INDIAN STATISTICAL INSTITUTE

CHENNAI CENTRE

M.STAT I. 2016-17 Semester II

Large Sample Statistical Methods

Final Examination

Numbers in brackets indicate points for each question. Total marks is 100.

Date: 2 May 2017 Duration: 3 hours

1 Suppose that$X_1, \cdots, X_n$ are i.i.d. observations whose mgf exists for some $t > 0$. Show that $X_{(n)} = O_P(\log(n))$, where $X_{(n)}$ is the largest order-statistic. [15]

2 Give examples of the following and justify your claim in each case [10]

(a) A sequence of random variables $\{X_n\}$ that converge in distribution to X but not in probability.

(b) A sequence of random variables $\{X_n\}$ that converge in probability to X but not with probability one.

3 Consider the simple linear regression model [15]

$$y_i = \beta x_i + \epsilon_i \quad i = 1, \cdots, n$$

with slope zero. ϵ_i are iid with mean 0 and variance σ^2. Find the asymptotic distribution of $\hat{\beta}$, the least squares estimator of β under suitable assumptions on x_i, namely, $\overline{x_n} \to 0$, max $\frac{x_i}{\sum x_j^2} \to 0$, $\frac{1}{n} \sum x_j^2 \to t < \infty$.

4 Let $X_1, \cdots, X_n$ be iid according to the Cauchy distribution [20]

$$f_\theta(x) = \frac{\theta}{\pi} \frac{1}{x^2 + \theta^2} \quad -\infty < x < \infty$$

(a) Show that the likelihood equation has unique root $\hat{\theta_n}$ that maximizes the likelihood function.

(b) Find the asymptotic distribution of $\hat{\theta_n}$.

5 Let $\Pi_N = \{x_{N1}, \cdots, x_{NN}\}, N = 1, 2, ...,$ be a sequence of finite populations such that Π_N has mean μ_N and variance σ_N^2. Let $\bar{X}_{nN}$ denote the mean of a random sample of size n drawn without replacement from the population Π_N. [20]

(a) Express $\bar{X}_{nN}$ as a two-sample simple linear rank statistic.

(b) Using part (a) and the Wald-Wolfowitz theorem, state a central limit theorem for $\bar{X}_{nN}$ as $n, N \to \infty$. State and verify the conditions required for the theorem to hold.

6 Suppose $X_1 \cdots, X_n$ are iid $\mathcal{N}(\theta, 1)$ and the parameter of interest is $p = P(X_1 \leq a)$ for fixed known constant a. The UMVU of p is [20]

$$\delta_{1n} = \Phi\left(\sqrt{\frac{n}{n-1}}(a - \bar{X})\right).$$

An alternative nonparametric estimator is

$$\delta_{2n} = \frac{1}{n}(\text{number of} X_i \leq a).$$

Find the asymptotic relative efficiency of δ_{2n} with respect to δ_{1n}.

———*** xXx ***———