INDIAN STATISTICAL INSTITUTE

CHENNAI CENTRE

M.STAT First Year

2014-15 Semester I

Large Sample Statistical Methods

Back paper Examination

Points for each question is in brackets. Total Points 100.
Students are allowed to bring 2 pages (front and back) of hand-written notes
Duration: 3 hours

1. (15) Suppose $X_1, \cdots, X_n$ are iid $U(\theta - 1/2, \theta + 1/2)$. Consider the one sample Wilcoxon statistic given by

$$W = \frac{1}{\binom{n}{2}} \sum_{i<j} I(X_i + X_j > 0)$$

for testing the hypothesis $\theta = 0$. Obtain the asymptotic distribution of W under the null hypothesis using the theory of U-statistics.

2. (15) Consider the simple linear regression model

$$y_i = \beta x_i + \epsilon_i \quad i = 1, \cdots, n$$

with slope zero. ϵ_i are iid with mean 0 and variance σ^2. Find the asymptotic distribution of $\hat{\beta}$, the least squares estimator of β under suitable assumptions on x_i, namely, $\overline{x_n} \to 0$, $\max \frac{x_i}{\sum x_j^2} \to 0$, $\frac{1}{n}\sum x_j^2 \to t < \infty$.

3. (15) Let $X_1, \cdots, X_n$ be iid according to the Cauchy distribution

$$f_\theta(x) = \frac{\theta}{\pi} \frac{1}{x^2 + \theta^2} \quad -\infty < x < \infty \tag{1}$$

   (a) Show that the likelihood equation has unique root $\hat{\theta_n}$ that maximizes the likelihood function.

   (b) Find the asymptotic distribution of $\hat{\theta_n}$.

4. (15) Let $X_i \sim \text{Bin}(p)$ and $T_n = \bar{X_n}$.

   (a) Find the variance stabilizing transformation h of T_n.

   (b) Find the asymptotic distribution of $\sqrt{n}(T_n(1 - T_n) - p(1 - p))$.

5. (15) Give examples of the following and justify your claim in each case

   (a) A sequence of random variables $\{X_n\}$ that converge in distribution to X but not in probability.

   (b) A sequence of random variables $\{X_n\}$ that converge in probability to X but not with probability one.

6. (15) State and prove Polya's theorem.

7. (10) A sequence of random variables $\{X_n\}$ is defined as X_n=the number of trials required to obtain the first success when the probability of success in each trial is $1/n$. Find the asymptotic distribution of X_n/n.