INDIAN STATISTICAL INSTITUTE
CHENNAI CENTRE
M.STAT First Year
2015-16 Semester II

Large Sample Statistical Methods
Final Examination

Points for each question is in brackets. Total Points 100.
Students are allowed to bring 4 pages (front and back) of hand-written notes
Duration: 3 hours

1. (15) If F is discrete, then show that $F_n \Rightarrow F$ iff for each x in the support of F, $P(X_n = x) \to P(X = x)$.

2. (10) A sequence of random variables X_n is defined as X_n=the number of trials required to obtain the first success when the probability of success in each trial is $1/n$. Find the asymptotic distribution of X_n/n.

3. (15) Consider the following sequence of $2n$ independent random variables:

$$Y_{i1}, Y_{i2} \sim \mathcal{N}(\mu_i, \sigma^2), \quad i = 1, \cdots, n.$$

Find the MLE of σ^2. Is the MLE consistent?

4. (15) Let $X_n \sim Bin(n, p)$.
   (a) Find the asymptotic distribution of $\sqrt{n}(X_n/n - p)$.
   (b) Find a consistent estimator of $p(1-p)$.
   (c) Use the results above and Slutsky's theorem to obtain approximate confidence intervals for p.

5. (15) Show that the Pearson chi-square statistic is a Wald statistic. Hence derive its asymptotic distribution under the null hypothesis.

6. (15) Dvoretzky, Kiefer, and Wolfowitz(1956) showed that when $X_1, \cdots, X_n$ are iid uniform(0,1), then there exists a finite positive constant C such that

$$P(D_n > d) \le C \exp\{-2nd^2\}$$

for all n, where D_n is the Kolmogorov Smirnov statistic. Using the above result show the following:
   (a) The result is true for $X_1, \cdots, X_n \overset{iid}{\sim} F$ for any distribution F.
   (b) $D_n \to 0$ with probability one.

7. (15) Consider the U-statistic $\frac{2}{n(n-1)} \sum_{1 \le i < j \le n} I(X_i + X_j > 0)$, where I denotes the indicator function. Find the asymptotic distribution of this statistic when F is symmetric about zero.