INDIAN STATISTICAL INSTITUTE

CHENNAI CENTRE

M.STAT I. 2015-16 Semester II

Large Sample Statistical Methods

Mid Semester Examination

Maximum Marks 100

Numbers in brackets indicate points for each question. Total marks is 110.
You can answer as much as you can. However, maximum you can score is 100.

Date: 04 March 2016 Duration: 3 hours

1 If F is discrete, then show that $F_n \Rightarrow F$ iff for each x in the support of F, $P(X_n = x) \to P(X = x)$. [15]

2 Let $X_n \Rightarrow X$ and $Y_n \xrightarrow{P} c$ where c is a constant. Show that $X_n + Y_n \Rightarrow X + c$. [15]

3 Let $X_i, i = 1, 2, \cdots, n$ be independent Uniform$(0, 2\theta)$. Find the asymptotic distribution of $(Q_p + Q_{1-p})/2$ where Q_p is the p-th quantile. For what value of p is the asymptotic variance minimized? [15]

4 Suppose that $X_1, \cdots, X_n$ are iid with Cauchy distribution

$$f_\theta(x) = \frac{\theta}{\pi}\frac{1}{(x^2+\theta^2)}, \quad -\infty < x < \infty.$$

Prove that the likelihood equation has a unique solution $\hat{\theta}_n$ and that this solution maximizes the likelihood function. Find the asymptotic distribution of $\hat{\theta}_n$ [15]

5 Let $X_i, i = 1, 2, \cdots, n$ be independent Exponential$(\frac{1}{\lambda})$ random variables with density function

$$f_X(x) = \frac{1}{\lambda}e^{-\frac{x}{\lambda}} \quad 0 < x < \infty$$

Find the large sample distribution of $\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\lambda}\right)$. [10]

6 $X_1, X_2, \cdots, X_n$ are iid Bernoulli(p). The parameter of interest is the population variance. Find the asymptotic distribution of the unbiased estimator s^2 in the following cases [15]

(a) when $p \neq 1/2$

(b) when $p = 1/2$

7 Suppose that $X_1, \cdots, X_n$ are iid with density $f_{\theta_0}(x)$ for θ_0 in an open interval $\Omega \subset R$, the model is identifiable and the support does not depend on θ. Suppose that $\tilde{\theta}_n$ is any $\sqrt{n}$-consistent estimator of θ_0. We set

$$\delta_n = \tilde{\theta}_n - \frac{l'(\tilde{\theta}_n)}{l''(\tilde{\theta}_n)}$$

where $l(\theta)$ is the loglikelihood function. Show that

$$\sqrt{n}(\delta_n - \theta_0) \Rightarrow \mathcal{N}(0, \frac{1}{I(\theta_0)})$$

under certain regularity conditions on $l(\theta)$ and its derivatives. State the conditions explicitly. [15]

8 F is logistic: $F(x) = (1 + \exp(-x))^{-1}, -\infty < x < \infty$. Show that the limiting distribution of the maximum is Gumbel, that is, $P(X_{nn} - \log n < t) \to e^{-e^{-t}}$ as $n \to \infty$ where X_{kn} is the k-th order statistic in a sample of size n from the distribution F. [10]

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