 observation in each interval for either process. (Computing interval/bandwidth small).*

Proof. Suppose we have m equal spaced intervals and the time points are $\pi(m) = \{0 = t_0 < t_1 < \cdots < t_m = T\}$, while the observation times for asset 1 is $\pi^{(1)}(m_1) = \{0 \leqslant t_0^{(1)} < t_1^{(1)} < \cdots < t_{m_1}^{(1)} \leqslant T\}$, and the observation time for asset 2 is $\pi^{(2)}(m_2) = \{0 \leqslant t_0^{(2)} < t_1^{(2)} < \cdots < t_{m_2}^{(2)} \leqslant T\}$. Under Assumption 3.6,$m_1 < m$ and $m_2 < m$.

With probability one, we have

$$\begin{aligned}\text{RLL} &= \sum_{i=1}^{m}\sum_{j=K_{i1}}^{K_{i2}}(P_i^{(1)*}-P_{i-1}^{(1)*})(P_j^{(2)*}-P_{j-1}^{(2)*}) \\ &= \sum_{i=1}^{m_1}\sum_{j=b_1}^{b_2}(P_{t_i^{(1)}}^{(1)*}-P_{t_{i-1}^{(1)}}^{(1)*})(P_j^{(2)*}-P_{j-1}^{(2)*})\end{aligned}$$

Where $b_1 = \max\{k : t_k \le t_{i-1}^{(1)}$ and there is at least one event between sample points $t_k^{(1)}-$ 1 and$t_k^{(1)}\}+1$ and $b_2 = \min\{k : t_k \ge t_i^{(2)}$ and there is at least one event between sample points $t_k^{(2)}-$ 1 and$t_k^{(2)}\}$ Since there is at most one observation for both assets, for each i, we have

$$\sum_{j=b_1}^{b_2}(P_j^{(2)*}-P_{j-1}^{(2)*}) = \sum_{j\in A_i}(P_j^{(2)*}-P_{j-1}^{(2)*})$$

Where $A_i = \{j|(t_{i-1}^{(1)}, t_i^{(1)}) \cap (t_{j-1}^{(2)}, t_j^{(2)}) \neq \emptyset\}$ Hence

$$\begin{aligned}\text{RLL} &= \sum_{i=1}^{m_1}(P_{t_i^{(1)}}^{(1)*}-P_{t_{i-1}^{(1)}}^{(1)*})\sum_{j=b_1}^{b_2}((P_j^{(2)*}-P_{j-1}^{(2)*}) \\ &= \sum_{i=1}^{m_1}(P_{t_i^{(1)}}^{(1)*}-P_{t_{i-1}^{(1)}}^{(1)*})\sum_{j\in A_i}((P_j^{(2)*}-P_{j-1}^{(2)*}) \\ &= \sum_{i=1}^{m_1}\sum_{j\in A_i}(P_{t_i^{(1)}}^{(1)*}-P_{t_{i-1}^{(1)}}^{(1)*})((P_j^{(2)*}-P_{j-1}^{(2)*}) \\ &= \text{CC}\end{aligned}$$

□

2.2 Bias of RLL

To find the Bias of RLL, first introduce two lemmas.

Lemma 2.2.1. *$W^{(1)}(t)$,$W^{(2)}(t)$ are two Brownian motions with $dW^{(1)}dW^{(2)} = \rho dt$, $T_1^{(i)} < T_2^{(i)}, i = 1, 2$ are random times for $W^{(i)}(t), i = 1, 2$ respectively,$\Delta W^{(i)} =$*

$W^{(i)}(T_2^{(i)}) - W^{(i)}(T_1^{(i)}), i = 1, 2$ *and* $\nu(I_1 \cap I_2)$ *denotes the length of the intersection between intervals* I_1 *and* I_2*, then,*

$$\mathrm{E}[\Delta W^{(1)} \Delta W^{(2)}] = \rho \, \mathrm{E}\left[\nu\left((T_1^{(1)}, T_1^{(1)}) \cap (T_1^{(2)}, T_2^{(2)})\right)\right].$$

Proof. See Appendix. □

Lemma 2.2.2. *(Symmetry) Let* $R_i^{(j)} = P_i^{(j)*} - P_{i-1}^{(j)*}, j = 1, 2$*, then*

$$\begin{aligned}
\mathrm{RLL} &= \sum_{i=1}^{m} \sum_{j=K_{i1}}^{K_{i2}} R_i^{(1)} R_j^{(2)} \\
&= \sum_{i=1}^{m} \left[\sum_{j=K_{i1}}^{i-1} R_i^{(1)} R_j^{(2)} + R_i^{(1)} R_i^{(2)} + \sum_{j=i+1}^{K_{i2}} R_i^{(1)} R_j^{(2)} \right] \\
&= \sum_{i=1}^{m} \sum_{j=K_{i1}}^{i-1} R_i^{(1)} R_j^{(2)} + \sum_{i=1}^{m} R_i^{(1)} R_i^{(2)} + \sum_{i=1}^{m} \sum_{j=i+1}^{K_{i2}} R_i^{(1)} R_j^{(2)}
\end{aligned}$$

Then

$$\sum_{i=1}^{m} \sum_{j=K_{i1}}^{i-1} R_i^{(1)} R_j^{(2)} = \sum_{i=1}^{m} \sum_{j=i+1}^{K_{i2}} R_j^{(1)} R_i^{(2)}.$$

Proof. See appendix. □

The following is the main theory for this subsection:

Theorem 2.2.3. *(Expectation and Bias of RLL) The expectation for RLL is:*

$$\begin{aligned}
&\mathrm{E(RLL)} \\
&= 1\rho\sigma_1\sigma_2 - \frac{1}{h}\rho\sigma_1\sigma_2\frac{\lambda_1^2\xi_2 + \lambda_2^2\xi_1}{\lambda_1\lambda_2(\lambda_1+\lambda_2)} \\
&+ \frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_1 h})e^{-\lambda_2 h} \\
&+ \frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_2 h}} - \frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_2 h})e^{-\lambda_1 h}. \qquad (2.2.1)
\end{aligned}$$

Hence the bias of RLL is

$$\begin{aligned}
&\mathrm{Bias(RLL)} \\
&= -\frac{1}{h}\rho\sigma_1\sigma_2\frac{\lambda_1^2\xi_2 + \lambda_2^2\xi_1}{\lambda_1\lambda_2(\lambda_1+\lambda_2)} \\
&+ \frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_1 h})e^{-\lambda_2 h} \\
&+ \frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_2 h}} - \frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_2 h})e^{-\lambda_1 h} \qquad (2.2.2)
\end{aligned}$$

Where $\xi_j = 1 - e^{-\lambda_j h}, j = 1, 2.$

Proof. See Appendix. □

Theorem 2.2.4. *While h goes to zero, the RLL estimator is asymptotically unbiased.*

Proof. See appendix. □

As an example for illustration, the following plot plots the theocratical values for scaled biases (scale T = 1) for RLL and RV. Note that RV's bias is: $-\frac{T}{h}\rho\sigma_1\sigma_2\frac{\lambda_1^2\xi_2+\lambda_2^2\xi_1}{\lambda_1\lambda_2(\lambda_1+\lambda_2)}$. The parameter values used are the same as the ones in the simulation study. We set the volatilities of the two processes as $\sigma_1 = 1$ and $\sigma_2 = 2$. Arrivals for process 1 are Poisson Process, with 1 arrival every minute on average. Arrivals for process

2 are Poisson Process with 1 arrival every 5 minutes on average, which means the arrival intensities are $\lambda_1 = 60$ and $\lambda_2 = 300$ in seconds. $T = 23,400$ seconds, with $h \in \{1800, 600, 300, 60, 30, 10, 1\}$ seconds.The correlation coefficient of the two processes runs from 0.1 to 0.9 with step equals to 0.1. $(\rho = 0.1 : 0.1 : 0.9)$.

The 9 panels in each figure represent the 9 values of ρ. The horizontal axis

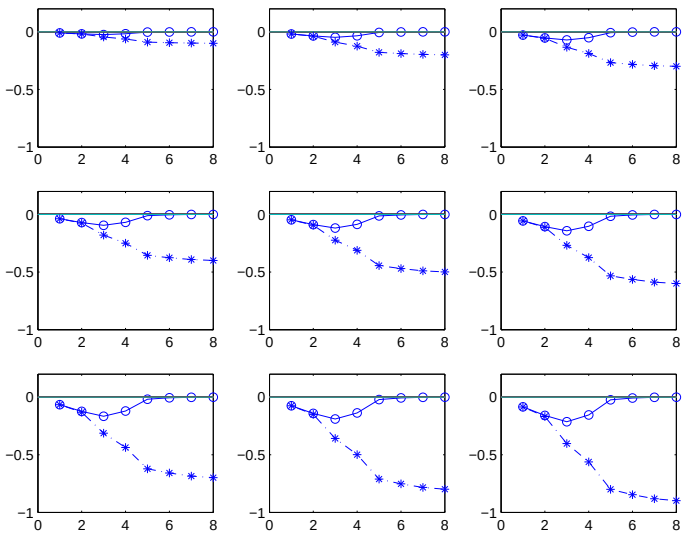


Figure 2.1: Bias: circle line is RLL, star line is RC

in each panel represents the index of the sampling frequency h in the set $h \in \{1800, 600, 300, 60, 30, 10, 1\}$.

2.3 Variance of RLL wigh no microstructure noise

To calculate the variance of RLL, we first concentrate the case without noise. For the simplicity reason, we state RLL is a different way. Different expressions also

lead us and readers to a better understanding of the structure of the estimator and characteristics as well.

The estimator is expressed in the following way:

$$\text{RLL} = \sum_{i=1}^{m}(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i)) + \sum_{i=1}^{m}(R_{L_{i2}} Z_i \mathbb{I}(L_{i2} > i)) + \sum_{i=1}^{m} R_i Z_i \tag{2.3.1}$$

This is true based on the symmetric theorem we provided in the earlier reading. By putting the estimator in this way, we can view the process of constructing the estimator in the following way:

- For any grid point $i, i = 1, ..., m$, the first process will have corresponding return R_i (this will be $P_i^{(1)} - P_{i-1}^{(1)}$it might be zero or not), and we look forward from that point (where time $t = ih$) for process 2, find the first event which gives a nonzero return of process 2. If that event happens in the interval of $((l-1)h, lh]$, then $L_{i1} = l$, where L_{i1} is a random variable. Notice that in this process we do not consider if there is any trade happening in the interval of $((i-1)h, ih]$ for process two, which will be considered in the third part of the estimator. Take the cross product of R_i and $Z_{L_{i1}}$, and sum all of them up from $i = 1$ to $i = m$.

- Switch the role of the two processes, and repeat the same procedure. That is for any grid point $i, i = 1, ...m$, we have $Z_i = P_i^{(2)} - P_{i-1}^{(2)}$ for process 2, and then we find L_{i2} for process 1 in the same manner as we find L_{i1} in the previous step. Take the cross product of Z_i and $R_{L_{i2}}$, and sum all of them up from $i = 1$ to $i = m$.

- Add a term of realized covariance $\sum_{i=1}^{m} R_i Z_i$ to the whole thing. This is done

for the reason that in the previous two steps, we omit the intervals $((i-1)h, ih]$ while we work on L_{i1} and L_{i2}. Now, by adding the realized covariance term, we add back what we left in the previous construction process.

Mathematically,the random variables L_{i1} and L_{i2} satisfy the following conditions:

$$L_{i1} = \min\{l > i : \text{at least one event for process 1 in the time interval } ((l-1)h, lh]\}$$

and

$$L_{i2} = \min\{l > i : \text{at least one event for process 2 in the time interval } ((l-1)h, lh]\}.$$

We can explore the distribution of these random variables a little bit. As for L_{i1}, express the probability as follows: $P(L_{i1} = l)$ =P(For process 2, nothing happened between $((i-1)h, (l-1)h]$, while at least one event happened in the interval of $((l-1)h, lh]$)=$P\left((l-i)h < \phi_{i-1}^{(2)} \leq (l-i+1)h\right)$, where $\phi_{i-1}^{(2)}$ is the waiting time for an event to happen from time point (i-1)h for process 2. Likewise, we can define $\phi_{i-1}^{(1)}$ as the waiting time for an event to happen from time point (i-1)h for process 1. Then $\phi_{i-1}^{(j)},\ j = 1, 2$ follows exponential distribution:

$$\phi_{i-1}^{(j)} \sim \exp(\lambda_{\mathrm{j}}) \quad j = 1, 2$$

with pdf:

$$f\left(\phi_{i-1}^{(j)} = t\right) = \lambda_j e^{-\lambda_j t},\ j = 1, 2.$$

Hence, we have the following probability function for $L_{ij},\ j = 1, 2$:

$$P(L_{i1} = l) = \int_{(l-i)h}^{(l-i+1)h} \lambda_2 e^{-\lambda_2 t} dt = e^{-\lambda_2 (l-i)h} - e^{-\lambda_2 (l-i+1)h} = \xi_2 e^{-\lambda_2 (l-i)h} \qquad (2.3.2)$$

and

$$P(L_{i2}=l)=\int_{(l-i)h}^{(l-i+1)h}\lambda_1 e^{-\lambda_1 t}dt=e^{-\lambda_1(l-i)h}-e^{-\lambda_1(l-i+1)h}=\xi_1 e^{-\lambda_1(l-i)h}. \quad (2.3.3)$$

By the way we decompose the estimator, it is not hard to imagine that there are more ways of decomposition. For example, we can express the estimator as:

$$\mathrm{RLL}=\sum_{i=1}^{m}(R_i Z_{L_{i1}})+\sum_{i=1}^{m}(R_{L_{i2}}Z_i)-\sum_{i=1}^{m}R_i Z_i$$

This is true because in the first two terms we double counted those intervals where both R_i and Z_i are not zeros. We subtract the realized covariance term to take care of that problem. Based on the principle of no double counting, we can also decompose the estimator into the following formats:

$$\mathrm{RLL}=\sum_{i=1}^{m}(R_i Z_{L_{i1}}\mathbb{I}(L_{i1}>i))+\sum_{i=1}^{m}(R_{L_{i2}}Z_i)$$

or

$$\mathrm{RLL}=\sum_{i=1}^{m}(R_i Z_{L_{i1}}+\sum_{i=1}^{m}(R_{L_{i2}}Z_i\mathbb{I}(L_{i2}>i))$$

For the calculation purpose, we will use the expression in (2.3.1) and simplify the notation as following:

$$\mathrm{RLL}=\mathrm{RLL}_1+\mathrm{RLL}_2+\mathrm{RC},$$

where $\mathrm{RLL}_1=\sum_{i=1}^{m}(R_i Z_{L_{i1}}\mathbb{I}(L_{i1}>i))$, $\mathrm{RLL}_2=\sum_{i=1}^{m}(R_{L_{i2}}Z_i\mathbb{I}(L_{i2}>i))$ and $\mathrm{RC}=\sum_{i=1}^{m}(R_{L_{i2}}Z_i)$ is the realized covariance term.

Now

$$\begin{aligned}\mathrm{Var}(\mathrm{RLL})&=\mathrm{Var}(\mathrm{RC})+\mathrm{Var}(\mathrm{RLL}_1)+\mathrm{Var}(\mathrm{RLL}_2)\\&\quad+2\mathrm{Cov}(\mathrm{RLL}_1,\mathrm{RC})+2\mathrm{Cov}(\mathrm{RLL}_2,\mathrm{RC})+2\mathrm{Cov}(\mathrm{RLL}_1,\mathrm{RLL}_2)\end{aligned} \quad (2.3.4)$$

To find the variance or RLL estimator, we need to calculate each part in the above equation. The proof outline is listed in the following subsections.

2.3.1 Variance of RC

For the realized covariance part, we use the result directly from "Covariance Measurement in the Presence of Non-Synchronous Trading and Market Microstructure Noise " by Griffin and Oomen (2006). To be consistent with our notations, adjust the interval length from 1 to T. Consequently we need to replace the M by $1/h$ to get consistent form in our case. Let $\xi_i = 1 - e^{-\lambda_i h}, i = 1, 2.$ $\beta_m = 1 - \frac{1}{h(\lambda_1+\lambda_2)}\left(\frac{\lambda_1}{\lambda_2}\xi_2 + \frac{\lambda_2}{\lambda_1}\xi_1\right)$. Then

$$\begin{aligned}\mathrm{Var(RC)} = h(1 + 2\rho^2 - \rho^2\beta_m^2)\sigma_1^2\sigma_2^2 - 4\frac{\rho^2\sigma_1^2\sigma_2^2}{(\lambda_1+\lambda_2)}\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right) \\ + 4\frac{\rho^2\sigma_1^2\sigma_2^2}{h(\lambda_1+\lambda_2)}\left(\frac{\xi_1\xi_2}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_2^2}\xi_2 + \frac{\lambda_2}{\lambda_1^2}\xi_1\right) \qquad (2.3.5)\end{aligned}$$

subject to a negligible term (refer to Griffin and Oomen (2006)). Not surprisingly, this term goes to zero while h goes to zero. (See Griffin and Oomen(2006)).

2.3.2 Var(RLL1) and Var(RLL2)

Using conditional expectations, we get the following expression. See Appendix B for the details.

$$
\begin{aligned}
&\mathrm{Var}(\mathrm{RLL}_1)\\
&= \sigma_1^2\sigma_2^2 h\frac{2e^{-\lambda_2 h}}{1-e^{-\lambda_2 h}} + \rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_2 h}\left(h - \frac{2\lambda_2}{\lambda_1(\lambda_1+\lambda_2)} + \frac{2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)\xi_1}{\lambda_1^2(\lambda_1+\lambda_2)^2 h}\right)\\
&+ 2\rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_2 h}\left(h^2 - \frac{\lambda_2 h\xi_1}{\lambda_1(\lambda_1+\lambda_2)}\right)\frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{\xi_2^2}\\
&+ 2\rho^2\sigma_1^2\sigma_2^2\xi_1^2 e^{-\lambda_2 h}e^{\lambda_1 h}\left(\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1} - \frac{1}{\lambda_1^2}\right)\\
&\cdot\frac{(m-1)e^{-(\lambda_1+\lambda_2)h} - me^{-2(\lambda_1+\lambda_2)h} + e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2}
\end{aligned}
\tag{2.3.6}
$$

Since Rll_2 and Rll_1 are symmetrical, we can get the following expression for $\mathrm{Var}(\mathrm{RLL}_2)$

$$
\begin{aligned}
&\mathrm{Var}(\mathrm{RLL}_2)\\
&= \sigma_1^2\sigma_2^2 h\frac{2e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}} + \rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_1 h}\left(h - \frac{2\lambda_1}{\lambda_2(\lambda_1+\lambda_2)} + \frac{2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)\xi_2}{\lambda_2^2(\lambda_1+\lambda_2)^2 h}\right)\\
&+ 2\rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_1 h}\left(h^2 - \frac{\lambda_1 h\xi_2}{\lambda_2(\lambda_1+\lambda_2)}\right)\frac{(m-1)e^{-\lambda_1 h} - me^{-2\lambda_1 h} + e^{-(m+1)\lambda_1 h}}{\xi_1^2}\\
&+ 2\rho^2\sigma_1^2\sigma_2^2\xi_2^2 e^{-\lambda_1 h}e^{\lambda_2 h}\left(\frac{\lambda_1 h}{\lambda_2(\lambda_1+\lambda_2)\xi_2} - \frac{1}{\lambda_2^2}\right)\\
&\cdot\frac{(m-1)e^{-(\lambda_1+\lambda_2)h} - me^{-2(\lambda_1+\lambda_2)h} + e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2}
\end{aligned}
\tag{2.3.7}
$$

Lemma 2.3.1. *When h is sufficiently small,*

$$
\begin{aligned}
&Var(RC) + Var(RLL_1) + Var(RLL_2)\\
&= 2\sigma_1^2\sigma_2^2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) + 2\frac{\rho^2\sigma_1^2\sigma_2^2}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}\right) + \text{Negligible term.}
\end{aligned}
\tag{2.3.8}
$$

Proof. We already prove that $\lim_{h\to 0} Var(RC) = 0$. Since $\lim_{h\to 0}(1-e^{-\lambda_j h})/h = \lambda_j,\ j = 1, 2,$

$$\lim_{h\to 0} \sigma_1^2\sigma_2^2 h \frac{2e^{-\lambda_2 h}}{1-e^{-\lambda_2 h}} = 2\frac{\sigma_1^2\sigma_2^2}{\lambda_2},$$

$$\lim_{h\to 0} \rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_2 h}\left(h - \frac{2\lambda_2}{\lambda_1(\lambda_1+\lambda_2)} + \frac{2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)\xi_1}{\lambda_1^2(\lambda_1+\lambda_2)^2 h}\right) = 2\rho^2\sigma_1^2\sigma_2^2\frac{\lambda_1}{(\lambda_1+\lambda_2)^2}$$

For any positive constant λ,

$$\begin{aligned}&\lim_{h\to 0} h^2\frac{(m-1)e^{-\lambda h} - me^{-2\lambda h} + e^{-(m+1)\lambda h}}{(1-e^{-\lambda h})^2}\\ &= \frac{(h-h^2)e^{-\lambda h} - he^{-2\lambda h} + h^2 e^{-\lambda-\lambda h}}{(1-e^{-\lambda h})^2} = \frac{\lambda + e^{-\lambda} - 1}{\lambda^2}\end{aligned}$$

Then,

$$\begin{aligned}&\lim_{h\to 0} 2\rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_2 h}\left(h^2 - \frac{\lambda_2 h\xi_1}{\lambda_1(\lambda_1+\lambda_2)}\right)\frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{\xi_2^2}\\ &= \lim_{h\to 0} 2\rho^2\sigma_1^2\sigma_2^2\left(1 - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\frac{\xi_1}{h}\right)\lim_{h\to 0} h^2\frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{\xi_2^2}\\ &= 2\rho^2\sigma_1^2\sigma_2^2\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\lambda_2 + e^{-\lambda_2} - 1}{\lambda_2^2}\\ &= 2\rho^2\sigma_1^2\sigma_2^2\frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)} + 2\rho^2\sigma_1^2\sigma_2^2\frac{\lambda_1(e^{-\lambda_2}-1)}{\lambda_2^2(\lambda_1+\lambda_2)},\end{aligned}$$

and,

$$\begin{aligned}
&\lim_{h\to 0} 2\rho^2\sigma_1^2\sigma_2^2\xi_1^2 e^{-\lambda_2 h}e^{\lambda_1 h}\left(\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}-\frac{1}{\lambda_1^2}\right)\\
&\cdot\frac{(m-1)e^{-(\lambda_1+\lambda_2)h}-me^{-2(\lambda_1+\lambda_2)h}+e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2}\\
&=\lim_{h\to 0} 2\rho^2\sigma_1^2\sigma_2^2\frac{\xi_1^2}{h^2}\left(\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}-\frac{1}{\lambda_1^2}\right)\lim_{h\to 0} h^2\\
&\cdot\frac{(m-1)e^{-(\lambda_1+\lambda_2)h}-me^{-2(\lambda_1+\lambda_2)h}+e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2}\\
&=-2\rho^2\sigma_1^2\sigma_2^2\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{(\lambda_1+\lambda_2)+e^{-(\lambda_1+\lambda_2)}-1}{(\lambda_1+\lambda_2)^2}\\
&=-2\rho^2\sigma_1^2\sigma_2^2\frac{\lambda_1}{(\lambda_1+\lambda_2)^2}+2\rho^2\sigma_1^2\sigma_2^2\frac{\lambda_1(1-e^{-(\lambda_1+\lambda_2)})}{(\lambda_1+\lambda_2)^3}
\end{aligned}$$

By symmetry, we can obtain the corresponding limit terms for $Var(RLL_2)$, and when we combine all these terms, we get:

$$\begin{aligned}
&\lim_{h\to 0}(Var(RLL_1)+Var(RLL_2))=2\sigma_1^2\sigma_2^2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)+2\frac{\rho^2\sigma_1^2\sigma_2^2}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}\right)\\
&+2\rho^2\sigma_1^2\sigma_2^2\left[\frac{(1-e^{-(\lambda_1+\lambda_2)})}{(\lambda_1+\lambda_2)^2}-\frac{\lambda_1(1-e^{-\lambda_2})}{\lambda_2^2(\lambda_1+\lambda_2)}-\frac{\lambda_2(1-e^{-\lambda_1})}{\lambda_1^2(\lambda_1+\lambda_2)}\right]
\end{aligned}$$

Hence,

$$\begin{aligned}
&Var(RC)+Var(RLL_1)+Var(RLL_2)\\
&=2T\sigma_1^2\sigma_2^2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)+2T\frac{\rho^2\sigma_1^2\sigma_2^2}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}\right)+\text{Negligible term},
\end{aligned}$$

Where the negligible term is:

$$2\rho^2\sigma_1^2\sigma_2^2\left[\frac{(1-e^{-(\lambda_1+\lambda_2)})}{(\lambda_1+\lambda_2)^2}-\frac{\lambda_1(1-e^{-\lambda_2})}{\lambda_2^2(\lambda_1+\lambda_2)}-\frac{\lambda_2(1-e^{-\lambda_1})}{\lambda_1^2(\lambda_1+\lambda_2)}\right]$$

.

□

2.3.3 Cov(RLL1, RC), and Cov(RLL2, RC)

Next, calculate $\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC})$, and $Cov(RLL_2, RC)$. Find the detail in Appendix B. The expressions are as follows:

$$\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC}) \\
&= \rho^2\sigma_1^2\sigma_2^2\xi_1^2\xi_2 \left(\frac{h}{\xi_1} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right)\left[\frac{h}{\xi_1\xi_2} - \frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1} + \frac{\lambda_2}{\lambda_1\xi_2}\right)\right] \\
&\cdot \frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{\xi_2^2} \\
&+ \rho^2\sigma_1^2\sigma_2^2\xi_1^2\xi_2 e^{(\lambda_1+\lambda_2)h}\left[\frac{h\lambda_2(1+e^{-\lambda_1 h})}{\lambda_1(\lambda_1+\lambda_2)\xi_1} - \frac{1}{\lambda_1^2}\left(1 + \frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2}\right)\right] \\
&\times \frac{(m-1)e^{-(\lambda_1+2\lambda_2)h} - me^{-2(\lambda_1+2\lambda_2)h} + e^{-(m+1)(\lambda_1+2\lambda_2)h}}{(1-e^{-(\lambda_1+2\lambda_2)h})^2}
\end{aligned} \tag{2.3.9}$$

Since Rll_2 and Rll_1 are symmetrical, we can get the following expression for $\mathrm{Cov}(\mathrm{RLL}_2, \mathrm{RC})$

$$\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_2, \mathrm{RC}) \\
&= \rho^2\sigma_1^2\sigma_2^2\xi_2^2\xi_1 \left(\frac{h}{\xi_2} - \frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right)\left[\frac{h}{\xi_1\xi_2} - \frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1\xi_2} + \frac{\lambda_1}{\lambda_2\xi_1}\right)\right] \\
&\cdot \frac{(m-1)e^{-\lambda_1 h} - me^{-2\lambda_1 h} + e^{-(m+1)\lambda_1 h}}{\xi_1^2} \\
&+ \rho^2\sigma_1^2\sigma_2^2\xi_2^2\xi_1 e^{(\lambda_1+\lambda_2)h}\left[\frac{h\lambda_1(1+e^{-\lambda_2 h})}{\lambda_2(\lambda_1+\lambda_2)\xi_2} - \frac{1}{\lambda_2^2}\left(1 + \frac{\lambda_1^2}{(\lambda_1+\lambda_2)^2}\right)\right] \\
&\times \frac{(m-1)e^{-(2\lambda_1+\lambda_2)h} - me^{-2(2\lambda_1+\lambda_2)h} + e^{-(m+1)(2\lambda_1+\lambda_2)h}}{(1-e^{-(2\lambda_1+\lambda_2)h})^2}
\end{aligned} \tag{2.3.10}$$

Lemma 2.3.2.

$$\lim_{h\to 0}\left[\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC}) + \mathrm{Cov}(\mathrm{RLL}_2, \mathrm{RC})\right] = 0. \tag{2.3.11}$$

Proof. In the expression of $\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC})$, the first term has limit:

$$\begin{aligned}
&\lim_{h\to 0} \rho^2\sigma_1^2\sigma_2^2\xi_1^2\xi_2 \left(h - \frac{\lambda_2\xi_1}{\lambda_1(\lambda_1+\lambda_2)}\right)\left[\frac{h}{\xi_1\xi_2} - \frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1} + \frac{\lambda_2}{\lambda_1\xi_2}\right)\right] \\
&\cdot \frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{\xi_2^2} \\
&= \rho^2\sigma_1^2\sigma_2^2 \lim_{h\to 0}\frac{\xi_1^2}{h^2} \cdot \lim_{h\to 0}\left(\frac{h}{\xi_1} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right) \cdot \lim_{h\to 0}\left[\frac{h}{\xi_1} - \frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1\xi_2}{\lambda_2\xi_1} + \frac{\lambda_2}{\lambda_1}\right)\right] \\
&\cdot \lim_{h\to 0} h^2 \frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{\xi_2^2}
\end{aligned}$$

Where, $\lim_{h\to 0}\left[\frac{h}{\xi_1} - \frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1\xi_2}{\lambda_2\xi_1} + \frac{\lambda_2}{\lambda_1}\right)\right] = 0$, and the other three limits are all constant, hence the limit of the first term is zero.

The limit for the second term in the expression of $\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC})$ is:

$$\begin{aligned}
&\lim_{h\to 0} \rho^2\sigma_1^2\sigma_2^2\xi_1^2\xi_2 e^{(\lambda_1+\lambda_2)h}\left[\frac{h\lambda_2(1+e^{-\lambda_1 h})}{\lambda_1(\lambda_1+\lambda_2)\xi_1} - \frac{1}{\lambda_1^2}\left(1 + \frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2}\right)\right] \\
&\cdot \frac{(m-1)e^{-(\lambda_1+2\lambda_2)h} - me^{-2(\lambda_1+2\lambda_2)h} + e^{-(m+1)(\lambda_1+2\lambda_2)h}}{(1-e^{-(\lambda_1+2\lambda_2)h})^2} \\
&= \rho^2\sigma_1^2\sigma_2^2 \lim_{h\to 0}\frac{\xi_1^2\xi_2}{h^2} \lim_{h\to 0}\left[\frac{h\lambda_2(1+e^{-\lambda_1 h})}{\lambda_1(\lambda_1+\lambda_2)\xi_1} - \frac{1}{\lambda_1^2}\left(1 + \frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2}\right)\right] \\
&\cdot \lim_{h\to 0} h^2 \frac{(m-1)e^{-(\lambda_1+2\lambda_2)h} - me^{-2(\lambda_1+2\lambda_2)h} + e^{-(m+1)(\lambda_1+2\lambda_2)h}}{(1-e^{-(\lambda_1+2\lambda_2)h})^2}
\end{aligned}$$

Where $\lim_{h\to 0}\frac{\xi_1^2\xi_2}{h^2} = 0$, and the other two limits are constant, so the limit of the second term is zero while h goes to zero. Hence $\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC})$ has limit zero. Likewise $\mathrm{Cov}(\mathrm{RLL}_2, \mathrm{RC})$, hence,

$$\lim_{h\to 0}\left[\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC}) + \mathrm{Cov}(\mathrm{RLL}_2, \mathrm{RC})\right] = 0.$$

□

2.3.4 Cov(RLL1, RLL2)

$$
\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RLL}_2) \\
&= \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 e^{-\lambda_1 h} C_1 \frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{\xi_2^2} \\
&- \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 C_2 \frac{(m-1)e^{-(\lambda_1+\lambda_2)h} - me^{-2(\lambda_1+\lambda_2)h} + e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2} \\
&+ \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 e^{-\lambda_2 h} D_1 \frac{(m-1)e^{-\lambda_1 h} - me^{-2\lambda_1 h} + e^{-(m+1)\lambda_1 h}}{\xi_1^2} \\
&- \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 D_2 \frac{(m-1)e^{-(\lambda_1+\lambda_2)h} - me^{-2(\lambda_1+\lambda_2)h} + e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2},
\end{aligned} \tag{2.3.12}
$$

where

$$
\begin{aligned}
C_1 &= \frac{h}{\xi_1}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{h^2}{\xi_1\xi_2} + \frac{\lambda_1 - \lambda_2}{\lambda_1^2(\lambda_1+\lambda_2)} - \frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_2} \\
C_2 &= \frac{1}{\lambda_1^2} + \frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)} \\
D_1 &= \frac{h}{\xi_2}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + \frac{h^2}{\xi_1\xi_2} + \frac{\lambda_2 - \lambda_1}{\lambda_2^2(\lambda_1+\lambda_2)} - \frac{\lambda_1 h}{\lambda_2(\lambda_1+\lambda_2)\xi_1} \\
D_2 &= \frac{1}{\lambda_2^2} + \frac{\lambda_1 h}{\lambda_2(\lambda_1+\lambda_2)}
\end{aligned}
$$

Lemma 2.3.3.

$$
\begin{aligned}
&\lim_{h\to 0} \mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RLL}_2) \\
&= \frac{\rho^2\sigma_1^2\sigma_2^2}{\lambda_1+\lambda_2}\left[\frac{e^{-\lambda_2}-1}{\lambda_2} + \frac{e^{-\lambda_1}-1}{\lambda_1} - (e^{-(\lambda_1+\lambda_2)}-1)\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2(\lambda_1+\lambda_2)}\right].
\end{aligned}
$$

Proof. The first term in the expression has limit:

$$
\begin{aligned}
&\lim_{h\to 0} \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 e^{-\lambda_1 h} C_1 \frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{\xi_2^2} \\
&= \rho^2\sigma_1^2\sigma_2^2 \lim_{h\to 0}\frac{\xi_1\xi_2}{h^2} \lim_{h\to 0} C_1 \lim_{h\to 0} h^2 \frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{\xi_2^2} \\
&= \rho^2\sigma_1^2\sigma_2^2 \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{\lambda_2 + e^{-\lambda_2} - 1}{\lambda_2^2}
\end{aligned}
$$

The second term has limit:

$$
\begin{aligned}
&- \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 C_2 \frac{(m-1)e^{-(\lambda_1+\lambda_2)h} - me^{-2(\lambda_1+\lambda_2)h} + e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2} \\
&= -\rho^2\sigma_1^2\sigma_2^2 \lim_{h\to 0}\frac{\xi_1\xi_2}{h^2}\lim_{h\to 0} C_2 \\
&\cdot \lim_{h\to 0} h^2 \frac{(m-1)e^{-(\lambda_1+\lambda_2)h} - me^{-2(\lambda_1+\lambda_2)h} + e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2} \\
&= -\rho^2\sigma_1^2\sigma_2^2 \frac{\lambda_2}{\lambda_1}\frac{(\lambda_1+\lambda_2)+e^{-(\lambda_1+\lambda_2)}-1}{(\lambda_1+\lambda_2)^2}
\end{aligned}
$$

Likewise, the limits for the third term and forth term are:

$$
\begin{aligned}
&\lim_{h\to 0} \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 e^{-\lambda_2 h} D_1 \frac{(m-1)e^{-\lambda_1 h} - me^{-2\lambda_1 h} + e^{-(m+1)\lambda_1 h}}{\xi_1^2} \\
&= \rho^2\sigma_1^2\sigma_2^2 \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\lambda_1+e^{-\lambda_1}-1}{\lambda_1^2}
\end{aligned}
$$

and,

$$
\begin{aligned}
&- \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 D_2 \frac{(m-1)e^{-(\lambda_1+\lambda_2)h} - me^{-2(\lambda_1+\lambda_2)h} + e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2} \\
&= -\rho^2\sigma_1^2\sigma_2^2 \frac{\lambda_1}{\lambda_2}\frac{(\lambda_1+\lambda_2)+e^{-(\lambda_1+\lambda_2)}-1}{(\lambda_1+\lambda_2)^2}
\end{aligned}
$$

Combine these terms, we have the limit for $\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RLL}_2)$:

$$
\begin{aligned}
&\lim_{h\to 0} \mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RLL}_2) \\
&= \frac{\rho^2\sigma_1^2\sigma_2^2}{\lambda_1+\lambda_2}\left[2 - \left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}\right) + \frac{e^{-\lambda_2}-1}{\lambda_2} + \frac{e^{-\lambda_1}-1}{\lambda_1} - \left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}\right)\frac{e^{-(\lambda_1+\lambda_2)}-1}{\lambda_1+\lambda_2}\right]
\end{aligned}
$$

□

2.3.5 Var(RLL)

Theorem 2.3.4. *After collecting all the variance and covariance terms, here is the expression of RLL's variance:*

$$
\begin{aligned}
&\mathrm{Var(RLL)}\\
&= h(1+2\rho^2-\rho^2\beta_m^2)\sigma_1^2\sigma_2^2 - 4\frac{\rho^2\sigma_1^2\sigma_2^2}{(\lambda_1+\lambda_2)}\left(\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}\right)\\
&+ 4\frac{\rho^2\sigma_1^2\sigma_2^2}{h(\lambda_1+\lambda_2)}\left(\frac{\xi_1\xi_2}{\lambda_1+\lambda_2}+\frac{\lambda_1}{\lambda_2^2}\xi_2+\frac{\lambda_2}{\lambda_1^2}\xi_1\right)+2\sigma_1^2\sigma_2^2 h\left[\frac{e^{-\lambda_2 h}}{1-e^{-\lambda_2 h}}+\frac{e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right]\\
&+ \rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_2 h}\left(h-\frac{2\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}+\frac{2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)\xi_1}{\lambda_1^2(\lambda_1+\lambda_2)^2 h}\right)\\
&+ \rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_1 h}\left(h-\frac{2\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}+\frac{2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)\xi_2}{\lambda_2^2(\lambda_1+\lambda_2)^2 h}\right)\\
&+ A_1\rho^2\sigma_1^2\sigma_2^2\frac{(m-1)e^{-\lambda_1 h}-me^{-2\lambda_1 h}+e^{-(m+1)\lambda_1 h}}{\xi_1^2}\\
&+ A_2\rho^2\sigma_1^2\sigma_2^2\frac{(m-1)e^{-\lambda_2 h}-me^{-2\lambda_2 h}+e^{-(m+1)\lambda_2 h}}{\xi_2^2}\\
&+ 2A_3\rho^2\sigma_1^2\sigma_2^2\frac{(m-1)e^{-(\lambda_1+\lambda_2)h}-me^{-2(\lambda_1+\lambda_2)h}+e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2}+o(h). \qquad (2.3.13)
\end{aligned}
$$

Subject to a negligible term in the $Var(RC)$ part, see Griffin and Oomen (2006). Where

$$
\begin{aligned}
A_1 &= 2e^{-\lambda_1 h}\left(h^2-\frac{\lambda_1 h\xi_2}{\lambda_2(\lambda_1+\lambda_2)}\right)\\
&\quad+\xi_2^2\xi_1\left(\frac{h}{\xi_2}-\frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right)\left(\frac{h}{\xi_1\xi_2}-\frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1\xi_2}+\frac{\lambda_1}{\lambda_2\xi_1}\right)\right)+\xi_1\xi_2 e^{-\lambda_2}C_2,
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= 2e^{-\lambda_2 h}\left(h^2-\frac{\lambda_2 h\xi_1}{\lambda_1(\lambda_1+\lambda_2)}\right)\\
&\quad+\xi_1^2\xi_2\left(\frac{h}{\xi_1}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right)\left(\frac{h}{\xi_1\xi_2}-\frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1}+\frac{\lambda_2}{\lambda_1\xi_2}\right)\right)+\xi_1\xi_2 e^{-\lambda_1}C_1,
\end{aligned}
$$

and,

$$A_3 = \xi_1^2 e^{(\lambda_1-\lambda_2)h}\left(\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1} - \frac{1}{\lambda_1^2}\right) + \xi_2^2 e^{(\lambda_2-\lambda_1)h}\left(\frac{\lambda_1 h}{\lambda_2(\lambda_1+\lambda_2)\xi_2} - \frac{1}{\lambda_2^2}\right) - \frac{\xi_1\xi_2}{2}(C_2+D_2),$$

The $o(h)$ term at the end is:

$$\rho^2\sigma_1^2\sigma_2^2\xi_1^2\xi_2 e^{(\lambda_1+\lambda_2)h}\left[\frac{h\lambda_2(1+e^{-\lambda_1 h})}{\xi_1\lambda_1(\lambda_1+\lambda_2)} - \frac{1}{\lambda_1^2}\left(1+\frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2}\right)\right] \cdot \frac{(m-1)e^{-(\lambda_1+2\lambda_2)h} - me^{-2(\lambda_1+2\lambda_2)h} + e^{-(m+1)(\lambda_1+2\lambda_2)h}}{(1-e^{-(\lambda_1+2\lambda_2)h})^2}$$
$$+ \rho^2\sigma_1^2\sigma_2^2\xi_2^2\xi_1 e^{(\lambda_1+\lambda_2)h}\left[\frac{h\lambda_1(1+e^{-\lambda_2 h})}{\xi_2\lambda_2(\lambda_1+\lambda_2)} - \frac{1}{\lambda_2^2}\left(1+\frac{\lambda_1^2}{(\lambda_1+\lambda_2)^2}\right)\right] \cdot \frac{(m-1)e^{-(2\lambda_1+\lambda_2)h} - me^{-2(2\lambda_1+\lambda_2)h} + e^{-(m+1)(2\lambda_1+\lambda_2)h}}{(1-e^{-(2\lambda_1+\lambda_2)h})^2}$$

This term is $o(h)$ due to Lemma 2.3.2.

By Lemma 2.3.1, and 2.3.2, and 2.3.12, we have the following theorem:

Theorem 2.3.5. *Without noise, when h is sufficiently small,*

$$\text{Var(RLL)} = 2\sigma_1^2\sigma_2^2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) + 2\frac{\rho^2\sigma_1^2\sigma_2^2}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}\right) + \text{Negligible term} \quad (2.3.14)$$

Refer to the subsections in this section for the details of the negligible terms. While h goes to zero, the main term is the same as the variance of Hayashi-Yoshida cumulative covariance estimator.

Based on the discussion of the mean and variance of RLL, we conclude that:

Theorem 2.3.6. *While there is no noise, when h is sufficiently small, the RMSE of RLL is:*

$$\text{RMSE(RLL)} = \sqrt{2\sigma_1^2\sigma_2^2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)+2\frac{\rho^2\sigma_1^2\sigma_2^2}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}\right)} + \text{Negligible term} \tag{2.3.15}$$

2.4 Variance of RLL with microstructure noise noise

To find the noise, we use the original notation of RLL. Recall that, with noise, RLL can be expressed as following(we use * to indicate noise terms corresponding to the no noise terms):

$$\begin{aligned}
\text{RLL}^* &= \sum_{i=1}^{m}\sum_{j=K_{i1}}^{K_{i2}}\left(P_i^{(1)*}-P_{i-1}^{(1)*}\right)\left(P_j^{(2)*}-P_{j-1}^{(2)*}\right) \\
&= \sum_{i=1}^{m}\sum_{j=K_{i1}}^{K_{i2}}\left[R_j+\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\right]\left[Z_i+\left(u_j^{(2)}-u_{j-1}^{(2)}\right)\right] \\
&= \sum_{i=1}^{m}\sum_{j=K_{i1}}^{K_{i2}} R_i^* Z_j^*
\end{aligned}$$

WhereK_{i1} and K_{i2} are random variables as before.

Then

$$\begin{aligned}
\text{RLL}^{*2} &= \left[\sum_{i=1}^{m}\sum_{j=K_{i1}}^{K_{i2}} R_i^* Z_j^*\right]^2 = \left[\sum_{i=1}^{m} R_i^* \sum_{l=K_{i1}}^{K_{i2}} Z_l^*\right]\left[\sum_{j=1}^{m} R_j^* \sum_{s=K_{j1}}^{K_{j2}} Z_s^*\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m}\left[R_i+\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\right]\left[\sum_{l=K_{i1}}^{K_{i2}} Z_l + \sum_{l=K_{i1}}^{K_{i2}}\left(u_l^{(2)}-u_{l-1}^{(2)}\right)\right] \\
&\cdot\left[R_j+\left(u_j^{(1)}-u_{j-1}^{(1)}\right)\right]\left[\sum_{s=K_{j1}}^{K_{j2}} Z_s + \sum_{s=K_{j1}}^{K_{j2}}\left(u_s^{(2)}-u_{s-1}^{(2)}\right)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m}\left[R_i+\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\right]\left[\bar{Z}_i+\left(u_{K_{i2}}^{(2)}-u_{K_{i1}-1}^{(2)}\right)\right] \\
&\cdot\left[R_j+\left(u_j^{(1)}-u_{j-1}^{(1)}\right)\right]\left[\bar{Z}_j+\left(u_{K_{j2}}^{(2)}-u_{K_{j1}-1}^{(2)}\right)\right]
\end{aligned}$$

Expand the above products and take expectation. Due to the independence between the R's and u's, Z's and $u^{(1)}$'s and $u^{(2)}$'s. Then we have,

$$\begin{aligned}
E\left[RLL^{*2}\right] &= E\sum_{i=1}^{m}\sum_{j=1}^{m} R_i\bar{Z}_i R_j\bar{Z}_j + E\sum_{i=1}^{m}\sum_{j=1}^{m} R_i\left(u^{(2)}_{K_{i2}} - u^{(2)}_{K_{i1}-1}\right) R_j\left(u^{(2)}_{K_{j2}} - u^{(2)}_{K_{j1}-1}\right) \\
&+ E\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u^{(1)}_i - u^{(1)}_{i-1}\right)\bar{Z}_i\left(u^{(1)}_j - u^{(1)}_{j-1}\right)\bar{Z}_j \\
&+ E\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u^{(1)}_i - u^{(1)}_{i-1}\right)\left(u^{(2)}_{K_{i2}} - u^{(2)}_{K_{i1}-1}\right)\left(u^{(1)}_j - u^{(1)}_{j-1}\right)\left(u^{(2)}_{K_{j2}} - u^{(2)}_{K_{j1}-1}\right) \\
&= E\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij} + E\sum_{i=1}^{m}\sum_{j=1}^{m} B_{ij} + E\sum_{i=1}^{m}\sum_{j=1}^{m} C_{ij} + E\sum_{i=1}^{m}\sum_{j=1}^{m} D_{ij}
\end{aligned}$$

Where

$$A_{ij} = R_i\bar{Z}_i R_j\bar{Z}_j,$$

$$B_{ij} = R_i\left(u^{(2)}_{K_{i2}} - u^{(2)}_{K_{i1}-1}\right) R_j\left(u^{(2)}_{K_{j2}} - u^{(2)}_{K_{j1}-1}\right),$$

$$C_{ij} = \left(u^{(1)}_i - u^{(1)}_{i-1}\right)\bar{Z}_i\left(u^{(1)}_j - u^{(1)}_{j-1}\right)\bar{Z}_j$$

and

$$D_{ij} = \left(u^{(1)}_i - u^{(1)}_{i-1}\right)\left(u^{(2)}_{K_{i2}} - u^{(2)}_{K_{i1}-1}\right)\left(u^{(1)}_j - u^{(1)}_{j-1}\right)\left(u^{(2)}_{K_{j2}} - u^{(2)}_{K_{j1}-1}\right).$$

Since we know that

$$E\left[RLL^*\right] = E\left[RLL\right],$$

i.e., the expectation of noise is zero. Then,

$$\begin{aligned}
\mathrm{Var}\left(\mathrm{RLL}^*\right) - \mathrm{Var}(\mathrm{RLL}) &= E\left[\mathrm{RLL}^{*2}\right] - \left(E\left[\mathrm{RLL}^*\right]\right)^2 - E\left[\mathrm{RLL}^2\right] + \left(E\left[\mathrm{RLL}\right]\right)^2 \\
&= E\left[\mathrm{RLL}^{*2}\right] - E\left[\mathrm{RLL}^2\right] \\
&= E\sum_{i=1}^{m}\sum_{j=1}^{m} B_{ij} + E\sum_{i=1}^{m}\sum_{j=1}^{m} C_{ij} + E\sum_{i=1}^{m}\sum_{j=1}^{m} D_{ij}
\end{aligned}$$

Next, calculate $E\sum_{i=1}^{m}\sum_{j=1}^{m}B_{ij}$. Since for $i \neq j$, R_i and R_j are independent, we have

$$
\begin{aligned}
& E\sum_{i=1}^{m}\sum_{j=1}^{m}B_{ij} \\
&= E\sum_{i=1}^{m}\sum_{j=1}^{m}R_iR_j\left(u_{K_{i2}}^{(2)}-u_{K_{i1}-1}^{(2)}\right)\left(u_{K_{j2}}^{(2)}-u_{K_{j1}-1}^{(2)}\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m}E\Big[E(R_iR_j \mid E_i^{(1)}\cap E_j^{(1)}) \\
&\cdot E\left(\left(u_{K_{i2}}^{(2)}-u_{K_{i1}-1}^{(2)}\right)\left(u_{K_{j2}}^{(2)}-u_{K_{j1}-1}^{(2)}\right) \mid K_{i1},K_{i2},K_{j1},K_{j2}\right)\Big] \\
&= \sum_{i=1}^{m}E\left[E(R_i^2 \mid E_i^{(1)})\right]E\left[E\left(\left(u_{K_{i2}}^{(2)}-u_{K_{i1}-1}^{(2)}\right)^2 \mid K_{i2},K_{i1}\right)\right] \\
&= \sum_{i=1}^{m}h\sigma_1^2\left[\sum_{l=i}^{\infty}\phi_2^2P(K_{i2}=l)+\sum_{s=-\infty}^{i}\phi_2^2P(K_{i1}=s)\right] \\
&= mh\sigma_1^2 2\phi_2^2 = 2\sigma_1^2\phi_2^2
\end{aligned}
$$

Since the underlying structure for the two process is symmetric, if we put RLL* as $\sum_{i=1}^{m}Z_i^*\bar{R}_i^*$ instead of $\sum_{i=1}^{m}R_i^*\bar{Z}_i^*$, then we can get the noise term with Z_i^2's and $(u^{(1)})^2$ similar to the B_{ij} terms, and that will give us the result : $2\sigma_2^2\phi_1^2$. Since $E\text{RLL}^* = E\sum_{i=1}^{m}Z_i^*\bar{R}_i^* = E\sum_{i=1}^{m}R_i^*\bar{Z}_i^*$, the expectation for the terms in both format with Z_i^2's and $(u^{(1)})^2$ must be the same, i.e.:

$$E\sum_{i=1}^{m}\sum_{j=1}^{m}C_{ij} = 2\sigma_2^2\phi_1^2.$$

At last, Calculate the noise term $E\sum_{i=1}^{m}\sum_{j=1}^{m}D_{ij}$. See Appendix for detail.

$$
\begin{aligned}
&\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m} D_{ij} \\
&= 4\frac{1}{h}\xi_1 e^{-\lambda_2 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_1^2 e^{-3\lambda_2 h}\phi_1^2\phi_2^2 \\
&+ 4\frac{1}{h}\xi_2 e^{-\lambda_1 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_2^2 e^{-3\lambda_1 h}\phi_1^2\phi_2^2 \\
&+ 4\frac{1}{h}\xi_1\xi_2\phi_1^2\phi_2^2 + 2(\frac{1}{h}-1)\xi_1^2\xi_2^2\phi_1^2\phi_2^2 \\
&- 4(\frac{1}{h}-1)\xi_1^2\xi_2 e^{-\lambda_2 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_1^2 e^{-2\lambda_2 h}\phi_1^2\phi_2^2 \\
&- 4(\frac{1}{h}-1)\xi_2^2\xi_1 e^{-\lambda_1 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_2^2 e^{-2\lambda_1 h}\phi_1^2\phi_2^2
\end{aligned}
$$

Collect all the parts, we get,

$$
\begin{aligned}
\mathrm{Var}\,(\mathrm{RLL}^*) = \mathrm{Var}(\mathrm{RLL}) &+ 2\sigma_1^2\phi_2^2 + 2\sigma_2^2\phi_1^2 \\
&+ 4\frac{1}{h}\xi_1 e^{-\lambda_2 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_1^2 e^{-3\lambda_2 h}\phi_1^2\phi_2^2 \\
&+ 4\frac{1}{h}\xi_2 e^{-\lambda_1 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_2^2 e^{-3\lambda_1 h}\phi_1^2\phi_2^2 \\
&+ 4\frac{1}{h}\xi_1\xi_2\phi_1^2\phi_2^2 + 2(\frac{1}{h}-1)\xi_1^2\xi_2^2\phi_1^2\phi_2^2 \\
&- 4(\frac{1}{h}-1)\xi_1^2\xi_2 e^{-\lambda_2 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_1^2 e^{-2\lambda_2 h}\phi_1^2\phi_2^2 \\
&- 4(\frac{1}{h}-1)\xi_2^2\xi_1 e^{-\lambda_1 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_2^2 e^{-2\lambda_1 h}\phi_1^2\phi_2^2 \qquad (2.4.1)
\end{aligned}
$$

Theorem 2.4.1. *With i.i.d noise, while h goes to zero,*

$$
\begin{aligned}
\lim_{h\to 0}\mathrm{Var}(\mathrm{RLL}^*) = 2\sigma_1^2\sigma_2^2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) &+ 2\frac{\rho^2\sigma_1^2\sigma_2^2}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}\right) \\
&+ 2(\sigma_1^2\phi_2^2+\sigma_2^2\phi_1^2) + 4(\lambda_1+\lambda_2)\phi_1^2\phi_2^2 \qquad (2.4.2)
\end{aligned}
$$

Proof. While h goes to zero,

$$
\lim_{h\to 0}\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m} D_{ij} = 4(\lambda_1+\lambda_2)\phi_1^2\phi_2^2.
$$

Combine this with the proof of **??**. □

Since the expectation of RLL are the same under the situation with and without microstructure noise, and the estimator is asymptotically unbiased, it follows that while h is sufficiently small, the RMSE of RLL* is

Theorem 2.4.2. *With i.i.d noise, while h is sufficiently small,*

$$\begin{aligned}&\mathrm{RMSE}(\mathrm{RLL}^*)\\ &= \sqrt{2\sigma_1^2\sigma_2^2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)+\frac{2\rho^2\sigma_1^2\sigma_2^2}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}\right)+2(\sigma_1^2\phi_2^2+\sigma_2^2\phi_1^2)+4(\lambda_1+\lambda_2)\phi_1^2\phi_2^2}\\ &+ \text{Negligible term}\end{aligned}$$

Chapter 3

Chapter 3 – Simulation Study

3.1 Basic Setup

First, we carry out the simulation with no microstructure noise. We shall compare the random lead-lag(RLL) estimator with the previous tick realized covariance(RC) estimator. In the simulation, we set the volatilities of the two processes as $\sigma_1 = 1$ and $\sigma_2 = 2$. Arrivals for process 1 are Poisson Process, with 1 arrival every minute on average. Arrivals for process 2 are Poisson Process with 1 arrival every 5 minutes on average, which means the arrival intensities are $\lambda_1 = 60$ and $\lambda_2 = 300$ in seconds. $T = 23,400$ seconds in a 6.5 hour (NYSE) trading day. The simulation sample size is 10000, which can be considered as the number of trading days. Estimates are calculated with $h \in \{1800, 600, 300, 60, 30, 10, 1\}$ seconds–that is, the sampling frequencies considered are 30 min, 10 min , 5min, 1 min, 30 sec, 10sec, and 1 sec. The correlation coefficient of the two processes runs from 0.1 to 0.9 with step equals to 0.1. $(\rho = 0.1 : 0.1 : 0.9)$.

3.2 Bias, Variance and RMSE when microstructure noise is absent

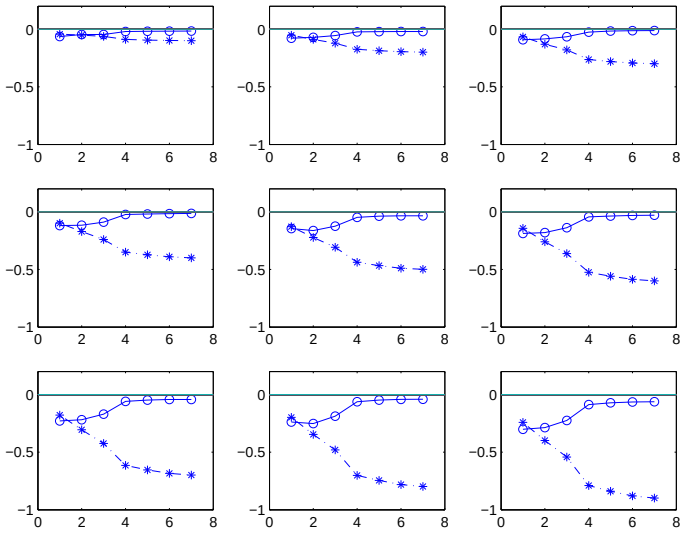


Figure 3.1: Bias: circle line is RLL, star line is RC

Figure 3.1, figure 3.2 and figure 3.3 plot bias, standard deviation and RMSE of the RLL(Red), RC(blue) estimators for various values of frequency and ρ. The 9 panels in each figure represent the 9 values of ρ. The horizontal axis in each panel represents the index of the sampling frequency h in the set $h \in \{1800, 600, 300, 60, 30, 10, 1\}$.

We found the following properties for the biases of RLL and RC:

- For all ρ's, if frequency is higher than 30 minutes, the bias of RLL estimator is lower than that of RC.

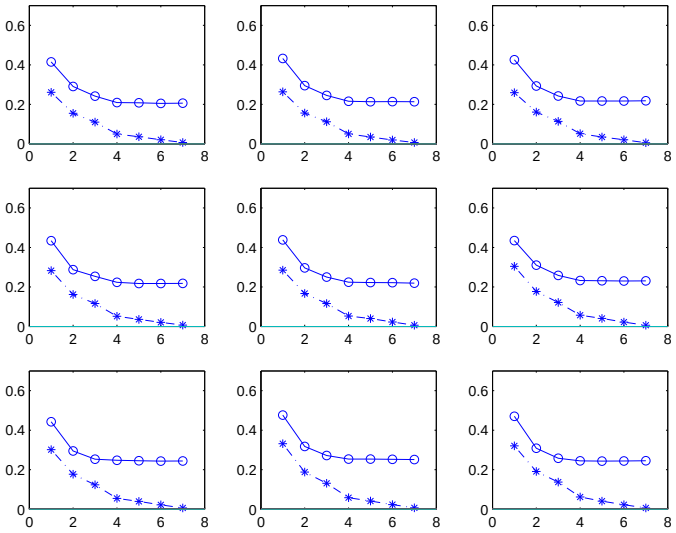


Figure 3.2: Standard Deviation: circle line is RLL, star line is RC

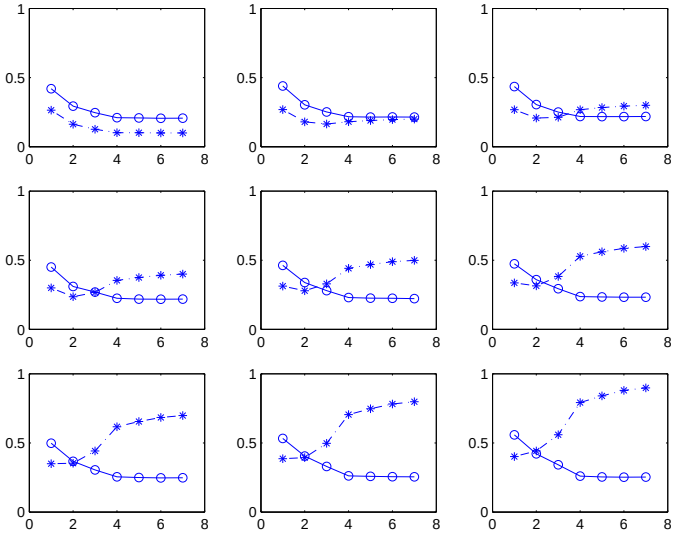


Figure 3.3: Root mean squared error(RMSE): circle line is RLL, star line is RC

- The RLL estimator is asymptotically unbiased as frequency increases goes to 0 while RC goes to zero as frequency gets high. Hence, it is highly biased.
- For all values of ρ, the minimum (taken over frequency) bias of RLL estimator is smaller than the minimum bias of RC.
- The difference among different values of bias for the RLL is very small after frequency hits 1 minute.

For the variance,

- The variance of RC is smaller than variance of RLL estimator for the same frequency.
- If we compare the variance of RLL estimator at frequency 1 min (because that is the optimal choice clearly) with variance of RC for frequency 30 min (because that has comparable bias to the optimal RLL estimate and also minimum bias among the RC estimates), then we find that the variance of the RLL estimate (1 min) is smaller than that of RC (30 min) for all ρ's.

For RMSE,

- For small ρ (≤ 0.3), the minimum rmse of RC is smaller than minimum rmse of RLL estimator. This is because RC is biased towards 0 and as frequency increases (h goes to 0), the variance gets very small. So when the parameter is itself close to zero, this is advantageous.

- For higher ρ, rmse of RC is minimized at freq 30 min, and that of RLL estimator is minimized as freq decreases, but as per the comment above it is enough to consider frequency 1 min. Rmse of RLL estimator at frequency 1 min is far smaller than rmse of RC at 30 min for all values of ρ greater than 0.3.

From these figures, we notice that the bias and variance of the RLL estimator both decrease with increasing frequency. Since the RLL estimator with the highest frequency coincide with Hayashi-Yoshida's CC estimator, the CC estimator is the best among all RLL estimators. However, the higher the frequency, the higher the computational time. After frequency 1 minute, the change in both bias and variance of the RLL estimator is negligible. So we can choose 1 minute to be the optimal frequency.

3.3 Bias, Variance and RMSE when microstructure noise is present

We carry out simulation under i.i.d. Gaussian microstructure noise with standard deviation $\xi_1 = \xi_2 = 120$. All the other conditions are the same as those in the case where no noise is present. Figure 3.4, figure 3.5 and figure 3.6 plot bias, standard deviation and RMSE of the RLL(circle), RC(star) estimators for various values of frequency and ρ while noise is present.

The performance of bias while microstructure noise is present is pretty similar to

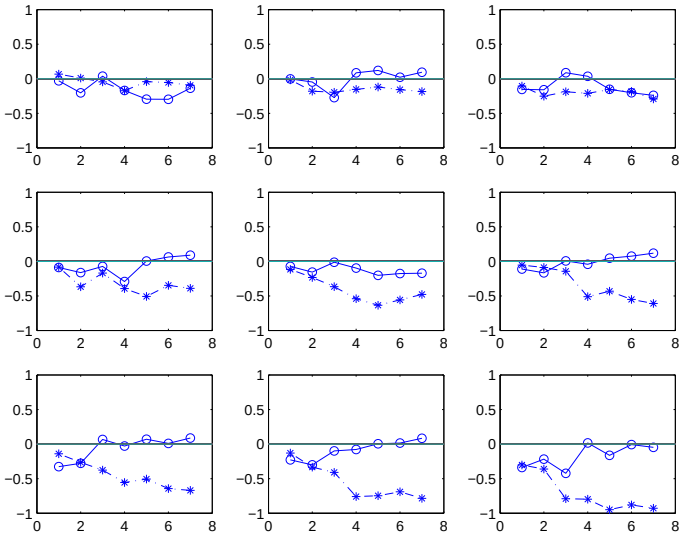


Figure 3.4: Bias: circle line is RLL, star line is RC

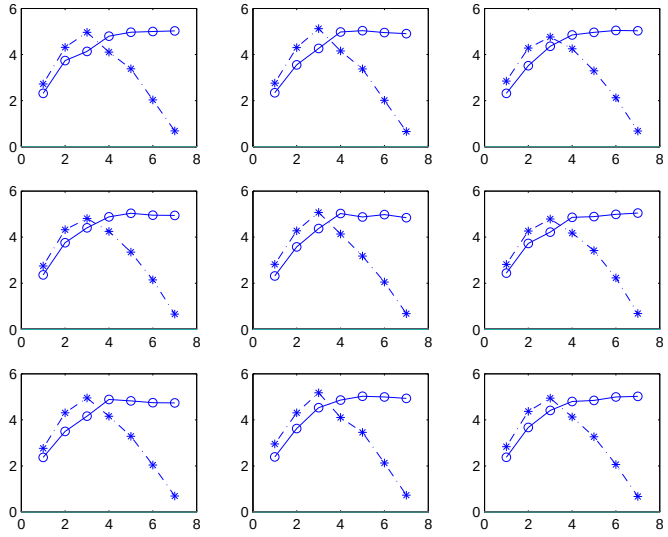


Figure 3.5: Standard Deviation: circle line is RLL, star line is RC

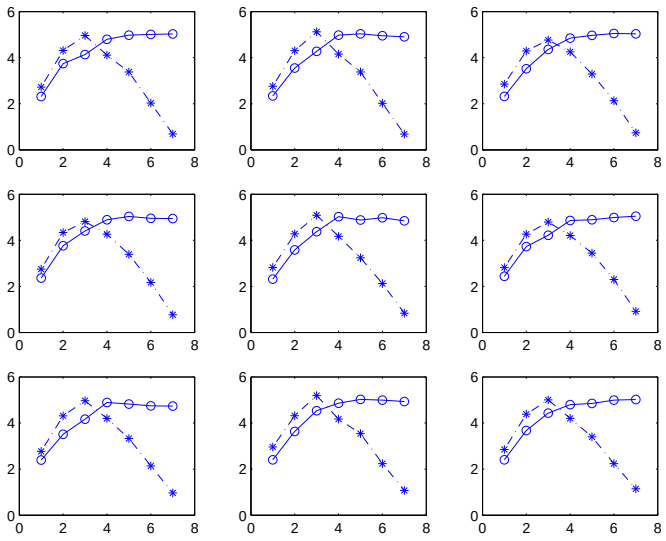


Figure 3.6: Root mean squared error(RMSE): circle line is RLL, star line is RC

those without noise. In addition to the observations made about bias in the no-noise setting, we observe that roughly, the bias of RLL decreases with increasing frequency while the bias of Previous Tick RC increases.

The performance of standard deviations of the estimators are very different from those under the situation of no noise present.

- The standard deviation of RLL increases with greater frequency, and it almost stays constant after the frequency hits 1 minute.
- The standard deviation of RC has a bell shape, for it increases with greater frequency first, and it hits the maximum value at around 5 minutes, then starts to decrease. It reaches the overall smallest value while the frequency is the

highest(1 second).

- While the frequency is lower than 5 minutes, the standard deviation of RLL is lower than that of RC; when the frequency is higher than 5 minutes, it is the other way around.

The performance of root mean square error (RMSE) is very similar to that of the standard deviation while noise is present. This is because the magnitude of bias is relatively much smaller than the magnitude of the standard deviation. Since $RMSE = \sqrt{\text{bias}^2 + \text{SD}^2}$, the value of RMSE should be very close to Standard deviation when bias is really small.

According to the discussion above, we will still choose 1 minute as our optimal frequency while microstructure noise is present. This gives us a good bias-variance trade-off.

Chapter 4

Chapter 4 – Real Data Example

4.1 Research Purpose and Market Introduction

We apply our RLL estimator and other estimators on DELL and APPLE transactions. The data are from Wharton Research Data Services (WRDS). The purpose is to estimate the correlation between the two stocks. Both companies trade on the National Association of Securities Dealers' Automated Quotation System (Nasdaq). Although we might hear about the Dow Jones Industrials every day, many of us have an imperfect knowledge of how the markets work. I would like to introduce a little bit about the capital markets, since that is where our data come from, and that is where lots of our research takes place. The concepts were introduced by Reilly and Brown (2005)

For different security markets, the major difference is between the primary markets and the secondary markets. The primary markets are where new issues of stocks

(equity) and bonds (fixed income) are sold by different entities, such as corporate stock issues of initial public offerings (IPOs). The secondary markets are where the outstanding issues are traded, and those issues were initially sold in the primary markets. The secondary markets provide liquidity and price discovery and also lots of other valuable information for investors and issuers. Like the example we take here, the data were originally from the secondary markets, and it studies the relationship between different stocks, which provides investors with important information for their financial applications, such as portfolio management.

We are not going to skip the fixed income markets and talk a little bit more about the equtiy markets. The secondary equity markets can have different trading systems. The major trading systems are the pure auction market and dealer market.The pure auction market is centralized, and it is also referred to as a price-driven or order-driven market, since for a given stock, a broker (without the ownership of the stock) works at a central location, to whom the interested sellers submit their ask price and buyers submit their bid price. The traded shares of stock are bought from the lowest ask price and sold at the highest bid price. On the other hand, in a dealer market, there are (ideally) lots of dealers who buy and sell the stock for their own accounts, and these activities provide market liquidity.

Also the equity markets can differ from their operations of exchange beyond trading systems. There are call markets and continuous markets. In a call market, all

the bid-and-ask prices are collected at discreet times, and for a given stock, a single price is arrived at to try to derive closest supply and demand quantities. In a continuous market, buyers and sellers trade at any time during the market open time.

Before we talk more about Nasdaq, where our data originally came from, we introduce the classification of the secondary equity market in the U.S.. According to Larry Harris (2003), the U.S. secondary equity markets can be classified as the following: primary listing markets, such as New York Stock Exchange (NYSE), Nasdaq National Market System (NMS), and etc.; regional markets, such as Boston Stock Exchange, Pacific Exchange and etc.; third-market dealers/brokers, such as Knight Trading Group, Nasdaq InterMarket, and etc; Alternative Trading Systems (ATSs) and Electronic Communications Networks (ENCs), such as BUIT, REDIBook, and etc.; and Electronic Crossing Systems (ECSs), such as POSIT, and etc..

In the following, we introduce two major U.S. Exchanges: NYSE Euronext and Nasdaq. According to the home page of NYSE Euronext, the exchange was first organized in 1972, when 24 brokers and merchants signed the Buttonwood Agreement. In 1817, the New York Stock and Exchange Board was established. The exchange was first incorporated as a not-for-profit organization on February 18th, 1971, and became a for-profit organization – the NYSE Group–in 2006. In 2007, the exchange was combined with Euronext N.V and became today's NYSE Euronext.

The National Association of Securities Dealers Automated Quotation System, better known as Nasdaq, is the biggest equity exchange in the world in terms of company listing, with almost 3,200 companies. It also has the largest trading volume among all the exchanges in the world. In 1971, the National Association of Securities Dealers (NASD) founded Nasdaq as the world's first electronic stock exchange. It is considered as an over-the-counter (OTC) market until the late 1980s. It grows to more of a stock market over the years. The highlighting companies are usually technology companies, like the ones we study: Apple inc. and Dell inc. See Wikipedia, the free encyclopedia.

4.2 Data Description

The data were queried from the Institute for the Study of Security Markets (ISSM) – a database contains tick-by-tick data covering the NYSE and AMEX between 1983 and 1992, and NASDAQ between 1987 and 1992. See Appendix D for data processing and a chart of trading days and part of the original Data. Since there are multiple prices at the same time for some time points for both trades, the first step of data cleaning is choosing the first observation of these multiple trades.

The next step is to check if there is any price jump between consecutive days, i.e., if the opening price is significantly different than the close price of the previous day. For 252 days, we use the first price of every day from day 2 to day 252 and the last price of every day from day 1 to day 251, then we use the first price and subtract it from the last price of the previous day to get the difference in price. We then plot

these differences.

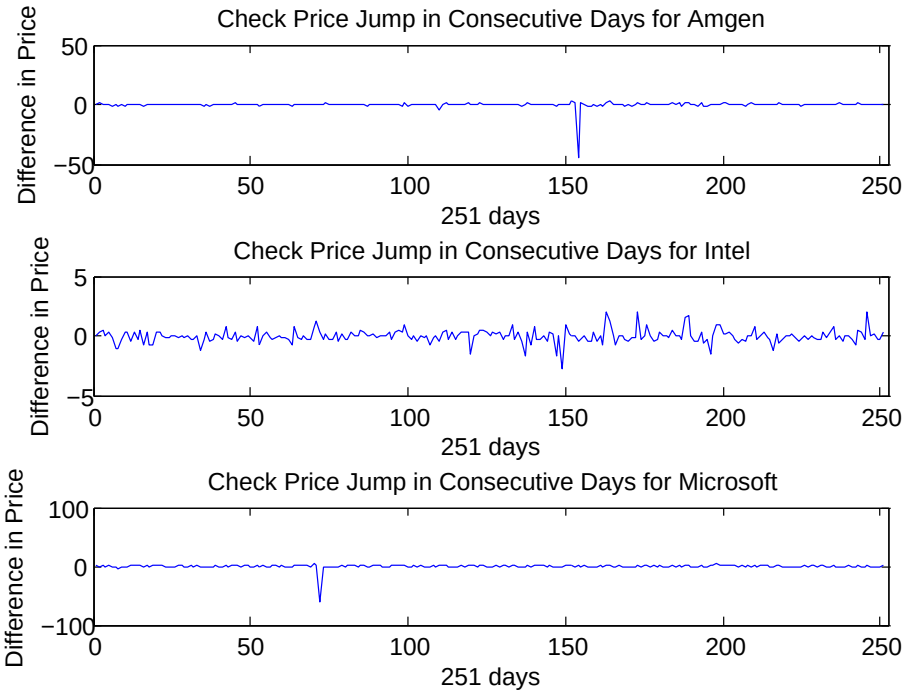


Figure 4.1: Detect Stock Split

From the plots, we can see that, for Intel, all the differences were essentially small. But for Microsoft (MSFT), there was one big price jump, which was caused by the difference between the $61 opening price on day 73(April 16th, 1990 in the original data) and the $120.75 end price on day 72 (April 12th, 1990 in the original data), which was almost twice as the opening price of the next trading day. According to Microsoft's web site, there was a 2-for-1 stock split announced on April 12, 1990, and the following is a Chart from the web site which "summarizes Mi-

crosoft's nine common stock splits since the initial public offering on March 13, 1986": (http://www.microsoft.com/msft/faq/stocksplit.mspx):

Table 4.1: Microsoft's nine common stock splits since the initial public offering on March 13, 1986.

Payable Date	Type of Split	Closing Price Before	Closing Price After
September 18, 1987	2 for 1	$114.50 (Sep 18)	$53.50 (Sept 21)
April 12, 1990	2 for 1	$120.75 (Apr 12)	$60.75 (Apr 16)
June 26, 1991	3 for 2	$100.75 (Jun 26)	$68.00 (Jun 27)
June 12, 1992	3 for 2	$112.50 (Jun 12)	$75.75 (Jun 15)
May 20, 1994	2 for 1	$97.75 (May 20)	$50.63 (May 23)
December 6, 1996	2 for 1	$152.875 (Dec 6)	$81.75 (Dec 9)
February 20, 1998	2 for 1	$155.13 (Feb 20)	$81.63 (Feb 23)
March 26, 1999	2 for 1	$178.13 (Mar 26)	$92.38 (Mar 29)
February 14, 2003	2 for 1	$48.30 (Feb 14)	$24.96 (Feb 18)

In a 2-for-1 split, one original share becomes 2 new shares; in a 3-for-2 split, one original share becomes 1.5 new shares. Since on April 12th, 1990, MSFT had a 2-for-1 share, the price almost went down to half, and so the total value of the outstanding common stock would remain the same. Usually companies split their stocks to attract more (especially relatively smaller) investors. This scenario commonly arises when the company does well, and the share prices increases, but till a point when the number of outstanding shares increase but price per share decreases. This seems true for the Microsoft data in our case (number of outstanding shares not available). Refer to the later graphs to see more details of the price changes.

We can see that for Amgen, there is a stock split too. From their web site (http://www.amgen.com/about/facts.html), we can see that In a 3-for-1 split, one

Table 4.2: Amgen's five common stock splits since the 1990.

Payable Date	Type of Split	Change
August 10, 1990	2 for 1	$88 to $44
September 10, 1991	3 for 1	$160 3/8 to $53 11/24
August 15, 1995	2 for 1	$94 1/8 to $47 1/16
February 26, 1999	2 for 1	$124 7/8 to $62 1/2
November 19, 1999	2 for 1	$94 11/16 to $47 11/32

original share becomes 3 new shares.

The following figure has three stocks' daily average price on the same plot.

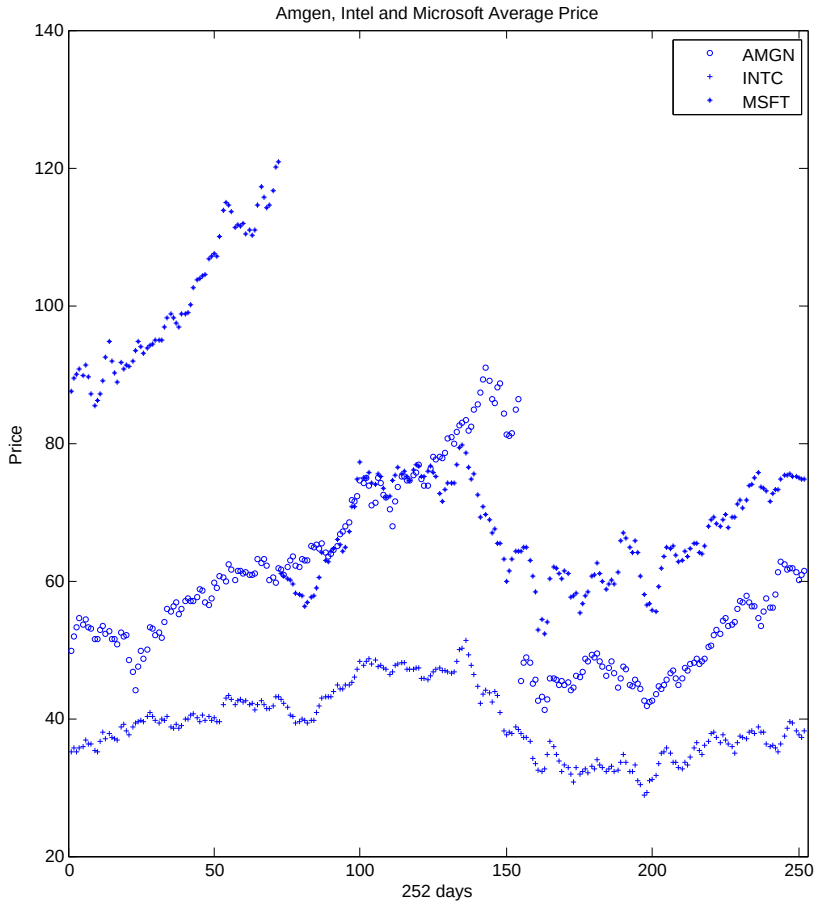


Figure 4.2: Average Daily Price for Three Stocks

We can see the stock split effect in the plot, and also we can see that if this effect is not considered, the three stocks prices seem to have changed in a more or less similar patten. This might be a general market trend during 1990. One can track the index price for more clues, and we will leave more market analysis for future work.

4.3 Some Results

After applying the covariance estimators, we come up with the following results in two different aspects. First, as frequency goes higher,our RLL estimator generates bigger estimates than the naive Realized Covariance estimator. This is due to the Epp's effect. The effect can be shown in the following table:

Table 4.3: Yearly Average Correlation Estimates

stocks	MSFT-INTC		MSFT-AMGN		INTC-AMGN	
method	RLL	RV	RLL	RV	RLL	RV
freq(sec)	yearly average estimates(1990)					
600	0.0576	0.1230	0.1034	0.11520	.0703	0.0972
300	0.0672	0.0618	0.1573	0.0414	0.0483	0.0423
60	0.0253	0.0090	0.0191	0.0005	0.0177	0.0044
30	0.0153	0.0054	0.0147	0.0020	0.0115	0.0010
10	0.0075	0.0013	0.0122	0.0025	0.0073	0.0027
1	0.0058	-0.0006	0.0085	-0.0002	0.0068	0.0008

Especially, while $h = 1$ second, the Realized Covariance estimates are much smaller than the RLL estimates. Actually, the Realized Covariance estimates for many days during the year are zero for this frequency. Figure 4.3 plots the yearly average correlations for the three stocks.

Second, we study a little bit more about the relationships among the stocks.

The upper panel of figure 4.4 shows that generally, the 1 minute RLL correlation estimates for Microsoft and Intel are bigger than the RV correlation estimates. The

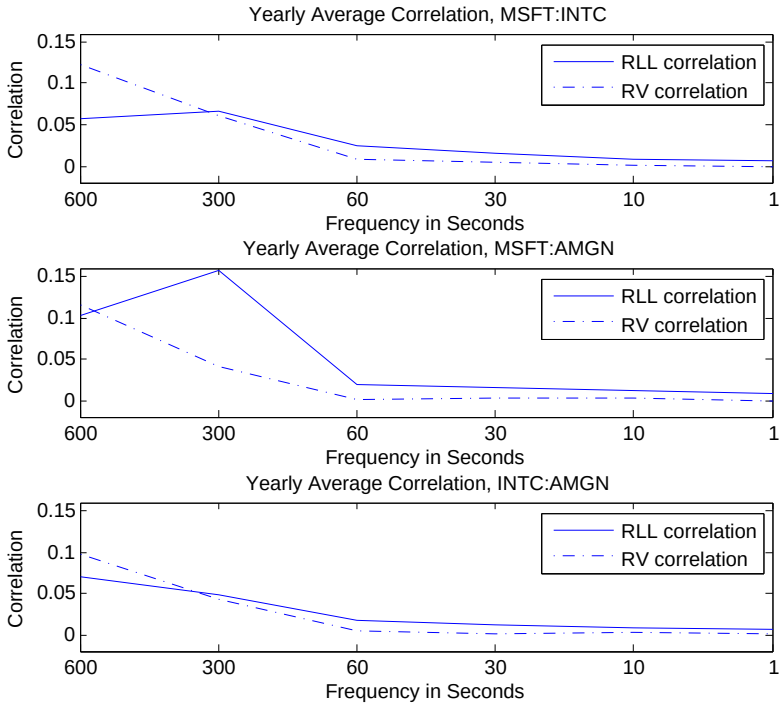


Figure 4.3: Yearly Average Correlations

lower panel shows that for frequency set as 1 minute, using RLL estimates, the correlation for Microsoft and Intel has a similar patten as the one for Amgen and Intel. This could probably be explained by the general market trend. Nevertheless, there are still more days with bigger correlations for Microsoft and Intel than the ones for Amgen and Intel. This is not surprising since Microsoft and Intel are both in the computer technology section, while Amgen is in the Pharmaceutical section. The

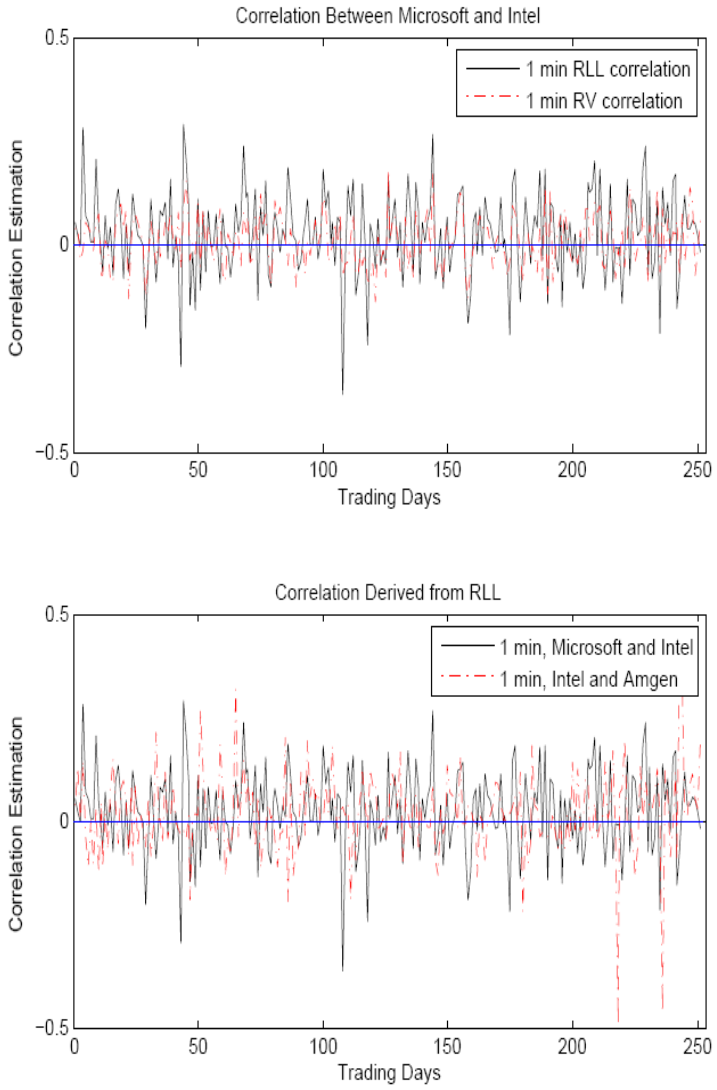


Figure 4.4: Correlation Plot

stocks in the same industry tend to have bigger correlations.

According to our real data application, we believe that our RLL estimator is a good tool for financial practice.

4.4 Conclusions and Future Work

Our simulation study support the theoretical proof. The real data example show that the RLL has nice properties over RC. The advantages of the RLL estimator over tick-based estimators like CC is that the former does not require exact timing of events and is computationally much faster. The variance can be reduced a lot by choosing a proper lower frequency at the cost of increasing the bias. However, unlike the other realized variance-type interval based estimators, the RLL estimator is asymptotically unbiased. So we can suitably choose a frequency that optimizes our bias-variance preferences. This is a unique feature of our estimator which is absent in any other estimator in the literature.

Here are some directions for ongoing and future research:

- We plan to apply variance reduction techniques like sub-sampling. Then we can compare the subsampled version of RLL with the subsampled version of Previous Tick RC and obtain the optimal frequency to gain good bias-variance trade-off.
- We can derive bootstrap estimates for confidence intervals for real data applications.
- We can also use alternative estimators based on bi-power variation and study the effect of noise and asynchronicity.

Chapter 5

Appendix

5.1 Proofs

'Previous tick' method is used to calculate the bias and variance of RLL, i.e. $P_t^{(j)} = p_{N_j(t)}^{(j)}$, where $N_j(t) = sup_n\{n|t_n^{(j)} \leq t\}$ as before. We adopt the methods used in the Griffin and Oomen (2006) paper and use the equivalent notation for proof of the theorems about RLL. Let $t_i^{(j)}$ denote the timing of the most recent observation (i.e. transaction / quote-update) of asset j prior to t = ih. Define $\tau_i^{(j)} = ih - t_i^{(j)}$ and $a_i = \max\{\tau_i^{(1)}, \tau_i^{(2)}\}$ and $bi = \min\{\tau_i^{(1)}, \tau_i^{(2)}\}$. Let $E_i^{(j)}$ denote the event that asset j trades at least once over the time interval $((i-1)h, ih]$. Let $\nu(I_1 \cap I_2)$ denote the length of the intersection between interval I_1 and I_2. We also use the short-hands $R_i = r_i^{(1)}$, $Z_i = r_i^{(2)}$, $u_i = u_{N_1(ih)}^{(1)}$, $v_i = u_{N_2(ih)}^{(2)}$ and $\xi_j = 1 - e^{-\lambda_j h}$ for simplicity of notation. In all the related proof of expectation, bias and variance, T is scaled to 1.

5.1.1 Proof of Lemma 2.2.1

Proof. I will only a proof for the following case, for all the other cases are similar.Suppose $T_1^{(1)} < T_1^{(2)} < T_2^{(1)} < T_2^{(2)}$, see Figure 5.1.

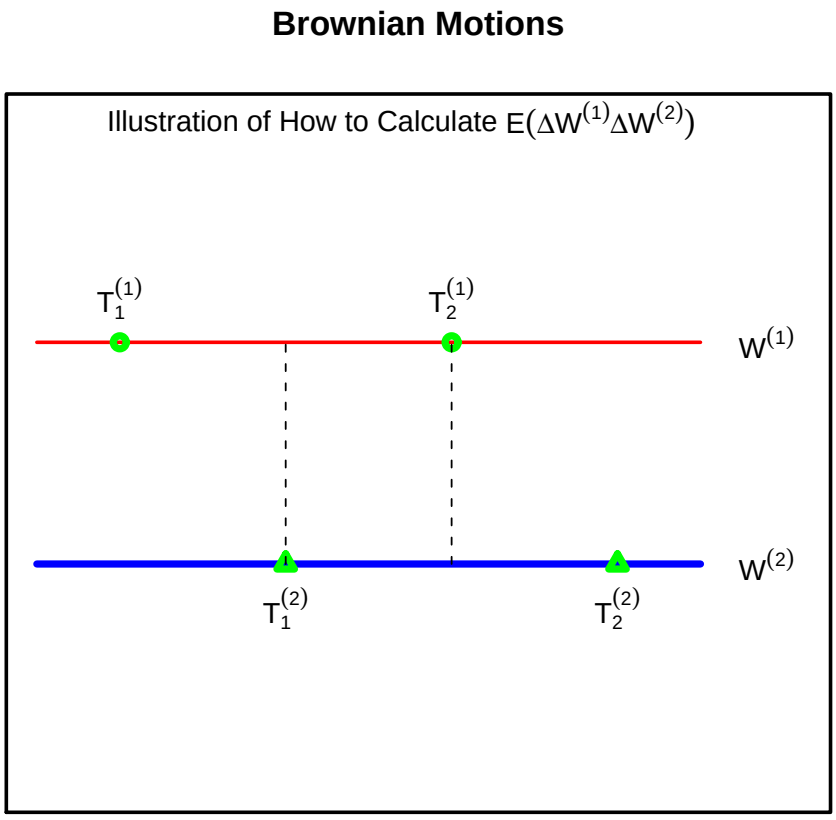


Figure 5.1: Calculate $E(\Delta W^{(1)}\Delta W^{(2)})$ while $T_1^{(1)} < T_1^{(2)} < T_2^{(1)} < T_2^{(2)}$

$$\Delta W^{(1)} = [W^{(1)}(T_2^{(1)}) - W^{(1)}(T_1^{(2)})] + [W^{(1)}(T_1^{(2)}) - W^{(1)}(T_1^{(1)})]$$

$$\Delta W^{(2)} = [W^{(2)}(T_2^{(2)}) - W^{(2)}(T_2^{(1)})] + [W^{(2)}(T_2^{(1)}) - W^{(2)}(T_1^{(2)})]$$

$$
\begin{aligned}
\mathrm{E}(\Delta W^{(1)}\Delta W^{(2)}) &= \mathrm{E}\left[\left(W^{(1)}(T_2^{(1)}) - W^{(1)}(T_1^{(2)})\right)\left(W^{(2)}(T_2^{(2)}) - W^{(2)}(T_2^{(1)})\right)\right] \\
&+ \mathrm{E}\left[\left(W^{(1)}(T_2^{(1)}) - W^{(1)}(T_1^{(2)})\right)\left(W^{(2)}(T_2^{(1)}) - W^{(2)}(T_1^{(2)})\right)\right] \\
&+ \mathrm{E}\left[\left(W^{(1)}(T_1^{(2)}) - W^{(1)}(T_1^{(1)})\right)\left(W^{(2)}(T_2^{(2)}) - W^{(2)}(T_2^{(1)})\right)\right] \\
&+ \mathrm{E}\left[\left(W^{(1)}(T_1^{(2)}) - W^{(1)}(T_1^{(1)})\right)\left(W^{(2)}(T_2^{(1)}) - W^{(2)}(T_1^{(2)})\right)\right] \\
&= 0 + \rho\,\mathrm{E}\left[length\ of\ (T_1^{(2)}, T_2^{(1)})\right] + 0 + 0 \\
&= \rho\,\mathrm{E}\left[\nu\left((T_1^{(1)}, T_1^{(1)}) \cap (T_1^{(2)}, T_2^{(2)})\right)\right]
\end{aligned}
$$

The zeros are due to the independence of Brownian motion increments. □

5.1.2 Proof of Lemma 2.2.2

Proof. See Figure 5.2. For a fixed closed trading day, for two stocks with observations known, first find the intervals containing at least one event for stock 1. Select the right end points of these intervals, and record them in an increasing manner, and list them as: $t_1, ..., t_T$, then

$$\sum_{i=1}^{m}\sum_{j=K_{i1}}^{i-1} R_i^{(1)}R_j^{(2)} = \sum_{l=1}^{T}\sum_{j=K_{t_l1}}^{t_l-1} R_{t_l}^{(1)}R_j^{(2)}$$

For a fixed l, at time $t_l h$, look backwards, find the most recent event that happened for stock 1 prior time $(t_l-1)h$, and we know that this event happens in $((t_{l-1}-1)h, t_{l-1}h]$. Hence $K_{t_l1} = t_{l-1}+1$, then,

$$\sum_{j=K_{t_l1}}^{t_l-1} R_{t_l}^{(1)}R_j^{(2)} = R_{t_l}^{(1)}R_{t_{l-1}+1}^{(2)} + R_{t_l}^{(1)}R_{t_{l-1}+2}^{(2)} + \cdots + R_{t_l}^{(1)}R_{t_l-1}^{(2)}$$

Now at time point $t_{l-1}h$, look forward, find the next event that happened for stock 1, and that event should be in $((t_l-1)h, t_l h]$, hence $K_{t_{l-1}+1,2} = t_l$, and this gives the

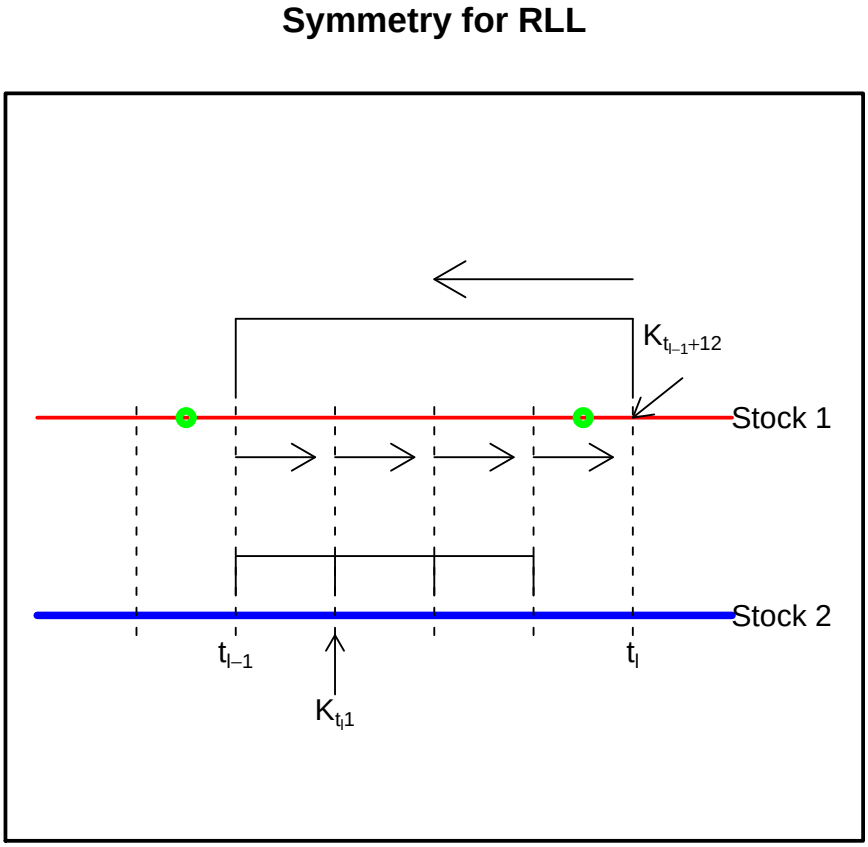


Figure 5.2: Prove Lemma 3.9

following:

$$R^{(1)}_{t_l} R^{(2)}_{t_{l-1}+1} = R^{(1)}_{K_{t_{l-1}+1,2}} R^{(2)}_{t_{l-1}+1}, \quad \ldots, \quad R^{(1)}_{t_l} R^{(2)}_{t_l-1} = R^{(1)}_{K_{t_l-1,2}} R^{(2)}_{t_l-1}.$$

Hence,

$$\sum_{j=K_{t_l 1}}^{t_l-1} R^{(1)}_{t_l} R^{(2)}_{j} = R^{(1)}_{K_{t_{l-1}+1,2}} R^{(2)}_{t_{l-1}+1} + \cdots + R^{(1)}_{K_{t_l-1,2}} R^{(2)}_{t_l-1}.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=K_{i1}}^{i-1} R_i^{(1)}R_j^{(2)} &= \sum_{l=1}^{T}\sum_{j=K_{t_l1}}^{t_l-1} R_{t_l}^{(1)}R_j^{(2)} \\
&= \sum_{l=1}^{T}\left(R_{K_{t_{l-1}+1,2}}^{(1)}R_{t_{l-1}+1}^{(2)} + \cdots + R_{K_{t_l-1,2}}^{(1)}R_{t_l-1}^{(2)}\right) \\
&= \sum_{i=1}^{m} R_{K_{i2}}^{(1)}R_i^{(2)} \\
&= \sum_{i=1}^{m}\sum_{j=i+1}^{K_{i2}} R_j^{(1)}R_i^{(2)}
\end{aligned}$$

□

5.1.3 Proof of Theory 2.2.3

$$\begin{aligned}
\text{RLL} &= \sum_{i=1}^{m}\sum_{j=K_{i1}}^{K_{i2}} (P_i^{(1)*} - P_{i-1}^{(1)*})(P_j^{(2)*} - P_{j-1}^{(2)*}) \\
&= \sum_{i=1}^{m}\sum_{j=K_{i1}}^{K_{i2}} (r_i^{(1)} + (u_i^{(1)} - u_{i-1}^{(1)}))(r_j^{(2)} + (u_j^{(2)} - u_{j-1}^{(2)})) \\
&= \sum_{i=1}^{m}(r_i^{(1)} + (u_i^{(1)} - u_{i-1}^{(1)}))(\sum_{j=K_{i1}}^{K_{i2}} r_j^{(2)} + (u_{K_{i2}}^{(2)} - u_{K_{i1}-1}^{(2)})) \\
&= \sum_{i=1}^{m}\sum_{j=K_{i1}}^{K_{i2}} r_i^{(1)}r_j^{(2)} + \sum_{i=1}^{m} r_i^{(1)}(u_{K_{i2}}^{(2)} - u_{K_{i1}-1}^{(2)}) \\
&+ \sum_{i=1}^{m}(u_i^{(1)} - u_{i-1}^{(1)})\sum_{j=K_{i1}}^{K_{i2}} r_j^{(2)} + \sum_{i=1}^{m}(u_i^{(1)} - u_{i-1}^{(1)})(u_{K_{i2}}^{(2)} - u_{K_{i1}-1}^{(2)}) \\
&= \sum_{i=1}^{m} A_i + \sum_{i=1}^{m} B_i + \sum_{i=1}^{m} C_i + \sum_{i=1}^{m} D_i
\end{aligned}$$

Where $A_i = \sum_{j=K_{i1}}^{K_{i2}} r_i^{(1)}r_j^{(2)}$,$B_i = r_i^{(1)}(u_{K_{i2}}^{(2)} - u_{K_{i1}-1}^{(2)})$, $C_i = (u_i^{(1)} - u_{i-1}^{(1)})\sum_{j=K_{i1}}^{K_{i2}} r_j^{(2)}$, and $D_i = (u_i^{(1)} - u_{i-1}^{(1)})(u_{K_{i2}}^{(2)} - u_{K_{i1}-1}^{(2)})$.

Hence

$$E(RLL) = E(\sum_{i=1}^{m} A_i) + E(\sum_{i=1}^{m} B_i) + E(\sum_{i=1}^{m} C_i) + E(\sum_{i=1}^{m} D_i)$$

Since

$$E(B_i|K_{i1}, K_{i2}) = E(r_i^{(1)})E(u_{K_{i2}}^{(2)} - u_{K_{i1}-1}^{(2)}) = 0,$$

$$E(C_i|K_{i1}, K_{i2}) = E(u_i^{(1)} - u_{i-1}^{(1)})E(\sum_{K_{i1}}^{K_{i2}} r_j^{(2)}) = 0,$$

$$E(D_i|K_{i1}, K_{i2}) = E(u_i^{(1)} - u_{i-1}^{(1)})E(u_{K_{i2}}^{(2)} - u_{K_{i1}-1}^{(2)}) = 0,$$

$E(RLL) = E(\sum_{i=1}^{m} A_i)$.

Now

$$A_i = \sum_{j=K_{i1}}^{K_{i2}} R_i Z_j = \sum_{j=K_{i1}}^{i-1} R_i Z_j + R_i Z_i + \sum_{j=i+1}^{K_{i2}} R_i Z_j$$

Hence

$$EA_i = E(R_i Z_i) + E(\sum_{j=K_{i1}}^{i-1} R_i Z_j) + E(\sum_{j=i+1}^{K_{i2}} R_i Z_j) = E(R_i Z_i) + II_i + I_i$$

Where $II_i = E(\sum_{j=K_{i1}}^{i-1} R_i Z_j)$ and $I_i = E(\sum_{j=i+1}^{K_{i2}} R_i Z_j) = E(R_i Z_i)$.

$$E(RLL) = \sum_{i=1}^{m} E(R_i Z_i) + \sum_{i=1}^{m} II_i + \sum_{i=1}^{m} I_i$$

Calculate $\sum_{i=1}^{m} E(R_i Z_i)$

To calculate $E(R_i Z_i)$ we determine the condition on a trade of asset R and Z in interval $((i-1)h, ih]$.

$$\begin{aligned} E(R_i Z_i) &= P\{E_i^{(1)} \cap E_i^{(2)}\} E(R_i Z_i | E_i^{(1)} \cap E_i^{(2)}) \\ &= \xi_1 \xi_2 \rho \sigma_1 \sigma_2 E(\nu((t_{i-1}^{(1)}, t_i^{(1)}) \cap (t_{i-1}^{(2)}, t_i^{(2)})) | E_i^{(1)} \cap E_i^{(2)}) \\ &= \xi_1 \xi_2 \rho \sigma_1 \sigma_2 E(b_{i-1} + h - a_i | a_i < h) \\ &= \rho \sigma_1 \sigma_2 (h - \frac{\lambda_1^2 \xi_2 + \lambda_2^2 \xi_1}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)}) \end{aligned} \tag{5.1.1}$$

The third equality follows Lemma 2.2.1. To derive $E(b_{i-1})$ and $E(a_i | a_i < h)$, we use that for exponential variable z_i with mean λ_i^{-1} the following holds:

$$F_{min}(u) = P(\min\{z_1, z_2\} < u) = 1 - P(z_1 > u \cap z_2 > u) = 1 - e^{-(\lambda_1 + \lambda_2)u},$$

$$F_{min}(u) = P(\max\{z_1, z_2\} < u) = P(z_1 < u \cap z_2 < u) = (1 - e^{-\lambda_1 u})(1 - e^{-\lambda_2 u}).$$

Thus,

$$E(b_{i-1}) = \int_0^\infty u(\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2)u} du = \frac{1}{\lambda_1 + \lambda_2}$$

and $\kappa_1 \equiv E(a_i | a_i < h)$ is equal to:

$$\begin{aligned} \kappa_1 &= \frac{1}{\xi_1 \xi_2} \int_0^h u(\lambda_1 e^{-u\lambda_1} + \lambda_2 e^{-u\lambda_2} - (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2)u}) du \\ &= h - \frac{h}{\xi_1 \xi_2} + \frac{1}{\lambda_1 + \lambda_2} (\frac{\lambda_1}{\xi_1 \lambda_2} + \frac{\lambda_2}{\xi_2 \lambda_1} + 1) \end{aligned}$$

$$
\begin{aligned}
& E(b_{i-1} + h - a_i | a_i < h) \\
&= h + E(b_{i-1}) - E(a_i | a_i < h) \\
&= h + \frac{1}{\lambda_1 + \lambda_2} - h + \frac{h}{\xi_1 \xi_2} - \frac{1}{\lambda_1 + \lambda_2}(\frac{\lambda_1}{\xi_1 \lambda_2} + \frac{\lambda_2}{\xi_2 \lambda_1} + 1) \\
&= \frac{h}{\xi_1 \xi_2} - \frac{\lambda_1^2 \xi_2 + \lambda_2^2 \xi_1}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2) \xi_1 \xi_2}
\end{aligned}
$$

Hence

$$
E(R_i Z_i) = \rho \sigma_1 \sigma_2 (h - \frac{\lambda_1^2 \xi_2 + \lambda_2^2 \xi_1}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)})
$$

$$
\begin{aligned}
\sum_{i=1}^{m} E(R_i Z_i) &= m \rho \sigma_1 \sigma_2 (h - \frac{\lambda_1^2 \xi_2 + \lambda_2^2 \xi_1}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)}) \\
&= \rho \sigma_1 \sigma_2 - \frac{1}{h} \frac{\lambda_1^2 \xi_2 + \lambda_2^2 \xi_1}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)}
\end{aligned} \tag{5.1.2}
$$

Calculate $\sum_{i=1}^{m} E(\sum_{j=i+1}^{K_{i2}} R_i Z_j) = \sum_{i=1}^{m} I_i$

First, recall the meaning of K_{i2}: for fixed R_i, look at the process $\{Z_i\}$, waiting for the first event happening after time point ih, say the first event happening in $((i+l-1)h, (i+l)h]$, then decide the value of $K_{i2} = l$. In other words, $K_{i2} = i + l$ means there is no event for process $\{Z_i\}$ in the interval of $((i-1)h, (i+l-1)h]$, but there is at least one event in the interval of $((i+l-1)h, (i+l)h]$. This leads to the

trivial result that

$$\sum_{j=i+1}^{K_{i2}} R_i Z_j = R_i Z_{K_{i2}}$$

Condition on $K_{i2} = i+l$, to calculate $E(R_i Z_{i+l})$ we condition more on $E_i^{(1)} \cap E_{i+l}^{(2)}$, by Lemma 2.2.1,

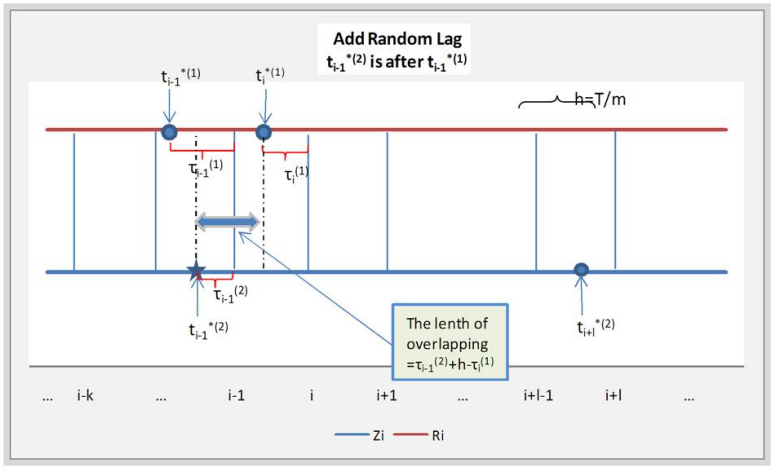
$$\begin{aligned}
&E(R_i Z_{i+l} | E_i^{(1)} \cap E_{i+l}^{(2)}, \text{ and } K_{i2} = i+l) \\
&= \rho\sigma_1\sigma_2 E(\tau_{i-1}^{(2)} + h - \tau_i^{(1)} | \tau_{i-1}^{(2)} \le \tau_{i-1}^{(1)}) + \rho\sigma_1\sigma_2 E(\tau_{i-1}^{(1)} + h - \tau_i^{(1)} | \tau_{i-1}^{(2)} > \tau_{i-1}^{(1)}) \quad (5.1.3)
\end{aligned}$$

Figure 5.3 shows the two different parts on the right side of Equation 5.1.3.

Let $z_1 = \tau_{i-1}^{(2)}$, $w_1 = \tau_{i-1}^{(1)}$ and $w_2 = \tau_i^{(1)}$, then $z_1, w_1 > 0$, and $0 < w_2 < h$.

Then,

$$\begin{aligned}
&E(\tau_{i-1}^{(2)} + h - \tau_i^{(1)} | \tau_{i-1}^{(2)} \le \tau_{i-1}^{(1)}) \\
&= E(z_1 + h - w_2 | z_1 \le w_1) \\
&= \int_0^h \int_0^\infty \int_{z_1}^\infty (z_1 + h - w_2) \lambda_1 e^{-\lambda_1 w_1} \lambda_2 e^{-\lambda_2 z_1} \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}} \, \mathrm{d}w_1 \mathrm{d}z_1 \mathrm{d}w_2 \\
&= \int_0^h \int_0^\infty (z_1 + h - w_2) \lambda_2 e^{-\lambda_2 z_1} \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}} \left(\int_0^\infty \lambda_1 e^{-\lambda_1 w_1} \, \mathrm{d}w_1 \right) \mathrm{d}z_1 \mathrm{d}w_2 \\
&= \int_0^h \int_0^\infty (z_1 + h - w_2) \lambda_2 e^{-(\lambda_1+\lambda_2) z_1} \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}} \, \mathrm{d}z_1 \mathrm{d}w_2 \\
&= \int_0^h \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}} \left(\int_0^\infty z_1 \lambda_2 e^{-(\lambda_1+\lambda_2) z_1} \, \mathrm{d}z_1 + \int_0^\infty (h - w_2) \lambda_2 e^{-(\lambda_1+\lambda_2) z_1} \, \mathrm{d}z_1 \right) \mathrm{d}w_2 \\
&= \int_0^h \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}} \left[\frac{\lambda_2}{(\lambda_1+\lambda_2)^2} + (h - w_2) \frac{\lambda_2}{\lambda_1+\lambda_2} \right] \mathrm{d}w_2 \\
&= \left[\frac{\lambda_2}{(\lambda_1+\lambda_2)^2} + \frac{h\lambda_2}{\lambda_1+\lambda_2} \right] \int_0^h \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}} \, \mathrm{d}w_2 - \frac{\lambda_2}{\lambda_1+\lambda_2} \frac{1}{1 - e^{-\lambda_1 h}} \int_0^h \lambda_1 w_2 e^{-\lambda_1 w_2} \, \mathrm{d}w_2 \\
&= \frac{\lambda_2}{(\lambda_1+\lambda_2)^2} + \frac{h\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} \frac{1}{1 - e^{-\lambda_1 h}} \left(h e^{-\lambda_1 h} + \frac{1}{\lambda_1} e^{-\lambda_1 h} - \frac{1}{\lambda_1} \right) \\
&= \frac{\lambda_2}{(\lambda_1+\lambda_2)^2} + \frac{h\lambda_2}{\lambda_1+\lambda_2} \frac{1}{1 - e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)} \quad (5.1.4)
\end{aligned}$$



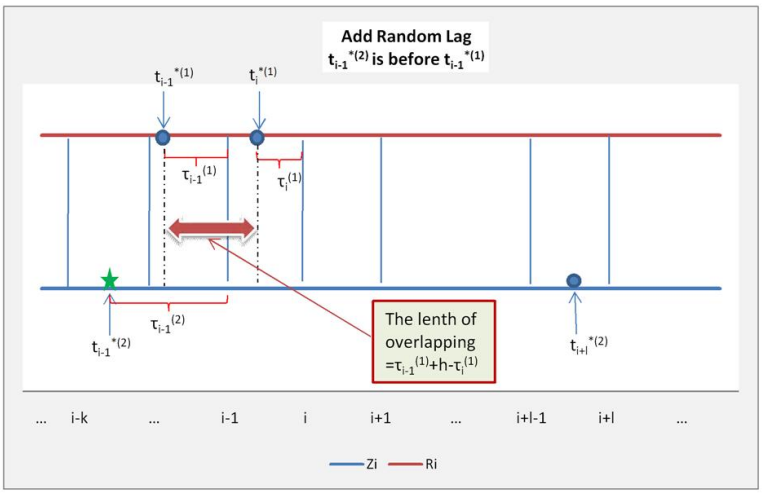


Figure 5.3: Calculate $E(R_i Z_{i+l} | E_i^{(1)} \cap E_{i+l}^{(2)})$

Provided

$$\begin{aligned}
&\int_0^\infty z_1\lambda_2 e^{-(\lambda_1+\lambda_2)z_1}\,\mathrm{d}z_1\\
&= -\frac{\lambda_2}{\lambda_1+\lambda_2}\int_0^\infty z_1\,\mathrm{d}e^{-(\lambda_1+\lambda_2)z_1}\\
&= -\frac{\lambda_2}{\lambda_1+\lambda_2}\left[z_1 e^{-(\lambda_1+\lambda_2)z_1}|_0^\infty - \int_0^\infty e^{-(\lambda_1+\lambda_2)z_1}\,\mathrm{d}z_1\right]\\
&= \frac{\lambda_2}{\lambda_1+\lambda_2}\int_0^\infty e^{-(\lambda_1+\lambda_2)z_1}\,\mathrm{d}z_1\\
&= \frac{\lambda_2}{(\lambda_1+\lambda_2)^2};
\end{aligned}$$

$$\int_0^\infty (h-w_2)\lambda_2 e^{-(\lambda_1+\lambda_2)z_1}\,\mathrm{d}z_1 = (h-w_2)\frac{\lambda_2}{\lambda_1+\lambda_2};$$

and

$$\begin{aligned}
&-\int_0^h \lambda_1 w_2 e^{-\lambda_1 w_2}\,\mathrm{d}w_2\\
&= \int_0^h w_2\,\mathrm{d}e^{-\lambda_1 w_2}\\
&= w_2 e^{-\lambda_1 w_2}|_0^h - \int_0^h e^{-\lambda_1 w_2}\,\mathrm{d}w_2\\
&= h e^{\lambda_1 h} + \frac{1}{\lambda_1}e^{-\lambda_1 h} - \frac{1}{\lambda_1}.
\end{aligned}$$

Let $\tau_{i-1}^{(1)} = w_1$,$\tau_i^{(1)} = w_2$, and $\tau_{i-1}^{(2)} = z_1$, then $w_1, z_1 > 0$,and $0 < w_2 < h$. Then

$$\begin{aligned}
&E(\tau_{i-1}^{(1)} + h - \tau_i^{(1)} | \tau_{i-1}^{(2)} > \tau_{i-1}^{(1)})\\
&= E(w_1 + h - w_2 | z_1 > w_1)\\
&= \int_0^h \int_0^\infty \int_0^{z_1} (w_1 + h - w_2)\lambda_1 e^{-\lambda_1 w_1}\lambda_2 e^{-\lambda_2 z_1}\frac{\lambda_1 e^{-\lambda_1 w_2}}{1-e^{-\lambda_1 h}}\,\mathrm{d}w_1\mathrm{d}z_1\mathrm{d}w_2
\end{aligned}$$

$$
\begin{aligned}
&\int_0^{z_1} (w_1 + h - w_2)\lambda_1 e^{-\lambda_1 w_1}\,\mathrm{d}w_1 \\
&= \int_0^{z_1} w_1\lambda_1 e^{-\lambda_1 w_1}\,\mathrm{d}w_1 + \int_0^{z_1} (h - w_2)\lambda_1 e^{-\lambda_1 w_1}\,\mathrm{d}w_1 \\
&= -\int_0^{z_1} w_1\,\mathrm{d}e^{-\lambda_1 w_1} - (h - w_2)e^{-\lambda_1 w_1}|_0^{z_1} \\
&= -\left(w_1 e^{-\lambda_1 w_1}|_0^{z_1} - \int_0^{z_1} e^{-\lambda_1 w_1}\,\mathrm{d}w_1\right) - (h - w_2)e^{-\lambda_1 z_1} + (h - w_2) \\
&= -z_1 e^{-\lambda_1 z_1} - \frac{1}{\lambda_1}e^{-\lambda_1 z_1} + \frac{1}{\lambda_1} - (h - w_2)e^{-\lambda_1 z_1} + (h - w_2) \\
&= -z_1 e^{-\lambda_1 z_1} - \left(\frac{1}{\lambda_1} + h - w_2\right)e^{-\lambda_1 z_1} + \left(\frac{1}{\lambda_1} + h - w_2\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&E(\tau_{i-1}^{(1)} + h - \tau_i^{(1)} | \tau_{i-1}^{(2)} > \tau_{i-1}^{(1)}) \\
&= \int_0^h \int_0^\infty -\lambda_2 z_1 e^{-(\lambda_1+\lambda_2)z_1} \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}}\,\mathrm{d}z_1\mathrm{d}w_2 \\
&+ \int_0^h \left[\int_0^\infty -\lambda_2 \left(\frac{1}{\lambda_1} + h - w_2\right)e^{-(\lambda_1+\lambda_2)z_1} + \lambda_2\left(\frac{1}{\lambda_1} + h - w_2\right)e^{\lambda_2 z_1}\,\mathrm{d}z_1\right] \\
&\cdot \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}}\,\mathrm{d}w_2
\end{aligned}
$$

The first part in the above equation equals to:

$$
\begin{aligned}
&\int_0^h \int_0^\infty -\lambda_2 z_1 e^{-(\lambda_1+\lambda_2)z_1} \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}}\,\mathrm{d}z_1\mathrm{d}w_2 \\
&= \int_0^h \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}}\left[\frac{\lambda_2}{\lambda_1 + \lambda_2}\int_0^\infty z_1\,\mathrm{d}e^{-(\lambda_1+\lambda_2)z_1}\right]\mathrm{d}w_2 \\
&= \int_0^h \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}}\frac{\lambda_2}{\lambda_1 + \lambda_2}\left[z_1 e^{-(\lambda_1+\lambda_2)z_1}|_0^\infty - \int_0^\infty e^{-(\lambda_1+\lambda_2)z_1}\,\mathrm{d}z_1\right]\mathrm{d}w_2 \\
&= -\frac{\lambda_2}{(\lambda_1 + \lambda_2)^2}\int_0^h \frac{\lambda_1 e^{-\lambda_1 w_2}}{1 - e^{-\lambda_1 h}}\,\mathrm{d}w_2 \\
&= -\frac{\lambda_2}{(\lambda_1 + \lambda_2)^2}
\end{aligned}
$$

To find out the second part, first calculate the following:

$$\begin{aligned}
&\int_0^\infty -\lambda_2\left(\frac{1}{\lambda_1}+h-w_2\right)e^{-(\lambda_1+\lambda_2)z_1}+\lambda_2\left(\frac{1}{\lambda_1}+h-w_2\right)e^{\lambda_2 z_1}\,\mathrm{d}z_1\\
&=\frac{\lambda_2}{\lambda_1+\lambda_2}\left(\frac{1}{\lambda_2}+h-w_2\right)e^{-(\lambda_1+\lambda_2)z_1}|_0^\infty-\left(\frac{1}{\lambda_2}+h-w_2\right)e^{-\lambda_2 z_1}|_0^\infty\\
&=-\frac{\lambda_2}{\lambda_1+\lambda_2}\left(\frac{1}{\lambda_2}+h-w_2\right)+\left(\frac{1}{\lambda_2}+h-w_2\right)\\
&=\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{1}{\lambda_2}+h-w_2\right)
\end{aligned}$$

Hence the second part is:

$$\begin{aligned}
&\int_0^h\left[\int_0^\infty -\lambda_2\left(\frac{1}{\lambda_1}+h-w_2\right)e^{-(\lambda_1+\lambda_2)z_1}+\lambda_2\left(\frac{1}{\lambda_1}+h-w_2\right)e^{\lambda_2 z_1}\,\mathrm{d}z_1\right]\\
&\cdot\frac{\lambda_1 e^{-\lambda_1 w_2}}{1-e^{-\lambda_1 h}}\,\mathrm{d}w_2\\
&=\int_0^h\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{1}{\lambda_1}+h-w_2\right)\frac{\lambda_1 e^{-\lambda_1 w_2}}{1-e^{-\lambda_1 h}}\,\mathrm{d}w_2\\
&=\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{1}{\lambda_1}+h\right)\int_0^h\frac{\lambda_1 e^{-\lambda_1 w_2}}{1-e^{-\lambda_1 h}}\,\mathrm{d}w_2+\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{1}{1-e^{-\lambda_1 h}}\int_0^h w_2\,\mathrm{d}e^{-\lambda_1 w_2}\\
&=\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{1}{\lambda_1}+h\right)+\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{1}{1-e^{-\lambda_1 h}}\left(w_2e^{-\lambda_1 w_2}|_0^h-\int_0^h e^{-\lambda_1 w_2}\,\mathrm{d}w_2\right)\\
&=\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{1}{\lambda_1}+h\right)+\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{1}{1-e^{-\lambda_1 h}}\left(he^{-\lambda_1 h}+\frac{1}{\lambda_1}e^{-\lambda_1 h}-\frac{1}{\lambda_1}\right)\\
&=\frac{\lambda_1 h}{\lambda_1+\lambda_2}\frac{1}{1-e^{-\lambda_1 h}}
\end{aligned}$$

Hence

$$E(\tau_{i-1}^{(1)}+h-\tau_i^{(1)}|\tau_{i-1}^{(2)}>\tau_{i-1}^{(1)})=-\frac{\lambda_2}{(\lambda_1+\lambda_2)^2}+\frac{\lambda_1 h}{\lambda_1+\lambda_2}\frac{1}{1-e^{-\lambda_1 h}}\tag{5.1.5}$$

Combine 5.1.4 and 5.1.5, equation 5.1.3 becomes:

$$
\begin{aligned}
&E(R_iZ_{i+l}|E_i^{(1)}\cap E_{i+l}^{(2)},\text{ and } K_{i2}=i+l)\\
&=\rho\sigma_1\sigma_2E(\tau_{i-1}^{(2)}+h-\tau_i^{(1)}|\tau_{i-1}^{(2)}\le\tau_{i-1}^{(1)})+\rho\sigma_1\sigma_2E(\tau_{i-1}^{(1)}+h-\tau_i^{(1)}|\tau_{i-1}^{(2)}>\tau_{i-1}^{(1)})\\
&=\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_1h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right] \qquad (5.1.6)
\end{aligned}
$$

The expectation of $R_iZ_j$will be:

$$
\begin{aligned}
&E(\sum_{j=i+1}^{K_{i2}}R_iZ_j)\\
&=\sum_{l=1}^{\infty}E(\sum_{j=i+1}^{K_{i2}}R_iZ_j|E_i^{(1)}\cap E_{i+l}^{(2)},\ and\ K_{i2}=i+l)P(E_i^{(1)}\cap E_{i+l}^{(2)},\ and\ K_{i2}=i+l)\\
&=\sum_{l=1}^{\infty}E(R_iZ_i+l|E_i^{(1)}\cap E_{i+l}^{(2)},\ and\ K_{i2}=i+l)P(E_i^{(1)}\cap E_{i+l}^{(2)},\ and\ K_{i2}=i+l)\\
&=\sum_{l=1}^{\infty}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_1h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_1h})(1-e^{-\lambda_2h})e^{-\lambda_2lh}\\
&=\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_1h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_1h})e^{-\lambda_2h} \qquad (5.1.7)
\end{aligned}
$$

Hence

$$
\sum_{i=1}^{m}E(\sum_{j=i+1}^{K_{i2}}R_iZ_j)=m\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_1h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_1h})e^{-\lambda_2h} \qquad (5.1.8)
$$

Calculate $\sum_{i=1}^{m}E(\sum_{j=K_{i1}}^{i-1}R_iZ_j)=\sum_{i=1}^{m}II_i$

By Lemma 2.2.2 (symmetry),

$$
E(\sum_{j=K_{i1}}^{i-1}R_iZ_j)=\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_2h}}-\frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_2h})e^{-\lambda_1h} \qquad (5.1.9)
$$

$$
\sum_{i=1}^{m}E(\sum_{j=K_{i1}}^{i-1}R_iZ_j)=m\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_2h}}-\frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_2h})e^{-\lambda_1h} \qquad (5.1.10)
$$

Note that $m=1/h$, this leads to the final expressions:

Final expression of E(RLL) **and** Bias(RLL)

Combine 5.1.2, 5.1.8 and 5.1.10, we will have the final expression of E(RLL):

$$\begin{aligned}
&\text{E(RLL)} \\
&= \rho\sigma_1\sigma_2 - \frac{1}{h}\rho\sigma_1\sigma_2\frac{\lambda_1^2\xi_2 + \lambda_2^2\xi_1}{\lambda_1\lambda_2(\lambda_1+\lambda_2)} \\
&+ \frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_1 h})e^{-\lambda_2 h} \\
&+ \frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_2 h}} - \frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_2 h})e^{-\lambda_1 h}
\end{aligned} \tag{5.1.11}$$

Bias:

$$\begin{aligned}
&\text{Bias(RLL)} \\
&= -\frac{1}{h}\rho\sigma_1\sigma_2\frac{\lambda_1^2\xi_2 + \lambda_2^2\xi_1}{\lambda_1\lambda_2(\lambda_1+\lambda_2)} \\
&+ \frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_1 h})e^{-\lambda_2 h} \\
&+ \frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_2 h}} - \frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_2 h})e^{-\lambda_1 h}
\end{aligned} \tag{5.1.12}$$

5.1.4 Proof of Theory 2.2.4

Proof. While h goes to zero,

$$\xi_j = 1 - e^{-\lambda_j h} = \lambda_j h + o(h), \quad j = 1, 2.$$

$$
\begin{aligned}
&\lim_{h\to 0}\left[-\frac{1}{h}\rho\sigma_1\sigma_2\frac{\lambda_1^2\xi_2+\lambda_2^2\xi_1}{\lambda_1\lambda_2(\lambda_1+\lambda_2)}\right]\\
&=-\rho\sigma_1\sigma_2\lim_{h\to 0}\frac{1}{h}\frac{\lambda_1(1-e^{-\lambda_2 h})}{\lambda_2(\lambda_1+\lambda_2)}-\rho\sigma_1\sigma_2\lim_{h\to 0}\frac{1}{h}\frac{\lambda_2(1-e^{-\lambda_1 h})}{\lambda_1(\lambda_1+\lambda_2)}\\
&=-\rho\sigma_1\sigma_2\lim_{h\to 0}\frac{\mathrm{d}\left[\lambda_1(1-e^{-\lambda_2 h})\right]/\mathrm{d}h}{\mathrm{d}\left[h\lambda_2(\lambda_1+\lambda_2)\right]/\mathrm{d}h}-\rho\sigma_1\sigma_2\lim_{h\to 0}\frac{\mathrm{d}\left[\lambda_2(1-e^{-\lambda_1 h})\right]/\mathrm{d}h}{\mathrm{d}\left[h\lambda_1(\lambda_1+\lambda_2)\right]/\mathrm{d}h}\\
&=-\rho\sigma_1\sigma_2\lim_{h\to 0}\frac{\lambda_1 e^{-\lambda_2 h}}{\lambda 1+\lambda_2}-\rho\sigma_1\sigma_2\lim_{h\to 0}\frac{\lambda_2 e^{-\lambda_1 h}}{\lambda 1+\lambda_2}\\
&=-\frac{\lambda_1}{\lambda 1+\lambda_2}\rho\sigma_1\sigma_2-\frac{\lambda_2}{\lambda 1+\lambda_2}\rho\sigma_1\sigma_2\\
&=-\rho\sigma_1\sigma_2
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_1 h})e^{-\lambda_2 h}\\
&=\rho\sigma_1\sigma_2\lim_{h\to 0}e^{-\lambda_2 h}-\rho\sigma_1\sigma_2\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\lim_{h\to 0}\frac{(1-e^{-\lambda_1 h})}{h}\lim_{h\to 0}e^{-\lambda_2 h}\\
&=\rho\sigma_1\sigma_2-\rho\sigma_1\sigma_2\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\lim_{h\to 0}\frac{\mathrm{d}\left[(1-e^{-\lambda_1 h})\right]/\mathrm{d}h}{\mathrm{d}h/\mathrm{d}h}\cdot 1\\
&=\rho\sigma_1\sigma_2-\rho\sigma_1\sigma_2\frac{\lambda_2}{\lambda_1+\lambda_2}\\
&=\rho\sigma_1\sigma_2\frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\rho\sigma_1\sigma_2\left[\frac{h}{1-e^{-\lambda_2 h}}-\frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2)}\right](1-e^{-\lambda_2 h})e^{-\lambda_1 h}\\
&=\rho\sigma_1\sigma_2\frac{\lambda_2}{\lambda_1+\lambda_2}
\end{aligned}
$$

Hence,

$$
\mathrm{Bias(RLL)}=-\rho\sigma_1\sigma_2+\rho\sigma_1\sigma_2\frac{\lambda_1}{\lambda_1+\lambda_2}+\rho\sigma_1\sigma_2\frac{\lambda_2}{\lambda_1+\lambda_2}=0
$$

□

5.1.5 Proofs for variance without microstructure noise

This section provides the detail calculation of the variance of RLL and its limit.

5.1.6 Proof of Theorem 2.3.4

To find Var(RLL), we decompose RLL as the following:

$$\begin{aligned}\mathrm{Var}(\mathrm{RLL}) = &\mathrm{Var}(\mathrm{RC}) + \mathrm{Var}(\mathrm{RLL}_1) + \mathrm{Var}(\mathrm{RLL}_2) \\ &+ 2\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC}) + 2\mathrm{Cov}(\mathrm{RLL}_2, \mathrm{RC}) + 2\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RLL}_2) \end{aligned} \quad (5.1.13)$$

We already re-stated the expression for Var(RC) in Subsection 3.3.1, and the calculation for the other parts are organized in the following sebsections. Some of the results are based on the following fact:

Fact 5.1.1. *If the random variables X, Y, Z , and T are from a multivariate normal distribution, then the following result holds:*

$$E(XYZT) = E(XY)E(ZT) + E(XZ)E(YT) + E(XT)E(YZ). \quad (5.1.14)$$

Var(RLL1)

Conditional on all the observed events, we have

$$\begin{aligned}
&\mathrm{Var}(\mathrm{RLL}_1) = \sum_{i=1}^{m}\sum_{j=1}^{m} \mathrm{Cov}\left(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i), R_j Z_{L_{j1}} \mathbb{I}(L_{j1} > j)\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \Big[E\left(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i) R_j Z_{L_{j1}} \mathbb{I}(L_{j1} > j)\right) \\
&- E\left(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) E\left(R_j Z_{L_{j1}} \mathbb{I}(L_{j1} > j)\right) \Big] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i R_j\right) E\left(Z_{L_{i1}} Z_{L_{j1}} \mathbb{I}(L_{i1} > i, L_{j1} > j)\right) \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i Z_{L_{j1}} \mathbb{I}(L_{j1} > j)\right) E\left(R_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) \\
&= \sum_{i=1}^{m} E\left(R_i^2\right) E\left(Z_{L_{i1}}^2 \mathbb{I}(L_{i1} > i)\right) + \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i Z_{L_{j1}} \mathbb{I}(L_{j1} > j)\right) E\left(R_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right)
\end{aligned}$$

Then take expectation with respect to the events times. The following notations will be used: $E_i^{(j)}$: there is at least one trade in the time interval $((i-1)h, ih]$ for process j, where j = 1, 2.

$$\begin{aligned}
&\mathrm{Var}(\mathrm{RLL}_1) = \sigma_1^2 \sigma_2^2 \sum_{i=1}^{m} E\left[\nu\left((t_{i-1}^{(1)}, t_i^{(1)}) \cdot (t_{i-1}^{(2)}, t_{L_{i1}}^{(2)})\right) \cdot P\left(E_i^{(1)}\right) P\left(E_{L_{i1}}^{(2)}\right)\right] \\
&+ \rho^2 \sigma_1^2 \sigma_2^2 \sum_{i=1}^{m}\sum_{j=1}^{m} E\Big[\nu\left((t_{i-1}^{(1)}, t_i^{(1)}) \cap (t_{j-1}^{(2)}, t_{L_{j1}}^{(2)})\right) \cdot \nu\left((t_{j-1}^{(1)}, t_j^{(1)}) \cap (t_{i-1}^{(2)}, t_{L_{i1}}^{(2)})\right) \\
&\cdot P\left(E_i^{(1)}\right) P\left(E_{L_{j1}}^{(2)}\right) P\left(E_j^{(1)}\right) P\left(E_{L_{i1}}^{(2)}\right)\Big] \\
&= (i) + (ii)
\end{aligned}$$

$$(i) = \sigma_1^2\sigma_2^2 \sum_{i=1}^{m} E\left[\nu\left((t_{i-1}^{(1)}, t_i^{(1)}) \cdot (t_{i-1}^{(2)}, t_{L_{i1}}^{(2)})\right) \cdot P\left(E_i^{(1)}\right) P\left(E_{L_{i1}}^{(2)}\right)\right]$$

$$= \sigma_1^2\sigma_2^2 \sum_{i=1}^{m} P\left(E_i^{(1)}\right) \sum_{l=i+1}^{\infty} P\left(E_l^{(2)}\right) E\left[\nu\left((t_{i-1}^{(1)}, t_i^{(1)}) \cdot (t_{i-1}^{(2)}, t_{L_{i1}}^{(2)})\right)\right]$$

$$= \sigma_1^2\sigma_2^2\xi_1\xi_2 \sum_{i=1}^{m}\sum_{l=i+1}^{\infty} e^{-\lambda_2(l-i)h}\left(h \cdot \frac{1}{\xi_1}\right)\left((l-i)h + \frac{h}{\xi_2}\right)$$

$$= \sigma_1^2\sigma_2^2\xi_1\xi_2 \sum_{i=1}^{m}\sum_{t=1}^{\infty} e^{-\lambda_2 th}\left(\frac{h^2}{\xi_1\xi_2} + \frac{h^2}{\xi_1} \cdot t\right)$$

$$= \sigma_1^2\sigma_2^2h^2 \sum_{i=1}^{m}\sum_{t=1}^{\infty} e^{-\lambda_2 th} + \sigma_1^2\sigma_2^2h^2\xi_2 \sum_{i=1}^{m}\sum_{t=1}^{\infty} te^{-\lambda_2 th}$$

$$= \sigma_1^2\sigma_2^2h^2 \sum_{i=1}^{m} \frac{e^{-\lambda_2 h}}{1-e^{-\lambda_2 h}} + \sigma_1^2\sigma_2^2h^2\xi_2 \sum_{i=1}^{m} \frac{e^{-\lambda_2 h}}{(1-e^{-\lambda_2 h})^2}$$

$$= 2m\sigma_1^2\sigma_2^2h^2 \frac{e^{-\lambda_2 h}}{1-e^{-\lambda_2 h}}$$

$$= 2\sigma_1^2\sigma_2^2h \frac{e^{-\lambda_2 h}}{1-e^{-\lambda_2 h}}$$

This is true because:

- $$E\left(\tau_i^{(j)} | 0 \le \tau_i^{(j)} < h\right) = \int_0^h t \cdot \frac{\lambda_1 e^{-\lambda_j t}}{1-e^{-\lambda_j h}} dt = \frac{1}{\lambda_j} - \frac{he^{\lambda_j h}}{1-e^{\lambda_j h}},$$

  $\forall i = 1, ..., m \; and \; j = 1, 2$. Hence,

$$E\left[\nu\left(t_{i-1}^{(1)}, t_i^{(1)}\right)\right] = E\left(\tau_{i-1}^{(1)} + h - \tau_i^{(1)} | 0 \le \tau_i^{(1)} < h\right)$$
$$= \frac{1}{\lambda_1} + h - \frac{1}{\lambda_1} + \frac{he^{\lambda_1 h}}{1-e^{\lambda_1 h}} = \frac{h}{1-e^{\lambda_1 h}}$$

- $$E\left[\nu\left(t_{i-1}^{(2)}, t_l^{(2)}\right)\right] = E\left(\tau_{i-1}^{(2)} + (l-i+1)h - \tau_l^{(2)} | 0 \le \tau_i^{(2)} < h\right)$$
$$= \frac{1}{\lambda_2} + (l-i+1)h - \frac{1}{\lambda_2} + \frac{he^{\lambda_2 h}}{1-e^{\lambda_2 h}} = (l-i)h + \frac{h}{1-e^{\lambda_2 h}}$$

- $$\sum_{t=1}^{\infty} e^{-\lambda_2 th} = \frac{e^{-\lambda_2 h}}{1 - e^{-\lambda_2 h}}$$

  Taking differential w.r.t λ_2 on both sides of the equation, and using simple algebra, we can get

$$\sum_{t=1}^{\infty} te^{-\lambda_2 th} = \frac{e^{-\lambda_2 h}}{(1 - e^{-\lambda_2 h})^2}$$

For part (ii), do the following:

$$(ii) = \rho^2\sigma_1^2\sigma_2^2 \sum_{i=1}^{m}\sum_{j=1}^{m} E\left[\nu\left((t_{i-1}^{(1)}, t_i^{(1)}) \cap (t_{j-1}^{(2)}, t_{L_{j1}}^{(2)})\right) \cdot \nu\left((t_{j-1}^{(1)}, t_j^{(1)}) \cap (t_{i-1}^{(2)}, t_{L_{i1}}^{(2)})\right)\right]$$

$$\cdot P\left(E_i^{(1)}\right) P\left(E_{L_{j1}}^{(2)}\right) P\left(E_j^{(1)}\right) P\left(E_{L_{i1}}^{(2)}\right)$$

$$= \rho^2\sigma_1^2\sigma_2^2 \sum_{i=1}^{m} E\left[\left(\nu\left((t_{i-1}^{(1)}, t_i^{(1)}) \cap (t_{i-1}^{(2)}, t_{L_{i1}}^{(2)})\right)\right)^2\right] P\left(E_i^{(1)}\right) P\left(E_{L_{i1}}^{(2)}\right)$$

$$+ 2\rho^2\sigma_1^2\sigma_2^2 \sum_{j=1}^{m}\sum_{i=1}^{j-1} E\left[\nu\left((t_{i-1}^{(1)}, t_i^{(1)}) \cap (t_{j-1}^{(2)}, t_{L_{j1}}^{(2)})\right) \cdot \nu\left((t_{j-1}^{(1)}, t_j^{(1)}) \cap (t_{i-1}^{(2)}, t_{L_{i1}}^{(2)})\right)\right]$$

$$\cdot P\left(E_i^{(1)}\right) P\left(E_{L_{j1}}^{(2)}\right) P\left(E_j^{(1)}\right) P\left(E_{L_{i1}}^{(2)}\right)$$

$$= (ii.a) + (ii.b)$$

Next find (ii.a), first, calculate $E\left((b_{i-1} + h - \tau_i^{(1)})^2 | 0 < \tau_i^{(1)} \leq h\right)$

$$E(b_{i-1}) = \frac{1}{\lambda_1 + \lambda_2},$$

and

$$Var(b_{i-1}) = \frac{1}{(\lambda_1 + \lambda_2)^2},$$

hence,

$$E(b_{i-1}^2) = \frac{2}{(\lambda_1 + \lambda_2)^2}.$$

We already have

$$E\left(\tau_i^{(1)}|0 \le \tau_i^{(1)} < h\right) = \frac{1}{\lambda_1} - \frac{he^{\lambda_1 h}}{1-e^{\lambda_1 h}}.$$

calculate:

$$E\left((\tau_i^{(1)})^2|0 \le \tau_i^{(1)} < h\right) = \int_0^h t^2 \frac{\lambda_1 e^{-\lambda_1 t}}{1-e^{-\lambda_1 h}} dt = \frac{2}{\lambda_1^2} - \frac{2he^{-\lambda_1 h}}{\lambda_1(1-e^{-\lambda_1 h})} - \frac{h^2 e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}.$$

Hence,

$$\begin{aligned}
&E\left((b_{i-1}+h-\tau_i^{(1)})^2|0 < \tau_i^{(1)} \le h\right) \\
&= E\left(b_{i-1}^2 + h^2 + (\tau_i^{(1)})^2 + ahb_{i-1} - 2b_{i-1}\tau_i^{(1)} - 2h\tau_i^{(1)}|0 < \tau_i^{(1)} \le h\right) \\
&= \frac{2}{(\lambda_1+\lambda_2)^2} + h^2 + \frac{2}{\lambda_1^2} - \frac{2he^{-\lambda_1 h}}{\lambda_1(1-e^{-\lambda_1 h})} - \frac{h^2 e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}} + \frac{2h}{\lambda_1+\lambda_2} \\
&- \frac{2}{\lambda_1(\lambda_1+\lambda_2)} + \frac{2he^{-\lambda_1 h}}{(\lambda_1+\lambda_2)(1-e^{-\lambda_1 h})} - \frac{2h}{\lambda_1} + \frac{2h^2 e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}} \\
&= \frac{h^2}{1-e^{-\lambda_1 h}} - \frac{2h\lambda_2}{\lambda_1(\lambda_1+\lambda_2)(1-e^{-\lambda_1 h})} + 2\frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_1^2(\lambda_1+\lambda_2)^2}.
\end{aligned}$$

Now we can get part (ii.a):

$(ii.a)$

$$\begin{aligned}
&= \rho^2\sigma_1^2\sigma_2^2 \sum_{i=1}^m E\left[\left(\nu\left((t_{i-1}^{(1)}, t_i^{(1)}) \cap (t_{i-1}^{(2)}, t_{L_{i1}}^{(2)})\right)\right)^2\right] P\left(E_i^{(1)}\right) P\left(E_{L_{i1}}^{(2)}\right) \\
&= \rho^2\sigma_1^2\sigma_2^2 \sum_{i=1}^m P\left(E_i^{(1)}\right) \sum_{l=i+1}^{\infty} P\left(L_{i1}=l\right) E\left((b_{i-1}+h-\tau_i^{(1)})^2|0 < \tau_i^{(1)} \le h\right) \\
&= \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 \sum_{i=1}^m \sum_{l=i+1}^{\infty} e^{-\lambda_2(l-i)h}\left[\frac{h^2}{1-e^{-\lambda_1 h}} - \frac{2h\lambda_2}{\lambda_1(\lambda_1+\lambda_2)(1-e^{-\lambda_1 h})} + \frac{2\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_1^2(\lambda_1+\lambda_2)^2}\right] \\
&= \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2\left[\frac{h^2}{1-e^{-\lambda_1 h}} - \frac{2h\lambda_2}{\lambda_1(\lambda_1+\lambda_2)(1-e^{-\lambda_1 h})} + \frac{2\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_1^2(\lambda_1+\lambda_2)^2}\right] \sum_{i=1}^m \sum_{t=1}^{\infty} e^{-\lambda_2 th} \\
&= m\rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_2 h}\left[h^2 - \frac{2h\lambda_2}{\lambda_1(\lambda_1+\lambda_2)} + 2\xi_1\frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_1^2(\lambda_1+\lambda_2)^2}\right] \\
&= \rho^2\sigma_1^2\sigma_2^2 e^{-\lambda_2 h}\left[h - \frac{2\lambda_2}{\lambda_1(\lambda_1+\lambda_2)} + 2\frac{\xi_1}{h}\frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_1^2(\lambda_1+\lambda_2)^2}\right].
\end{aligned}$$

Next, find (ii.b). In the expression, we already assumed $i < j$, and we need to discuss the following cases: Case 1: When $i < L_{i1} < j$, the interval $((j-1)h, jh]$ for process 1 does not intersect with the interval $((i-1)h, L_{i1}h]$ for process 2, i.e. $\nu\left[\left(t_{j-1}^{(1)}, t_j^{(1)}\right) \cap \left(t_{i-1}^{(2)}, t_{L_{i1}}^{(2)}\right)\right] = 0$. Hence the corresponding expression for (ii.b) equals zero under this scenario.

Case 2: When $L_{i1} \geq j$, it has to be $L_{i1} = L_{j1}$, and we have $P(L_{i1} \geq j) = e^{-\lambda_2(j-i)h}$.The corresponding expectation term inside the summations is:

$$E\left[\nu\left(\left(t_{i-1}^{(1)}, t_i^{(1)}\right) \cap \left(t_{j-1}^{(2)}, t_{L_{j1}}^{(2)}\right)\right) \cdot \nu\left(\left(t_{j-1}^{(1)}, t_j^{(1)}\right) \cap \left(t_{i-1}^{(2)}, t_{L_{j1}}^{(2)}\right)\right)\right]$$

Conditional on $L_{j1} = l$, where $l > j$, we have the expectation equals to:

$$E\left[\left(b_{i-1} + h - \tau_i^{(1)}\right)\left(\tau_{j-1}^{(1)} + h - \tau_i^{(1)}\right) \mid 0 < \tau_i^{(1)} \leq h, 0 < \tau_j^{(1)} \leq h, L_{j1} = l\right].$$

Case 2.a: While $0 < \tau_{j-1}^{(1)} \leq (j-i-1)h$,

$$\left(b_{i-1} + h - \tau_i^{(1)} \mid 0 < \tau_i^{(1)} \leq h, 0 < \tau_{j-1}^{(1)} \leq (j-i-1)h, L_{j1} = l\right)$$

is independent of

$$\left(\tau_{j-1}^{(1)} + h - \tau_j^{(1)} \mid 0 < \tau_i^{(1)} \leq h, 0 < \tau_j^{(1)} \leq h, 0 < \tau_{j-1}^{(1)} \leq (j-i-1)h, L_{j1} = l\right),$$

hence,

$$
\begin{aligned}
&E\Big[\Big(b_{i-1}+h-\tau_i^{(1)}\Big)\Big(\tau_{j-1}^{(1)}+h-\tau_i^{(1)}\Big) \mid 0<\tau_i^{(1)}\le h, 0<\tau_j^{(1)}\le h,\\
&0<\tau_{j-1}^{(1)}\le (j-i-1)h, L_{j1}=l\Big]\\
&=E\left[\Big(b_{i-1}+h-\tau_i^{(1)}\Big) \mid 0<\tau_i^{(1)}\le h, L_{j1}=l\right]\\
&\cdot E\left[\Big(\tau_{j-1}^{(1)}+h-\tau_j^{(1)}\Big) \mid 0<\tau_i^{(1)}\le h, 0<\tau_j^{(1)}\le h, 0<\tau_{j-1}^{(1)}\le (j-i-1)h, L_{j1}=l\right]\\
&=\left(\frac{1}{\lambda_1+\lambda_2}+h-\frac{1}{\lambda_1}+\frac{he^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right)\\
&\cdot\left(\frac{1}{\lambda_1}-\frac{(j-i-1)he^{-\lambda_1(j-i-1)h}}{1-e^{-\lambda_1(j-i-1)h}}+h-\frac{1}{\lambda_1}+\frac{he^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right)\\
&=\frac{h^2}{\xi_1^2}-\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}-\frac{h^2(j-i-1)e^{-\lambda_1(j-i-1)h}}{\xi_1(1-e^{-\lambda_1(j-i-1)h})}\\
&+\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\frac{(j-i-1)he^{-\lambda_1(j-i-1)h}}{1-e^{-\lambda_1(j-i-1)h}}.
\end{aligned}
$$

This is true, because

$$
P\left(\tau_{j-1}^{(1)} \mid 0<\tau_{j-1}^{(1)}\le (j-i-1)h\right)=\frac{\lambda_1 e^{-\lambda_1 t}}{1-e^{-\lambda_1(j-i-1)h}},
$$

so

$$
\begin{aligned}
E\left[\tau_{j-1}^{(1)} \mid 0<\tau_{j-1}^{(1)}\le (j-i-1)h\right] &= \int_0^{(j-i-1)h} t\cdot\frac{\lambda_1 e^{-\lambda_1 t}}{1-e^{-\lambda_1(j-i-1)h}}dt\\
&=\frac{1}{\lambda_1}-\frac{(j-i-1)he^{-\lambda_1(j-i-1)h}}{1-e^{-\lambda_1(j-i-1)h}} \qquad (5.1.15)
\end{aligned}
$$

Case 2.b: $\tau_{j-1}^{(1)} > (j-i-1)h$, then $\tau_{j-1}^{(1)} = \tau_i^{(1)} + (j-i-1)h$, then we have the

expectation as following:

$$
\begin{aligned}
&E\Big[\left(b_{i-1}+h-\tau_i^{(1)}\right)\left(\tau_{j-1}^{(1)}+h-\tau_i^{(1)}\right) \mid 0<\tau_i^{(1)}\le h, 0<\tau_j^{(1)}\le h,\\
&\tau_{j-1}^{(1)}>(j-i-1)h, L_{j1}=l\Big]\\
&=E\left[\tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h\right]E\left[b_{i-1}+h\right]-E\left[\left(\tau_i^{(1)}\right)^2 \mid 0<\tau_i^{(1)}\le h\right]\\
&+E\left[(j-i)h-\tau_j^{(1)} \mid 0<\tau_j^{(1)}\le h\right]E\left[b_{i-1}+h-\tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h\right]\\
&=\left(\frac{1}{\lambda_1}-\frac{he^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right)\left(\frac{1}{\lambda_1+\lambda_2}+h\right)-\frac{2}{\lambda_1^2}+\frac{2he^{-\lambda_1 h}}{\lambda_1(1-e^{-\lambda_1 h})}+\frac{h^2e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\\
&+\left[(j-i)h-\frac{1}{\lambda_1}+\frac{he^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right]\left[\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\\
&=-\frac{1}{\lambda_1^2}+\frac{h^2e^{-\lambda_1 h}}{(1-e^{-\lambda_1 h})^2}+\frac{(j-i)h^2}{1-e^{-\lambda_1 h}}-\frac{(j-i)h\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}
\end{aligned}
$$

Combine Case 2.1 and 2.2 with their corresponding probabilities.

$$
\begin{aligned}
&E\left[\left(b_{i-1}+h-\tau_i^{(1)}\right)\left(\tau_{j-1}^{(1)}+h-\tau_i^{(1)}\right) \mid 0<\tau_i^{(1)}\le h, 0<\tau_j^{(1)}\le h, L_{j1}=l\right]\\
&=\Big[\frac{h^2}{\xi_1^2}-\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}-\frac{h^2(j-i-1)e^{-\lambda_1(j-i-1)h}}{\xi_1(1-e^{-\lambda_1(j-i-1)h})}\\
&+\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\frac{(j-i-1)he^{-\lambda_1(j-i-1)h}}{1-e^{-\lambda_1(j-i-1)h}}\Big]\left(1-e^{-\lambda_1(j-i-1)h}\right)\\
&+\left[-\frac{1}{\lambda_1^2}+\frac{h^2e^{-\lambda_1 h}}{(1-e^{-\lambda_1 h})^2}+\frac{(j-i)h^2}{1-e^{-\lambda_1 h}}-\frac{(j-i)h\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\cdot e^{-\lambda_1(j-i-1)h}\\
&=\frac{h^2}{\xi_1^2}-\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}+\left(\frac{\lambda_2 he^{-\lambda_1 h}}{\lambda_1(\lambda_1+\lambda_2)\xi_1}-\frac{1}{\lambda_1^2}\right)e^{-\lambda_1(j-i-1)h}
\end{aligned}
$$

Hence part (ii.b) has the following expression:

$$(ii.b) = 2\rho^2\sigma_1^2\sigma_2^2\sum_{j=1}^{m}\sum_{i=1}^{j-1}E\Big[\nu\Big((t_{i-1}^{(1)},t_i^{(1)})\cap(t_{j-1}^{(2)},t_{L_{j1}}^{(2)})\Big)\cdot\nu\Big((t_{j-1}^{(1)},t_j^{(1)})\cap(t_{i-1}^{(2)},t_{L_{i1}}^{(2)})\Big)\Big]$$

$$\cdot P\left(E_i^{(1)}\right)P\left(E_{L_{j1}}^{(2)}\right)P\left(E_j^{(1)}\right)P\left(E_{L_{i1}}^{(2)}\right)$$

$$= 2\rho^2\sigma_1^2\sigma_2^2\xi_1^2\sum_{j=1}^{m}\sum_{i=1}^{j-1}\left[\frac{h^2}{\xi_1^2}-\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}+\left(\frac{\lambda_2 h e^{-\lambda_1 h}}{\lambda_1(\lambda_1+\lambda_2)\xi_1}-\frac{1}{\lambda_1^2}\right)e^{-\lambda_1(j-i-1)h}\right]$$

$$\cdot\sum_{l=j+1}^{\infty}P(L_{j1}=l)P(L_{i1}\geq j)$$

$$= 2\rho^2\sigma_1^2\sigma_2^2\xi_1^2\sum_{j=1}^{m}\sum_{i=1}^{j-1}\left[\frac{h^2}{\xi_1^2}-\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}+\left(\frac{\lambda_2 h e^{-\lambda_1 h}}{\lambda_1(\lambda_1+\lambda_2)\xi_1}-\frac{1}{\lambda_1^2}\right)e^{-\lambda_1(j-i-1)h}\right]$$

$$\cdot\sum_{l=j+1}^{\infty}P(L_{i1}=l)$$

$$= 2\rho^2\sigma_1^2\sigma_2^2\xi_1^2e^{-\lambda_2 h}\left[\frac{h^2}{\xi_1^2}-\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}\right]\sum_{j=1}^{m}\sum_{i=1}^{j-1}e^{-\lambda_2(j-i)h}$$

$$+ 2\rho^2\sigma_1^2\sigma_2^2\xi_1^2e^{\lambda_1 h-\lambda_2 h}\left[\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}-\frac{1}{\lambda_1^2}\right]\sum_{j=1}^{m}\sum_{i=1}^{j-1}e^{-(\lambda_1+\lambda_2)(j-i)h}$$

$$= 2\rho^2\sigma_1^2\sigma_2^2\xi_1^2e^{-\lambda_2 h}\left[\frac{h^2}{\xi_1^2}-\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}\right]\frac{(m-1)e^{-\lambda_2 h}-me^{-2\lambda_2 h}+e^{-(m+1)\lambda_2 h}}{(1-e^{-\lambda_2 h})^2}$$

$$+ 2\rho^2\sigma_1^2\sigma_2^2\xi_1^2e^{\lambda_1 h-\lambda_2 h}\left[\frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_1}-\frac{1}{\lambda_1^2}\right]$$

$$\cdot\frac{(m-1)e^{-(\lambda_1+\lambda_2)h}-me^{-2(\lambda_1+\lambda_2)h}+e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2}$$

This is true because for any constant $c>0$,

$$\sum_{j=1}^{m}\sum_{i=1}^{j-1}e^{(j-i)c}=\sum_{i=1}^{m-1}(m-i)e^{-ic}=\frac{(m-1)e^{-c}-me^{-2c}+e^{-(m+1)c}}{(1-e^{-c})^2}.$$

Combine all the terms, we get the expression for $\mathrm{Var}(\mathrm{RLL}_1)$ as in equation (2.3.6).

Cov(RLL1, RC)

Conditional on all the observed events, we have

$$\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC}) = \sum_{i=1}^{m}\sum_{j=1}^{m} \mathrm{Cov}\left(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i), R_j Z_j\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \left[E\left(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i) R_j Z_j\right) - E\left(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) E\left(R_j Z_j\right)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i R_j\right) E\left(Z_{L_{i1}} Z_j \mathbb{I}(L_{i1} > i)\right) + \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i Z_j\right) E\left(R_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) \\
&= \sum_{i=1}^{m} E\left(R_i^2\right) E\left(Z_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) + \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i Z_j\right) E\left(R_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right)
\end{aligned}$$

The First term equals zero, since $L_{i1} > i$, then Z_j is independent of $Z_{L_{i1}}$, hence $E\left(Z_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) = 0$. Hence,

$$\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC}) = \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i Z_j\right) E\left(R_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) \\
&= \sum_{i=1}^{m} E\left(R_i Z_i\right) E\left(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) + \sum_{i=1}^{m}\sum_{j=1}^{i-1} E\left(R_i Z_j\right) E\left(R_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) \\
&+ \sum_{j=1}^{m}\sum_{i=1}^{j-1} E\left(R_i Z_j\right) E\left(R_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) \\
&= (i) + (ii) + (iii)
\end{aligned}$$

$$(i) = \sum_{i=1}^{m} E\left(R_i Z_i\right) E\left(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) = 0$$

This is true because if $Z_i = 0$, then it is obvious that the expectation equals zero. Otherwise, $Z_i \neq 0$, then $L_{i1} = i$, and then $\mathbb{I}(L_{i1} > i) = 0$, hence the expectation is zero.

$$(ii) = \sum_{i=1}^{m}\sum_{j=1}^{i-1} E\left(R_i Z_j\right) E\left(R_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) = 0$$

This is true because if $t_j^{(1)} < t_j^{(2)}$, then R_i and Z_j's intervals do not intersect with each other, hence $E\left(R_i Z_j\right) = 0$. If $t_j^{(1)} \geq t_j^{(2)}$, then R_j and $Z_{L_{i1}}$'s intervals do not intersect with each other, hence

$E\left(R_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) = 0$. Both case make part (ii) equal zero. Hence

$\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC}) = (iii)$.

For part (iii), if $i < L_{i1} \leq j-1$, then the interval of Z_j is not going to intersect the interval of R_i, hence $E(R_i Z_j) = 0$, and $(iii) = 0$. For the case $L_{i1} > j-1$, since we need $Z_j \neq 0$, we have $L_{i1} = j$. Then take expectation with respect to the events times, and we have:

$$
\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC}) \\
&= \rho^2 \sigma_1^2 \sigma_2^2 \sum_{j=1}^{m} \sum_{i=1}^{j-1} E\left[\nu\left((t_{i-1}^{(1)}, t_i^{(1)}) \cap (t_{j-1}^{(2)}, t_j^{(2)})\right) \cdot \nu\left((t_{j-1}^{(1)}, t_j^{(1)}) \cap (t_{j-1}^{(2)}, t_j^{(2)})\right)\right] \\
&\cdot P\left(E_i^{(1)}\right) P\left(E_j^{(1)}\right) P\left(L_{i1} = j\right) \\
&= \rho^2 \sigma_1^2 \sigma_2^2 \xi_1^2 \xi_2 \sum_{j=1}^{m} \sum_{i=1}^{j-1} e^{-\lambda_2 (j-i) h} \\
&\cdot E\left[\left(b_{i-1} + h - \tau_i^{(1)} \mid 0 < \tau_i^{(1)} \leq h\right)\left(b_{j-1} + h - a_j \mid 0 < a_j \leq h\right)\right]
\end{aligned}
$$

Case 1: $b_{j-1} \leq (j-i-1)h$, then $P(b_{j-1} \leq (j-i-1)h) = 1 - e^{-(\lambda_1+\lambda_2)(j-i-1)h}$. In this case, $\left(b_{i-1} + h - \tau_i^{(1)} \mid 0 < \tau_i^{(1)}, b_{j-1} \leq (j-i-1)h\right)$ is independent of $(b_{j-1} + h - a_j \mid 0 < a_j \leq h, b_{j-1} \leq (j-i-1)h)$. Similar as equation (5.1.15)

$$
E\left[b_{j-1} \mid 0 < b_{j-1} \leq (j-i-1)h\right] = \frac{1}{\lambda_1 + \lambda_2} - \frac{(j-i-1)h e^{-(\lambda_1+\lambda_2)(j-i-1)h}}{1 - e^{-(\lambda_1+\lambda_2)(j-i-1)h}}.
$$

Hence the expectation inside of the summation signs is:

$$
\begin{aligned}
&E\Big[\Big(b_{i-1}+h-\tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h, b_{j-1}\le (j-i-1)h\Big)\\
&\cdot (b_{j-1}+h-a_j \mid 0<a_j\le h, b_{j-1}\le (j-i-1)h)\Big]\\
&= E\left[\Big(b_{i-1}+h-\tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h\Big)\right]\\
&\cdot E\left[(b_{j-1}+h-a_j \mid 0<a_j\le h, b_{j-1}\le (j-i-1)h)\right]\\
&= \left[\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\left[\frac{1}{\lambda_1+\lambda_2}-\frac{(j-i-1)he^{-(\lambda_1+\lambda_2)(j-i-1)h}}{1-e^{-(\lambda_1+\lambda_2)(j-i-1)h}}\right.\\
&\left.+h-h+\frac{h}{\xi_1\xi_2}-\frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1}+\frac{\lambda_2}{\lambda_1\xi_2}+1\right)\right]\\
&= \left[\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\left[\frac{h}{\xi_1\xi_2}-\frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1}+\frac{\lambda_2}{\lambda_1\xi_2}\right)\right]\\
&-\left[\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\frac{(j-i-1)he^{-(\lambda_1+\lambda_2)(j-i-1)h}}{1-e^{-(\lambda_1+\lambda_2)(j-i-1)h}}
\end{aligned}
$$

Case 2: $b_{j-1} > (j-i-1)h$, then $P(b_{j-1} > (j-i-1)h) = e^{-(\lambda_1+\lambda_2)(j-i-1)h}$, and $b_{j-1} = (j-i-1)h + \tau_i^{(1)}$ in this case. The expectation is:

$$
\begin{aligned}
&E\Big[\Big(b_{i-1}+h-\tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h, b_{j-1}> (j-i-1)h\Big)\\
&\cdot (b_{j-1}+h-a_j \mid 0<a_j\le h, b_{j-1}> (j-i-1)h)\Big]\\
&= E\left[(b_{i-1}+h)\right]E\left[\tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h\right]-E\left[\left(\tau_i^{(1)}\right)^2 \mid 0<\tau_i^{(1)}\le h\right]\\
&+E\left[b_{i-1}+h-\tau_i^{(1)} \mid \tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h\right]E\left[(j-i)h-a_j \mid 0<a_j\le h\right]\\
&= \left(\frac{1}{\lambda_1+\lambda_2}+h\right)\left(\frac{1}{\lambda_1}-\frac{he^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right)-\frac{2}{\lambda_1^2}+\frac{2he^{-\lambda_1 h}}{\lambda_1(1-e^{-\lambda_1 h})}\\
&+\frac{h^2e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}+\left(\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right)\\
&\cdot\left[(j-i-1)h+\frac{h}{\xi_1\xi_2}-\frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1}+\frac{\lambda_2}{\lambda_1\xi_2}+1\right)\right]
\end{aligned}
$$

Combine Case 1 and Case 2 with their corresponding probabilities:

$$
\begin{aligned}
&E\left[\left(b_{i-1}+h-\tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h\right)\left(b_{j-1}+h-a_j \mid 0<a_j\le h\right)\right] \\
&=\left[\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\left[\frac{h}{\xi_1\xi_2}-\frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1}+\frac{\lambda_2}{\lambda_1\xi_2}\right)\right] \\
&\cdot\left(1-e^{-(\lambda_1+\lambda_2)(j-i-1)h}\right) \\
&-\left[\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\frac{(j-i-1)he^{-(\lambda_1+\lambda_2)(j-i-1)h}}{1-e^{-(\lambda_1+\lambda_2)(j-i-1)h}}\left(1-e^{-(\lambda_1+\lambda_2)(j-i-1)h}\right) \\
&+\left[\left(\frac{1}{\lambda_1+\lambda_2}+h\right)\left(\frac{1}{\lambda_1}-\frac{he^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right)-\frac{2}{\lambda_1^2}+\frac{2he^{-\lambda_1 h}}{\lambda_1(1-e^{-\lambda_1 h})}+\frac{h^2e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right] \\
&\cdot e^{-(\lambda_1+\lambda_2)(j-i-1)h}+\left(\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right) \\
&\cdot\left[(j-i-1)h+\frac{h}{\xi_1\xi_2}-\frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1}+\frac{\lambda_2}{\lambda_1\xi_2}+1\right)\right]e^{-(\lambda_1+\lambda_2)(j-i-1)h} \\
&=\left[\frac{h}{1-e^{-\lambda_1 h}}-\frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\left[\frac{h}{\xi_1\xi_2}-\frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1}+\frac{\lambda_2}{\lambda_1\xi_2}\right)\right] \\
&+\left[\frac{h\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\frac{1+e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}-\frac{1}{\lambda_1^2}\left(1+\frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2}\right)\right]e^{-(\lambda_1+\lambda_2)(j-i-1)h}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RC}) = (iii) \\
&= \rho^2\sigma_1^2\sigma_2^2\xi_1^2\xi_2 \left[\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\left[\frac{h}{\xi_1\xi_2} - \frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1} + \frac{\lambda_2}{\lambda_1\xi_2}\right)\right] \\
&\sum_{j=1}^{m}\sum_{i=1}^{j-1} e^{-\lambda_2(j-i)h} \\
&+ \rho^2\sigma_1^2\sigma_2^2\xi_1^2\xi_2 e^{(\lambda_1+\lambda_2)h}\left[\frac{h\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\frac{1+e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}} - \frac{1}{\lambda_1^2}\left(1+\frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2}\right)\right] \\
&\sum_{j=1}^{m}\sum_{i=1}^{j-1} e^{-(\lambda_1+2\lambda_2)(j-i)h} \\
&= \rho^2\sigma_1^2\sigma_2^2\xi_1^2\xi_2 \left[\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\left[\frac{h}{\xi_1\xi_2} - \frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_2\xi_1} + \frac{\lambda_2}{\lambda_1\xi_2}\right)\right] \\
&\cdot \frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{(1-e^{-\lambda_2 h})^2} \\
&+ \rho^2\sigma_1^2\sigma_2^2\xi_1^2\xi_2 e^{(\lambda_1+\lambda_2)h}\left[\frac{h\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\frac{1+e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}} - \frac{1}{\lambda_1^2}\left(1+\frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2}\right)\right] \\
&\cdot \frac{(m-1)e^{-(\lambda_1+2\lambda_2)h} - me^{-2(\lambda_1+2\lambda_2)h} + e^{-(m+1)(\lambda_1+2\lambda_2)h}}{(1-e^{-(\lambda_1+2\lambda_2)h})^2}
\end{aligned}
$$

Cov(RLL1, RLL2)

Conditional on all the event times, we will have:

$$
\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RLL}_2) = \mathrm{Cov}\left(\sum_{i=1}^{m} R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i), \sum_{i=1}^{m} R_{L_{i2}} Z_i \mathbb{I}(L_{i2} > i)\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \mathrm{Cov}\left(R_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i), R_{L_{j2}} Z_j \mathbb{I}(L_{j2} > j)\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i R_{L_{j2}} \mathbb{I}(L_{j2} > j)\right) E\left(Z_j Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right) \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i Z_j\right) E\left(R_{L_{j2}} Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\mathbb{I}(L_{j2} > j)\right)
\end{aligned}
$$

For the first term in the equation above, we discuss the following cases:

(i). While $i = j$, we have $E\left(R_i R_{L_{i2}} \mathbb{I}(L_{i2} > i)\right) E\left(Z_i Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\right)$, since $L_{i2} > i$, R_i is independent of $R_{L_{i2}}$, so the expectation is zero.

(ii). While $i < j$, $L_{j2} > j > i$, then $E\left(R_i R_{L_{j2}} \mathbb{I}(L_{j2} > j)\right) = 0$.

(iii). While $i > j$, $L_{i1} > i > j$, then $E\left(Z_j Z_{L_{j1}} \mathbb{I}(L_{i1} > i)\right) = 0$. Hence the first term is zero, and the expression of the covariance reduces to:

$$\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RLL}_2) = \sum_{i=1}^{m}\sum_{j=1}^{m} E\left(R_i Z_j\right) E\left(R_{L_{j2}} Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\mathbb{I}(L_{j2} > j)\right) \\
&= \sum_{i=1}^{m} E\left(R_i Z_i\right) E\left(R_{L_{i2}} Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\mathbb{I}(L_{i2} > i)\right) \\
&+ \sum_{j=1}^{m}\sum_{i=1}^{j-1} E\left(R_i Z_j\right) E\left(R_{L_{j2}} Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\mathbb{I}(L_{j2} > j)\right) \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{i-1} E\left(R_i Z_j\right) E\left(R_{L_{j2}} Z_{L_{i1}} \mathbb{I}(L_{i1} > i)\mathbb{I}(L_{j2} > j)\right)
\end{aligned}$$

The first term in the above expression equals zero. This is true because when $Z_i = 0$, $E(R_i Z_i) = 0$; while $Z_i \neq 0$, $L_{i1} = i$, hence $\mathbb{I}(L_{i1} > i) = 0$, and then the expectation equals zero. Take expectation with respect to the event time, then,

$$\begin{aligned}
&\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RLL}_2) \\
&= \rho^2 \sigma_1^2 \sigma_2^2 \xi_1 \xi_2 \sum_{j=1}^{m}\sum_{i=1}^{j-1} E\left[(b_{i-1} + h - \tau_i^{(1)})(\tau_{j-1}^{(1)} + h - \tau_j^{(2)}) \mid 0 < \tau_i^{(1)} \leq h, 0 < \tau_j^{(2)} \leq h\right] \\
&\cdot P(L_{i1} > j-1) P(E_{L_{j2}}^{(1)}) \\
&+ \rho^2 \sigma_1^2 \sigma_2^2 \xi_1 \xi_2 \sum_{i=1}^{m}\sum_{j=1}^{i-1} E\left[(b_{j-1} + h - \tau_j^{(2)})(\tau_{i-1}^{(2)} + h - \tau_i^{(1)}) \mid 0 < \tau_j^{(2)} \leq h, 0 < \tau_i^{(1)} \leq h\right] \\
&\cdot P(L_{j2} > i-1) P(E_{L_{i1}}^{(2)}) = (i) + (ii)
\end{aligned}$$

This is true because that,if $L_{i1} \leq j-1$, then Z_j's interval is not going to overlap R_i's interval, hence $E(R_i Z_j) = 0$. If $L_{i1} > j-1$, then L_{i1} has to be j. Likewise, if $L_{j2} \leq i-1$, then R_i's interval is not going to overlap Z_j's interval, hence $E(R_i Z_j) = 0$. If $L_{j2} > i-1$, then L_{j2} has to be i. Notice that (i) and (ii) are symmetric, so we will work on term (i) only in the following.

To find the expectation

$$E\left[(b_{i-1} + h - \tau_i^{(1)})(\tau_{j-1}^{(1)} + h - \tau_j^{(2)}) \mid 0 < \tau_i^{(1)} \leq h, 0 < \tau_j^{(2)} \leq h\right]$$

in the summation sign in the above equation for term (i), consider the following cases:

Case 1: $\tau_{j-1}^{(1)} < (j-i-1)h$, which has probability $P\left(\tau_{j-1}^{(1)} < (j-i-1)h\right) = 1 - e^{-\lambda_1(j-i-1)h}$, then,$(b_{i-1} + h - \tau_i^{(1)} \mid 0 < \tau_i^{(1)} \leq h, \tau_{j-1}^{(1)} < (j-i-1)h)$ is independent of $(\tau_{j-1}^{(1)} + h - \tau_j^{(2)} \mid 0 < \tau_j^{(2)} \leq h, \tau_{j-1}^{(1)} < (j-i-1)h)$. Hence,

$$\begin{aligned}
&E\left[(b_{i-1} + h - \tau_i^{(1)})(\tau_{j-1}^{(1)} + h - \tau_j^{(2)}) \mid 0 < \tau_i^{(1)} \leq h, 0 < \tau_j^{(2)} \leq h, \tau_{j-1}^{(1)} < (j-i-1)h\right] \\
&= E\left[(b_{i-1} + h - \tau_i^{(1)} \mid 0 < \tau_i^{(1)} \leq h, \tau_{j-1}^{(1)} < (j-i-1)h)\right] \\
&\cdot E\left[(\tau_{j-1}^{(1)} + h - \tau_j^{(2)}) \mid 0 < \tau_j^{(2)} \leq h, \tau_{j-1}^{(1)} < (j-i-1)h\right] \\
&= \left[\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2} + \frac{h}{1-e^{-\lambda_2 h}} - \frac{(j-i-1)he^{-\lambda_1(j-i-1)h}}{1-e^{-\lambda_1(j-i-1)h}}\right]
\end{aligned}$$

Case 2: $\tau_{j-1}^{(1)} \geq (j-i-1)h$, which has probability

$$P\left(\tau_{j-1}^{(1)} \geq (j-i-1)h\right) = e^{-\lambda_1(j-i-1)h},$$

then,$\tau_{j-1}^{(1)} = (j-i-1)h + \tau_i^{(1)}$. Hence,

$$
\begin{aligned}
&E\Big[(b_{i-1}+h-\tau_i^{(1)})(\tau_i^{(1)}+(j-i)h-\tau_j^{(2)}) \mid 0<\tau_i^{(1)}\le h, 0<\tau_j^{(2)}\le h,\\
&\tau_{j-1}^{(1)} \ge (j-i-1)h\Big]\\
&= E\left[(b_{i-1}+h)\right] E\left[\tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h\right] - E\left[\left(\tau_i^{(1)}\right)^2 \mid 0<\tau_i^{(1)}\le h\right]\\
&+ E\left[b_{i-1}+h-\tau_i^{(1)} \mid \tau_i^{(1)} \mid 0<\tau_i^{(1)}\le h\right] E\left[(j-i)h-\tau_j^{(2)} \mid 0<\tau_j^{(2)}\le h\right]\\
&= \left(\frac{1}{\lambda_1+\lambda_2}+h\right)\left(\frac{1}{\lambda_1}-\frac{he^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right) - \frac{2}{\lambda_1^2} + \frac{2he^{-\lambda_1 h}}{\lambda_1(1-e^{-\lambda_1 h})} + \frac{h^2e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\\
&+ \left(\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right)\left[(j-i)h - \frac{1}{\lambda_2} - \frac{he^{-\lambda_2 h}}{1-e^{-\lambda_2 h}}\right]
\end{aligned}
$$

Combine Case 1 and Case 2:

$$
\begin{aligned}
&E\left[(b_{i-1}+h-\tau_i^{(1)})(\tau_{j-1}^{(1)}+h-\tau_j^{(2)}) \mid 0<\tau_i^{(1)}\le h, 0<\tau_j^{(2)}\le h\right]\\
&= \left[\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right]\left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2} + \frac{h}{1-e^{-\lambda_2 h}} - \frac{(j-i-1)he^{-\lambda_1(j-i-1)h}}{1-e^{-\lambda_1(j-i-1)h}}\right]\\
&\cdot (1-e^{-\lambda_1(j-i-1)h})\\
&+ \left[\left(\frac{1}{\lambda_1+\lambda_2}+h\right)\left(\frac{1}{\lambda_1}-\frac{he^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right) - \frac{2}{\lambda_1^2} + \frac{2he^{-\lambda_1 h}}{\lambda_1(1-e^{-\lambda_1 h})} + \frac{h^2e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}\right]\\
&\cdot e^{-\lambda_1(j-i-1)h}\\
&+ \left(\frac{h}{1-e^{-\lambda_1 h}} - \frac{\lambda_2}{\lambda_1(\lambda_1+\lambda_2)}\right)\left[(j-i)h - \frac{1}{\lambda_2} - \frac{he^{-\lambda_2 h}}{1-e^{-\lambda_2 h}}\right] e^{-\lambda_1(j-i-1)h}\\
&= \frac{\lambda_1-\lambda_2}{\lambda_1^2(\lambda_1+\lambda_2)} + \frac{h}{\xi_1}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right) - \frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_2} + \frac{h^2}{\xi_1\xi_2} - \left(\frac{1}{\lambda_1^2} + \frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)}\right)\\
&\cdot e^{-\lambda_1(j-i-1)h}\\
&= C_1 - C_2 e^{-\lambda_1(j-i-1)h}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(i) &= \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 \sum_{j=1}^{m}\sum_{i=1}^{j-1} E\Big[(b_{i-1}+h-\tau_i^{(1)})(\tau_{j-1}^{(1)}+h-\tau_j^{(2)}) \\
&\quad \mid 0<\tau_i^{(1)}\le h, 0<\tau_j^{(2)}\le h\Big] P(L_{i1}>j-1)P(E_{L_{j2}}^{(1)}) \\
&= \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 \sum_{j=1}^{m}\sum_{i=1}^{j-1} \left(C_1 - C_2 e^{-\lambda_1(j-i-1)h}\right) \sum_{l=j+1}^{\infty} P(L_{j2}=l)P(L_{i1}\ge j) \\
&= \rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 e^{-\lambda_1 h} \sum_{j=1}^{m}\sum_{i=1}^{j-1} e^{-\lambda_2(j-i)h}\left(C_1 - C_2 e^{-\lambda_1(j-i-1)h}\right) \\
&= C_1\rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 e^{-\lambda_1 h} \sum_{j=1}^{m}\sum_{i=1}^{j-1} e^{-\lambda_2(j-i)h} - C_2\rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 \sum_{j=1}^{m}\sum_{i=1}^{j-1} e^{-(\lambda_1+\lambda_2)(j-i)h} \\
&= C_1\rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 e^{-\lambda_1 h}\frac{(m-1)e^{-\lambda_2 h} - me^{-2\lambda_2 h} + e^{-(m+1)\lambda_2 h}}{(1-e^{-\lambda_2 h})^2} \\
&\quad - C_2\rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 \frac{(m-1)e^{-(\lambda_1+\lambda_2)h} - me^{-2(\lambda_1+\lambda_2)h} + e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2}
\end{aligned}
$$

By symmetry, we can get term (ii):

$$
\begin{aligned}
(ii) &= D_1\rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 e^{-\lambda_2 h}\frac{(m-1)e^{-\lambda_1 h} - me^{-2\lambda_1 h} + e^{-(m+1)\lambda_1 h}}{(1-e^{-\lambda_1 h})^2} \\
&\quad - D_2\rho^2\sigma_1^2\sigma_2^2\xi_1\xi_2 \frac{(m-1)e^{-(\lambda_1+\lambda_2)h} - me^{-2(\lambda_1+\lambda_2)h} + e^{-(m+1)(\lambda_1+\lambda_2)h}}{(1-e^{-(\lambda_1+\lambda_2)h})^2}
\end{aligned}
$$

Combine (i) and (ii), we get $\mathrm{Cov}(\mathrm{RLL}_1, \mathrm{RLL}_2)$ as in equation (2.3.12), where

$$
\begin{aligned}
C_1 &= \frac{h}{\xi_1}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right) + \frac{h^2}{\xi_1\xi_2} + \frac{\lambda_1-\lambda_2}{\lambda_1^2(\lambda_1+\lambda_2)} - \frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)\xi_2} \\
C_2 &= \frac{1}{\lambda_1^2} + \frac{\lambda_2 h}{\lambda_1(\lambda_1+\lambda_2)} \\
D_1 &= \frac{h}{\xi_2}\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right) + \frac{h^2}{\xi_1\xi_2} + \frac{\lambda_2-\lambda_1}{\lambda_2^2(\lambda_1+\lambda_2)} - \frac{\lambda_1 h}{\lambda_2(\lambda_1+\lambda_2)\xi_1} \\
D_2 &= \frac{1}{\lambda_2^2} + \frac{\lambda_1 h}{\lambda_2(\lambda_1+\lambda_2)}.
\end{aligned}
$$

5.1.7 Noise terms for $\mathrm{Var}(\mathrm{RLL}_{\mathrm{noise}})$

For the reason of simplicity, we calculate $E\sum_{i=1}^{m}\sum_{j=1}^{m}D_{ij}$ by changing it to an equivalent symmetric way. By using the earlier symmetric form for RLL in the variance calculation (without noise), we represent RLL$*$ as:

$$\begin{aligned}
\mathrm{RLL}^* &= \sum_{i=1}^{m}\left[R_i + u_i^{(1)} - u_{i-1}^{(1)}\right]\left[Z_{L_{i1}} + u_{L_{i1}}^{(2)} - u_i^{(2)}\right]\mathbb{I}(L_{i1} > i) \\
&+ \sum_{i=1}^{m}\left[R_{L_{i2}} + u_{L_{i2}}^{(1)} - u_i^{(1)}\right]\left[Z_i + u_i^{(2)} - u_{i-1}^{(2)}\right]\mathbb{I}(L_{i2} > i) \\
&+ \sum_{i=1}^{m}\left[R_i + u_i^{(1)} - u_{i-1}^{(1)}\right]\left[Z_i + u_i^{(2)} - u_{i-1}^{(2)}\right] \\
&= \mathrm{RLL}_1^* + \mathrm{RLL}_2^* + \mathrm{RV}^*
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{VarRLL}^* &= \mathrm{Var}(\mathrm{RLL}_1^*) + \mathrm{Var}(\mathrm{RLL}_2^*) + \mathrm{Var}(\mathrm{RV}^*) \\
&+ 2\mathrm{Cor}(\mathrm{RLL}_1^*, \mathrm{RLL}_2^*) + 2\mathrm{Cor}(\mathrm{RLL}_1^*, \mathrm{RV}^*) + 2\mathrm{Cor}(\mathrm{RLL}_2^*, \mathrm{RV}^*)
\end{aligned}$$

The noise term corresponding to $\sum_{i=1}^{m}\sum_{j=1}^{m}D_{ij}$ comes from the above terms form with cross products only including u_i's. The part in $\mathrm{Var}(\mathrm{RLL}_1^*)$ is:

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_i^{(1)} - u_{i-1}^{(1)}\right)\left(u_j^{(1)} - u_{j-1}^{(1)}\right)\left(u_{L_{i1}}^{(2)} - u_i^{(2)}\right)\left(u_{L_{j1}}^{(2)} - u_j^{(2)}\right)\mathbb{I}(L_{i1} > i)\mathbb{I}(L_{j1} > j)$$

Take expectation:

$$\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\left(u_j^{(1)}-u_{j-1}^{(1)}\right)\left(u_{L_{i1}}^{(2)}-u_i^{(2)}\right)\left(u_{L_{j1}}^{(2)}-u_j^{(2)}\right)$$

$$\cdot\,\mathbb{I}(L_{i1}>i)\mathbb{I}(L_{j1}>j)$$

$$=\mathrm{E}\left[\sum_{i=1}^{m}\left(u_i^{(1)}-u_{i-1}^{(1)}\right)^2\left(u_{L_{i1}}^{(2)}-u_i^{(2)}\right)^2\mathbb{I}(L_{i1}>i)\right]$$

$$+2\mathrm{E}\Big[\sum_{j=1}^{m}\left(u_{j-1}^{(1)}-u_{j-2}^{(1)}\right)\left(u_j^{(1)}-u_{j-1}^{(1)}\right)\left(u_{L_{(j-1)1}}^{(2)}-u_i^{(2)}\right)\left(u_{L_{j1}}^{(2)}-u_j^{(2)}\right)$$

$$\cdot\,\mathbb{I}(L_{(j-1)1}>j-1)\mathbb{I}(L_{j1}>j)\Big]$$

$$=\mathrm{Part(a)}+\mathrm{Part(b)}$$

Where

$$\mathrm{Part(a)}=\sum_{i=1}^{m}\mathrm{P}(\mathrm{E_i^{(1)}})\mathrm{E}\left[\left(u_i^{(1)}-u_{i-1}^{(1)}\right)^2|E_i^{(1)}\right]$$

$$\cdot\,\mathrm{P}(L_{i1}>i)\mathrm{E}\left[\left(u_{L_{i1}}^{(2)}-u_i^{(2)}\right)^2|L_{i1}>1\right]$$

$$=m\xi_1 2\phi_1^2 e^{-\lambda_2 h}2\phi_2^2=4\frac{1}{h}\xi_1 e^{-\lambda_2 h}\phi_1^2\phi_2^2$$

and

$$\text{Part(b)} = 2\sum_{j=2}^{m} \mathrm{E}\left[\left(u_{j-1}^{(1)} - u_{j-2}^{(1)}\right)\left(u_{j}^{(1)} - u_{j-1}^{(1)}\right)\right]$$

$$\cdot \mathrm{E}\left[\left(u_{L_{(j-1)1}}^{(2)} - u_{i}^{(2)}\right)\left(u_{L_{j1}}^{(2)} - u_{j}^{(2)}\right)\mathbb{I}(L_{(j-1)1} > j-1)\mathbb{I}(L_{j1} > j)\right]$$

$$= 2\sum_{j=2}^{m} \mathrm{P}(\mathrm{E}_{\mathrm{j-1}}^{(1)})\mathrm{P}(\mathrm{E}_{\mathrm{j}}^{(1)})\mathrm{E}\left[\left(u_{j-1}^{(1)} - u_{j-2}^{(1)}\right)\left(u_{j}^{(1)} - u_{j-1}^{(1)}\right)|E_{j-1}^{(2)} \cap E_{j}^{(1)}\right]$$

$$\cdot \mathrm{P}(\mathrm{L_{(j-1)1} = j\ \&\ L_{j1} > j})\mathrm{E}\left[\left(u_{j}^{(2)} - u_{j-1}^{(2)}\right)\left(u_{L_{j1}}^{(2)} - u_{j}^{(2)}\right)|L_{(j-1)1} = j\ \&\ L_{j1} > j\right]$$

$$+ 2\sum_{j=2}^{m} \mathrm{P}(\mathrm{E}_{\mathrm{j-1}}^{(1)})\mathrm{P}(\mathrm{E}_{\mathrm{j}}^{(1)})\mathrm{E}\left[\left(u_{j-1}^{(1)} - u_{j-2}^{(1)}\right)\left(u_{j}^{(1)} - u_{j-1}^{(1)}\right)|E_{j-1}^{(2)} \cap E_{j}^{(1)}\right]$$

$$\cdot \mathrm{P}(\mathrm{L_{(j-1)1} > j\ \&\ L_{j1} > j})$$

$$\cdot \mathrm{E}\left[\left(u_{L_{(j-1)1}}^{(2)} - u_{j-1}^{(2)}\right)\left(u_{L_{j1}}^{(2)} - u_{j}^{(2)}\right)|L_{(j-1)1} > j\ \&\ L_{j1} > j\right]$$

$$= 2(m-1)\xi_1^2(-\phi_1^2)\xi_2 e^{-\lambda_2(j-j+1)h}e^{-\lambda_2 h}(-\phi_2^2) + 2(m-1)\xi_1^2(-\phi_1^2)e^{-2\lambda_2 h}\phi_2^2$$

$$= -2(\frac{1}{h} - 1)\xi_1^2 e^{-3\lambda_2 h}\phi_1^2\phi_2^2$$

This is true because $\mathrm{P(L_{(j-1)1} > j\ \&\ L_{j1} > j) = P(L_{(j-1)1} > j)} = e^{-2\lambda_2 h}$

i.e.

$$\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_{i}^{(1)} - u_{i-1}^{(1)}\right)\left(u_{j}^{(1)} - u_{j-1}^{(1)}\right)\left(u_{L_{i1}}^{(2)} - u_{i}^{(2)}\right)\left(u_{L_{j1}}^{(2)} - u_{j}^{(2)}\right)$$

$$\cdot \mathbb{I}(L_{i1} > i)\mathbb{I}(L_{j1} > j) = 4\frac{1}{h}\xi_1 e^{-\lambda_2 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h} - 1)\xi_1^2 e^{-3\lambda_2 h}\phi_1^2\phi_2^2$$

By the symmetry property, the expectation of the part in $\mathrm{Var(RLL_2^*)}$ is

$$\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_{L_{i2}}^{(1)} - u_{i}^{(1)}\right)\left(u_{L_{j2}}^{(1)} - u_{j}^{(1)}\right)\left(u_{i}^{(2)} - u_{i-1}^{(2)}\right)\left(u_{j}^{(2)} - u_{j-1}^{(2)}\right)$$

$$\cdot \mathbb{I}(L_{i2} > i)\mathbb{I}(L_{j2} > j) = 4\frac{1}{h}\xi_2 e^{-\lambda_1 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h} - 1)\xi_2^2 e^{-3\lambda_1 h}\phi_1^2\phi_2^2$$

Next, the expectation of the part in $\mathrm{Var}(\mathrm{RV}^*)$ is

$$\begin{aligned}
&\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\left(u_j^{(1)}-u_{j-1}^{(1)}\right)\left(u_i^{(2)}-u_{i-1}^{(2)}\right)\left(u_j^{(2)}-u_{j-1}^{(2)}\right)\\
&=\sum_{i=1}^{m}\mathrm{P}(\mathrm{E}_{\mathrm{i}}^{(1)})\mathrm{P}(\mathrm{E}_{\mathrm{i}}^{(2)})\mathrm{E}\left[\left(u_i^{(1)}-u_{i-1}^{(1)}\right)^2\left(u_i^{(2)}-u_{i-1}^{(2)}\right)^2|E_i^{(1)}\cap E_i^{(2)}\right]\\
&+2\sum_{j=2}^{m}\mathrm{P}(\mathrm{E}_{\mathrm{j}}^{(1)})\mathrm{P}(\mathrm{E}_{\mathrm{j}}^{(2)})\mathrm{P}(\mathrm{E}_{\mathrm{j-1}}^{(1)})\mathrm{P}(\mathrm{E}_{\mathrm{j-1}}^{(2)})\\
&\cdot\mathrm{E}\Big[\left(u_{j-1}^{(1)}-u_{j-2}^{(1)}\right)\left(u_j^{(1)}-u_{j-1}^{(1)}\right)\left(u_{j-1}^{(2)}-u_{j-1}^{(2)}\right)\left(u_j^{(2)}-u_{j-1}^{(2)}\right)\\
&|E_{j-1}^{(1)}\cap E_j^{(1)}\cap E_{(j-1)}^{(2)}\cap E_j^{(2)}\Big]\\
&=m\xi_1\xi_2(2\phi_1^2)(2\phi_2^2)+2(m-1)\xi_2^2\xi_2^2(-\phi_1^2)(-\phi_2^2)\\
&=4\frac{1}{h}\xi_1\xi_2\phi_1^2\phi_2^2+2(\frac{1}{h}-1)\xi_1^2\xi_2^2\phi_1^2\phi_2^2
\end{aligned}$$

Next, find the expectation of the part in $\mathrm{Cor}(\mathrm{RLL}_1^*,\mathrm{RLL}_2^*)$:

$$\begin{aligned}
&\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\left(u_{L_{j2}}^{(1)}-u_j^{(1)}\right)\left(u_{L_{i1}}^{(2)}-u_i^{(2)}\right)\left(u_j^{(2)}-u_{j-1}^{(2)}\right)\\
&\cdot\mathbb{I}(L_{i1}>i)\mathbb{I}(L_{j2}>j)
\end{aligned}$$

Discuss the following cases:

Case A: $i=j$,

$$\mathrm{E}\left[\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\left(u_{L_{i2}}^{(1)}-u_i^{(1)}\right)|E_i^{(1)}\cap L_{i2}>i\right]=0,$$

hence the whole expectation is zero.

Case B: $i<j$,

$$\mathrm{E}\left[\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\left(u_{L_{j2}}^{(1)}-u_j^{(1)}\right)|E_i^{(1)}\cap L_{j2}>j\right]=0,$$

hence the whole expectation is zero.

Case C: $i > j$,

$$\mathrm{E}\left[\left(u_j^{(2)} - u_{j-1}^{(2)}\right)\left(u_{L_{i1}}^{(2)} - u_i^{(1)}\right)|E_j^{(2)} \cap L_{i1} > i\right] = 0,$$

hence the whole expectation is zero.

Hence, the expectation of the part in $\mathrm{Cor}(\mathrm{RLL}_1^*, \mathrm{RLL}_2^*)$ is zero.

Next, find the expectation of the part in $\mathrm{Cor}(\mathrm{RLL}_1^*, \mathrm{RV}^*)$:

$$\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_i^{(1)} - u_{i-1}^{(1)}\right)\left(u_j^{(1)} - u_{j-1}^{(1)}\right)\left(u_{L_{i1}}^{(2)} - u_i^{(2)}\right)\left(u_j^{(2)} - u_{j-1}^{(2)}\right)\mathbb{I}(L_{i1} > i)$$

Discuss the following cases:

Case A: $i = j$,

$$\mathrm{E}\left[\left(u_i^{(2)} - u_{i-1}^{(2)}\right)\left(u_{L_{i1}}^{(2)} - u_i^{(2)}\right)|E_i^{(2)} \cap L_{i1} > i\right] = 0,$$

hence the whole expectation is zero.

Case B: $j < i$, when $j < i - 1$,

$$\mathrm{E}\left[\left(u_i^{(1)} - u_{i-1}^{(1)}\right)\left(u_j^{(1)} - u_{j-1}^{(1)}\right)|E_i^{(1)} \cap E_j^{(1)}\right] = 0,$$

and when $j = i - 1$,

$$\mathrm{E}\left[\left(u_{i-1}^{(2)} - u_{i-2}^{(2)}\right)\left(u_{L_{i1}}^{(2)} - u_i^{(2)}\right)|E_{i-1}^{(2)} \cap L_{i1} > i\right] = 0,$$

hence the whole expectation is zero when $j < i$.

Case C: $j > i$, when $j > i + 1$,

$$\mathrm{E}\left[\left(u_i^{(1)} - u_{i-1}^{(1)}\right)\left(u_j^{(1)} - u_{j-1}^{(1)}\right)|E_i^{(1)} \cap E_j^{(1)}\right] = 0,$$

and when $j = i + 1$,

$$\begin{aligned}
&\mathrm{E}\sum_{i=1}^{m-1}\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\left(u_{i+1}^{(1)}-u_i^{(1)}\right)\left(u_{L_{i1}}^{(2)}-u_i^{(2)}\right)\left(u_{i+1}^{(2)}-u_i^{(2)}\right)\mathbb{I}(L_{i1}>i)\\
&=\sum_{i=1}^{m-1}\mathrm{P}(\mathrm{E}_{\mathrm{i}}^{(1)})\mathrm{P}(\mathrm{E}_{\mathrm{i+1}}^{(1)})\mathrm{P}(\mathrm{L_{i1}=i+1})\\
&\cdot\mathrm{E}\left[\left(-u_i^{(1)2}\right)\left(u_{i+1}^{(2)}-u_i^{(2)}\right)^2|E_i^{(1)}\cap E_{i+1}^{(1)}\cap L_{i1}=i+1\right]\\
&+\sum_{i=1}^{m-1}\mathrm{P}(\mathrm{E}_{\mathrm{i}}^{(1)})\mathrm{P}(\mathrm{E}_{\mathrm{i+1}}^{(1)})\mathrm{P}(\mathrm{L_{i1}>i+1})\mathrm{E}\left[\left(-u_i^{(1)2}\right)u_i^{(2)2}|E_i^{(1)}\cap E_{i+1}^{(1)}\cap L_{i1}>i+1\right]\\
&=(m-1)\xi_1^2\xi_2e^{-\lambda_2h}(-\phi_1^2)(2\phi_2^2)+(m-1)\xi_1^2e^{-2\lambda_2h}(-\phi_1^2)\phi_2^2\\
&=-2(\frac{1}{h}-1)\xi_1^2\xi_2e^{-\lambda_2h}\phi_1^2\phi_2^2-(\frac{1}{h}-1)\xi_1^2e^{-2\lambda_2h}\phi_1^2\phi_2^2
\end{aligned}$$

Hence the expectation of the part in $\mathrm{Cor}(\mathrm{RLL}_1^*,\mathrm{RV}^*)$ is:

$$\begin{aligned}
&\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_i^{(1)}-u_{i-1}^{(1)}\right)\left(u_j^{(1)}-u_{j-1}^{(1)}\right)\left(u_{L_{i1}}^{(2)}-u_i^{(2)}\right)\left(u_j^{(2)}-u_{j-1}^{(2)}\right)\mathbb{I}(L_{i1}>i)\\
&=-2(\frac{1}{h}-1)\xi_1^2\xi_2e^{-\lambda_2h}\phi_1^2\phi_2^2-(\frac{1}{h}-1)\xi_1^2e^{-2\lambda_2h}\phi_1^2\phi_2^2
\end{aligned}$$

Lastly, by symmetry property, the expectation of the part in $\mathrm{Cor}(\mathrm{RLL}_2^*,\mathrm{RV}^*)$ is:

$$\begin{aligned}
&\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_{L_{i2}}^{(1)}-u_i^{(1)}\right)\left(u_j^{(1)}-u_{j-1}^{(1)}\right)\left(u_i^{(2)}-u_{i-1}^{(2)}\right)\left(u_j^{(2)}-u_{j-1}^{(2)}\right)\mathbb{I}(L_{i2}>i)\\
&=-2(\frac{1}{h}-1)\xi_2^2\xi_1e^{-\lambda_1h}\phi_1^2\phi_2^2-(\frac{1}{h}-1)\xi_2^2e^{-2\lambda_1h}\phi_1^2\phi_2^2
\end{aligned}$$

Collecting all the terms, we have

$$
\begin{aligned}
&\mathrm{E}\sum_{i=1}^{m}\sum_{j=1}^{m} D_{ij} \\
&= 4\frac{1}{h}\xi_1 e^{-\lambda_2 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_1^2 e^{-3\lambda_2 h}\phi_1^2\phi_2^2 \\
&+ 4\frac{1}{h}\xi_2 e^{-\lambda_1 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_2^2 e^{-3\lambda_1 h}\phi_1^2\phi_2^2 \\
&+ 4\frac{1}{h}\xi_1\xi_2\phi_1^2\phi_2^2 + 2(\frac{1}{h}-1)\xi_1^2\xi_2^2\phi_1^2\phi_2^2 \\
&- 4(\frac{1}{h}-1)\xi_1^2\xi_2 e^{-\lambda_2 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_1^2 e^{-2\lambda_2 h}\phi_1^2\phi_2^2 \\
&- 4(\frac{1}{h}-1)\xi_2^2\xi_1 e^{-\lambda_1 h}\phi_1^2\phi_2^2 - 2(\frac{1}{h}-1)\xi_2^2 e^{-2\lambda_1 h}\phi_1^2\phi_2^2
\end{aligned}
$$

5.2 Data pre-processing and summary

The Intel(INTC), Microsoft (MSFT), and Amgen (AMGN) data were Nasdaq trades from 1990. They all contain 252 trading days, and the days coincided. The following is a table of trading days:

The following are parts of the original DELL, APPLE and MICROSOFT data sets:

For all these data sets, observation 250 (27DEC1990)had a really late start time, which was 10:01:59. All the other observations were all before 09:45:00, hence we remove day 250 and trim all the data sets to the time interval (09:45:00, 16:00:00) during the remaining 251 trading days.

The following figures show the daily trading frequencies and daily price changes for all these three stocks.

Trading Days for Amgen, Intel and Microsoft in 1990 **1**
23:32 Monday, August 4, 2008

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
1	.	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1
2	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	.	1	1	1	1
3	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1
4	.	1	1	1	1	1	.	.	1	1	1	1	.	.	.	1	1	1	1	1	.	.	1
5	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1
6	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.
7	.	1	1	.	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1
8	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1
9	.	.	.	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.
10	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1
11	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	.	1
12	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.	1	1	1	1	1	.	.

	24	25	26	27	28	29	30	31
	N	N	N	N	N	N	N	N
1	1	1	1	.	.	1	1	1
2	.	.	1	1	1	.	.	.
3	.	.	1	1	1	1	1	.
4	.	1	1	1	.	.	1	.
5	1	1	.	.	.	1	1	1
6	.	1	1	1	1	1	.	.
7	1	1	1	1	.	.	1	1
8	1	.	.	1	1	1	1	1
9	1	1	1	1	1	.	.	.
10	1	1	1	.	.	1	1	1
11	.	.	1	1	1	1	1	.
12	1	.	1	1	1	.	.	1

Figure 5.4: Dell has the same trading days

Original Microsoft Data 13:22 Sunday, August 3, 2008 **1**

Obs	symbol	date	time	price	DateTime
1	MSFT	02JAN1990	9:30:05	87.250	02JAN90:09:30:05
2	MSFT	02JAN1990	9:30:36	86.875	02JAN90:09:30:36
3	MSFT	02JAN1990	9:30:49	87.000	02JAN90:09:30:49
4	MSFT	02JAN1990	9:31:09	87.250	02JAN90:09:31:09
5	MSFT	02JAN1990	9:31:17	86.875	02JAN90:09:31:17
6	MSFT	02JAN1990	9:31:29	87.000	02JAN90:09:31:29
7	MSFT	02JAN1990	9:31:31	87.250	02JAN90:09:31:31
8	MSFT	02JAN1990	9:31:33	86.875	02JAN90:09:31:33
9	MSFT	02JAN1990	9:31:49	87.000	02JAN90:09:31:49
10	MSFT	02JAN1990	9:31:54	86.875	02JAN90:09:31:54
11	MSFT	02JAN1990	9:31:55	86.875	02JAN90:09:31:55
12	MSFT	02JAN1990	9:31:56	86.875	02JAN90:09:31:56
13	MSFT	02JAN1990	9:31:56	86.875	02JAN90:09:31:56
14	MSFT	02JAN1990	9:31:58	87.250	02JAN90:09:31:58
15	MSFT	02JAN1990	9:31:58	87.250	02JAN90:09:31:58
16	MSFT	02JAN1990	9:32:27	87.250	02JAN90:09:32:27
17	MSFT	02JAN1990	9:32:31	87.250	02JAN90:09:32:31
18	MSFT	02JAN1990	9:32:45	87.000	02JAN90:09:32:45
19	MSFT	02JAN1990	9:32:54	87.000	02JAN90:09:32:54
20	MSFT	02JAN1990	9:33:10	87.250	02JAN90:09:33:10
21	MSFT	02JAN1990	9:33:10	87.250	02JAN90:09:33:10
22	MSFT	02JAN1990	9:33:21	86.875	02JAN90:09:33:21
23	MSFT	02JAN1990	9:33:23	86.875	02JAN90:09:33:23
24	MSFT	02JAN1990	9:33:34	87.000	02JAN90:09:33:34
25	MSFT	02JAN1990	9:34:03	86.875	02JAN90:09:34:03
26	MSFT	02JAN1990	9:34:26	86.750	02JAN90:09:34:26
27	MSFT	02JAN1990	9:34:39	86.750	02JAN90:09:34:39
28	MSFT	02JAN1990	9:34:54	86.750	02JAN90:09:34:54
29	MSFT	02JAN1990	9:35:11	86.750	02JAN90:09:35:11
30	MSFT	02JAN1990	9:35:46	86.500	02JAN90:09:35:46
31	MSFT	02JAN1990	9:35:48	86.500	02JAN90:09:35:48
32	MSFT	02JAN1990	9:36:03	86.500	02JAN90:09:36:03
33	MSFT	02JAN1990	9:36:12	86.500	02JAN90:09:36:12
34	MSFT	02JAN1990	9:36:22	86.500	02JAN90:09:36:22
35	MSFT	02JAN1990	9:36:36	86.500	02JAN90:09:36:36
36	MSFT	02JAN1990	9:36:42	86.500	02JAN90:09:36:42
37	MSFT	02JAN1990	9:36:51	86.500	02JAN90:09:36:51
38	MSFT	02JAN1990	9:36:52	86.500	02JAN90:09:36:52

Figure 5.5: Parts of the Microsoft Original Data

Original Intel Data 23:32 Monday, August 4, 2008 **1**

Obs	symbol	date	time	price	DateTime
1	INTC	02JAN1990	9:30:08	34.500	02JAN90:09:30:08
2	INTC	02JAN1990	9:30:15	34.250	02JAN90:09:30:15
3	INTC	02JAN1990	9:30:18	34.250	02JAN90:09:30:18
4	INTC	02JAN1990	9:30:20	34.500	02JAN90:09:30:20
5	INTC	02JAN1990	9:30:22	34.500	02JAN90:09:30:22
6	INTC	02JAN1990	9:30:37	34.250	02JAN90:09:30:37
7	INTC	02JAN1990	9:30:38	34.250	02JAN90:09:30:38
8	INTC	02JAN1990	9:31:33	34.500	02JAN90:09:31:33
9	INTC	02JAN1990	9:32:28	34.250	02JAN90:09:32:28
10	INTC	02JAN1990	9:32:29	34.250	02JAN90:09:32:29
11	INTC	02JAN1990	9:33:00	34.250	02JAN90:09:33:00
12	INTC	02JAN1990	9:33:10	34.500	02JAN90:09:33:10
13	INTC	02JAN1990	9:33:14	34.250	02JAN90:09:33:14
14	INTC	02JAN1990	9:33:15	34.250	02JAN90:09:33:15
15	INTC	02JAN1990	9:33:35	34.250	02JAN90:09:33:35
16	INTC	02JAN1990	9:33:46	34.250	02JAN90:09:33:46
17	INTC	02JAN1990	9:34:00	34.500	02JAN90:09:34:00
18	INTC	02JAN1990	9:34:08	34.250	02JAN90:09:34:08
19	INTC	02JAN1990	9:34:25	34.500	02JAN90:09:34:25
20	INTC	02JAN1990	9:35:01	34.250	02JAN90:09:35:01
21	INTC	02JAN1990	9:36:52	34.500	02JAN90:09:36:52
22	INTC	02JAN1990	9:37:00	34.250	02JAN90:09:37:00
23	INTC	02JAN1990	9:38:56	34.500	02JAN90:09:38:56
24	INTC	02JAN1990	9:39:10	34.250	02JAN90:09:39:10
25	INTC	02JAN1990	9:41:18	34.250	02JAN90:09:41:18
26	INTC	02JAN1990	9:41:27	34.250	02JAN90:09:41:27
27	INTC	02JAN1990	9:41:27	34.250	02JAN90:09:41:27
28	INTC	02JAN1990	9:41:33	34.250	02JAN90:09:41:33
29	INTC	02JAN1990	9:41:38	34.250	02JAN90:09:41:38
30	INTC	02JAN1990	9:41:41	34.250	02JAN90:09:41:41
31	INTC	02JAN1990	9:41:43	34.250	02JAN90:09:41:43
32	INTC	02JAN1990	9:43:53	34.500	02JAN90:09:43:53
33	INTC	02JAN1990	9:44:17	34.250	02JAN90:09:44:17
34	INTC	02JAN1990	9:44:33	34.125	02JAN90:09:44:33
35	INTC	02JAN1990	9:44:57	34.250	02JAN90:09:44:57
36	INTC	02JAN1990	9:46:40	34.125	02JAN90:09:46:40
37	INTC	02JAN1990	9:47:00	34.125	02JAN90:09:47:00
38	INTC	02JAN1990	9:49:38	34.250	02JAN90:09:49:38

Figure 5.6: Parts of the Intel Original Data

Original Amgen Data 23:32 Monday, August 4, 2008 **1**

Obs	symbol	date	time	price	DateTime
1	AMGN	02JAN1990	9:30:17	49.25	02JAN90:09:30:17
2	AMGN	02JAN1990	9:30:18	49.25	02JAN90:09:30:18
3	AMGN	02JAN1990	9:30:22	49.25	02JAN90:09:30:22
4	AMGN	02JAN1990	9:30:22	49.25	02JAN90:09:30:22
5	AMGN	02JAN1990	9:30:23	48.75	02JAN90:09:30:23
6	AMGN	02JAN1990	9:30:28	48.75	02JAN90:09:30:28
7	AMGN	02JAN1990	9:30:30	49.25	02JAN90:09:30:30
8	AMGN	02JAN1990	9:31:53	49.25	02JAN90:09:31:53
9	AMGN	02JAN1990	9:32:23	49.25	02JAN90:09:32:23
10	AMGN	02JAN1990	9:32:23	49.25	02JAN90:09:32:23
11	AMGN	02JAN1990	9:32:24	49.25	02JAN90:09:32:24
12	AMGN	02JAN1990	9:32:24	49.25	02JAN90:09:32:24
13	AMGN	02JAN1990	9:32:33	49.00	02JAN90:09:32:33
14	AMGN	02JAN1990	9:33:15	49.25	02JAN90:09:33:15
15	AMGN	02JAN1990	9:33:26	49.25	02JAN90:09:33:26
16	AMGN	02JAN1990	9:37:50	49.25	02JAN90:09:37:50
17	AMGN	02JAN1990	9:38:18	49.25	02JAN90:09:38:18
18	AMGN	02JAN1990	9:39:52	49.25	02JAN90:09:39:52
19	AMGN	02JAN1990	9:40:09	48.75	02JAN90:09:40:09
20	AMGN	02JAN1990	9:40:24	48.75	02JAN90:09:40:24
21	AMGN	02JAN1990	9:41:30	49.25	02JAN90:09:41:30
22	AMGN	02JAN1990	9:41:59	49.25	02JAN90:09:41:59
23	AMGN	02JAN1990	9:42:39	48.75	02JAN90:09:42:39
24	AMGN	02JAN1990	9:44:59	49.00	02JAN90:09:44:59
25	AMGN	02JAN1990	9:44:59	49.00	02JAN90:09:44:59
26	AMGN	02JAN1990	9:45:27	48.50	02JAN90:09:45:27
27	AMGN	02JAN1990	9:46:17	48.50	02JAN90:09:46:17
28	AMGN	02JAN1990	9:46:22	49.00	02JAN90:09:46:22
29	AMGN	02JAN1990	9:46:31	48.50	02JAN90:09:46:31
30	AMGN	02JAN1990	9:46:42	48.75	02JAN90:09:46:42
31	AMGN	02JAN1990	9:46:59	48.75	02JAN90:09:46:59
32	AMGN	02JAN1990	9:47:12	48.50	02JAN90:09:47:12
33	AMGN	02JAN1990	9:47:21	48.50	02JAN90:09:47:21
34	AMGN	02JAN1990	9:47:31	48.50	02JAN90:09:47:31
35	AMGN	02JAN1990	9:47:45	48.50	02JAN90:09:47:45
36	AMGN	02JAN1990	9:47:50	48.50	02JAN90:09:47:50
37	AMGN	02JAN1990	9:47:54	48.75	02JAN90:09:47:54
38	AMGN	02JAN1990	9:48:02	48.75	02JAN90:09:48:02

Figure 5.7: Parts of the Amgen Original Data

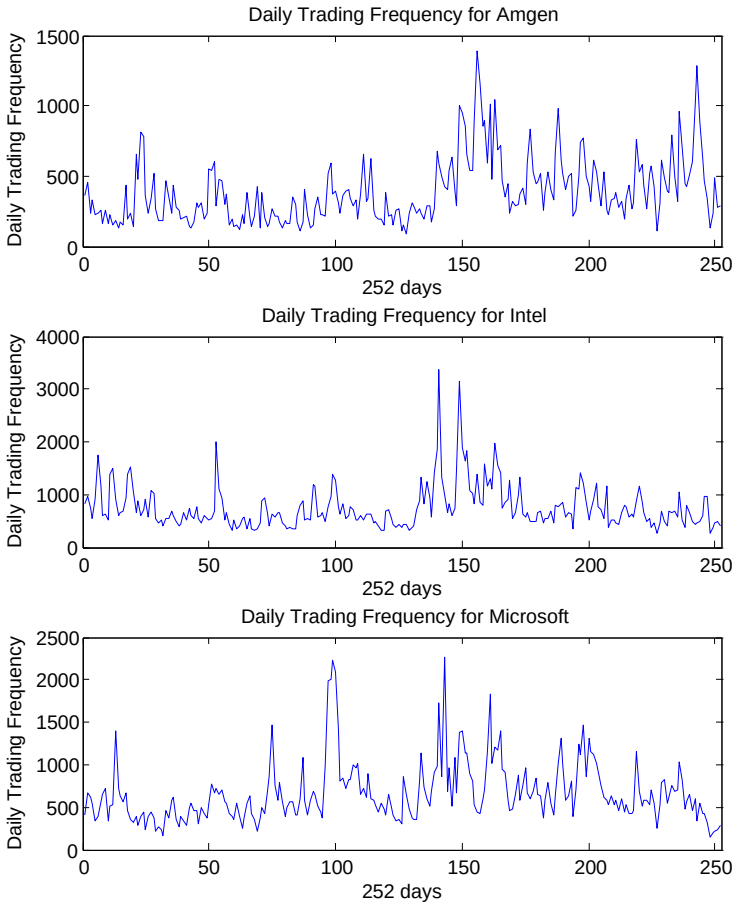


Figure 5.8: Daily Trading Frequency for Three Stocks

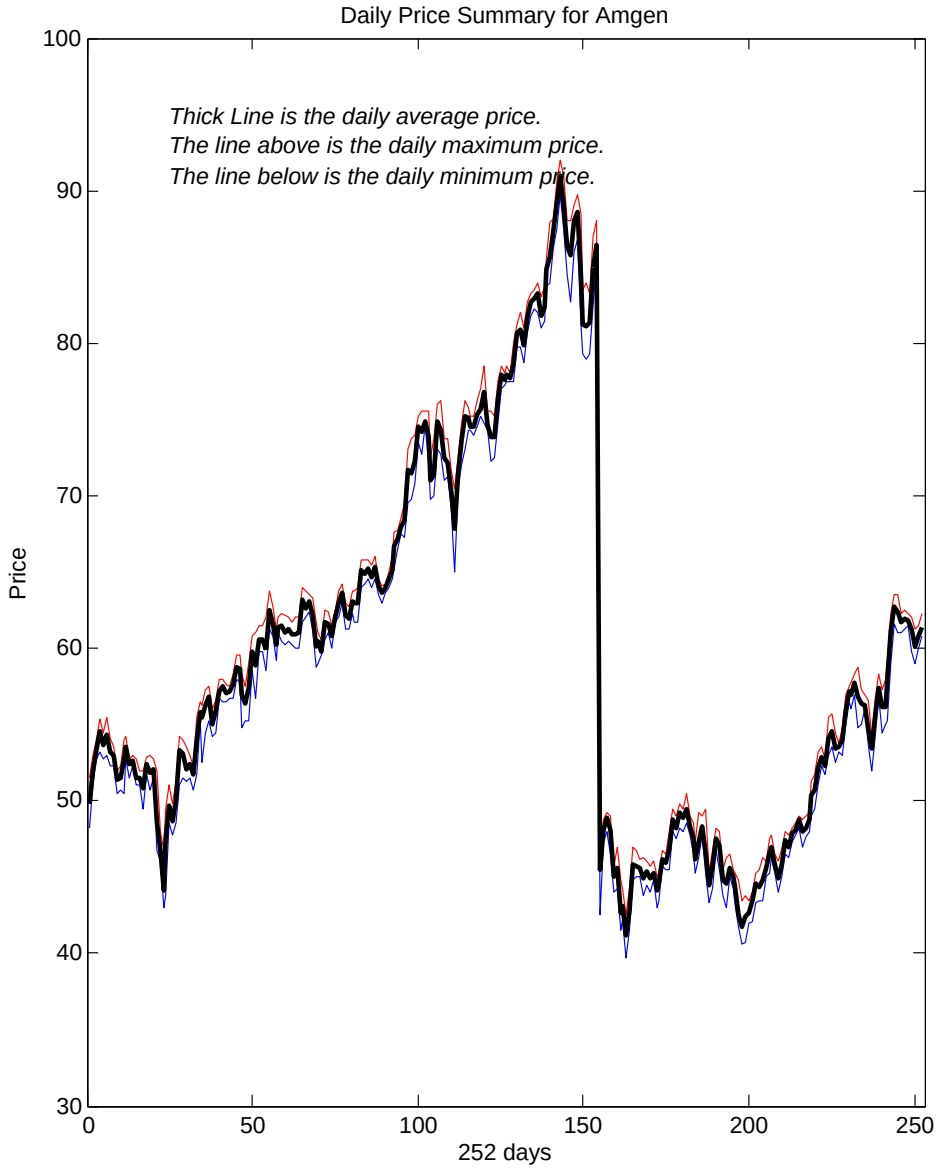


Figure 5.9: Amgen Trading Days Price Summary

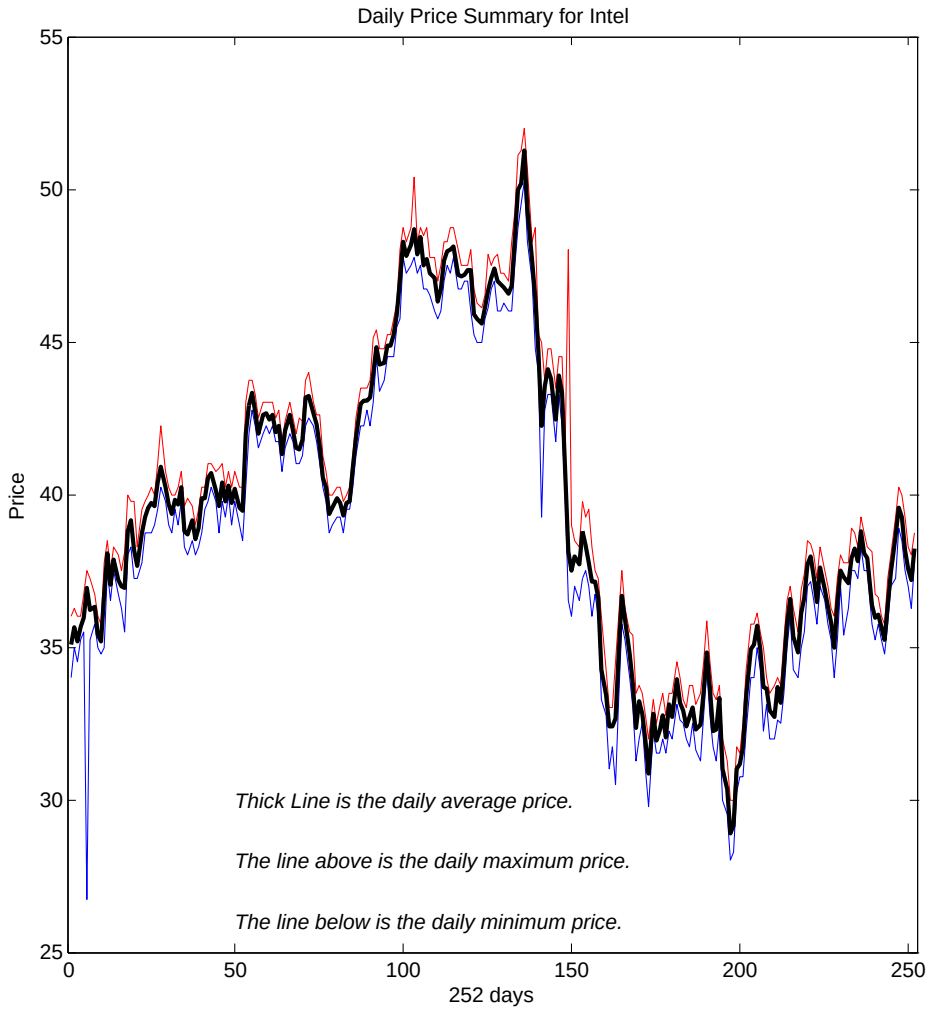


Figure 5.10: Intel Trading Days Price Summary

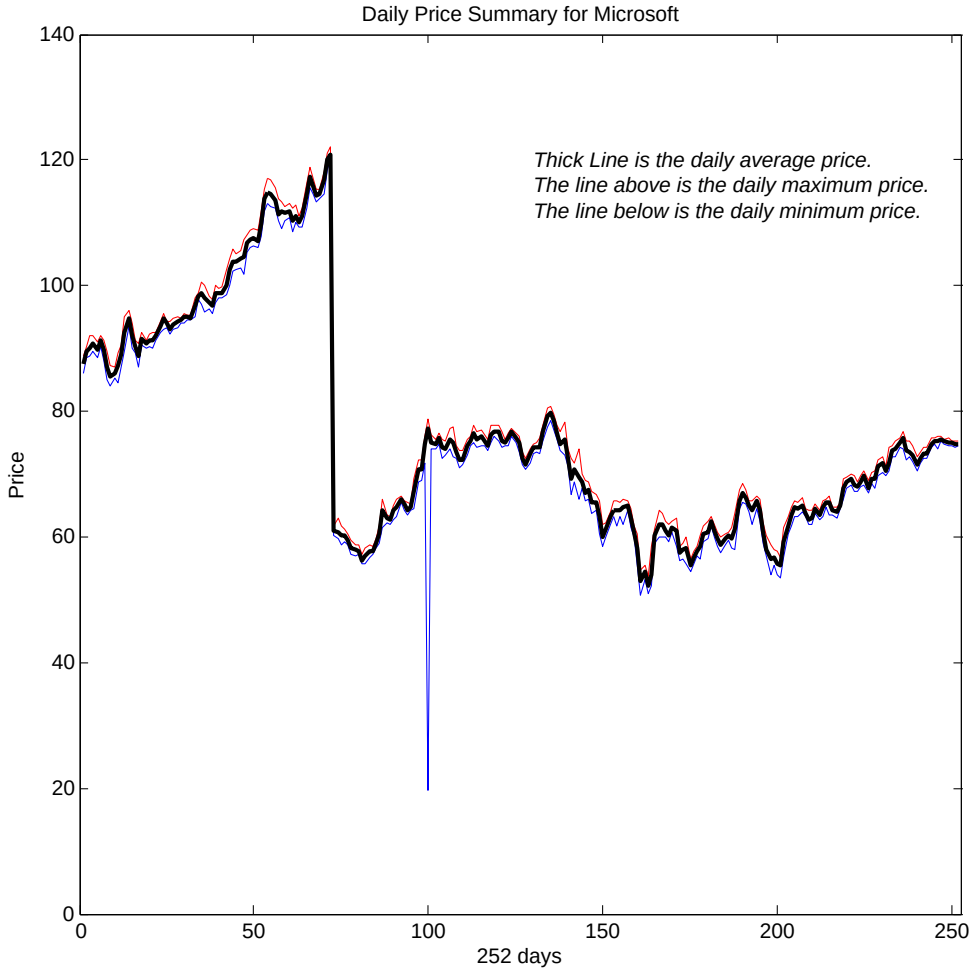


Figure 5.11: Microsoft Trading Days Price Summary

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