Indian Statistical Institute, Bangalore
B. Math II, Second Semester, 2019-20
Mid-semester Examination, Statistics II

28.02.20 **Maximum Score 100** **Duration: 3 Hours**

1. (8+2) Suppose U, V, W are independent normal variables, with U and V being $N(\mu, 1)$ and W being $\mathcal{N}(0, 1)$. Let $X_1 = U + W$ and $X_2 = V + W$. In other words, a common error of measurement W contaminates both U and V. Show that the joint distribution of (X_1, X_2) belongs to a one parameter Exponential family. What is the natural parameter?

2. (5+5+5) Let $X_1, \cdots, X_n$ be a sample from a population with density $p(x, \theta)$ given by
$$p(x, \theta) = \frac{1}{\sigma} exp(-\frac{x-\mu}{\sigma}), \quad \mu \leq x < \infty$$
Here $\theta = (\mu, \sigma)$ with $-\infty < \mu < \infty, \sigma > 0$.
   (a) Find a sufficient statistic for μ when σ is known.
   (b) Find a sufficient statistic for σ when μ is known.
   (c) Find a sufficient statistic for θ.

3. (7+6+7)
   (a) Let $\mathcal{U}$ denote the class of all unbiased estimators of $q(\theta)$ and $\mathcal{Z}$ denote the class of all statistics with expectation zero. Prove that $U \in \mathcal{U}$ has minimum variance in $\mathcal{U}$ iff $\mathrm{E}(UZ) = 0 \ \forall Z \in \mathcal{Z}$.
   (b) Suppose $S(X)$ is sufficient for θ. Let $T(X)$ be an estimator for $q(\theta)$. Let $T_1(X) = E[T(X)|S(X)]$. Show that the mean square error of $T_1(X)$ is smaller than or equal to that of $T(X)$.
   (c) Suppose $S(X)$ is complete sufficient for θ. Let $S_1(X)$ be a function of $S(X)$ which is unbiased for $q(\theta)$. Then show that $S_1(X)$ is UMVUE for $q(\theta)$.

4. (15) Consider the regression model:
$$y_i = bx_i + e_i, 1 \leq i \leq n,$$
where x_i's are fixed non-zero real numbers and e_i's are independent random variables with mean zero and equal variance. Consider estimators of the form $\sum_{i=1}^{n} a_i y_i$ (where a_i 's are non random real numbers) that are unbiased for b. Show that the least squares estimator of b has the minimum variance in this class of estimators.

5. (8+2) Suppose $(X_1, \cdots, X_n) \sim i.i.d.Unif([0, \theta])$. Show that the Pareto prior, given below as π, is a conjugate family of distributions and find the posterior mean under Pareto(α, β) prior.
$$\pi(\theta|\alpha, \beta) = \frac{\alpha\beta^{\alpha}}{\theta^{\alpha+1}}, \theta \geq \beta > 0$$
Note: $E(\theta|\alpha, \beta) = \frac{\alpha\beta}{\alpha-1}$ for the above distribution.

6. (7+8) Let $\theta > 0$ be an unknown parameter, and $X_1, X_2, \cdots, X_n$ be a random sample from the distribution with density
$$f(x) = 2x/\theta^2, \quad 0 \leq x \leq \theta.$$
Find the maximum likelihood estimator of θ and its mean squared error.

7. (8+7) Let $X_1, \cdots, X_n$ be iid Bernoilli(p). Show that $\bar{X}$ attains the Cramer-Rao lower bound. Hence show that $\bar{X}$ is the UMVUE of p.