INDIAN STATISTICAL INSTITUTE

CHENNAI CENTRE

M.STAT I. 2013-14 Semester I

Large Sample Statistical Methods Final Examination

Numbers in [ ] denote points for each question. Total points is 110. You can answer any part of any question. However, the maximum you can score is 100.

Duration: 3 hours

1 [10] A sequence X_n of random variables converge in distribution to a random variable X.

(a) Construct random variable Y_n and Y such that $Y_n \stackrel{d}{=} X_n$, $Y \stackrel{d}{=} X$ and $Y_n \stackrel{wp1}{\longrightarrow} Y$. (No proof required. Just define the random variables.)

(b) Show that when $Y_n \stackrel{wp1}{\longrightarrow} Y$, for a continuous function g, $g(Y_n) \stackrel{wp1}{\longrightarrow} g(Y)$.

(c) Conclude that $g(X_n)$ converge in distribution to $g(X)$

2 [15] Let $X_1, \cdots, X_n$ be iid according to Poisson distribution.

(a) Find the asymptotic relative efficiency of δ_{1n}=(number of $X_i = 0)/n$ to $\delta_{2n} = \exp\{-\bar{X}_n\}$ as estimators of $e^{-\lambda}$.

(b) Are any of them asymptotically efficient? Justify your answer.

3 [20] Let $X_1, \cdots, X_n$ are independent Bernoulli(p). The parameter of interest is the population variance.

(a) Find the asymptotic distribution of the unbiased estimator s^2 and the mle $m_2 = (n-1)s^2/n$ when $p \neq 1/2$,

(b) Find the asymptotic distribution of s^2 and m_2 when $p = 1/2$.

4 [20] In the context of hypothesis testing, answer the following:

(a) Show that the likelihood ratio test is consistent.

(b) Consider the shrinking set of alternatives $\theta_{1n} = c/\sqrt{n}$, where c is a constant, and null $\theta_0 = 0$. Suppose the test statistics has asymptotic normal distribution given by $\frac{\sqrt{n}(T_n - \mu(\theta_{1n}))}{\sigma(\theta_{1n})} \Rightarrow \mathcal{N}(0,1)$ for some μ and σ where μ is differentiable at 0 and σ is continuous at 0. Show that the tests that reject for large values of T_n and are asymptotically of level α have asymptotic power $1 - \Phi(z_\alpha - h\mu'(0)/\sigma(0))$.

5 [15] The observations Y_i have gamma distribution $\Gamma(\gamma, 1/\tau)$. The parameter of interest is $1/\tau$ and τ has the conjugate prior density $\Gamma(g, \alpha)$. Determine the limit distribution of the posterior mean and verify that it is asymptotically efficient.

continued on back

6 [20] Consider the U-statistic

$$U_n = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} X_i X_j$$

for estimating $\theta = \mu^2$ where $X_1, \cdots, X_n$ are iid with mean μ and variance σ^2.

(a) Find the projection $\hat{U}_n$ of U_n explicitly

(b) Show that $\sqrt{n}(\hat{U}_n - \theta) \Rightarrow \mathcal{N}(0, 4\mu^2\sigma^2)$.

(c) Express $R_n = U_n - \hat{U}_n$ as a U-statistic and show that $E(R_n^2) = O(n^{-2})$

(d) Conclude that $\sqrt{n}(U_n - \theta) \Rightarrow \mathcal{N}(0, 4\mu^2\sigma^2)$.

7 [10] Answer the following

(a) Show that, for the class of continuous F's, the exact distribution of the Kolmogorov-Smirnov statistic does not depend on F.

(b) Define D_{n+} as the one-sided Kolmogorov-Smirnov distance. Show that nD_{n+}^2 is asymptotically distributed as Exponential(2).

$X = \sqrt{n}(s^2 - \sigma^2) \Rightarrow N(0, 4(1-2p)^2)$.

$m_2 = \frac{n-1}{n} s^2$.

$X = \sqrt{n}\left(\frac{n}{n-1} m_2 - \sigma^2\right)$

$= \sqrt{n} \cdot \left[\frac{n}{n-1}(m_2 - \sigma_2) + \frac{1}{n-1}\sigma^2\right]$

$= \left(\frac{n}{n-1}\right) Y + \frac{\sqrt{n}}{n-1}\sigma^2$.

$Y = \left(\frac{n-1}{n}\right) X - \frac{\sigma^2}{\sqrt{n}}$

$X = n(s^2 - \sigma^2) \Rightarrow \frac{1}{4}(\chi^2 - 1)$

$n\left(\frac{n}{n-1} m_2 - \sigma^2\right)$

$= \frac{n}{n-1}\left[n(m_2 - \sigma^2) + \sigma^2\right]$

$= \frac{n}{n-1} Y + \frac{n}{n-1}\left(\frac{1}{4}\right)$

$\Rightarrow \frac{1}{4}(\chi^2 - 1) + \frac{1}{4} = \frac{1}{4}\chi^2$.

$\mu_4 =$

$n[\mu^2 + \sigma^2] - n^2\left(\mu^2 + \frac{\sigma^2}{n}\right)$

$\hat{U}_n - \theta = 2\mu(\bar{X} - \mu)$

$\sqrt{n}(\bar{X} - \mu) \Rightarrow N(0, \sigma^2)$